Addressing Job Processing Variability Through Redundant Execution and Opportunistic Checkpointing: A Competitive Analysis

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ABSTRACT

The completion times of jobs in a computing cluster may be influenced by a variety of factors including job size and machine processing variability. In this paper, we explore online resource allocation policies, combining size-dependent scheduling with redundant execution and opportunistic checkpointing so as to minimize the overall job flowtime, i.e., average delays. We introduce a simplified model for the job service capacity of a computing cluster leveraging redundant execution/checkpointing. In this setting, we propose two resource allocation algorithms, SRPT+R and LAPS+R(β) subject to checkpointing overhead not exceeding the number of jobs which are processed. We provide new theoretical performance bounds for these algorithms: SRPT+R is shown to be $o(1/\epsilon)$ competitive under $(1+\epsilon)$-speed resource augmentation, while LAPS+R(β) is shown to be $o(\frac{1}{\beta\epsilon})$ competitive under $(2+2\beta+2\epsilon)$-speed resource augmentation.

Categories and Subject Descriptors

C.4 [**Performance of Systems**]: Design studies; Modeling techniques; I.1.2 [**Algorithms**]: Analysis of algorithms

Keywords

Job Scheduling, Redundancy, Optimization, Competitive Analysis, Potential Function, Dual-Fitting

1. INTRODUCTION

Job traces from large-scale computing clusters indicate that the job completion time can vary substantially [8, 9]. This variability has many sources including variability in job size and in machine processing capacity. Furthermore, the job profiles in production clusters are becoming increasingly diverse as small latency-sensitive jobs coexist with large batch processing applications which take hours to months to complete [47]. Moreover, with the size of today's computing clusters continuing to grow, component failures and resource contention have become a common phenomenon in cloud infrastructure [23, 30]. As a result, the machine service capacity may fluctuate significantly over the lifetime of a job. The same job may experience a far higher response time when executed at a different time on the same server [19]. These two dimensions of variability make efficient job scheduling for fast response time (also referred to as job flowtime) on large-scale computing clusters challenging.

To tackle variability in job size, various schedulers have been proposed, which can provide efficient resource sharing among heterogeneous applications. Widely deployed schedulers to-date include the Fair scheduler [3] and the Capacity scheduler [2]. It is well known that the SRPT scheduler (Shortest Remaining Processing Time) is optimal for minimizing the overall/ average job flowtime [18] on a single machine in the clairvoyant setting. As such, many works have aimed to extend SRPT scheduling to yield efficient scheduling algorithms in the multiprocessor setting with the objective of reducing job flowtimes for different systems and programming frameworks [20, 32, 33, 49]. Under SRPT, job sizes are known to the job scheduler upon arrival and smaller job are given priority. When only the distribution of job sizes is known, it has been shown in [4] that, Gittins index-based policy is optimal for minimizing the expected job flowtime under the Poisson Arrival Process again on a single server. The Gittins index depends on knowing the service given to-date to each job and gives priority to the job with the highest index. If the job size distribution belongs to the New-Better-than-Used-in-Expectation (NBUE) class, the Gittins index policy reduces to the First-Come-First-Served discipline. If the job size distribution is of the Decreasing-Hazard-Rate (DHR) class, this policy reduces to the Least-Attained-Service scheme. Extensions of the the Gittins index policy to the multiple-processor setting is not available.

To deal with component failures and resource contention, computing clusters can exploit redundant execution wherein multiple copies of the same job execute on available machines until the first completes. With redundancy, it is expected that one copy of the same job may complete quickly avoiding long completion times. The Google MapReduce system has shown that redundancy can decrease the average job flowtime by 44% [16]. Many other cloud computing systems apply simple heuristics exploiting redundancy and they also proven to be efficient for reducing job flowtimes via practical deployments [1, 7, 9, 13, 16, 28, 48].

Recently, researchers have started to investigate the effectiveness of scheduling redundant copies from a queuing perspective [14, 19, 35, 36, 39, 42]. These works assume a specific distribution of the job execution time where jobs follow the same distribution. However, they do not directly account for whether job response time variability is due to the variability in job size or in machine

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processing speed. Indeed, if there is no variability in machine service capacity, scheduling multiple copies of the same job may not help at all and redundancy is a waste of resource.

With the aforementioned observations in mind, in this paper, we explore the impact of variability in both job size and machine service capacity. We consider algorithms which can prioritize job scheduling and dynamically vary the number of redundant copies so as to minimize the overall job flowtime. We propose a simple model for a cluster's service capacity accounting for redundant execution. To take full advantage of redundancy, we also allow opportunistic checkpointing [34]. Upon checkpointing, the state of the redundant copy which has made the most progress is propagated and cloned to other copies. Therefore, all the redundant copies can be in the same state and proceed to execute based on the new state after checkpointing. This allows the system to opportunistically exploit variability in machine service speed. Checkpointing is a valuable tool in parallel scheduling systems to preempt, migrate and perform dynamic partitioning [40] on its active jobs. In our design, checkpointing occurs only upon job arrival, to or departure from the system, so as to limit overheads.

Most previous work studying job scheduling assume that clusters are working in the non-multitasking mode, i.e., a server (CPU Core) can only serve one job at any time. However, multitasking is a reasonable model of current scheduling policies in CPUs, web servers, routers, etc [15, 41, 43]. In a multitasking cluster, a server may run multiple jobs simultaneously and jobs can share resources with different proportions. In this paper, we will also study scheduling algorithms, which determine checkpointing times, the number of redundant copies between successive checkpoints as well as the fraction of resource share in both the multitasking and non-multitasking modes.

Our Results

In a non-multitasking cluster, we propose the SRPT+R Algorithm where redundancy is used only when the number of active jobs is less than the number of servers. For clusters allowing multitasking, we design the LAPS+R(β) Algorithm, which shares resources among a fixed fraction of the active jobs with priority given to jobs which arrived most recently. This paper has made the following technical contributions, which are also summarized in Fig. 1:

- After reviewing the related work in Section 2, in Section 3, we propose a framework to model the impact of machine processing variability in a system exploiting redundancy/checkpointing
- We apply a new dual fitting framework in Section 4 to show that SRPT+R without multitasking is $(1+\epsilon)$-speed, $o(\frac{1}{\epsilon})$-competitive in terms of the overall job flowtime. To the best of our knowledge, this is the first work to analyze the performance bound of scheduling algorithms with redundancy via a dual fitting approach.
- We show in Section 5 that the LAPS+R(β) Algorithm is $(2+2\beta+2\epsilon)$-speed, $o(\frac{1}{\beta\epsilon})$-competitive by adopting the potential function argument. Moreover, our setting of the potential function is quite different from that in the previous work, e.g., [17].

In our analysis, we assume resource augmentation [29], which is necessary to achieve lower bounds for job scheduling on multiprocessors.

2. RELATED WORK

In this section, we begin by giving a brief introduction to existing job schedulers. Then, we review the related work on redundant execution in large-scale computing clusters.

The design of job schedulers for large-scale computing clusters is currently an active research area [11, 12, 32, 33, 46, 49]. In particular, several works have derived performance bounds on algorithms geared at minimizing the total job completion time by formulating an approximate linear programming problem [11, 12, 46]. By contrast, other works, e.g., [32, 33, 49] derive performance bounds for algorithms with respect to the the total job flowtime. Leonardia et al. show in [31] that there is a strong lower bound on any online randomized algorithm for the job scheduling problem on multiple unit-speed processors with the objective to minimize the overall job flowtime. Based on this lower bound, [32, 33, 49] extended the SRPT scheduler to design algorithms that minimize the overall flowtime consisting of multiple small tasks with precedence constraints. [11, 12, 32, 33, 46, 49] are conducted in the clairvoyant setting, i.e., the job size is known once the job arrives. For the non-clairvoyant setting, [24–26] designed several multitasking algorithms in which machines are allocated to all active jobs and priority is given to the most-recently-arrived jobs.

All of the above studies assume accurate knowledge of machine service capacity and do not address dynamic scheduling of redundant copies for a job.

Production clusters and big data computing frameworks have adopted various approaches which use redundancy to speedup job processing. The initial Google MapReduce system launched redundant copies when a job is close to its completion [16]. Hadoop adopts another solution, i.e., LATE, which schedules a redundant copy for an active task only if its estimated progress rate is below certain threshold [1]. By comparison, Microsoft Mantri [9] schedules a new copy for a running task if its progress is slow and the total resource consumption is expected to decrease once a new redundant copy is made.

Researchers have proposed different schemes to take advantage of redundancy. Chen *et al.* propose a smart redundancy scheme in [13] to accurately estimate the task progress rate and launch redundant copies accordingly. Ananthanarayanan *et al.* propose in [7] to use redundancy for very small jobs when the extra loading is not high. As an extension to [7], Ananthanarayanan further present GRASS [8], which carefully schedules redundant copies for approximation jobs. Moreover, Ren *et al.* propose Hopper [38] to allocate computing slots based on the virtual job size, which is larger than the actual size. Hopper can immediately schedule a redundant copy once the progress rate of a task is detected to be slow. No performance characterization has been developed for these heuristics.

Recently, Xu *et al.* developed several optimization frameworks to study the design of scheduling algorithms utilizing redundancy [44, 45]. The proposed algorithms in [44] require the knowledge of exact distribution of the task response time. Xu *et al.* also analyze performance bounds of the proposed algorithm which extends the SRPT Scheduler in [45] using a resource augmentation argument. A fundamental limitation is that their resultant bounds are not scalable as they increase linearly in the number of machines.

