Kelvin-Helmholtz Instability

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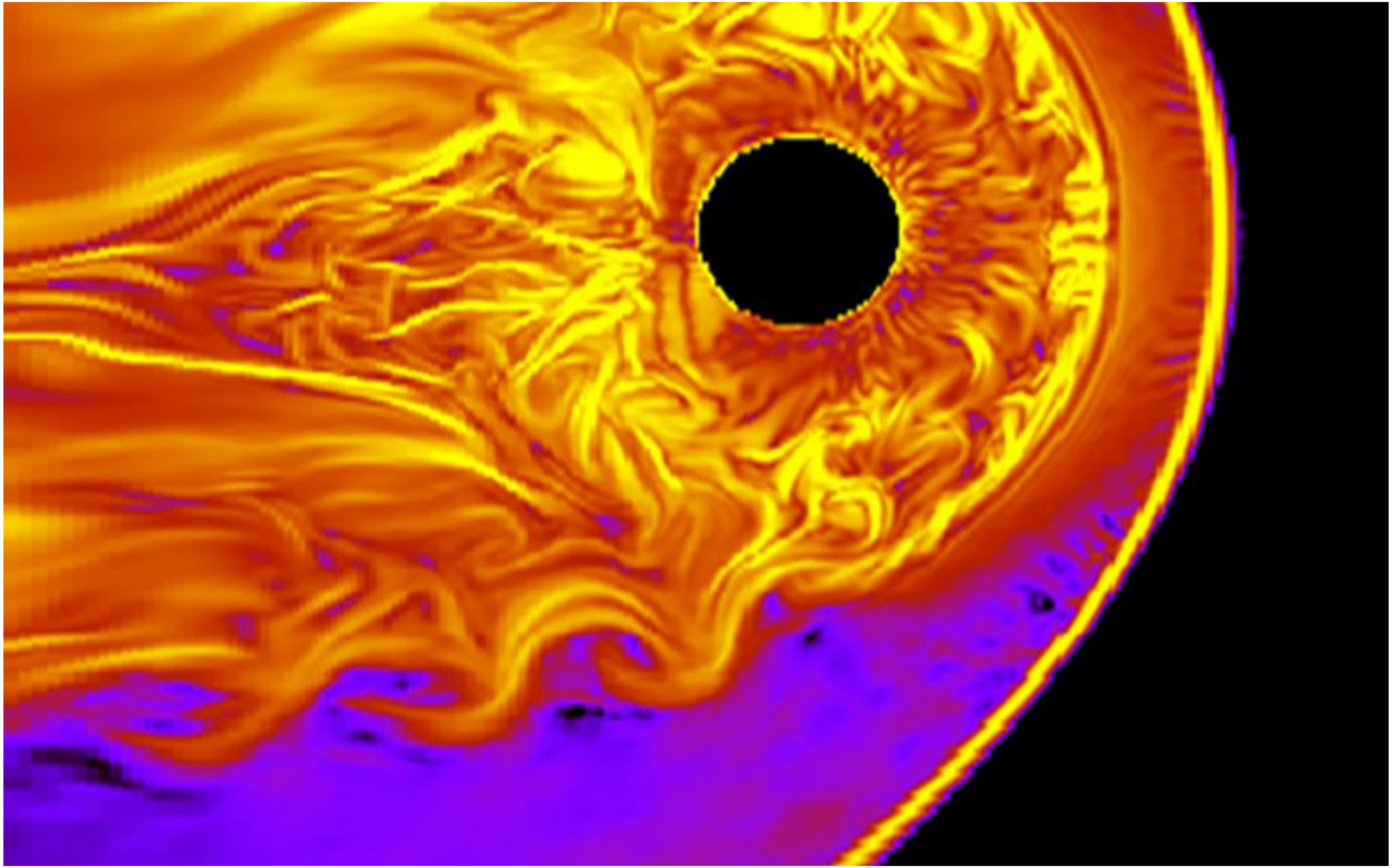
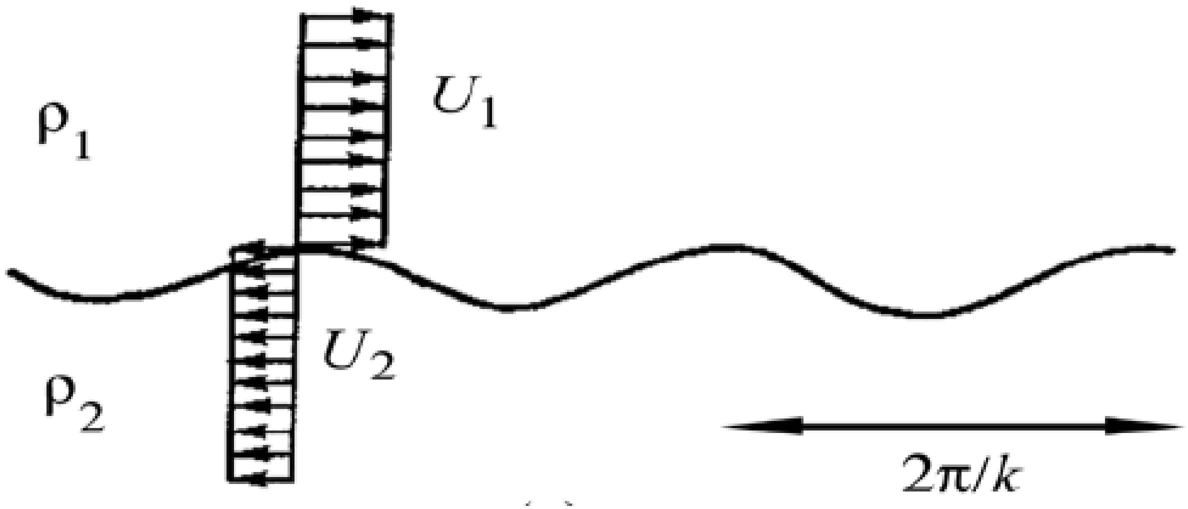


