Convex Functions, Transformations

Huanle Xu

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1 Convex Optimization Problem

All convex optimization problems have the form of:

$$\begin{aligned} \min_x \ & f(x) \\ \text{s.t. } & g_i(x) \le 0, \quad \forall i = 1, 2, \cdots, m \\ & x \in X, \end{aligned}$$

where $f(x)$ and $g_i(x)$ are convex functions and $X \in \mathbb{R}^n$ is a convex set.

2 Convex functions

Let $f : \mathbb{R}^n \to \mathbb{R}$, we say that f is convex if

$$f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2), \tag{1}$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in [0, 1]$. We say that f is concave if $-f$ is convex.

Proposition 1. *Consider the optimization problem*

$$\begin{aligned} \min_x \ & f(x) \\ s.t. \ & x \in S, \end{aligned}$$

where $S \in \mathbb{R}^n$ is a convex set and f is a convex function. Then, any local minimizer is also a global minimizer.

2.1 Convexity-Preserving Transformations

The following hold:

- (Non-Negative Combinations) Let $f_1, \cdots, f_m$ be convex functions, and let $\alpha_1, \cdots, \alpha_m \ge 0$. Then, the function $\sum_{i=1}^m \alpha_i f_i$ is also convex.
- (Pointwise Supremum) Let $\{f_i\}_{i \in I}$ be an arbitrary family of convex functions on $\mathbb{R}^n$. Then, the pointwise supremum $f = \sup_{i \in I} f_i$ is also convex.
- (Composition with an Increasing Convex Function) Let f be a convex function, and let $g : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. Then, the function $g(f(x))$ is convex on $\mathbb{R}^n$.

2.2 Differentiable Convex Functions

When f is a differentiable function, we can characterize its convexity via its gradient.

Theorem 1. *Let $f : \Omega \to R$ be a differentiable function on the open set $\Omega \in \mathbb{R}^n$, and let $S \subset \Omega$ be convex. Then, f is convex on S iff*

$$f(x_1) \geq f(x_2) + (\nabla f(x_2))^T (x_1 - x_2), \tag{2}$$

for all $x_1, x_2 \in S$.

Theorem 2. *Let $f : S \to \mathbb{R}$ be a twice continuously differentiable function on the open convex set $S \subset \mathbb{R}^n$. Then, f is convex on S iff $\nabla^2 f(\overline{x})$ is positive semidefinite for all $\overline{x} \in S$.*

3 Some Useful Inequalities

Let us begin with Jensen's inequality, which can be viewed as a generalization of (1).

Proposition 2. ***(Jensen's Inequality)*** *Let f be a convex function. Then, for any $x_1, x_2, \cdots, x_k \in dom(f)$ and $\alpha_1, \alpha_2, \cdots, \alpha_k \in [0, 1]$ such that $\sum_{i=1}^{k} \alpha_k = 1$, we have*

$$f(\sum_{i=1}^{k} \alpha_i x_i) \leq \sum_{i=1}^{k} \alpha_i f(x_i). \tag{3}$$

Proposition 3. *For all $x_1, x_2, \cdots, x_n \in \mathbb{R}_+$, the following holds:*

$$(\prod_{i=1}^{n} x_i)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i. \tag{4}$$

4 Examples of Convex Functions

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = \log\left(\sum_{i=1}^{n} exp(x_i)\right)$. We compute

$$\frac{\partial^2 f}{\partial x_i x_j} = \begin{cases} \frac{exp(x_i)}{\sum_{i=1}^{n} exp(x_i)} - \frac{exp(2x_i)}{(\sum_{i=1}^{n} exp(x_i))^2} & \text{if} \quad i = j, \\ -\frac{exp(x_i + x_j)}{(\sum_{i=1}^{n} exp(x_i))^2} & \text{if} \quad i \neq j. \end{cases}$$

2. Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be given by $f(x) = (\prod_{i=1}^{n} x_i)^{1/n}$. We compute

$$\frac{\partial^2 f}{\partial x_i x_j} = \begin{cases} -(n-1)\frac{(\prod_{i=1}^{x} x_i)^{1/n}}{n^2 x_i^2} & \text{if} \quad i = j, \\ \frac{(\prod_{i=1}^{x} x_i)^{1/n}}{n^2 x_i x_j} & \text{if} \quad i \neq j. \end{cases}$$