Another body of work related to this paper focuses on the study of scheduling algorithms for jobs with intermediate parallelizability. In these works, e.g., [5, 10, 17, 22, 27], jobs are parallelizable and the service rate can be arbitrarily scaled. In particular, Samuli *et al.* present several optimal scheduling policies for different capacity regions in [5] but for the transient case only. [10], [17] and [22] propose similar algorithms which allow multitasking for jobs wherein priority is given to the most-recently-arrived-jobs. These works de-

Algorithm	Minimize	Machine Speed	Bound	Technique	Support Multitasking	Account for Machine Variability	Account for Job Parallelizability
SRPT+R	$\sum_{j=1}^{N} f_j$	$1+\epsilon$	$o(\frac{1}{\epsilon})$	Dual Fitting	N	Y	N
LAPS+R(β)	$\sum_{j=1}^{N} f_j$	2+ $2\beta+2\epsilon$	$o(\frac{1}{\beta\epsilon})$	Potential Function	Y	Y	N
WLAPS(β), [Gupta10]	$\sum_{j=1}^{N} f_j^2$	$2+\epsilon$	$o(\frac{1}{\beta\epsilon})$	Potential Function	Y	N	Y
LAPS(β), [Edmonds12]	$\sum_{j=1}^{N} f_j$	$1+\beta+\epsilon$	$o(\frac{1}{\beta\epsilon})$	Potential Function	Y	N	Y
Intermediate -SRPT, [Moseley14]	$\sum_{j=1}^{N} f_j$	1	$o(1)\cdot 4^{1/(1-\alpha)}\cdot \log P$	Potential Function	Y	N	Y

Figure 1: Summary of the results for different algorithms where f_j denotes the job flowtime. Note that only our proposed algorithms, i.e., the SRPT+R and LAPS+R(β) address the issue of variability in machine service capacity. The other three algorithms, i.e., WLAPS(β), LAPS(β) and Intermediate-SRPT, consider that job can be parallelizable, and the speedup curve is $\Gamma(x)=x^{\alpha}$, where x denotes the number of allocated processors. The bound for the last algorithm, i.e., Intermediate-SRPT contains a factor, $\log P$, where P is the ratio between the largest and the smallest job size.

velop competitive performance bounds with respect to the total job flowtime by adopting potential function arguments. Sungjin *et al.* also provide a competitive bound for the SRPT-based parallelizable algorithm under multitasking in [27]. One limitation of [27] is that the resulting bound is potentially very large [1]. By contrast, this paper is motivated by the setting where there is variability in machine service capacity and we consider algorithms in both the multitasking and non-multitasking modes.

3. SYSTEM MODEL

Consider a computing cluster which consists of M servers (CPU cores or machines) where the servers are indexed from 1 to M. Job j arrives to the cluster at time a_j and the job arrival process, $(a_1, a_2, \cdots, a_N)$, is an arbitrary deterministic time sequence. In addition, job j has a workload which requires p_j units of time to complete when processed on a machine at *unit* speed. Job j completes at time c_j and its flowtime f_j, is denoted by $f_j = c_j - a_j$. In this paper, we focus on minimizing the overall job flowtime, i.e., $\sum_{j=1}^{N} f_j$.

The service capacity of machines are assumed to be identically distributed random processes with stationary increments. To be specific, we let $S_i = (S_i(t)|t \geq 0)$ be a random process where $S_i(t,\tau) = S_i(\tau) - S_i(t)$ denote the *cumulative* service delivered by machine i in the interval $(t,\tau]$. The service capacity of a machine has unit mean speed and a peak rate of Δ, so for all $\tau > t \geq 0$, we have $S_i(t,\tau] \leq (\tau - t)\cdot\Delta$ almost surely and $\mathbb{E}\big[S_i(t,\tau]\big] = \tau - t$.

In this paper, we assume preemption and migration are allowed, namely, the scheduler can preempt a running job and subsequently resume execution on the same or other servers later on. Below, we will first introduce a service model where each server can only serve one job at a time. Subsequently, we will discuss a service model allowing multitasking, i.e., where a server can execute multiple jobs simultaneously.

3.1 Job Processing in a Non-Multitasking Cluster

As discussed in the introduction, in this paper, our aim is to mitigate the impact of service variability by (possibly) varying the number of redundant copies of a job along with appropriate checkpointing of service progress. In fact, we shall make the following assumption across the system:

[1] In [27], when $\alpha \to 0$, the competitive bound approaches ∞.

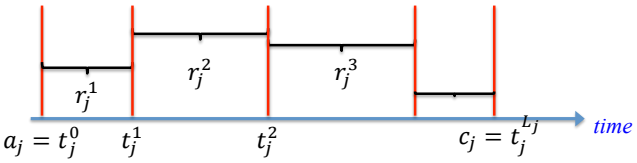

Figure 2: The service process of job j.

ASSUMPTION 1. *A job j can be checkpointed only if there is an arrival to, or departure from, the system which under the scheduling and redundancy policy results in a change in the number of redundant copies for the job.*

REMARK 1. *We refer to Assumption 1 as a scalability assumption. On the one hand, it limits the checkpointing overheads in the system. On the other, if the number of redundant copies of a job is modified, checkpointing should be applied to make the most of the allocated resources, i.e., start the processing of the possibly redundant copies at the most advanced state amongst the previously executing copies.*

As illustrated in Fig. 2, one can view the service process of job j in a non-multitasking cluster by dividing its service period (from its arrival to its completion) into several subintervals, i.e., $\big\{(t_j^{k-1}, t_j^k]\big\}_k$ where t_j^k denotes the time when the kth checkpointing of job j occurs. The job arrival and completion times are also considered as checkpointing times, i.e., $t_j^0 = a_j$ and $t_j^{L_j} = c_j$ if job j experiences (L_j+1) checkpoints. During in $(t_j^{k-1}, t_j^k]$, job j is running on r_j^k redundant servers. Thus, together $\boldsymbol{t_j} = \big(t_j^k | k = 0, 1, \cdots, L_j\big)$ and $\boldsymbol{r_j} = \big(r_j^k | k = 1, 2, \cdots, L_j\big)$ capture the checkpoint times and the scheduled redundancy for job j.

We will let $g(r,t)$ denote the cumulative service delivered to a job on r redundant machines and checkpointed at the end of an interval of duration t. Clearly, $g(r,t)$ is equivalent to the amount of work processed by the redundant copy which has made the most progress. In this paper, we make the following assumption for $g(r,t)$:

ASSUMPTION 2. *We shall model (approximate) the cumulative service capacity under redundant execution, $g(r,t)$, by its mean, i.e.,*

$$g(r,t) = \mathbb{E}\Big[\max_{i=1,2,\cdots,r} S_i(0,t]\Big]. \quad (1)$$

REMARK 2. *Assumption 2 essentially replaces the service capacity of the system with the mean but accounts for the mean gains one might expect when there are redundant copies executed.*

The following lemmas illustrate two important properties of $g(r,t)$:

LEMMA 1. *For a fixed t, $\{g(r,t)\}_r$ is a concave sequence, i.e., $g(r,t) - g(r-1,t) \leq g(r-1,t) - g(r-2,t)$.*

Lemma 1 states that the marginal increase of the mean service capacity in the number of redundant executions is decreasing.

LEMMA 2. *For all $r \in \mathbb{N}$ and $r \leq M$, $g(r,t) \leq \min\{\Delta t, rt\}$.*

Lemma 2 states that the mean service capacity under redundant execution can grow at most linearly in the redundancy, rt, and is bounded by the peak service capacity of any single redundant copy, Δt.

Given Assumption 2, the last checkpoint time for job j, $t_j^{L_j}$, is also the completion time c_j and satisfies the following equation:

$$\sum_{k=1}^{L_j} g(r_j^k, t_j^k - t_j^{k-1}) = p_j. \quad (2)$$

In the sequel, we shall also make use of the speedup function, $h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t)$, defined as follows:

$$h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t) = \begin{cases} \frac{g(r_j^k, t_j^k - t_j^{k-1})}{t_j^k - t_j^{k-1}} & t \in (t_j^{k-1}, t_j^k], \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

The speedup function captures the speedup that redundant execution is delivering in a checkpointing interval relative to a job execution on a unit speed machine. (2) can be reformulated in terms of the speedup as follows:

$$\int_{a_j}^{c_j} h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, \tau) d\tau = p_j. \quad (4)$$

REMARK 3. *Note that the speedup depends, not only on the number of redundant copies being executed, but also, on all the times when checkpointing occurs. In this sense, $h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t)$ is not a causal function. However, in the following sections, $h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t)$ will be a convenient notation to study competitive performance bounds for our proposed algorithms.*

3.2 Job Processing in a Multitasking Cluster

With multitasking, a server can run several jobs simultaneously and the service a job receives on a server is proportional to the fraction of processing resource it is assigned.

