

Sparsity and Model Selection

Tool: Subset selection, LASSO

The Analytics Edge: Modern day datasets often have many predictor variables that might help predict an output of interest. For example in cancer diagnosis, we need to identify which genes (among many possibilities) can predict cancer. Likewise in economics, different types of variables have been proposed by researchers to predict the economic growth of countries. In these settings, it is important to find which variables are relevant. Techniques for subset selection play a major role here to identify sparse predictive models.

Overview

We will focus on some more recent ideas for regression and classification that have been particularly designed for problems with larger datasets and many predictor variables. Our interest in these problems is in the predictive power of the models where the objective is to get good out of sample (test set) predictions but there are many possible predictor variables. While it is possible to get good in sample predictions by getting more complicated models, the concern is that there is a chance of overfitting the data.

1. Simpler models often tend to work better for out of sample predictions and so we will penalize models for excessive model complexity.
2. With the increase in computational power, we can use the data by partitioning it into training, validation and test sets. The training set is used to estimate the model parameters. The validation set is used to do model selection while the test set is the evaluation set on which we will simply evaluate or check how the model performs.

Bias-variance tradeoff

We will discuss this in the context of linear regression. Suppose we have a training set with observations $\{(x_i, y_i)\}$ for $i = 1, \dots, n$ where $x_i \in \mathbb{R}^p$ and $y_i \in \mathbb{R}$. Assume that the true model is given as:

$$y = f(x) + \epsilon,$$

where ϵ is the noise term with mean 0 and variance σ^2 .

Suppose we use a least squares method with linear regression to develop a model $\hat{f}(x)$ to approximate $f(x)$ where the predicted values from the model are $\hat{y} = \hat{f}(x)$. While we choose the fit of the model to minimize the sum of squared errors in the training set, we would also like it to do well in points outside the training set (generalizable to the test set).

Suppose (x_0, y_0) is a random test observation that might not necessarily be observed in the training set. We would really like to have a model, that would give a small value for

$$E \left[(y_0 - \hat{f}(x_0))^2 \right]$$

where the expectation is taken over all possible realizations from the true model. A key equality:

$$E \left[(y_0 - \hat{f}(x_0))^2 \right] = \text{Variance}[\hat{f}(x_0)] + \sigma^2 + E \left[(f(x_0) - \hat{f}(x_0))^2 \right].$$

The test mean squared error is decomposed into three terms:

1. Variance of the estimator
2. Variance of the error term (irreducible error)
3. Square of the bias of the estimator.

A complex model will typically have high variance of the estimator but low bias. On the other hand a simple model will have low variance of the estimator but a high bias. The key is to find the right balance between simplicity and complexity.

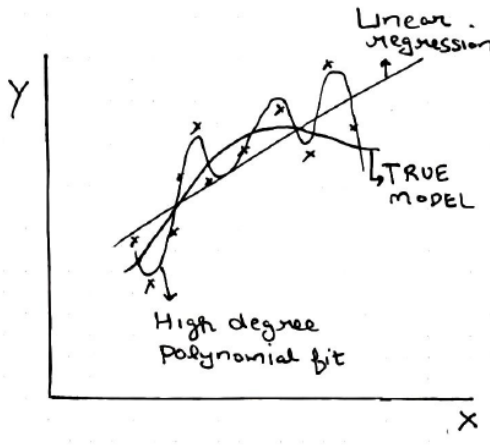


Figure 0.1: Linear regression (low variance but high bias), High degree polynomial regression (high variance but low bias)

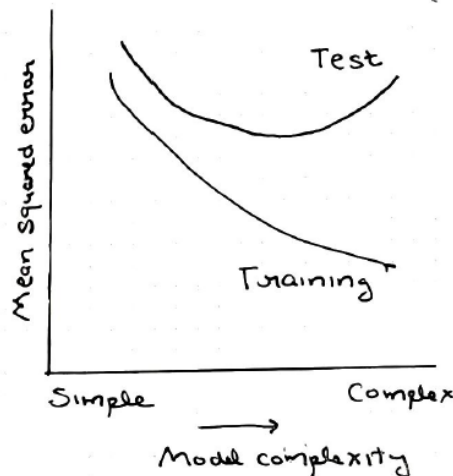


Figure 0.2: As the model gets more complex, we would get lower training set error. However the test error often has a U-shaped behavior.

k-Fold Cross Validation

1. Divide the data randomly into k subsets (folds) of (roughly) equal size.
2. Start with the first fold as a validation set and use the remaining $k - 1$ folds to fit the model.
3. Compute the error of the fitted model in the held-out fold.
4. Repeat steps 2 and 3 by using the second, third and so on folds as the hold-out fold with the remaining $k - 1$ folds to fit the model.
5. Average the error across all the k fitted models to estimate the cross-validation error.

Some of the commonly used choices for k are 5 or 10.

When we set $k = n$, this reduces to the leave one out cross validation scheme.