Introduction

- Two fluids with different velocities flow parallel to each other
- Solid oscillating line represents their interaction surface
- For the purpose of the calculations, the discontinuity of the velocity is allowed



Introduction (cont.)

- For the fluid on top:

$$\vec{q}_1 = \nabla\Phi_1, \quad \text{and} \quad \nabla^2\Phi_1 = 0, \quad z > 0 \tag{5.2.1}$$

- With Bernoulli's equation:

$$\frac{\partial\Phi_1}{\partial t} + \frac{(\nabla\Phi_1)^2}{2} = -\frac{p_1}{\rho_1} + C_1 \tag{5.2.2}$$

- For the fluid on the bottom:

$$\vec{q}_2 = \nabla\Phi_2, \quad \text{and} \quad \nabla^2\Phi_2 = 0 \quad z < 0 \tag{5.2.3}$$

- With Bernoulli's equation:

$$\frac{\partial\Phi_2}{\partial t} + \frac{(\nabla\Phi_2)^2}{2} = -\frac{p_2}{\rho_2} + C_2 \tag{5.2.4}$$

Kinematic Condition

- Let there be a disturbance on the interface whose vertical displacement is $z = \zeta(x, t)$. And let the interface $F = z - \zeta(x, t) = 0$. On the surface the kinematic conditions are:

$$\frac{\partial F}{\partial t} + \nabla\Phi_1 \cdot \nabla F = 0 \tag{5.2.5}$$

$$\frac{\partial F}{\partial t} + \nabla\Phi_2 \cdot \nabla F = 0 \tag{5.2.6}$$

- Which can be written as:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \Phi_1}{\partial x}\frac{\partial \zeta}{\partial x} = \frac{\partial \Phi_1}{\partial z}, \quad z = \zeta, \tag{5.2.7}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \Phi_2}{\partial x}\frac{\partial \zeta}{\partial x} = \frac{\partial \Phi_2}{\partial z}, \quad z = \zeta, \tag{5.2.8}$$

Dynamic condition

- Using Bernoulli's equation for both fluids and that the pressures of both fluids must be equal at z = ζ, the dynamic condition can be written as:

$$p_1 = p_2, \quad \text{or} \quad z = \zeta. \tag{5.2.9}$$

$$\rho_1 \left(\frac{\partial \Phi_1}{\partial t} + \frac{(\nabla \Phi_1)^2}{2} - C_1 \right) = \rho_2 \left(\frac{\partial \Phi_2}{\partial t} + \frac{(\nabla \Phi_2)^2}{2} - C_2 \right), \quad \text{or} \quad z = \zeta. \tag{5.2.10}$$

- With boundary conditions:

$$\nabla \Phi_1 \to U_1 \vec{i}, \quad z \to \infty, \tag{5.2.11}$$

$$\nabla \Phi_2 \to U_2 \vec{i}, \quad z \to -\infty. \tag{5.2.12}$$

Linearization

- Now considering infinitesimal disturbances:

$$\Phi_1 = U_1 x + \phi_1, \quad and \quad \Phi_2 = U_2 x + \phi_2, \tag{5.2.14}$$

- With:

$$\nabla^2 \phi_1 = 0, \quad z > 0, \tag{5.2.15}$$

$$\nabla^2 \phi_2 = 0, \quad z < 0, \tag{5.2.16}$$

- The kinematic condition becomes:

$$\frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x} = \frac{\partial \phi_1}{\partial z}, \quad z = 0, \tag{5.2.18}$$

$$\frac{\partial \zeta}{\partial t} + U_2 \frac{\partial \zeta}{\partial x} = \frac{\partial \phi_2}{\partial z}, \quad z = 0, \tag{5.2.19}$$

- And the dynamic condition becomes:

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} \right), \quad z = 0. \tag{5.2.20}$$

