PROXIMITY SPACES AND DE VRIES DUALITY

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Abstract. We detail the theory of proximity spaces in the direction of compactifications and the Smirnov construction, in which a bijection between compactifications of a Tychonoff space and proximities compatible with this space is given. We then describe the process of De Vries [3] by forming a category of "de Vries algebras" and "de Vries morphisms" and showing their dual equivalence to the category of compact Hausdorff spaces and continuous functions.

Contents

0. Introduction

In the early 1940s, Samuel Eilenberg and Saunders Mac Lane developed category theory as a way to relate different areas of mathematics. In particular, they were interested in the relation between topological and algebraic structures. Eilenberg and Mac Lane presented their work in their 1945 paper "General Theory of Natural Equivalences" [5]. In Section 1, we describe some of the basic ideas of category theory, setting the stage for us to prove the dual equivalence of two categories.

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We switch our attention in Section 2 to the idea of proximity spaces. Riesz [15] started the discussion of proximities in 1908 with his "theory of enchainment". In 1950, Efremovič [4] rediscovered the idea and developed a set of axioms for two subsets to be "near". Such a relation on a set X is called a proximity (sometimes called an EF-proximity) and the set X along with the proximity is called a "proximity space". This notion of nearness is a natural generalization of metric space theory, as can be seen in the relevant examples in Section 2. Efremovič went further to show that a proximity on a set induces a topology. We continue, in Section 2, to develop the fundamental properties of proximities on topological spaces. Toward the end of the section, we examine the correlation of a proximity on a space to the compactifications of the space. [11]

In 1939, Alexandroff [1] developed the idea of "ends" while studying extensions of topological spaces. He also asked the question, "Which topological spaces admit a separated proximity compatible with the given topology?" Smirnov [16], in 1952, used the ends of Alexandroff to obtain a compactification of a Tychonoff space. This compactification is often referred to as the "Smirnov Compactification" and the development leading up to it the "Smirnov Construction". This construction is the heart of Section 3. Smirnov also gave an answer to Alexandroff's query with "Smirnov's Compactification Theorem", given in this section. We use the remainder of Section 3 to prove another famous result of Smirnov known as "Smirnov's Theorem" which states there is a bijection between the compactifications of a Tychonoff space and the proximities on that space "compatible" with the topology on the space.

In Sections 4 and 5, we trace the Ph.D. thesis of de Vries [3]. De Vries developed the idea of "complete compingent algebras", which we call "de Vries algebras". These algebras are complete Boolean algebras with an added structure. De Vries algebras along with a set of mappings between them, which we call "de Vries morphisms", form a category. In Section 5, we use the results of Section 1 to prove de Vries Duality Theorem, showing that the category $\mathbf{CPT_2}$ of compact Hausdorff spaces and continuous functions is dually equivalent to the category of de Vries algebras and de Vries morphisms.

We will use the following notations throughout this paper.

$\mathcal{P}(X)$	The power set of X
$\tau(X)$	The topology on the space X
$\operatorname{cl} A$	The closure of A
$\operatorname{int} A$	The interior of A
id_X	The identity function on X
$f\|_X$	The restriction of f to X
$\mathrm{C}(X)$	The set of continuous functions from X to $[0,1]$
$\mathrm{C}(X,Y)$	The set of continuous functions from X to Y
$\mathrm{a}\,\mathcal{F}$	The adherent points of the filter $\mathcal{F}$
$\mathcal{N}(p)$	The set of all neighborhoods of p
$\mathcal{RO}(X)$	The set of all regular open subsets of X
$\mathrm{Cl}\,X$	The set of all clopen subsets of X

1. Categories

Definition 1.1. A **category** $\mathfrak{C}$ consists of three entities:

(1) A class of objects denoted $\mathrm{ob}\,\mathfrak{C}$.
(2) A class of mappings between objects of $\mathfrak{C}$ known as morphisms. For $A, B \in \mathrm{ob}\,\mathfrak{C}$, we denote the morphisms from A to B as $\mathrm{M}(A, B)$.
(3) A binary operation $\circ$ such that for any $A, B, C \in \mathrm{ob}\,\mathfrak{C}$, the operator $\circ$ maps $\mathrm{M}(A, B) \times \mathrm{M}(B, C)$ to $\mathrm{M}(A, C)$ and satisfies the following:
  (a) (Associativity) For any $A, B, C, D \in \mathrm{ob}\,\mathfrak{C}$, $f \in \mathrm{M}(A, B)$, $g \in \mathrm{M}(B, C)$, and $h \in \mathrm{M}(C, D)$, $h \circ (g \circ f) = (h \circ g) \circ f$. In other words, the following commutes.

$$\begin{tikzcd} A \arrow[r, "g\circ f"] \arrow[dr, bend right, "f"'] & C \arrow[dr, bend left, "h"] & \\ & B \arrow[u, "g"'] \arrow[r, "h\circ g"'] \arrow[ul, phantom, "\circ"] & D \arrow[ul, phantom, "\circ"] \end{tikzcd}$$

  (b) (Identity) For any $A, B \in \mathrm{ob}\,\mathfrak{C}$ there is a morphism $1_A \in \mathrm{M}(A, A)$ and a morphism $1_B \in \mathrm{M}(B, B)$ such that for any $f \in \mathrm{M}(A, B)$, $1_B \circ f = f = f \circ 1_A$.