We will model a cluster allowing multitasking as follows. Comparing with the service model in Subsection 3.1, we include another variable x_j^k, to characterize the fraction of resource assigned to job j in the kth subinterval, i.e., $(t_j^{k-1}, t_j^k]$. Here, we assume that job j shares the same fraction of processing resource on all the machines on which it is being executed. Let $\boldsymbol{x_j} = \left(x_j^k | k = 1, 2, \cdots, L_j\right)$ and we define another speedup function, $\widetilde{h}_j(\boldsymbol{t_j}, \boldsymbol{x_j}, \boldsymbol{r_j}, t)$, as follows:

$$\widetilde{h}_j(\boldsymbol{t_j}, \boldsymbol{x_j}, \boldsymbol{r_j}, t) = \begin{cases} x_j^k \cdot h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t) & t \in (t_j^{k-1}, t_j^k], \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Paralleling (4), the completion time of job j, c_j must satisfy the following equation:

$$\int_{a_j}^{c_j} \widetilde{h}_j(\boldsymbol{t_j}, \boldsymbol{x_j}, \boldsymbol{r_j}, \tau) d\tau = p_j. \quad (6)$$

In the sequel, we will design and analyze algorithms in both the multitasking mode and the non-multitasking mode.

3.3 Competitive Performance Metrics

In this paper, we will study algorithms for scheduling, which involves determining checkpointing times, the number of redundant copies for jobs between successive checkpoints and in the multitasking setting the fraction of resource shares. Note that, when there is no variability in the machine's service capacity, our problem reduces to job scheduling on multiple unit-speed processors with the objective of minimizing the overall flowtime. This has been proven to be NP-hard even when preemption and migration are allowed and previous work [27,29] has adopted a resource augmentation analysis. Under such analysis, the performance of the optimal algorithm on M unit-speed machines is compared with that of the proposed algorithms on M δ-speed machines where $\delta > 1$.

The following definition characterizes the competitive performance of an online algorithm using resource augmentation.

DEFINITION 1. *[29] An online algorithm is δ-speed, c-competitive if the algorithm's objective is within a factor of c of the optimal solution's objective when the algorithm is given δ resource augmentation.*

In this paper, we also adopt the resource augmentation setup to bound the competitive performance of our proposed algorithms. With resource augmentation, the service capacity in each checkpointing interval under our algorithms is scaled by δ. Similarly, the value of the speedup functons, i.e., $h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t)$ and $\widetilde{h}_j(\boldsymbol{t_j}, \boldsymbol{x_j}, \boldsymbol{r_j}, t)$, under our algorithms is δ times that under the optimal algorithm of the same variables.

4. ALGORITHM DESIGN FOR CLUSTERS WITH NON-MULTITASKING PROCESSORS

In a non-multitasking cluster, each server can only serve one job at a time. Before going into the details of algorithm design, we first state the optimal problem formulation. For ease of illustration, we let $\boldsymbol{y}_j = (\boldsymbol{t_j}, \boldsymbol{r_j}, L_j)$ denote the *checkpointing trajectory* of job j and $\boldsymbol{y} = (\boldsymbol{y}_j | j = 1, 2, \cdots, N)$ that for all jobs. Moreover, let $\mathbb{1}(A)$ denote the indicator function that takes value 1 if A is true and 0 otherwise. The optimal problem formulation is as follows:

$$\min_{\boldsymbol{y}} \sum_{j=1}^{N} (c_j - a_j) \quad \text{(OPT)}$$

such that (a), (b), (c), (d) are satisfied

(a) Job completion: The completion time of job j, c_j, satisfies: $\int_{a_j}^{c_j} h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t) dt = p_j, \ \forall j$.

(b) Resource constraint: The total number of redundant executions at any time $t \geq 0$ is no larger than the number of machines, M, i.e., $\sum_{j:a_j \leq t} \sum_{k=1}^{L_j} r_j^k \cdot \mathbb{1}(t \in (t_j^{k-1}, t_j^k]) \leq M, \ \forall t$.

(c) Checkpoint trajectory: The number of checkpoints for each job is between 2 and $2N$ since there are $2N$ job arrivals and departures, i.e., $L_j \in \{1, 2, \cdots, 2N-1\}$. Given a feasible L_j, the checkpoint times of job j, $\boldsymbol{t}_j$, satisfy: $\boldsymbol{t}_j \in \mathcal{T}_j^{L_j+1}$

where $\mathcal{T}_j^{L_j+1} = \{(t_0, t_1, \cdots, t_{L_j}) \in \mathbb{R}^{L_j+1} | a_j = t_0 < t_1 < \cdots < t_{L_j} = c_j\}$. Moreover, the number of redundant copies must be integer valued, i.e., $\boldsymbol{r}_j \in \mathbb{N}^{L_j}$.

(d) Checkpointing overhead constraint: Job checkpoints must satisfy Assumption 1, i.e., for $0 \le k \le L_j$, $t_j^k \in \{a_j\}_j \cup \{c_j\}_j$ and for $2 \le k \le L_j$, $r_j^{k-1} \ne r_j^k$.

Since the OPT problem is NP-Hard, we will propose a heuristic for job scheduling, named, SRPT+R, which is a simple extension of the SRPT scheduler [18].

4.1 SRPT+R Algorithm

Let $p_j(t)$ denote the amount of the unprocessed work for job j at time t and $n(t)$ denote the number of active jobs at time t. In this section, we will assume without loss of generality that jobs have been ordered such that $p_1(t) \le p_2(t) \le \cdots \le p_{n(t)}(t)$.

At a high level, the algorithm works as follows. When $n(t) \ge M$, the M jobs with smallest $p_j(t)$, i.e., Job 1 to M are each assigned to a server while the others wait. If $n(t) < M$, the job with the smallest $p_j(t)$, i.e., Job 1, is scheduled on $M - \lfloor \frac{M}{n(t)} \rfloor (n(t)-1)$ machines while the others are scheduled on $\lfloor \frac{M}{n(t)} \rfloor$ machines each. Here, $\lfloor x \rfloor$ represents the largest integer which does not exceed x.

The corresponding pseudo-code is exhibited in Algorithm 1 shown in the panel below. In Algorithm 1, *checkpointing* denotes the set of jobs needs to be checkpointed and $M(t)$ denotes the set of machines which will be reassigned new jobs at time t.

Algorithm 1: SRPT+R Algorithm

while *A job arrives to or departure from the system* **do**
  Sort the jobs in the order such that $p_1(t) \le p_2(t) \le \cdots \le p_{n(t)}(t)$ and count the number of redundant copies being executed for job j, r_j ;
  Initialize $M(t)$ to be the set of idle machines, $numofcheckpoints = 0$ and $checkpointing = \emptyset$;
  for $j = 1, 2, \cdots, n(t)$ **do**
    Compute the number of redundant executions, $r_j(t)$ based on SRPT+R scheduling for job j;
    if $r_j(t) \ne r_j$ **then**
      $numofcheckpoints$++;
      $M(t) =$ $M(t) \cup \{$the set of machines job j is running on$\}$;
      $checkpointing = checkpointing \cup \{j\}$;
  for $k = 1, 2, \cdots, numofcheckpoints$ **do**
    $j = checkpointing[k]$;
    Checkpoint job j and assigns its redundant executions to $r_j(t)$ machines which are uniformly chosen at random from $M(t)$;

4.2 Performance guarantee for SRPT+R and our techniques

We will let OPT and SR denote the cost, i.e., the overall job flowtime, achieved by an optimal scheduling policy, and our proposed SRPT+R Algorithm respectively. Our main result, characterizing the competitive performance of SRPT+R, is given in the following theorem:

THEOREM 1. *SRPT+R is $(1+\epsilon)$-speed, $o(\frac{1}{\epsilon})$-competitive with respect to the total job flowtime.*

We shall prove Theorem 1 by adopting the online dual fitting approach. The first step is to formulate a minimization problem whose cost serves as an approximation to the optimal cost, OPT with a guarantee that it is within a constant of OPT. We then formulate the dual problem for the approximation and exploit the fact that a feasible solution to this dual problem gives a lower bound on its cost, which in turn is a constant times the cost of the proposed algorithm.

Dual fitting was first developed by [6, 21] and is now widely used for the analysis of online algorithms [25, 26]. In particular, [6] and [25, 26] address linear objectives, and use the dual-fitting approach to derive competitive bounds for traditional scheduling algorithms without redundancy. By contrast, [21] focus on a convex objective in the multitasking mode. By comparison, in our work, we include integer constraints associated with the non-multitasking mode. Moreover, our setting of dual variables is quite different from that in these works.

4.3 Proof of Theorem 1

To prove Theorem 1, we shall both approximate the objective of OPT relax Constraint (d) in OPT to obtain the following problem P1:

$$\min_{\boldsymbol{y}} \sum_{j=1}^{N} \int_{a_j}^{\infty} \frac{(t - a_j + 2p_j)}{p_j} \cdot h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t) dt \tag{P1}$$

$$\text{s.t.} \int_{a_j}^{c_j} h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t) dt \ge p_j, \quad \forall j,$$

$$\sum_{j: a_j \le t} \sum_{k=1}^{L_j} r_j^k \cdot \mathbb{1}(t \in (t_j^{k-1}, t_j^k]) \le M, \quad \forall t,$$

$$L_j \in \{1, 2, \cdots, 2N-1\}, \quad \boldsymbol{t_j} \in \mathcal{T}_j^{L_j+1}, \quad \boldsymbol{r_j} \in \mathbb{N}^{L_j}, \ \forall j.$$

The following lemma shows that the optimal cost of P1, denoted by $P1$, is not far from that achieved by the optimal policy, OPT.