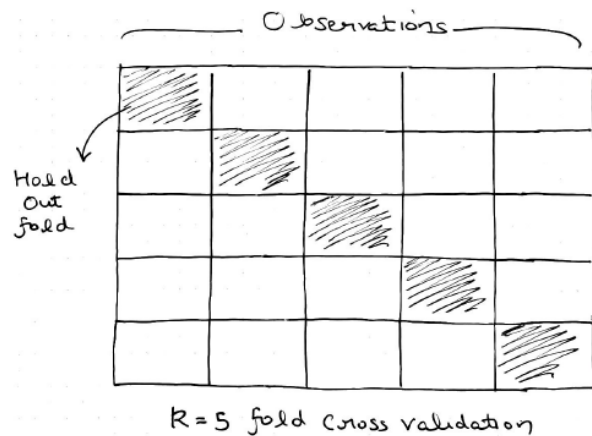


Figure 0.3: k-fold cross validation.

Subset selection

In classical statistical prediction, we have:

$$\text{Number of observations } n \gg \text{Number of predictors } p$$

However, there are many problems in the era of the big data age where:

$$\text{Number of observations } n < \text{ or } \approx \text{Number of predictors } p$$

For example in cancer diagnosis in terms of genes, the number of genes is very large and one needs to check among them to identify the genes that predict the chances of getting cancer. Note that when $n < p$ or $n \approx p$, there is a significant risk of over-fitting and there might be no longer a unique fit too. In such cases, one needs to decide which variables are important in making predictions and drop those variables that are less useful. We are also interested in identifying from a large set of variables, a much smaller subset that has the strongest effects on the response (this is fundamentally a combinatorial problem).

Question: Given a set of observations $\{(x_i, y_i)\}$ for $i = 1 \dots, n$ with $x_i \in \mathbb{R}^p$, what is the best subset of predictors for the output?

A possible algorithm is provided below:

Algorithm:

1. Let M_0 denote a null model with no predictors (only the intercept). This model would simply predict the mean of the training set for each observation for a linear regression model (or the fraction of ones in logistic regression).
2. For $k = 1, \dots, p$, do:
 - (a) Fit all $\binom{p}{k}$ models that contains exactly k predictors and the intercept.
 - (b) Pick the best among these models by choosing the model with the minimum sum of squared errors for linear regression or maximum log-likelihood for logistic regression. Call this model M_k
3. Choose the best model among M_0, M_1, \dots, M_p using cross-validation errors or adjusted R-squared (linear regression) or AIC (logistic regression).

The complexity of this algorithm is exponential since we are solving for $O(2^p)$ linear or logistic regressions.

Forward stepwise selection

We discuss a computationally efficient algorithm to solve this problem in comparison to the best subset selection but this method is not guaranteed to solve the problem to optimality. The method is based on a greedy algorithm.

1. Let M_0 denote a null model with no predictors (only the intercept). This model would simply predict the mean of the training set for each observation (or the fraction of ones in logistic regression).
2. For $k = 1, \dots, p$, do:
 - (a) Fit all models that augment the predictors in model M_{k-1} with exactly one more predictor (a total of $p - k + 1$ models are fit in step k).
 - (b) Pick the best among these models by choosing the model with the minimum sum of squared errors for linear regression or maximum log-likelihood for logistic regression. Call this model M_k
3. Choose the best model among M_0, M_1, \dots, M_p using cross-validation errors or adjusted R-squared (linear regression) or AIC (logistic regression).

The complexity of this algorithm is much lower since we are solving for $O(p^2)$ linear or logistic regressions. On the other hand there is no guarantee that the greedy algorithm will always find the optimal solution for any fixed k .

Least absolute shrinkage and selection operator (LASSO)

In standard linear regression, the problem we solve is:

$$\min_{\beta_0, \beta_1, \dots, \beta_p} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_{i1} - \dots - \beta_p x_{ip})^2.$$

LASSO modifies the linear regression model by **accounting for model complexity** as follows where $\lambda \geq 0$ is a parameter:

$$\min_{\beta_0, \beta_1, \dots, \beta_p} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_{i1} - \dots - \beta_p x_{ip})^2 + \lambda \sum_{j=1}^p |\beta_j|.$$

Here $\lambda \geq 0$ is a tuning parameter that provides a tradeoff between fitting the data (first term) with model complexity (second term in terms of non-zero values of the beta coefficients).

1. When $\lambda = 0$, LASSO reduces to standard linear regression.
2. When $\lambda \uparrow \infty$, the second term dominates and LASSO will make all the beta coefficients for the predictor variables go to zero.

The LASSO method for linear regression was popularized in 1996 by the paper “Regression Shrinkage and Selection via the lasso” by Tibshirani. The paper has over 25000 citations as of August 2018. The objective coefficient in LASSO is convex and tries to roughly promote sparsity. One of the advantages of LASSO is that **since it is convex, the local optimum is the global optimum** and there are efficient ways to solve the problem to optimality. However this objective function is not differentiable unlike standard linear regression.