Henceforth, when we say "the category of A and B" we mean the category whose objects are the elements in A and whose morphisms between those objects are the elements in B. Unless otherwise stated, we will assume that composition is normal function composition.

Examples 1.2. The following are examples of categories.

(1) **SET** The category of sets and functions.
(2) $\mathbf{CPT_2}$ The category of compact Hausdorff spaces and continuous functions.

(3) **BA** The category of Boolean algebras and Boolean algebra homomorphisms.
(4) **TYCH** The category of Tychonoff spaces and continuous functions.
(5) $\mathbf{ZDCPT_2}$ The category of zero dimensional compact Hausdorff spaces and continuous functions.

Definition 1.3. Let $\mathfrak{B}$ and $\mathfrak{C}$ be categories. A **covariant functor** (respectively **contravariant functor**) F is a correspondence from $\mathfrak{B}$ to $\mathfrak{C}$ such that

(1) For each $B \in \operatorname{ob} \mathfrak{B}$ there is a unique $F(B) \in \operatorname{ob} \mathfrak{C}$
(2) For $A, B \in \operatorname{ob} \mathfrak{B}$ and $f \in \mathrm{M}(A, B)$, there is a unique $F(f) \in \mathrm{M}(F(A), F(B))$ (resp. $F(f) \in \mathrm{M}(F(B), F(A))$).
(3) For all $B \in \operatorname{ob} \mathfrak{B}$, $F(1_B) = 1_{F(B)}$
(4) For all $A, B, C \in \operatorname{ob} \mathfrak{B}$, $f \in \mathrm{M}(A, B)$, and $g \in \mathrm{M}(B, C)$, $F(g \circ f) = F(g) \circ F(f)$ (resp. $F(g \circ f) = F(f) \circ F(g)$). In other words, the following respective diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[dl, phantom, "\circ"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)"'] & F(B) \arrow[d, "F(g)"] \arrow[dl, phantom, "\circ"] \\ & F(C) \end{tikzcd} \qquad \begin{tikzcd} F(A) & F(B) \arrow[l, "F(f)"'] \arrow[dl, phantom, "\circ"] \\ & F(C) \arrow[u, "F(g)"'] \arrow[ul, "F(g\circ f)"] \end{tikzcd}$$

Definition 1.4. Let $\mathfrak{B}$ and $\mathfrak{C}$ be categories and F, G be covariant functors (respectively contravariant functors) from $\mathfrak{B}$ to $\mathfrak{C}$. A correspondence $\eta : F \to G$ is a **natural transformation** if it maps each $A \in \operatorname{ob} \mathfrak{B}$ to a unique $\eta(A) \in \mathrm{M}(F(A), G(A))$ such that for any $B \in \operatorname{ob} \mathfrak{B}$ and $f \in \mathrm{M}(A, B)$, $\eta(B) \circ F(f) = G(f) \circ \eta(A)$ (resp. $\eta(A) \circ F(f) = G(f) \circ \eta(B)$). In other words, the following respective diagram commutes.

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[r, "\eta(A)"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circ"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta(B)"'] & G(B) \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[r, "\eta(A)"] \arrow[dr, phantom, "\circ"] & G(A) \\ F(B) \arrow[r, "\eta(B)"'] \arrow[u, "F(f)"] & G(B) \arrow[u, "G(f)"'] \end{tikzcd}$$

Example 1.5. Consider the category **TYCH**. For $X \in \operatorname{ob} \mathbf{TYCH}$, let βX be the Stone-Cech compactification of X. For $X, Y \in \operatorname{ob} \mathbf{TYCH}$ and $f \in \mathrm{M}(X, Y)$, let βf be the continuous extension of f from βX to βY. Let $e(X) : X \to \beta X$ be the inclusion map. Then $e : 1_{\mathbf{TYCH}} \to \beta$ is a natural transformation and we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "e(X)"] \arrow[d, "f"'] \arrow[dr, phantom, "\circ"] & \beta X \arrow[d, "\beta(f)"] \\ Y \arrow[r, "e(Y)"'] & \beta Y \end{tikzcd}$$

Definition 1.6. Let $\mathfrak{C}$ be a category and $A, B \in \operatorname{ob}\mathfrak{C}$. The morphism $f \in \mathrm{M}(A, B)$ is an **isomorphism** if there is a $g \in \mathrm{M}(B, A)$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

Definition 1.7. Let $\mathfrak{B}$ and $\mathfrak{C}$ be categories and F, G be functors (either covariant or contravariant) from $\mathfrak{B}$ to $\mathfrak{C}$. A natural transformation $\eta : F \to G$ is a **natural isomorphism** if for all $B \in \operatorname{ob}\mathfrak{B}$, $\eta(B)$ is an isomorphism.

If a natural isomorphism exists between two categories $\mathfrak{B}$ and $\mathfrak{C}$, we write $\mathfrak{B} \cong \mathfrak{C}$.

Definition 1.8. Two categories $\mathfrak{B}$ and $\mathfrak{C}$ are said to be **equivalent** (respectively **dually equivalent**) if there exist covariant functors (resp. contravariant functors) F from $\mathfrak{B}$ to $\mathfrak{C}$ and G from $\mathfrak{C}$ to $\mathfrak{B}$ as well as natural isomorphisms $\eta : F \circ G \to 1_{\mathfrak{C}}$ and $\zeta : 1_{\mathfrak{B}} \to G \circ F$.

When two categories are dually equivalent, we will often say the two categories are dual or have duality.