LEMMA 3. *$P1$ is upper bounded by $\big(1+2\Delta\big)OPT$, i.e., $P1 \le \big(1+2\Delta\big)OPT$.*

Let $\boldsymbol{\alpha} = (\alpha_j | j = 1, 2, \cdots, N)$ and $\boldsymbol{\beta} = (\beta(t) | t \in \mathbb{R}^+)$ denote the Lagrangian dual variables corresponding to the first and second constraint in P1 respectively. The Lagrangian function associated with P1 can be written as shown in (7) with the dual problem for P1 given by:

$$\max_{\boldsymbol{\alpha} \ge \boldsymbol{0}, \boldsymbol{\beta} \ge \boldsymbol{0}} \min_{\boldsymbol{y}} \quad \Phi(\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \tag{D1}$$

$$\text{s.t.} \quad L_j \in \{1, 2, \cdots, 2N-1\}, \quad \boldsymbol{r_j} \in \mathbb{N}^{L_j}, \quad \boldsymbol{t_j} \in \mathcal{T}_j^{L_j+1}.$$

Applying weak duality theory for continuous programs [37], we can conclude that the optimal value to D1 is a lower bound for $P1$. Moreover, the objective of D1 can be reformulated in (8).

It is still difficult to solve D1 as it involves a minimization of a complex nonlinear objective function of integer valued variables. However, it follows from Lemma 2 that $r_j^k \ge \mathbb{1}(t \in (t_j^{k-1}, t_j^k]) \cdot h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t)$ for all j and $t \ge a_j$ thus, we have that,

$$\sum_{k=1}^{L_j} r_j^k \cdot \mathbb{1}(t \in (t_j^{k-1}, t_j^k]) \ge \sum_{k=1}^{L_j} \mathbb{1}(t \in (t_j^{k-1}, t_j^k]) \cdot h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t) = h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t), \tag{9}$$

where the last equality is due to that $h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t) = 0$ for $t > c_j = t_j^{L_j}$. Based on (9), it can be readily shown that the second term on the R.H.S of $\Phi(\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ in (8) is lower bounded by: $\int_0^\infty \sum_{j: a_j \le t} \Big[\Big(\frac{t-a_j}{p_j} + 2 - \alpha_j + \beta(t)\Big) \cdot h_j(\boldsymbol{t_j}, \boldsymbol{r_j}, t)\Big] dt$. As a

$$\Phi(\boldsymbol{y},\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{j=1}^{N}\int_{a_j}^{\infty}\frac{(t-a_j+2p_j)}{p_j}\cdot h_j(\boldsymbol{t_j},\boldsymbol{r_j},t)dt - \sum_{j=1}^{N}\alpha_j\big(\int_{a_j}^{\infty}h_j(\boldsymbol{t_j},\boldsymbol{r_j},t)dt - p_j\big) + \int_0^{\infty}\beta(t)\cdot\big(\sum_{j:a_j\le t}\sum_{k=1}^{L_j}r_j^k\cdot\mathbb{1}(t\in(t_j^{k-1},t_j^k]) - M\big)dt, \tag{7}$$

$$\Phi(\boldsymbol{y},\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_j\alpha_j p_j - M\int_0^{\infty}\beta(t)dt + \int_0^{\infty}\sum_{j:a_j\le t}\Big[(\frac{t-a_j}{p_j}+2-\alpha_j)h_j(\boldsymbol{t_j},\boldsymbol{r_j},t) + \beta(t)\sum_{k=1}^{L_j}r_j^k\cdot\mathbb{1}(t\in(t_j^{k-1},t_j^k])\Big]dt. \tag{8}$$

result, for a fixed α_j and $\beta(t)$ such that for all $t \ge a_j$,

$$\frac{t-a_j}{p_j} + 2 - \alpha_j + \beta(t) \ge 0, \tag{10}$$

the minimum of $\Phi(\boldsymbol{y},\boldsymbol{\alpha},\boldsymbol{\beta})$ can be attained by setting all $\boldsymbol{r_j}$ to $\mathbf{0}$ and $\boldsymbol{t_j} = (a_j, c_j)$. In this solution, there are no other checkpoints for job j other than the job arrival and departure.

Therefore, restricting $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to satisfy (10) would give a lower bound on D1 and result in the following optimization problem:

$$\begin{aligned} \max_{\boldsymbol{\alpha},\boldsymbol{\beta}} \quad & \sum_j\alpha_j p_j - M\int_0^{\infty}\beta(t)dt \qquad \text{(P2)}\\ s.t. \quad & \alpha_j - \beta(t) \le \frac{t-a_j}{p_j} + 2, \ \ \forall j, \ \ t \ge a_j,\\ & \alpha_j \ge 0, \ \ \forall j, \qquad \beta(t) \ge 0, \ \ \forall t. \end{aligned}$$

Based on Lemma 3, we conclude that $P2 \le P1 \le \big(1+2\Delta\big)OPT$ where $P2$ is the optimal cost for P2.

Next, we shall find a setting of dual variables, i.e., $(\boldsymbol{\alpha},\boldsymbol{\beta})$ that is feasible for P2. Moreover, the cost of this setting should relate to SR. To achieve this, we set the dual variables in the sequel based on another scheduling policy, SRPT, which uses a $(1+\epsilon)$-speed resource argumentation.

4.3.1 Setting of dual variables

Observe that SRPT+R and SRPT only differ when the number of active jobs is less than M and that when this is the case SRPT only assigns a single machine to each active job. Since SRPT+R maintains the same scheduling order and each job is scheduled with at least the same number of copies as SRPT, we conclude that the cost of SRPT, denoted by $SRPT$, is a lower bound for SR.

In this section, we let $n(t)$ and $p_j(t)$ denote the number of active jobs and the size of the remaining workload of job j under SRPT with $(1+\epsilon)$-speed resource argumentation respectively.

Let $\Theta_j = \{k : a_k \le a_j \le c_k\}$, i.e., the set of jobs that are active when job j arrives and $A_j = \{k \ne j : k \in \Theta_j \text{ and } p_k(a_j) \le p_j\}$, i.e., jobs whose residual workload upon job j's arrival is less than job j's processing requirement, and let $\rho_j = |A_j|$. Our setting of dual variables is illustrated in the following lemma.

LEMMA 4. *For all $j \in \{1,2,\cdots,N\}$ and $t \ge 0$, let*

$$\alpha_j = \frac{1}{(1+\epsilon)p_j}\sum_{k=1}^{\rho_j}\Big(\lfloor\frac{n(a_j)-k}{M}\rfloor - \lfloor\frac{n(a_j)-k-1}{M}\rfloor\Big)p_k(a_j) + \frac{1}{1+\epsilon}\Big(\lfloor\frac{n(a_j)-\rho_j-1}{M}\rfloor + 1\Big), \tag{11}$$

and

$$\beta(t) = \frac{1}{(1+\epsilon)M}n(t). \tag{12}$$

This setting is feasible for Problem P2, i.e., $\alpha_j, \beta(t) \ge 0$ and $\alpha_j - \beta(t) \le \frac{t-a_j}{p_j} + 2$.

4.3.2 Performance bound of SRPT+R

To bound the cost associated with the choices for the dual variables in Lemma 4, we first show a lemma, which quantifies the total job flowtime under SRPT in the transient case when there are no job arrivals after time t.

LEMMA 5. *Suppose there are $n(t)$ jobs in the system at time t such that their remaining workload are $p_1(t) \le p_2(t) \le \cdots, \le p_{n(t)}(t)$ and there are no further arrivals after time t, then, the overall remaining job flowtime under SRPT scheduling, $F(t)$, is given by:*

$$F(t) = \frac{1}{1+\epsilon}\sum_{j=1}^{n(t)}(\lfloor\frac{n(t)-j}{M}\rfloor + 1)p_j(t). \tag{13}$$

Based on Lemma 5, if job j never arrives to the cluster and the subsequent jobs do not enter the cluster, the overall remaining job flowtime at time a_j under SRPT is given by:

$$F_j'(a_j) = \frac{1}{1+\epsilon}\sum_{k=1}^{n(a_j)-1}(\lfloor\frac{n(a_j)-1-k}{M}\rfloor + 1)p_k(a_j). \tag{14}$$

By contrast, when job j arrives at time a_j but the subsequent jobs do not enter the cluster, the overall remaining job flowtime at time a_j under SRPT is given by:

$$\begin{aligned} F_j(a_j) = & \frac{1}{1+\epsilon}\sum_{k=1}^{\rho_j}(\lfloor\frac{n(a_j)-k}{M}\rfloor + 1)p_k(a_j)\\ & + \frac{1}{1+\epsilon}\Big(\lfloor\frac{n(a_j)-\rho_j-1}{M}\rfloor + 1\Big)p_j\\ & + \frac{1}{1+\epsilon}\sum_{k=\rho_j+1}^{n(a_j)}(\lfloor\frac{n(a_j)-k}{M}\rfloor + 1)p_k(a_j), \end{aligned} \tag{15}$$

therefore, one can view α_j as the difference of (15) and (14), i.e., the incremental increase of the overall job flowtime caused by the arrival of job j and divided by $(1+\epsilon)p_j$. As a result, $\sum_j p_j\alpha_j$ corresponds to the overall job flowtime under SRPT, i.e., $\sum_j\alpha_j p_j = SRPT$.

Moreover, $(1+\epsilon)M\beta(t)$ is the number of active jobs in the cluster at time t, so, $M\int_0^{\infty}\beta(t) = \frac{1}{1+\epsilon}SRPT$. Therefore, we have $\sum_j\alpha_j p_j - M\int_0^{\infty}\beta(t)dt = \frac{\epsilon}{1+\epsilon}SRPT$.

