

Convex Functions, Transformations

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1 Convex Optimization Problem

All convex optimization problems have the form of:

$$\begin{aligned} \min_x & f(x) \\ \text{s.t. } & g_i(x) \leq 0, \quad \forall i = 1, 2, \dots, m \\ & x \in X, \end{aligned}$$

where $f(x)$ and $g_i(x)$ are convex functions and $X \in \mathbb{R}^n$ is a convex set.

2 Convex functions

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we say that f is convex if

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2),$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in [0, 1]$. We say that f is concave if $-f$ is convex.

Proposition 1. *Consider the optimization problem*

$$\begin{aligned} \min_x & f(x) \\ \text{s.t. } & x \in S, \end{aligned}$$

where $S \in \mathbb{R}^n$ is a convex set and f is a convex function. Then, any local minimizer is also a global minimizer.

2.1 Convexity-Preserving Transformations

The following hold:

- (Non-Negative Combinations) Let f_1, \dots, f_m be convex functions, and let $\alpha_1, \dots, \alpha_m \geq 0$. Then, the function $\sum_{i=1}^m \alpha_i f_i$ is also convex.
- (Pointwise Supremum) Let $\{f_i\}_{i \in I}$ be an arbitrary family of convex functions on \mathbb{R}^n . Then, the pointwise supremum $f = \sup_{i \in I} f_i$ is also convex.
- (Composition with an Increasing Convex Function) Let f be a convex function, and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing convex function. Then, the function $g(f(x))$ is convex on \mathbb{R}^n .

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2.2 Differentiable Convex Functions

When f is a differentiable function, we can characterize its convexity via its gradient.

Theorem 1. *Let $f : \Omega \rightarrow \mathbb{R}$ be a differentiable function on the open set $\Omega \in \mathbb{R}^n$, and let $S \subset \Omega$ be convex. Then, f is convex on S iff*

$$f(x_1) \geq f(x_2) + (\nabla f(x_2))^T(x_1 - x_2), \quad (1)$$

for all $x_1, x_2 \in S$.

Theorem 2. *Let $f : S \rightarrow \mathbb{R}$ be a twice continuously differentiable function on the open convex set $S \subset \mathbb{R}^n$. Then, f is convex on S iff $\nabla^2 f(\bar{x})$ is positive semidefinite for all $\bar{x} \in S$.*