Example 1.9 (Stone Duality [17])**.** For $B \in \operatorname{ob}\mathbf{BA}$, let $S(B)$ denote the Stone space of B. For $A, B \in \operatorname{ob}\mathbf{BA}$ and $f \in \mathrm{M}(A, B)$, define $S(f) : S(B) \to S(A)$ by $S(f)(\mathcal{U}) = \{a \in A : f(a) \in \mathcal{U}\}$. The mapping S is a contravariant functor from $\mathbf{BA}$ to $\mathbf{ZDCPT_2}$.

For $X \in \operatorname{ob}\mathbf{ZDCPT_2}$, the clopen sets of X, $\mathrm{Cl}(X)$, form a Boolean algebra. For $X, Y \in \operatorname{ob}\mathbf{ZDCPT_2}$ and $f \in \mathrm{M}(X, Y)$, define $\mathrm{Cl}(f) : \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ by $\mathrm{Cl}(f)(U) = f^{\leftarrow}[U]$. The mapping Cl is a contravariant functor from $\mathbf{ZDCPT_2}$ to $\mathbf{BA}$.

Define $\eta : A \to \mathrm{Cl}(S(A))$ by $\eta_A(a) = \{\mathcal{U} \in S(A) : a \in \mathcal{U}\}$ and $\zeta_X : S(\mathrm{Cl}(X)) \to X$ by $\zeta_X(\mathcal{U}) = \bigcap \mathcal{U}$.

For any $A \in \operatorname{ob}\mathbf{BA}$, η_A is a Boolean isomorphism and for any $X \in \operatorname{ob}\mathbf{ZDCPT_2}$, ζ_X is a homeomorphism.

For $A, B \in \operatorname{ob}\mathbf{BA}$; $f \in \mathrm{M}(A, B)$; $X, Y \in \operatorname{ob}\mathbf{ZDCPT_2}$; and $g \in \mathrm{M}(X, Y)$, we have the following diagram.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\circ"] & \mathrm{Cl}(S(A)) \arrow[d, "\mathrm{Cl}(S(f))"] \\ B \arrow[r, "\eta_B"'] & \mathrm{Cl}(S(B)) \end{tikzcd} \qquad \begin{tikzcd} S(\mathrm{Cl}(X)) \arrow[r, "\zeta_X"] \arrow[d, "S(\mathrm{Cl}(g))"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "g"] \\ S(\mathrm{Cl}(Y)) \arrow[r, "\zeta_Y"'] & Y \end{tikzcd}$$

These diagrams commute so η and ζ are natural isomorphisms. Therefore, **BA** and $\mathbf{ZDCPT_2}$ are dually equivalent.

2. Proximities

Definition 2.1. For a set X, a binary relation δ on $\mathcal{P}(X)$ is a **proximity** on X if, for all $A, B \subseteq X$, the following conditions are satisfied.

(P1) $A \,\delta\, B \Rightarrow B \,\delta\, A$
(P2) $A \,\delta\, B \cup C \Leftrightarrow A \,\delta\, B$ or $A \,\delta\, C$
(P3) $A \,\delta\, B \Rightarrow A \neq \varnothing$ and $B \neq \varnothing$
(P4) $A \,\not\delta\, B \Rightarrow \exists C \subseteq X$ s.t. $A \,\not\delta\, C$ and $B \,\not\delta\, X \setminus C$
(P5) $A \cap B \neq \varnothing \Rightarrow A \,\delta\, B$

The (P4) axiom is called the strong axiom. If $A \,\delta\, B$, we say A is "near" B. For simplicity, we will often write $x \,\delta\, A$ instead of $\{x\} \,\delta\, A$.

Definition 2.2. A proximity δ is called **separated** if, for any $x, y \in X$, it also satisfies

(P6) $x \,\delta\, y \Rightarrow x = y$

Remark 2.3. A proximity δ on a pseudo-metric space is separated.

Example 2.4. Let (X, d) be a metric space. Define, for $A, B \subseteq X$, $A \,\delta\, B$ if and only if $d(A, B) = 0$. The relation δ is a proximity on X.

Example 2.5. Let (X, d) be a metric space and $r \in \mathbb{R}$ such that $r \geq 0$. Define, for $A, B \subseteq X$, $A \,\delta\, B$ if and only if $d(A, B) \leq r$. The relation δ is a proximity on X.

Definition 2.6. A **proximity space** is a pair (X, δ) where X is a set and δ is a proximity on X.

Proposition 2.7. *Let (X, δ) be a proximity space. For all $A, B, C, D \subseteq X$ and $x \in X$,*

(1) $\varnothing \neq A \subseteq B \Rightarrow A \,\delta\, B$
(2) $A \,\delta\, B, A \subseteq C,$ *and* $B \subseteq D \Rightarrow C \,\delta\, D$
(3) $x \,\delta\, A$ *and* $x \,\delta\, B \Rightarrow A \,\delta\, B$.

Proof.

(1) Suppose $\varnothing \neq A \subseteq B$. Then $A \cap B \neq \varnothing$. By (P5), $A \,\delta\, B$.
(2) Suppose $A \,\delta\, B$. By (P2), $A \cup (C \setminus A) \,\delta\, B$ and again by (P2), $A \cup (C \setminus A) \,\delta\, B \cup (D \setminus B)$. But $A \subseteq C$ and $B \subseteq D$ implies $C = A \cup (C \setminus A)$ and $D = B \cup (D \setminus B)$. Hence $C \,\delta\, D$.
(3) By way of contradiction, suppose $x \,\delta\, A$, $x \,\delta\, B$, and $A \,\not\delta\, B$. By (P4), there is a $C \subseteq X$ such that $A \,\not\delta\, C$ and $B \,\not\delta\, X \setminus C$. Either $x \in C$ or $x \in X \setminus C$. Suppose, without loss of generality,

that $x \in C$. Then by 2.7(2), since $x \,\delta\, A$, we have $A \,\delta\, C$, a contradiction.