Based on Lemma 3, we conclude that $\frac{\epsilon}{1+\epsilon}SR \le \frac{\epsilon}{1+\epsilon}SRPT \le P2 \le P1 \le \big(1+2\Delta\big)\cdot OPT$. This implies $SR \le o(\frac{1}{\epsilon})OPT$ and completes the proof of Theorem 1. □

5. ALGORITHM DESIGN FOR MULTITASKING PROCESSORS

In this section, we design a scheduling algorithm for clusters supporting multitasking, so in addition to checkpoint times and redundancy, there is another decision set of variables, $\boldsymbol{x} = (\boldsymbol{x_j} : j = 1, 2, \cdots, N)$ where $\boldsymbol{x_j} = (x_j^k | k = 1, 2, \cdots, L_j)$, i.e, the fractions of resource shares to be allocated to each job during checkpointing intervals. To be specific, we propose the LAPS+R(β) Algorithm which is an extension of LAPS (the Latest Arrival Processor Sharing Algorithm). In LAPS, resources are shared among active jobs in the cluster and priority is given to those which arrived most recently. However, the LAPS Algorithm proposed in [17] only considers resource shares without redundant executions, our LAPS+R(β) Algorithm generalizes this to allow redundant copies of jobs to be varied dynamically. In this section, we assume, without loss of generality, that jobs have been ordered such that $a_1 \geq a_2 \geq \cdots \geq a_{n(t)}$, i.e., from the oldest to the most recently arrived.

Algorithm 2: LAPS+R(β) Algorithm

while *A job arrives to or departure from the system* **do**
  Sort the jobs in the order such that $a_1 \geq a_2 \geq \cdots \geq a_{n(t)}$ and count the number of redundant copies being executed for job j, r_j ;
  Initialize $M(t)$ to be the set of machines which are not fully occupied, $numofcheckpoints = 0$ and $checkpointing = \emptyset$;
  for $j = 1, 2, \cdots, n(t)$ **do**
    Compute the number of redundant executions, $r_j(t)$ and the fraction of resource share $x_j(t)$, based on the LAPS+R(β) scheduling policy for job j;
    if $r_j(t) \neq r_j$ **then**
      $numofcheckpoints$++;
      $M(t) = M(t) \cup \{$the set of machines job j is running on$\}$;
      $checkpointing = checkpointing \cup \{j\}$;
  for $k = 1, 2, \cdots, numofcheckpoints$ **do**
    $j = checkpointing[k]$;
    Checkpoint job j and assigns its redundant executions to $r_j(t)$ machines which are uniformly chosen at random from $M(t)$ with a resource share of $x_j(t)$;
    if *The machines that job j is executing on are fully occupied* **then**
      Remove these machines from $M(t)$;

5.1 LAPS+R(β) Algorithm

The algorithm depends on the parameter $\beta \in (0, 1)$. Say, $\beta = 1/2$, then the algorithm essentially schedules the $\frac{1}{2}n(t)$ most-recently-arrived jobs. If there are fewer than M such jobs, they are each assigned an roughly equal number of servers for execution without multitasking. If $\frac{1}{2}n(t) > M$, each job will roughly get a share of $\frac{M}{\frac{1}{2}n(t)}$ on some machine.

For a given number of active jobs $n(t)$, let parameter β, $z \in \mathbb{N}$, $\alpha \in \{0, 1, \cdots, M-1\}$ and $\gamma \in [0, 1)$ such that $\beta n(t) = zM + \alpha + \gamma$.

The LAPS+R(β) Algorithm operates as follows. At time t, if $z = 0$, jobs indexed from $(n(t) - \alpha)$ to $(n(t) - 1)$ are scheduled on $\lfloor \frac{M}{\alpha+1} \rfloor$ machines each, and Job $n(t)$ is scheduled on the remaining $(M - \alpha \lfloor \frac{M}{\alpha+1} \rfloor)$ machines. In this case, there is no multitasking. By contrast, if $z \geq 1$, jobs indexed from $(n(t) - zM - \alpha)$ to $(n(t) - 1)$ are each assigned a single machine and get a resource share of $\frac{1}{z+1}$. And, Job $n(t)$ is scheduled on $(M - \alpha)$ machines with a $\frac{1}{z+1}$ share of its resources.

The corresponding pseudo-code exhibited in Algorithm 2 is shown in the panel above. In Algorithm 2, $checkpointing$ denotes the set of jobs that need to be checkpointed and $M(t)$ denotes the set of machines which will be reassigned new jobs at time t.

5.2 Performance guarantee for LAPS+R(β) and our techniques

Let OPT and LR denote the cost of the optimal scheduling policy and LAPS+R(β) respectively. The main result in this section, characterizing the competitive performance of LAPS+R(β), is given in the following theorem:

THEOREM 2. *LAPS+R(β) is $(2+2\beta+2\epsilon)$-speed $o(\frac{1}{\beta\epsilon})$-competitive with respect to the total job flowtime.*

The dual fitting approach fails in this setting to prove Theorem 2, so we adopt the use of a potential function. The main idea of this method is to find a potential function which combines the optimal schedule and LAPS+R(β). Let $LR(t)$ and $OPT(t)$ denote the accumulated job flowtime under LAPS+R(β) with a $(2+2\beta+2\epsilon)$-speed resource augmentation and the optimal schedule, respectively. We define a potential function, $\Lambda(t)$, whose evolution may have jumps as well as continuous change. $\Lambda(t)$ should satisfy the following three properties:

- Boundary Condition: $\Lambda(0) = \Lambda(\infty) = 0$.
- Jump Condition: the potential function may have jumps only when a job arrives or completes under the LAPS+R(β) schedule, and if present, it must be decreased.
- Drift Condition: with a $(2 + 2\beta + 2\epsilon)$-speed resource augmentation, for any time t not corresponding to a jump, and some constant c_1 and c_2, we have that,

$$\frac{d\Lambda(t)}{dt} \leq -\frac{\beta\epsilon}{c_1} \cdot \frac{dLR(t)}{dt} + \frac{1}{c_2} \cdot \frac{dOPT(t)}{dt}. \qquad (16)$$

By integrating (16) and accounting for the negative jump and the boundary condition, one can see that the existence of such a potential function guarantees that $LR \leq \frac{c_1}{c_2} \cdot \frac{1}{\beta\epsilon} OPT$ under a $(2+2\beta+2\epsilon)$-speed resource augmentation.

Potential function is widely used to derive performance bounds with resource augmentation for online parallel scheduling algorithms e.g., [17, 27]. However, since we need to deal with redundancy and checkpointing, our proposal for the potential function is quite different from that in [17] and [27] where they only address sublinear speedup.

5.3 Proof of Theorem 2

To prove this theorem, we shall propose a potential function, $\Lambda(t)$, which satisfies all the three properties specified above.

5.3.1 *Defining potential function, $\Lambda(t)$*

Consider a checkpointing trajectory for job j under LAPS+R(β) and the optimal schedule, denoted by $(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j)$ and $(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*)$ respectively. Let $\psi^*(t)$ be the jobs that are still active at time t under the optimal scheduling and denote by $\psi(t)$ the set of jobs that are active under LAPS+R(β). Thus, we have that, $|\psi(t)| = n(t)$. Further, let $n_j(t)$ denote the number of jobs which are active at time t and arrive no later than job j under LAPS+R(β). Define the cumulative service difference between the two schedules for job j at time t, i.e., $\pi_j(t)$, as follows:

$$\pi_j(t) = \max\Big[\int_{a_j}^{t} \widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, \tau)d\tau - \int_{a_j}^{t} \widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, \tau)d\tau\ ,\ 0\Big], \tag{17}$$

Let $\delta = 2 + 2\beta + 2\epsilon$ and define

$$f(n_j(t)) = \begin{cases} 1 & \beta n_j(t) \le M, \\ \frac{M}{\beta n_j(t)} & \text{otherwise.} \end{cases} \tag{18}$$

Note that $f(n_j(t))$ takes the minimum of 1 and $\frac{M}{\beta n_j(t)}$ where the latter is roughly the total resource LAPS+R(β) would allocate to job j if $n_j(t)$ jobs were active at time t. We define a potential function, $\Lambda(t)$, whose evolution may have jumps as well as continuous change. Our potential function is given by

$$\Lambda(t) = \sum_{j \in \psi(t)} \Lambda_j(t), \tag{19}$$

where $\Lambda_j(t)$ is the ratio between (17) and (18), i.e., $\Lambda_j(t) = \frac{\pi_j(t)}{\delta \cdot f(n_j(t))}$.

5.3.2 *Changes in $\Lambda(t)$ caused by job arrival, departure and job processing*

Clearly, our potential function satisfies the boundary condition. Indeed, since each job is completed under LAPS+R(β), thus, $\psi(t)$ will eventually be empty, and $\Lambda(0) = \Lambda(\infty) = 0$.

Let us consider possible jump times. When job j arrives to the system at time a_j, $\pi_j(a_j) = 0$ and $f(n_j(t))$ does not change for all $k \neq j$. Therefore, we conclude that the job arrival does not change the potential function $\Lambda(t)$. When a job leaves the system under LAPS+R(β), $f(n_j(t))$ can only increase if job j is active at time t, leading to a decrease in $\Lambda_j(t)$. As a consequence, the job arrival or departure does not cause any increase in the potential function, $\Lambda(t)$, thus, the jump condition on the potential function is satisfied.