Normal Mode Analysis

- Assuming a sinusoidal disturbance and satisfying the Laplace equation:

$$\zeta = Ae^{i(kx-\omega t)} \tag{5.2.21}$$

$$\phi_1 = \bar{\phi}_1 \, e^{-kz} e^{i(kx-\omega t)} \tag{5.2.22}$$

$$\phi_2 = \bar{\phi}_2 \, e^{kz} e^{i(kx-\omega t)} \tag{5.2.23}$$

- After substituting the three expressions above into the kinematic condition equations and the dynamic condition equation, the three equations can be solved as a system of equations. This leads to an eigenvalue problem with the eigenvalue:

$$\omega = k\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm ik\frac{\sqrt{\rho_1 \rho_2}|U_1 - U_2|}{\rho_1 + \rho_2} \tag{5.2.26}$$

Normal Mode Analysis

- The disturbances are converted by the average velocity:

$$\frac{\partial \phi_1}{\partial x}|_0 + \frac{\partial \phi_2}{\partial x}|_0 = k(U_1 - U_2)\zeta = 2U\zeta$$

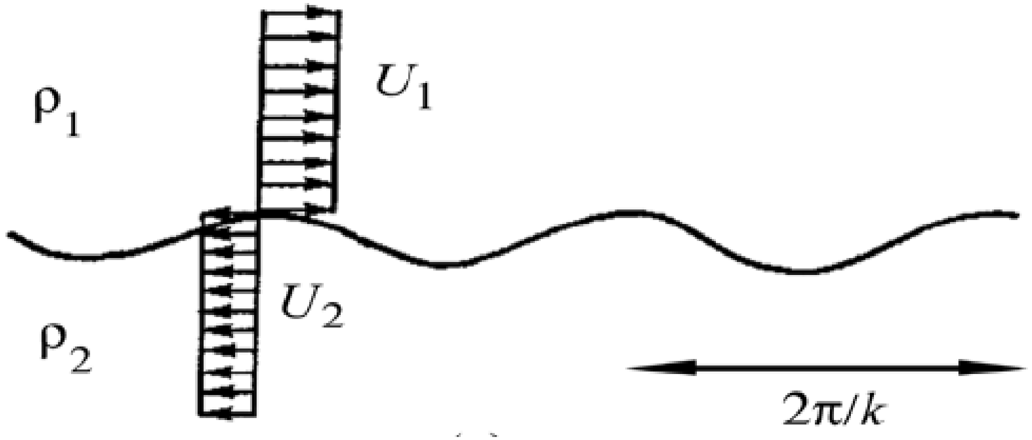
- Which is positive near the crests and negative near the troughs.

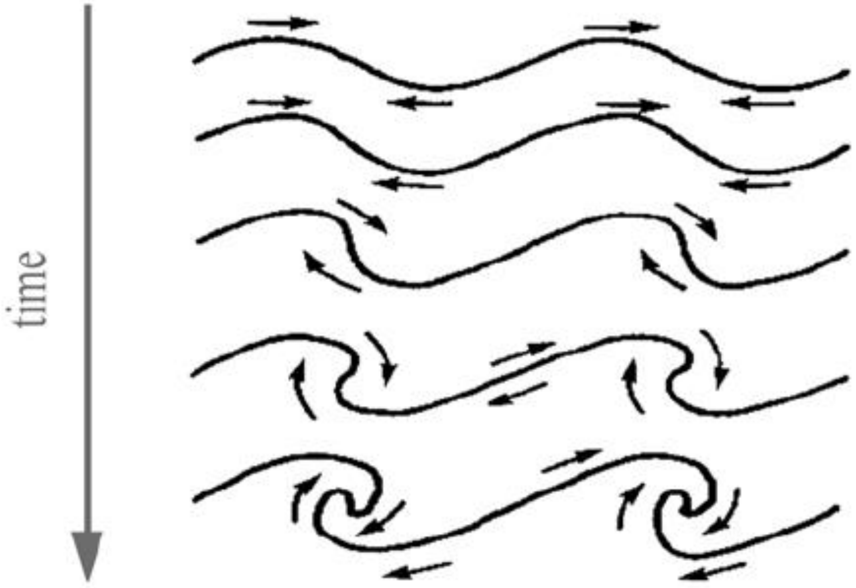
- The calculations up to this point have neglected the effects of gravity and surface tension. Considering these effects, the eigenvalue condition becomes:

$$\frac{\omega}{k} = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[-\frac{\rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + \frac{g}{k} \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} + \frac{Tk}{\rho_1 + \rho_2} \right]^{1/2}$$

Effect on the Flow

- Resulting structures travel at the average velocity of the upper and lower flows
- Unstable solutions grow in time





Laboratory Experiment

- Experiment featured on YouTube by “Sixty Symbols”

- https://www.youtube.com/watch?v=8foMwq2yJPo&t=240s

Questions?

Sources

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