$\square$

Theorem 2.8. *The (P4) axiom is equivalent to*

(P4') $A \not\delta B \Rightarrow \exists C, D \subseteq X$ s.t. $A \not\delta X \setminus C$, $B \not\delta X \setminus D$, and $C \not\delta D$

Proof. Suppose the (P4) axiom holds and $A \not\delta B$. By (P4), there is a $D \subseteq X$ such that $A \not\delta D$ and $B \not\delta X \setminus D$. By (P1), $A \not\delta D$ implies $D \not\delta A$. Applying (P4) again, there is a $C \subseteq X$ such that $D \not\delta C$ and $A \not\delta X \setminus C$. Then (P1) yields $C \not\delta D$ and all of the conditions of C and D are satisfied.

Now, suppose (P4') holds and $A \not\delta B$. By (P4'), there is a $D, E \subseteq X$ such that $D \not\delta E$. By 2.7(1), $D \subseteq X \setminus E$. Let $C = X \setminus D$. Then $A \not\delta X \setminus D$ and thus $A \not\delta C$. Also, $B \not\delta X \setminus E$. As $D = X \setminus C$, by 2.7(2), $B \not\delta X \setminus C$ and we are done. $\square$

Proposition 2.9. *Let (X, δ) be a proximity space. For $A \subseteq X$, $\operatorname{cl} A = \{x \in X : x \,\delta\, A\}$ defines a closure operator on X.*

Proof. Suppose $A, B \subseteq X$. We must verify the Kuratowski closure axioms.

$\operatorname{cl} \varnothing = \varnothing$: By (P3), $x \not\delta \varnothing$ for all $x \in X$.

$A \subseteq \operatorname{cl} A$: Let $x \in A$. 2.7(1), $x \,\delta\, A$. So $x \in \operatorname{cl} A$.

$\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$: Let $x \in \operatorname{cl}(A \cup B)$. Then $x \,\delta\, (A \cup B)$. By (P2), $x \,\delta\, A$ or $x \,\delta\, B$. So $x \in \operatorname{cl} A$ or $x \in \operatorname{cl} B$. Thus $x \in \operatorname{cl} A \cup \operatorname{cl} B$. Inclusion the other way is the converse of this argument.

$\operatorname{cl}(\operatorname{cl} A) = \operatorname{cl} A$: Just as $A \subseteq \operatorname{cl} A$, we have $\operatorname{cl} A \subseteq \operatorname{cl}(\operatorname{cl} A)$. To show the other inclusion, suppose $x \notin \operatorname{cl} A$. Then $x \not\delta A$. So, by (P4), $\exists C \subset X$ such that $x \not\delta C$ and $A \not\delta X \setminus C$. Let $y \in A$. Then $y \not\delta X \setminus C$. By 2.7(1), $y \in C$. Hence $\operatorname{cl} A \subseteq C$. Since $x \not\delta C$, by 2.7(1), $x \not\delta \operatorname{cl} A$. Thus $x \notin \operatorname{cl}(\operatorname{cl} A)$. $\square$

Corollary 2.10. *Let (X, δ) be a proximity space. The set $\tau(\delta) = \{X \setminus \operatorname{cl} A : A \subseteq X\}$ is a topology on X.*

Corollary 2.11. *Let (X, δ) be a proximity space and $A \subseteq X$. Then $A \in \tau(\delta)$ if and only if $x \not\delta X \setminus A$ for all $x \in A$.*

We say the topology $\tau(\delta)$ is the topology on X induced by δ. Henceforth, when we talk about the topology on a proximity space, we mean this induced topology unless otherwise stated.

Definition 2.12. Let X, Y be spaces such that X is embedded in Y and δ a proximity on Y. Define, for $A, B \subseteq X$, $A \,\delta|_X\, B$ if and only if $A \,\delta\, B$. We say $\delta|_X$ is the **subspace proximity** of δ on X.

Proposition 2.13. *Let X, Y be spaces such that X is embedded in Y and δ a proximity on Y. Then $\delta|_X$ is a proximity on X and the topology induced by $\delta|_X$ on X is the subspace topology of $\tau(\delta)$ on X.*

Proof. It is clear that axioms (P1) through (P4) are satisfied for $\delta|_X$ on $\mathcal{P}(X)$. For (P5), let $A, B \subseteq X$ such that $A \not\delta|_X B$. Then $A \not\delta B$. By (P5), there exists a $C \subseteq Y$ such that $A \not\delta C$ and $B \not\delta Y \setminus C$. So $A \not\delta C \cap X$ and, as $X \setminus (C \cap X) \subseteq Y \setminus C$, $B \not\delta X \setminus (C \cap X)$. Thus $A \not\delta|_X C \cap X$ and $B \not\delta|_X X \setminus (C \cap X)$. Therefore, (P5) is satisfied and $\delta|_X$ is a proximity on X.