To choose the λ values, we use a grid of possible values and compute the cross-validation error for each value of λ . We can then choose the λ with the smallest cross-validation error. You can then refit the final model using all the observations for the selected value of λ .

Related formulation: Best subset selection problem

$$\begin{aligned} \min_{\beta_0, \beta_1, \dots, \beta_p} \quad & \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_{i1} - \dots - \beta_p x_{ip})^2 \\ \text{s.t.} \quad & \sum_{j=1}^p \mathbb{I}(\beta_j \neq 0) \leq k \end{aligned}$$

where $\mathbb{I}(\beta_j \neq 0) = 1$ if $\beta_j \neq 0$ and 0 otherwise. This can be reformulated as a quadratic integer optimization problem as follows:

$$\begin{aligned} \min_{\beta_0, \beta_1, \dots, \beta_p} \quad & \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_{i1} - \dots - \beta_p x_{ip})^2 \\ \text{s.t.} \quad & \sum_{j=1}^p z_j \leq k \\ & z_j \in \{0, 1\}, \quad j = 1, \dots, p \\ & -Mz_j \leq \beta_j \leq Mz_j, \quad j = 1, \dots, p, \end{aligned}$$

where M is a sufficiently large positive value.

Cross-country growth regression: Application in econometrics

Economists are interested in understanding the factors such as economic policy, political and other factors that are linked to the rate of economic growth. Economists have studied this problem by identifying a few factors at a time. For example some of these factors include the initial GDP, degree of capitalism, population growth, equipment investment - each of which has been proposed to try and explain the rate of economic growth in countries. In an influential paper in 1991, “Economic growth in a cross-section of countries”, Quarterly Journal of Economics, Robert Barro used data from a various countries over the period 1960 to 1985 to show that the growth rate is positively related to school enrollment rates and negatively related to the initial 1960 level of real per capita GDP. For example, this might be partly argued by the fact that poorer countries with low ratios of capital to labor have higher growth rates. This paper had close to 20000 citations as of August 2018.

However there have been many such variables that have been proposed and so it is hard to often know which variables are really correlated with growth. While there is a proliferation of possible explanatory variables and little guidance from economic theory often on how to choose among these variables, it is possible to use ideas on subset selection from linear regression to aid towards this. The goal of these methods is to help and identify variables that are most important using cross-country growth data. Economists have often found that it is possible for one of the variables (say x_1) to be significant in a regression when x_2 and x_3 are included but it becomes insignificant when a new variable x_4 is included. In this case, it is useful to obtain some guidance on which variables are really important and to validate if such a dependence on economic growth is really robust.

We will use a dataset that was used in the paper: I just ran two million regressions by Sala-I-Martin and Model uncertainty in cross country growth regression by Fernandez et. al. The dataset has 41 possible explanatory variables with 72 countries. The data description is provided next. Note that if you try all 2^{41} possible combinations, it leads to around 2 trillion possibilities.

1. Country: Country name in abbreviation
2. y numeric: Economic growth 1960-1992 as from the Penn World Tables Rev 6.0
3. Abslat numeric: Absolute latitude
4. Spanish numeric: Spanish colony dummy
5. French numeric: French colony dummy
6. Brit numeric: British colony dummy
7. WarDummy numeric: War dummy
8. LatAmerica numeric: Latin America dummy
9. SubSahara numeric: Sub-Sahara dummy
10. OutwarOr numeric: Outward Orientation
11. Area numeric: Area surface
12. PrScEnroll numeric: Primary school enrolment
13. LifeExp numeric: Life expectancy
14. GDP60 numeric: Initial GDP in 1960
15. Mining numeric: Fraction of GDP in mining
16. EcoOrg numeric: Degree of capitalism
17. YrsOpen numeric: Number of years having an open economy
18. Age numeric: Age
19. Buddha numeric: Fraction Buddhist
20. Catholic numeric: Fraction Catholic
21. Confucian numeric: Fraction Confucian
22. EthnoL numeric: Ethnolinguistic fractionalization
23. Hindu numeric: Fraction Hindu
24. Jewish numeric: Fraction Jewish
25. Muslim numeric: Fraction Muslim
26. PrExports numeric: Primary exports 1970
27. Protestants numeric: Fraction Protestants
28. RuleofLaw numeric: Rule of law
29. Popg numeric: Population growth
30. WorkPop numeric: workers per inhabitant
31. LabForce numeric: Size of labor force
32. HighEnroll numeric: Higher education enrolment
33. PublEduPct numeric: Public education share
34. RevnCoup numeric: Revolutions and coups
35. PolRights numeric: Political rights
36. CivLib numeric: Civil liberties
37. English numeric: Fraction speaking English
38. Foreign numeric: Fraction speaking foreign language
39. RFEXDist numeric: Exchange rate distortions
40. EquipInv numeric: Equipment investment
41. NequipInv numeric: Non-equipment investment
42. stdBMP numeric: stand. dev. of black market premium
43. BIMktPm numeric: black market premium

Figure 0.4: Data description.