Beside job arrivals and departures under LAPS+R(β), there are no other events leading to changes in $f(n_j(t))$ and thus changes in $\Lambda_j(t)$ depend only on $\pi_j(t)$, see definition of $\Lambda_j(t)$. Specifically, for all $t \notin \{a_j\}_j \cup \{c_j\}_j$, we have that,

$$\frac{d\Lambda(t)}{dt} = \sum_{j \in \psi(t)} \frac{d\Lambda_j(t)}{dt} = \sum_{j \in \psi(t)} \frac{d\pi_j(t)/dt}{\delta \cdot f(n_j(t))},$$

where we let $\frac{d\pi_j(t)}{dt} = \lim_{\tau \to 0^+} \frac{\pi_j(t+\tau) - \pi_j(t)}{\tau}$ and thus $\frac{d\pi_j(t)}{dt}$ exists for all $t \ge 0$. Moreover, we have that,

$$\begin{aligned} \frac{d\pi_j(t)}{dt} \le\ & \mathbb{1}(j \in \psi^*(t))\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t) \\ & - \mathbb{1}(j \notin \psi^*(t))\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t), \end{aligned} \tag{20}$$

indeed, either $j \in \psi^*(t)$ so job j has not completed under the optimal policy and the drift is bounded by the first term in (20), or $j \notin \psi^*(t)$ and the job has completed under the optimal policy, the difference term in (17) is positive and its derivative is given by the the second term in (20). Therefore, for all $t \notin \{a_j\}_j \cup \{c_j\}_j$, we have the following upper bound:

$$\begin{aligned} \frac{d\Lambda(t)}{dt} \le & \sum_{j \in \psi(t)} \frac{\mathbb{1}(j \in \psi^*(t))\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)}{\delta \cdot f(n_j(t))} \\ & - \sum_{j \in \psi(t)} \frac{\mathbb{1}(j \notin \psi^*(t))\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t)}{\delta \cdot f(n_j(t))} \\ \le & \underbrace{\sum_{j \in \psi^*(t)} \frac{\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)}{\delta \cdot f(n_j(t))}}_{\Gamma^*(t)} - \underbrace{\sum_{j \in \psi(t) \setminus \psi^*(t)} \frac{\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t)}{\delta \cdot f(n_j(t))}}_{\Gamma(t)}, \end{aligned} \tag{21}$$

where $\psi(t) \setminus \psi^*(t)$ contains all the jobs that are in $\psi(t)$ but not in $\psi^*(t)$. For ease of illustration, let $\Gamma^*(t)$ and $\Gamma(t)$ denote the two terms on the R.H.S. of (21). In the sequel, we bound these two terms.

5.3.3 *An upper bound of $\Gamma^*(t)$*

When $\beta n_j(t) \ge M$, we have $f(n_j(t)) = M/\beta n_j(t)$, thus, $\frac{\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)}{f(n_j(t))} = \frac{\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)}{M/\beta n_j(t)}$. By contrast, when $\beta n_j(t) \le M$, we have that $f(n_j(t)) = 1$, so $\frac{\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)}{f(n_j(t))} = \widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)$, which is bounded by Δ based on Lemma 2. Therefore, we have that $\frac{\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)}{\delta f(n_j(t))} \le \frac{1}{\delta}\Big(\frac{\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)}{M/\beta n_j(t)} + \Delta\Big)$ and

$$\begin{aligned} \Gamma^*(t) &\le \sum_{j \in \psi^*(t)} \frac{1}{\delta}\Big(\frac{\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)}{M/\beta n_j(t)} + \Delta\Big) \\ &\le \frac{\Delta|\psi^*(t)|}{\delta} + \sum_{j \in \psi^*(t)} \beta n(t) \frac{\widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t)}{\delta M} \\ &\le \Delta|\psi^*(t)|/\delta + \beta n(t)/\delta, \end{aligned} \tag{22}$$

where the last inequality is due to that $\sum_{j \in \psi^*(t)} \widetilde{h}_j(\boldsymbol{t}_j^*, \boldsymbol{x}_j^*, \boldsymbol{r}_j^*, t) \le \sum_{j \in \psi^*(t)} \sum_k x_j^k r_j^k \cdot \mathbb{1}(t \in (t_j^{k-1}, t_j^k]) \le M$ for all t.

5.3.4 *An upper bound of $\Gamma(t)$*

First, $\Gamma(t)$ ban be represented as:

$$\Gamma(t) = \sum_{j \in \psi^*(t) \cap \psi(t)} \frac{\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t)}{\delta f(n_j(t))} - \sum_{j \in \psi(t)} \frac{\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t)}{\delta f(n_j(t))}. \tag{23}$$

To get an upper bound of $\Gamma(t)$, we will consider two cases. In particular, $\beta n(t) = zM + \alpha + \gamma$, we consider the case where $z = 0$ and that where $z \ge 1$.

Case 1: Suppose $z = 0$, in this case, $\lceil \beta n(t) \rceil \le M$. Since $n_j(t) \le n(t)$ for all $1 \le j \le n(t)$, it then follows that $\beta n_j(t) \le M$ and $f(n_j(t)) = 1$, which implies, for all $j \in \psi^*(t) \cap \psi(t)$, $\frac{\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t)}{\delta f(n_j(t))} \le \Delta$ since $\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t) \le \delta\Delta$ as we are using a δ-speed resource augmentation. Thus, the first term on the R.H.S. of (23) is upper bounded by:

$$\sum_{j \in \psi^*(t) \cap \psi(t)} \frac{\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t)}{\delta f(n_j(t))} \le \sum_{j \in \psi^*(t) \cap \psi(t)} \Delta \le |\psi^*(t)|\Delta. \tag{24}$$

Consider $j \in \psi(t)$ where $n(t) - \alpha \le j \le n(t)$ and $t \in [t_j^{k-1}, t_j^k)$ for some $k \in \{1, 2, \cdots, L_j\}$. Then, the number of redundant executions for job j, $r_j^k \ge \lfloor \frac{M}{\alpha+1} \rfloor \ge 1$. Thus, it follows that, $\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t) \ge \delta$ and $\frac{\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t)}{\delta f(n_j(t))} \ge 1$. Combining (23) and (24), we then have that,

$$\Gamma(t) \le \Delta|\psi^*(t)| - (\alpha + 1) \le \Delta|\psi^*(t)| - \beta n(t), \tag{25}$$

Case 2: Suppose $z \ge 1$, then, $\lceil \beta n(t) \rceil > M$ and $\frac{M}{\beta n(t)} \ge \frac{1}{z+1}$. Similarly, we consider job $j \in \psi(t)$ where $n(t) - kM - \alpha \le j \le n(t)$ and $t \in (t_j^{k-1}, t_j^k]$. Based on the scheduling policy of LAPS+R(β), we have that, $x_j^k = \frac{1}{z+1}$ and $r_j^k \ge 1$. Therefore, $\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t)$ is bounded by:

$$\frac{\delta}{z+1} \le \widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t) \le \frac{\delta \cdot \Delta}{z+1}. \tag{26}$$

Moreover, we have that, $\min[1\ ,\ \frac{M}{\beta n_j(t)}] \geq \min[1\ ,\ \frac{M}{\beta n(t)}] \geq \frac{1}{z+1}$. Therefore, it follows that,

$$\frac{1}{z+1} \leq \min\left[1\ ,\ \frac{M}{\beta n_j(t)}\right] \leq f(n_j(t)) \leq \frac{M}{\beta n_j(t)}. \quad (27)$$

Combining (26) and (27), we have that, for all $n(t) - kM - \alpha \leq j \leq n(t) - 1$,

$$\frac{1/(z+1)}{M/\beta n_j(t)} \leq \frac{\widetilde{h}_j(\boldsymbol{t}_j, \boldsymbol{x}_j, \boldsymbol{r}_j, t)}{\delta f(n_j(t))} \leq \Delta. \quad (28)$$

Substituting (28) into (23), it then follows that,

$$\begin{aligned}\Gamma(t) &\leq \sum_{j\in\psi^*(t)\cap\psi(t)} \Delta - \sum_{j=n(t)-zM-\alpha}^{n(t)} \frac{1/(z+1)}{M/\beta n_j(t)} \\ &\leq \Delta|\psi^*(t)| - \frac{\beta z M(n(t) - \frac{zM}{2} - \frac{\alpha}{2})}{M(z+1)} \\ &\leq \Delta|\psi^*(t)| - \beta(\frac{1}{2} - \frac{\beta}{4})n(t),\end{aligned} \quad (29)$$

where the second inequality is due to that $n_j(t) = j$ and the last inequality is because $zM + \alpha \leq \beta n(t)$ and $\frac{z}{z+1} \geq 1/2$. Based on **Case 1** and **Case 2**, we have that $\Gamma(t) \leq \Delta|\psi^*(t)| - \beta(\frac{1}{2} - \frac{\beta}{4})n(t)$.

5.3.5 *Bounding the performance of LAPS+R(β)*

Combining (21) and (22), we have the following upper bound for the drift, $\frac{d\Lambda(t)}{dt}$:

$$\begin{aligned}\frac{d\Lambda(t)}{dt} &\leq \Gamma^*(t) + \Gamma(t) \\ &\leq \Delta|\psi^*(t)|/\delta + \beta n(t)/\delta + \Delta|\psi^*(t)| - \beta(\frac{1}{2} - \frac{\beta}{4})n(t). \\ &= \frac{(\delta+1)\Delta}{\delta}|\psi^*(t)| + \frac{\beta(1 - \delta(\frac{1}{2} - \frac{\beta}{4}))}{\delta} n(t) \\ &\leq \frac{(\delta+1)\Delta}{\delta}|\psi^*(t)| - \frac{\beta\epsilon}{2\delta} n(t),\end{aligned} \quad (30)$$

where the last inequality is due to $\delta = 2 + 2\beta + 2\epsilon$ and $\delta(\frac{1}{2} - \frac{\beta}{4}) \geq 1 + \frac{\epsilon}{2}$.