Now, to show $\tau(\delta|_X)$ is the subspace topology, let $U \in \tau(\delta)$ and $x \in U \cap X$. As $x \in U$, by 2.11, $x \not\delta Y \setminus U$. Then $x \not\delta (Y \setminus U) \cap X$. But $(Y \setminus U) \cap X = X \setminus (U \cap X)$ and so $x \not\delta X \setminus (U \cap X)$. Thus, $x \not\delta|_X X \setminus (U \cap X)$. Since this holds for all $x \in U \cap X$, 2.11 yields $U \cap X \in \tau(\delta|_X)$.

For the reverse inclusion, let $U \in \tau(\delta|_X)$. Then $x \not\delta|_X (X \setminus U)$ for all $x \in U$ and thus $x \not\delta X \setminus U$ for all $x \in U$. So $U \subseteq Y \setminus \mathrm{cl}_{\tau(\delta)}(X \setminus U)$. But then $X \setminus U \subseteq \mathrm{cl}_{\tau(\delta)}(X \setminus U) \cap X$. As $X \setminus U$ is closed in X, $X \setminus U = \mathrm{cl}_{\tau(\delta)}(X \setminus U) \cap X$. Now,

$$\begin{aligned}
\mathrm{int}_{\tau(\delta)}(Y \setminus (X \setminus U)) \cap X &= (Y \setminus \mathrm{cl}_{\tau(\delta)}(X \setminus U)) \cap X \\
&= X \setminus (\mathrm{cl}_{\tau(\delta)}(X \setminus U) \cap X) \\
&= X \setminus (X \setminus U) \\
&= U
\end{aligned}$$

Hence $\tau(\delta|_X)$ is in the subspace topology of $\tau(\delta)$ on X. □

Definition 2.14. Let (X, δ) be a proximity space and X a topological space. Then we say δ is **compatible** with $\tau(X)$ if and only if $\tau(\delta) = \tau(X)$.

Proposition 2.15. *Let (X, δ) be a proximity space. For all $A, B \subseteq X$,*

$$A \,\delta\, B \Leftrightarrow \mathrm{cl}\, A \,\delta\, \mathrm{cl}\, B$$

Proof. Suppose $A\delta B$. It follows immediately from 2.7(2) that $\mathrm{cl}\, A\delta\mathrm{cl}\, B$.

Conversely, suppose $A \not\delta B$. Then by (P4), there is a $C \subseteq X$ such that $A \not\delta C$ and $B \not\delta X \setminus C$. Let $x \in X$ such that $x \,\delta\, A$. Then, by (4), $x \not\delta C$. So $x \in X \setminus C$. Hence $\mathrm{cl}\, A \subseteq X \setminus C$. Since $X \setminus C \not\delta B$ then, by 2.7(2), $\mathrm{cl}\, A \not\delta B$. Applying this argument again, we get $\mathrm{cl}\, A \not\delta \mathrm{cl}\, B$. □

Definition 2.16. Let (X, δ) be a proximity space and $A, B \subseteq X$. We say B is a **δ-neighborhood** of A and write $A \ll B$ if and only if $A \not\delta X \setminus B$.

In the above definition, we may also say A is "surrounded by" B.

Proposition 2.17. *Let (X,δ) be a proximity space. For all $A,B \subseteq X$,*

(Q1) $\varnothing \ll \varnothing$
(Q2) $A \ll B \Rightarrow A \subseteq B$
(Q3) $A \ll B \Rightarrow X \setminus B \ll X \setminus A$
(Q4) $A \ll B \cap C \Leftrightarrow A \ll B$ *and* $A \ll C$
(Q5) $A \ll B \Rightarrow \exists C \subseteq X$ *s.t.* $A \ll C \ll B$

Futhermore, if (X,δ) is a separated proximity space then, for all $x,y \in X$,

(Q6) $x \neq y \Leftrightarrow \{x\} \ll X \setminus \{y\}$

Proof.

(Q1) By (P3) we have that $\varnothing \not\delta X$ and so $\varnothing \ll \varnothing$.
(Q2) Suppose $A \ll B$. Then $A \not\delta X \setminus B$ and by (P5) we get $A \cap (X \setminus B) = \varnothing$ which implies $A \subseteq B$.
(Q3) Suppose $A \ll B$. Then $A \not\delta X \setminus B$. By (P1), $X \setminus B \not\delta A$ so $X \setminus B \not\delta X \setminus (X \setminus A)$; therefore, $X \setminus B \ll X \setminus A$.
(Q4) Suppose $A \not\delta X \setminus (B \cap C)$. As $B \cap C = (X \setminus B) \cup (X \setminus C)$ then by (P2), $A \not\delta X \setminus B$ and $A \not\delta X \setminus C$. Hence $A \ll B$ and $A \ll C$.
(Q5) Suppose $A \not\delta X \setminus B$. By (P4), there is a $D \subseteq X$ such that $A \not\delta D$ and $X \setminus D \not\delta X \setminus B$ which, by (Q3), yields $B \not\delta D$. Letting $C = X \setminus D$ gives the result.
(Q6) This follows directly from (P6).

□

Proposition 2.18. *Let (X,δ) be a proximity space. For all $A,B,C,D \subseteq X$,*

(1) $A \subseteq B \ll C \subseteq D \Rightarrow A \ll D$
(2) $A \ll B \Rightarrow \operatorname{cl} A \ll \operatorname{int} B$

Proof.

(1) By way of contradiction, suppose that $A \subseteq B \ll C \subseteq D$ and $A \not\ll D$. Then $A \,\delta\, X \setminus D$. By 2.7(2), $B \,\delta\, X \setminus C$. But this means $B \not\ll C$, a contradiction.
(2) Suppose $A \ll B$. Then $A \not\delta X \setminus B$. By 2.15, $\operatorname{cl} A \not\delta \operatorname{cl}(X \setminus B)$ and $\operatorname{cl}(X \setminus B) = X \setminus \operatorname{int} B$. Hence $\operatorname{cl} A \ll \operatorname{int} B$.

□

Proposition 2.19. *Let (X,δ) be a proximity space and $\{A_i\}_{i=1}^n$, $\{B_i\}_{i=1}^n$ finite families of subsets of X such that $A_i \ll B_i$ for $i = 1,\dots,n$.*

(1) $\bigcap_{i=1}^n A_i \ll \bigcap_{i=1}^n B_i$
(2) $\bigcup_{i=1}^n A_i \ll \bigcup_{i=1}^n B_i$

Proof.

(1) We induct on n. For $n = 1$, the result is clear. Assume $\bigcap_{i=1}^{n-1} A_i \ll \bigcap_{i=1}^{n-1} B_i$. By 2.18(1), $\bigcap_{i=1}^{n} A_i \ll \bigcap_{i=1}^{n-1} B_i$. Also, $\bigcap_{i=1}^{n} A_i \subseteq A_n \ll B_n$ implies $\bigcap_{i=1}^{n} A_i \ll B_n$. So by (Q4), $\bigcap_{i=1}^{n} A_i \ll \bigcap_{i=1}^{n} B_i$.

(2) By (Q3), for all i, $X \setminus B_i \ll X \setminus A_i$. By 2.19(1), $\bigcap_{i=1}^{n}(X \setminus B_i) \ll \bigcap_{i=1}^{n}(X \setminus A_i)$. Hence $X \setminus (\bigcup_{i=1}^{n} B_i) \ll X \setminus (\bigcup_{i=1}^{n} A_i)$. So $\bigcup_{i=1}^{n} A_i \ll \bigcup_{i=1}^{n} B_i$.

□

Proposition 2.20. *Let X be a set and $\ll$ a binary relation on $\mathcal{P}(X)$ satisfying (Q1) through (Q5). Then the binary relation δ defined by*

$$A \,\delta\, B \Leftrightarrow A \not\ll X \setminus B$$

is a proximity on X.

Proof.
(P1): Let $A, B \subseteq X$ and suppose $A \,\delta\, B$. Then $A \not\ll X \setminus B$. By (Q3), $B \not\ll X \setminus A$. So $B \,\delta\, A$.

(P2): Let $A, B, C \subseteq X$ and suppose $A \,\delta\, (B \cup C)$. Then $A \not\ll X \setminus (B \cup C)$. So $A \not\ll (X \setminus B) \cap (X \setminus C)$. By (Q4), $A \not\ll X \setminus B$ or $A \not\ll X \setminus C$. Hence $A \,\delta\, B$ or $A \,\delta\, C$.

(P3): Let $B \subseteq X$. By (Q1), $\varnothing \ll \varnothing$ and by (Q3), $X \ll X$. So we have $B \subseteq X \ll X$. By 2.18(1), $B \ll X$. Hence $B \not\delta\, \varnothing$.

(P4): Let $A, B \subseteq X$ and suppose $A \not\delta\, B$. Then $A \ll X \setminus B$. By (Q5), there is a $C \subseteq X$ such that $A \ll C \ll X \setminus B$. So $A \not\delta\, X \setminus C$ and $C \not\delta\, B$.

(P5): Let $A, B \subseteq X$ and suppose $A \ll X \setminus B$. By (Q2), $A \subseteq X \setminus B$. So $A \cap B = \varnothing$. □

Proposition 2.21. *If the binary relation $\ll$ in 2.20 also satisfies (Q6), then δ, as defined in 2.20, is a separated proximity.*

Proof. The result is immediate. □

Theorem 2.22. *Let (X, δ) be a proximity space. Define the set*

$$\tau(\ll) = \{U \subseteq X : x \in U \Rightarrow x \ll U\}$$

Then $\tau(\ll)$ is a topology on X and $\tau(\ll) = \tau(\delta)$.

Proof. Since $\tau(\delta)$ is a topology, it is enough to show that $\tau(\ll) = \tau(\delta)$.

Let $U \in \tau(\delta)$. Take $x \in U$. Then $x \notin X \setminus U = cl_{\tau(\delta)}(X \setminus U)$. Thus $x \not\delta\, X \setminus U$ and so $x \ll U$. Hence $U \in \tau(\ll)$.

Conversely, let $U \in \tau(\ll)$. Take $x \in U$. Then $x \ll U$ which implies $x \not\delta\, X \setminus U$. Therefore, $x \notin cl_{\tau(\delta)}(X \setminus U)$. So $x \in \mathrm{int}_{\tau(\delta)}\, U$. Hence $U \in \tau(\delta)$. □

Since $\ll$ is uniquely determined by δ and $\tau(\delta) = \tau(\ll)$, we will use these two proximity definitions interchangeably.