Based on (30), and our three requirements on the potential function, see Section 5.2, we have that,

$$\begin{aligned}0 = \Lambda(\infty) - \Lambda(0) &\leq \int_0^\infty \frac{d\Lambda(t)}{dt} dt \\ &\leq \frac{(\delta+1)\Delta}{\delta} \int_0^\infty |\psi^*(t)| dt - \frac{\beta\epsilon}{2\delta} \int_0^\infty n(t) dt \\ &= \frac{(\delta+1)\Delta}{\delta} OPT - \frac{\beta\epsilon}{2\delta} LR,\end{aligned} \quad (31)$$

where the first inequality is due to the possibly negative jumps in the potential function, $\Lambda(t)$. (31) implies that $LR \leq \frac{2(\delta+1)\Delta}{\beta\epsilon}$, this completes the proof of Theorem 2. □

6. CONCLUSIONS AND FUTURE DIRECTIONS

This paper is a first attempt to address the impact of two key sources of variability in cloud computing clusters: job size and machine processing variability. Our primary aim and contribution was to, contribute to the fundamental understanding of job scheduling and redundant execution/checkpointing algorithms towards mitigating impact of variability on response time. As need for delivering predictable service on shared cluster and computing platforms grows, approaches, such as ours, will likely be an element of possible solutions. Extensions of this work to non-clairvoyant scenarios, the case of jobs with associated task graphs etc, are likely next steps towards developing the foundational theory and associated algorithms to address this problem.

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APPENDIX

A. PROOFS OF ALL LEMMAS

A.1 Proof of Lemma 1

PROOF. For a fixed t, let $F_{S_i(t)}(x)$ and $F_{H_r(t)}(x)$ denote the cumulative density function of $S_i(t)$ and $H_r(t)$ respectively where $H_r(t) = \max_{i=1,2,\cdots,r} S_i(t)$. Note that $F_{H_r(t)}(x) = F^r_{S_i(t)}(x)$. We then have: $g(r,t) = \mathbb{E}\big[H_r(t)\big] = \int_0^\infty (1 - F^r_{S_i(t)}(x))dx$, which further implies that,

$$\begin{aligned} g(r,t) - g(r-1,t) &= \int_0^\infty F^{r-1}_{S_i(t)}(x) \cdot (1 - F_{S_i(t)}(x))dx \\ &\le \int_0^\infty F^{r-2}_{S_i(t)}(x) \cdot (1 - F_{S_i(t)}(x))dx \\ &= g(r-1,t) - g(r-2,t). \end{aligned}$$

This completes the proof. □

A.2 Proof of Lemma 2

PROOF. As shown in the proof of Lemma 1, $g(r,t) = \int_0^\infty (1 - F^r_{S_i(t)}(x))dx$. Therefore, it follow that,

$$\begin{aligned} \int_0^\infty (1 - F^r_{S_i(t)}(x))dx &= \int_0^\infty (1 - F_{S_i(t)}(x)) \sum_{l=0}^{r-1} (F^l_{S_i(t)}(x))dx \\ &\ge r \int_0^\infty (1 - F_{S_i(t)}(x)) F^{r-1}_{S_i(t)}(x)dx \\ &= r(g(r,t) - g(r-1,t)), \end{aligned}$$

which implies $g(r,t) \le \frac{r}{r-1} g(r-1,t)$. Hence, $g(r,t) \le r g(1,t)$. Moreover, we have $g(1,t) = \mathbb{E}\big[S_i(t)\big] = t$. Thus, we have:

$$g(n,t) \le nt. \tag{32}$$

Since $g_i(t) = S_i(t) \le \Delta t$, therefore, it follows that,

$$\mathbb{E}\Big[\max_{i=1,2,\cdots,n} g_i(t)\Big] \le \Delta t. \tag{33}$$

The result follows from (32) and (33). This completes the proof. □

A.3 Proof of Lemma 3

PROOF. Consider an optimal solution to OPT, $\boldsymbol{y}^*$, whose corresponding job completion time for job j is denoted by c^*_j. Thus, for all $j = 1, 2, \cdots, N$, $\boldsymbol{y}^*$ and c^*_j satisfy:

$$\int_{a_j}^{c^*_j} h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t)dt = p_j. \tag{34}$$

Moreover, it follows that $h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t) = 0$ for all $t \ge c^*_j$, thus, we have: $\int_{a_j}^\infty h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t)dt = p_j$, and it follows that,

$$\begin{aligned} &\int_{a_j}^\infty \frac{1}{p_j}(t - a_j) h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t)dt = \int_{a_j}^{c^*_j} \frac{1}{p_j}(t - a_j) h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t)dt \\ &\le \int_{a_j}^{c^*_j} \frac{1}{p_j}(c^*_j - a_j) h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t)dt = c^*_j - a_j. \end{aligned} \tag{35}$$

Following Lemma 2, it can be readily shown that $h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t) \le \Delta$. Therefore, we have that,

$$p_j = \int_{a_j}^{c^*_i} h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t)dt \le \Delta(c^*_j - a_j). \tag{36}$$

Combining (35) and (36), we have:

$$\int_{a_j}^\infty \frac{(t - a_j + 2p_j)}{p_j} \cdot h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t)dt \le (1 + 2\Delta)(c^*_j - a_j).$$

Since the optimal solution to OPT must be feasible for P1, it follows that,

$$\begin{aligned} P1 &\le \sum_{j=1}^N \int_{a_j}^\infty \frac{(t - a_j + 2p_j)}{p_j} \cdot h_j(\boldsymbol{t^*_j}, \boldsymbol{r^*_j}, t)dt \\ &\le (1 + 2\Delta) \sum_{j=1}^N (c^*_j - a_j) = (1 + 2\Delta) OPT. \end{aligned} \tag{37}$$

This completes the proof. □

A.4 Proof of Lemma 4

PROOF. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both nonnegative, we only need to show $\alpha_j - \beta(t) \le \frac{t - a_j}{p_j} + 2$ for all j and $t \ge a_j$. Suppose $n(a_j) = zM + q > M$, indeed, note that the multiplicative factor for $p_k(a_j)$ in (11) is nonzero only when $k = lM + q$ for some $l = 0, 1, \cdots, z$. Therefore, (11) can be rewritten as:

$$\alpha_j = \frac{\sum_{k=0}^z p_{kM+q}(a_j) \mathbb{1}(kM + q \le \rho_j)}{(1+\epsilon)p_j} + \frac{\Big(\lfloor \frac{n(a_j) - \rho_j - 1}{M} \rfloor + 1\Big)}{1 + \epsilon}. \tag{38}$$

For ease of illustration, let Ω_1 and Ω_2 denote the two terms on the R.H.S of (38) respectively.

If $n(a_j) \le M$, then $\Omega_1 = 0$ and we have that $\alpha_j = \frac{1}{1+\epsilon}$, so the result follows. Let us thus consider the case where $n(a_j) > M$ as follows:

Case I: Suppose the jobs in Θ_j are completed at time t. As depicted in Fig. 3, if there are no job arrivals after time a_j, then, jobs indexed by $km + q$ where k is a non-negative integer, are all processed on Machine q. Since the service capacity of Machine q is $(1 + \epsilon)(t - a_j)$ during $(a_j, t]$, thus, it follows that,

$$t - a_j \ge \frac{1}{1 + \epsilon} \sum_{k=0}^z p_{kM+q}(a_j). \tag{39}$$

In contrast, if there are other job arrivals after time a_j, since we assume Θ_j must have completed by time t, Machine q needs to process an amount of work which exceeds $\sum_{k=0}^z p_{kM+q}(a_j)$, therefore, (39) still holds. Thus, we have that,

$$\begin{aligned} \frac{t - a_j}{p_j} &\ge \frac{1}{(1+\epsilon)p_j} \sum_{k=0}^z p_{kM+q}(a_j) \\ &= \frac{\sum_{k=0}^z p_{kM+q}(a_j) \mathbb{1}(kM + q \le \rho_j)}{(1+\epsilon)p_j} \\ &+ \frac{\sum_{k=0}^z p_{kM+q}(a_j) \mathbb{1}(kM + q \ge \rho_j + 1)}{(1+\epsilon)p_j} \\ &\overset{(i)}{\ge} \Omega_1 + \frac{\sum_{k=0}^z \mathbb{1}(kM + q \ge \rho_j + 1)}{(1+\epsilon)} \\ &= \alpha_j \ge \alpha_j - \beta(t) - 2, \end{aligned} \tag{40}$$

where (i) is due to that $p_{km+q}(a_j) \ge p_j$ when $km + q > \rho_j$.