Proposition 2.23. *Let $(X, \ll)$ be a proximity space. For all $A, B \subseteq X$ such that $A \ll B$, there is a $U \in \tau(\ll)$ such that $A \ll U \ll B$.*

Proof. Suppose $A \ll B$. By (Q5), there is a $C \subseteq X$ such that $A \ll C \ll B$. By 2.18(2), $\operatorname{cl} A \subseteq \operatorname{int} C \in \tau(\delta)$. Letting $U = \operatorname{int} C$ gives the result. $\square$

Proposition 2.24. *Let $(X, \ll)$ be a proximity space. For all $A, B \subseteq X$ such that $A \ll B$, there is a $U \in \mathcal{RO}(X)$ such that $A \ll U \ll B$.*

Proof. Suppose $A \ll B$. By (Q5), there is a $C \subseteq X$ such that $A \ll C \ll B$. By 2.18(2), $\operatorname{cl} A \subseteq \operatorname{int} C$. So we have $A \subseteq \operatorname{cl} A \ll \operatorname{int} C \subseteq \operatorname{int}\operatorname{cl} C$. By 2.18(1), $A \ll \operatorname{int}\operatorname{cl} C$. Similarly, since $C \ll B$ then $\operatorname{cl} C \ll \operatorname{int} B$. So we have $\operatorname{int}\operatorname{cl} C \subseteq \operatorname{cl} C \ll \operatorname{int} B \ll B$. By 2.18(1), $\operatorname{int}\operatorname{cl} C \ll B$. Therefore, $A \ll \operatorname{int}\operatorname{cl} C \ll B$. $\square$

Proposition 2.25. *Let (X, δ) be a proximity space. The space $(X, \tau(\delta))$ is Hausdorff if and only if δ is separated.*

Proof. Suppose (X, δ) is Hausdorff and $x \,\delta\, y$. Then by the definition of $\tau(\delta)$, $x \in \operatorname{cl}\{y\} = \{y\}$. Therefore, $x = y$.

Conversely, suppose $x \,\delta\, y \Rightarrow x = y$. Take $x, y \in X$ such that $x \neq y$. Then $x \not\delta\, y$. By (P4'), there are $C, D \subseteq X$ such that $x \ll C$, $y \ll D$, and $C \ll X \setminus D$. By 2.23, there are $U, V \in \tau(\delta)$ such that $x \ll U \ll C$ and $y \ll V \ll D$. By 2.7(1), $x \in U \subseteq C$ and $y \in V \subseteq D$. Also, $C \ll X \setminus D$ implies $C \subseteq X \setminus D$ and so $U \cap V \subseteq C \cap D = \varnothing$. $\square$

Definition 2.26. Let X be a space. For all $A, B \subseteq X$, A and B are **completely separated** if and only if there is a function $f \in \mathrm{C}(X)$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.

Theorem 2.27. *Let $(X, \ll)$ be a proximity space and $A, B \subseteq X$. Then $A \ll B$ implies A and $X \setminus B$ are completely separated.*

Proof. Let $A, B \subseteq X$ and $A \ll B$. Let

$$Q = \{z/2^n : n \in \mathbb{N}, z \in \mathbb{Z}\} \cap (0, 1)$$

We will induct on n to show that there is a family $\{U_i \in \tau(\delta)\}_{i \in Q}$ which forms a $\ll$-chain between A and B.

For $n = 1$, $Q \cap (0, 1) = 1/2$. By 2.23, there is a $U_{1/2} \in \tau(\delta)$ such that $A \ll U_{1/2} \ll B$.

Now, let $U_0 = A$ and $U_1 = B$. Suppose that

$$U_0 \ll U_{1/2^n} \ll \cdots \ll U_{(2^n-1)/2^n} \ll U_1$$

Then for each $k \in \mathbb{Z} \cap [0, 2^n - 1]$, $U_{k/2^n} \ll U_{(k+1)/2^n}$. By 2.23, there is a $U_{(2k+1)/2^{n+1}} \in \tau(\delta)$ such that $U_{k/2^n} \ll U_{(2k+1)/2^{n+1}} \ll U_{(k+1)/2^n}$. Therefore

$$U_0 \ll U_{1/2^{n+1}} \ll U_{2/2^{n+1}} \ll \cdots \ll U_{(2^{n+1}-2)/2^{n+1}} \ll U_{(2^{n+1}-1)/2^{n+1}}$$

So, there is a family $\{U_i \in \tau(\delta)\}_{i \in Q}$ which forms a $\ll$-chain between A and B. So, for all $i, j \in Q$, $i < j$ implies $U_i \ll U_j$. Then by 2.18(2), $U_i \subseteq \operatorname{cl} U_i \ll U_j$.

We proceed as in the proof of Urysohn's Lemma [19]. Define $f : X \to [0,1]$ by

$$f(x) = \begin{cases} \inf\{i : x \in U_i\} & : x \in U_i \text{ for some } i \in Q \\ 1 & : x \notin U_i \text{ for all } i \in Q \end{cases}$$

As $A \subseteq U_0$, $f[A] \subseteq \{0\}$. Also, note that $X \setminus B \subseteq X \setminus U_1$. Therefore, $f[X \setminus B] \subseteq \{1\}$.

All that is remains is show that $f \in \mathrm{C}(X)$.