Case II: Suppose jobs indexed from 1 to κ in Θ_j have completed where $\kappa \le \rho_j$. Let $\kappa = z_1 M + q_1$. An argument similar to that for Case I shows that by time t, it follows that,

$$t - a_j \ge \frac{1}{1 + \epsilon} \sum_{k=0}^{z_1} p_{kM+q_1}(a_j). \tag{41}$$

One the one hand, when $q \le q_1$, we have that,

$$\begin{aligned}\Omega_1 &= \frac{1}{(1+\epsilon)p_j}\sum_{k=0}^{z_1} p_{kM+q}(a_j) \\ &\quad + \frac{\sum_{k=z_1+1}^{z} p_{kM+q}(a_j)\mathbb{1}(kM+q \le \rho_j)}{(1+\epsilon)p_j} \\ &\overset{\text{(ii)}}{\le} \frac{1}{(1+\epsilon)p_j}\sum_{k=0}^{z_1} p_{kM+q_1}(a_j) + \frac{\sum_{k=z_1+1}^{z}\mathbb{1}(kM+q \le \rho_j)}{1+\epsilon} \\ &= \frac{1}{(1+\epsilon)p_j}\sum_{k=0}^{z_1} p_{kM+q_1}(a_j) + \frac{\sum_{k=0}^{z}\mathbb{1}(\kappa < kM+q \le \rho_j)}{1+\epsilon},\end{aligned} \tag{42}$$

where (ii) is due to jobs with index smaller than ρ_j in Θ_j have remaining workload no less than p_j. On the other, when $q > q_1$, we have that,

$$\begin{aligned}\Omega_1 &= \frac{\sum_{k=0}^{z_1-1} p_{kM+q}(a_j)}{(1+\epsilon)p_j} + \frac{\sum_{k=z_1}^{z} p_{kM+q}(a_j)\mathbb{1}(kM+q \le \rho_j)}{(1+\epsilon)p_j} \\ &\le \frac{1}{(1+\epsilon)p_j}\sum_{k=0}^{z_1} p_{kM+q_1}(a_j) + \frac{\sum_{k=z_1}^{z}\mathbb{1}(kM+q \le \rho_j)}{1+\epsilon} \\ &= \frac{1}{(1+\epsilon)p_j}\sum_{k=0}^{z_1} p_{kM+q_1}(a_j) + \frac{\sum_{k=0}^{z}\mathbb{1}(\kappa < kM+q \le \rho_j)}{1+\epsilon}.\end{aligned} \tag{43}$$

Therefore, it follows that,

$$\begin{aligned}\Omega_1 &\le \frac{1}{(1+\epsilon)p_j}\sum_{k=0}^{z_1} p_{kM+q_1}(a_j) + \frac{\sum_{k=0}^{z}\mathbb{1}(\kappa < kM+q \le \rho_j)}{1+\epsilon} \\ &\le \frac{1}{(1+\epsilon)p_j}\sum_{k=0}^{z_1} p_{kM+q_1}(a_j) + \frac{\lceil\frac{\rho_j-\kappa}{M}\rceil}{1+\epsilon} \\ &\overset{\text{(iii)}}{\le} \frac{t-a_j}{p_j} + \frac{1}{1+\epsilon}\lceil\frac{\rho_j-\kappa}{M}\rceil,\end{aligned} \tag{44}$$

where $\lceil x\rceil$ denotes the smallest integer which is no less than x and (iii) is due to (41). Based on (44), we then have that,

$$\begin{aligned}\alpha_j &\le \frac{t-a_j}{p_j} + \frac{1}{1+\epsilon}\lceil\frac{\rho_j-\kappa}{M}\rceil + \frac{\left(\lfloor\frac{n(a_j)-\rho_j-1}{M}\rfloor+1\right)}{1+\epsilon} \\ &\le \frac{t-a_j}{p_j} + \frac{1}{1+\epsilon}\left(\lfloor\frac{n(a_j)-\kappa}{M}\rfloor+2\right) \\ &\le \frac{t-a_j}{p_j} + \beta(t) + 2,\end{aligned} \tag{45}$$

where the last inequality is due to that $\beta(t) \ge \frac{1}{1+\epsilon}\left(\lfloor\frac{n(a_j)-\kappa}{M}\rfloor\right)$ since the number of active jobs at time t, $n(t)$, is no less than $n(a_j)-\kappa$.

Case III: Suppose jobs indexed from 1 to κ in Θ_j have completed where $\kappa = z_1M + q_1 > \rho_j$. In this case, (41) still holds. Moreover, an argument similar to that for (44) shows that,

$$\begin{aligned}\Omega_1 &\le \frac{1}{(1+\epsilon)p_j}\sum_{k=0}^{z_1} p_{kM+q_1}(a_j) - \lfloor\frac{\kappa-\rho_j}{M}\rfloor \\ &\le \frac{t-a_j}{p_j} - \frac{1}{1+\epsilon}\lfloor\frac{\kappa-\rho_j}{M}\rfloor.\end{aligned} \tag{46}$$

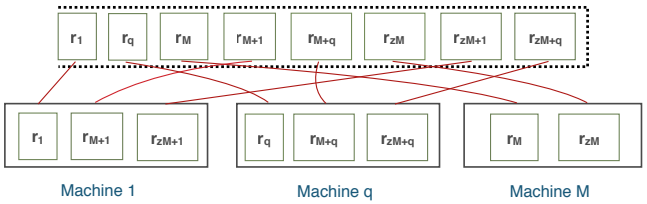

Figure 3: The scheduling process of SRPT at time a_j where $n(a_j) = zM + q$ and there are no further job arrivals after a_j. Jobs are sorted based on the remaining size, which is denoted by r_j for job j, i.e., $r_j = p_j(a_j)$. Jobs indexed by $kM + i$ for some integer valued k and i are assigned to machine i.

Therefore, it follows that,

$$\begin{aligned}\alpha_j &\le \frac{t-a_j}{p_j} - \frac{1}{1+\epsilon}\lfloor\frac{\kappa-\rho_j}{M}\rfloor + \frac{1}{1+\epsilon}\lceil\frac{n(a_j)-\rho_j}{M}\rceil \\ &\le \frac{t-a_j}{p_j} + \frac{1}{1+\epsilon}\left(\lfloor\frac{n(a_j)-\kappa}{M}\rfloor+2\right) \\ &\le \frac{t-a_j}{p_j} + \beta(t) + 2.\end{aligned} \tag{47}$$

Thus, we conclude that, for all the three cases above, the constraint on α_j and $\beta(t)$ is satisfied. This completes the proof. □

A.5 Proof of Lemma 5

PROOF. Let $f_j(t)$ denote the remaining flowtime for job j at time t. Thus, the job completion time, c_j is equal to $f_j(t) + t$. Since we have indexed jobs such that $p_1(t) \le p_2(t) \le \cdots \le p_{n(t)}(t)$, under SRPT, it follows that $c_1 \le c_2 \le \cdots \le c_{n(t)}$. When $n(t) \le M$, (13) follows immediately since all jobs can be scheduled simultaneously and $f_j(t)$ is equal to $p_j(t)/(1+\epsilon)$.

Let us then consider the case where $n(t) > M$. Let $n(t) = zM + q$ where $z \ge 1$, $0 \le q \le M-1$ and z, q are non-negative integers. We first show that for all k such that $M \le k \le n(t)$, the following result holds:

$$\sum_{j=k-M+1}^{k} f_j(t) = \frac{1}{1+\epsilon}\sum_{j=1}^{k} p_j(t) \tag{48}$$

As illustrated in Fig. 4, at any time between t and c_1, there are $(k-M)$ jobs waiting to be processed among those k jobs which complete first. Hence, the accumulated waiting time in this period is $(k-M)f_1(t)$. Similarly, at any time between c_1 and c_2, there are $(k-M-1)$ jobs waiting to be processed and they contribute $(k-M-1)\cdot(c_2-c_1) = (k-M-1)\cdot(f_2(t)-f_1(t))$ waiting time. Hence, the total waiting time of the k jobs is given by:

$$\sum_{j=0}^{k-M-1}(k-M-j)\cdot(f_{j+1}(t)-f_j(t)) = \sum_{j=1}^{k-M} f_j(t). \tag{49}$$

Therefore, the total remaining flowtime for these k jobs is as follows:

$$\sum_{j=1}^{k} f_j(t) = \frac{1}{1+\epsilon}\sum_{j=1}^{k} p_j(t) + \sum_{j=1}^{k-M} f_j(t). \tag{50}$$

By shifting terms in (50), we have:

$$\sum_{j=k-M+1}^{k} f_j(t) = \frac{1}{1+\epsilon}\sum_{j=1}^{k} p_j(t).$$

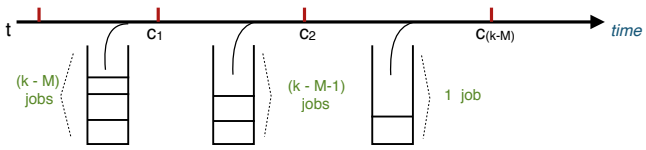

Figure 4: The number of jobs waiting to be processed in different time periods where $k > M$.

Summing up all job flowtime, it follows that:

$$\begin{aligned}\sum_{j=1}^{n(t)} f_j(t) &= \sum_{j=1}^{q} f_j(t) + \sum_{k=1}^{z} \sum_{j=(k-1)M+q+1}^{kM+q} f_j(t) \\ &\overset{(i)}{=} \frac{1}{1+\epsilon}\Big(\sum_{j=1}^{q} p_j(t) + \sum_{k=1}^{z} \sum_{j=1}^{kM+q} p_j(t)\Big) \\ &= \frac{1}{1+\epsilon} \sum_{j=1}^{n(t)} \Big(\Big\lfloor \frac{n(t)-j}{M} \Big\rfloor + 1\Big) p_j(t),\end{aligned} \tag{51}$$

where on the R.H.S. of (i), the first term is due to that the flowtime of the first q jobs is equal to their remaining job size and the second term is due to that $\sum_{j=(k-1)M+q+1}^{kM+q} f_j(t) = \frac{1}{1+\epsilon} \sum_{j=1}^{kM+q} p_j(t)$. This completes the proof. $\square$