For all $0 < b \leq 1$, $f^{\leftarrow}[[0,b)] = \bigcup_{q<b} U_q$ which is $\tau(\delta)$-open. For all $0 \leq a < 1$, $f^{\leftarrow}[(a,1]] = \bigcup_{q>a}(X \setminus \operatorname{cl} U_q)$ which is also $\tau(\delta)$-open. As $\{[0,b) : b \leq 1\} \cup \{(a,1] : 0 \leq a\}$ is a subbase for the topology on $[0,1]$, $f \in \mathrm{C}(X)$. $\square$

Corollary 2.28. *Let (X, δ) be a proximity space. Then $(X, \tau(\delta))$ is completely regular.*

Proof. Let $A \subseteq X$ be closed and $p \in X \setminus A$. Then $p \ll X \setminus A$. So $f(p) = 0$ and $f[A] \subseteq \{1\}$. $\square$

Corollary 2.29. *Let (X, δ) be a separated proximity space. Then $(X, \tau(\delta))$ is Tychonoff.*

Proof. By 2.25, a separated proximity is Hausdorff. A completely regular Hausdorff space is Tychonoff. $\square$

Theorem 2.30. *Let X be a Tychonoff space. For $A, B \subseteq X$, define $A \not\delta B$ if and only if A and B are completely separated in X. Then δ is a separated proximity on X and δ is compatible with $\tau(X)$.*

Proof. First, we will show that δ is a proximity on X. Let $A, B, C \subseteq X$.

(P1): Suppose $A \not\delta B$. Then there is an $f \in \mathrm{C}(X)$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. Define $g = 1 - f$. Then $f \in \mathrm{C}(X)$ and $f[B] \subseteq \{0\}$ and $f[A] \subseteq \{1\}$. Hence $B \not\delta A$.

(P2): Suppose $A \not\delta B \cup C$. Then there is an $f \in \mathrm{C}(X)$ such that $f[A] \subseteq \{0\}$ and $f[B \cup C] \subseteq \{1\}$. But then $f[B] \subseteq \{1\}$ and $f[C] \subseteq \{1\}$. Hence $A \not\delta B$ and $A \not\delta C$.

Conversely, suppose $A \not\delta B$ and $A \not\delta C$. Then there are $g, h \in \mathrm{C}(X)$ such that $g[A] \subseteq \{0\}$, $g[B] \subseteq \{1\}$, $h[A] \subseteq \{0\}$, and $h[C] \subseteq \{1\}$.

Define $f(x) = \min\{(g+f)(x), 1\}$. Then $f \in \mathrm{C}(X)$, $f[A] \subseteq \{0\}$, $f[B \cup C] \subseteq \{1\}$. Hence $A \not\delta B \cup C$.

(P3): Suppose $A = \varnothing$ and $B \subseteq X$. Let $f(x) = \mathrm{id}_X$. Then $f \in \mathrm{C}(X)$, $f[B] \subseteq \{1\}$, and $f[A] = \varnothing \subseteq \{0\}$. Thus $A \not\delta B$.

(P4): Suppose $A \not\delta B$. Then there is an $f \in \mathrm{C}(X)$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. Set $C = \{x \in X : 1/2 \leq f(x) \leq 1\}$. Then set

$$g(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 1 & 1/2 < x \leq 1 \end{cases}$$

Then $(g \circ f)[A] \subseteq \{0\}$, $(g \circ f)[C] \subseteq \{1\}$, and $(g \circ f) \in \mathrm{C}(X)$. Hence $A \not\delta C$. Also, it is clear that $X \setminus C \not\delta B$.

(P5): Suppose $A \not\delta B$. Then there is an $f \in \mathrm{C}(X)$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. Then $A \cap B = \varnothing$.

(P6): Since X is Hausdorff, it follows from 2.25 that δ is separated.

Now we will show that $\tau(X) = \tau(\delta)$.

$\tau(X) \subseteq \tau(\delta)$: Let $U \in \tau(X)$. Take $x \in U$. Then $x \notin X \setminus U$ which is closed. Since X is completely regular, there is an $f \in \mathrm{C}(X)$ such that $f(x) = 0$ and $f[X \setminus U] \subseteq \{1\}$. Thus $x \not\delta X \setminus U$ for all $x \in U$. By 2.11, $U \in \tau(\delta)$.

$\tau(\delta) \subseteq \tau(X)$: Let $U \in \tau(\delta)$. Take $x \in U$. By 2.11, $x \not\delta X \setminus U$. Therefore, there is an $f \in \mathrm{C}(X)$ such that $f(x) = 0$ and $f[X \setminus U] \subseteq \{1\}$. Thus $f^{\leftarrow}[[0, 1/2)]$ is an open neighborhood of x and is contained in U. Therefore, $U \in \tau(X)$. □

Proposition 2.31. *Let X be a normal space. For all $A, B \subseteq X$, $\mathrm{cl}\, A \cap \mathrm{cl}\, B = \varnothing$ if and only if A and B are completely separated.*

Proof. Suppose A and B are completely separated. Then there is an $f \in \mathrm{C}(X)$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. So $\mathrm{cl}\, f[A] \subseteq \{0\}$ and $\mathrm{cl}\, f[B] \subseteq \{1\}$. Hence $\mathrm{cl}\, f[A] \cap \mathrm{cl}\, f[B] = \varnothing$. Since $\mathrm{cl}\, A \subseteq f^{\leftarrow}[\mathrm{cl}\, f[A]]$ and $\mathrm{cl}\, B \subseteq f^{\leftarrow}[\mathrm{cl}\, f[B]]$, we have $\mathrm{cl}\, A \cap \mathrm{cl}\, B = \varnothing$.

The converse follows from Urysohn's Lemma [19]. □

Proposition 2.32. *Let $(X, \tau(\ll))$ be a compact space. For all $A, B \subseteq X$,*

$$\mathrm{cl}\, A \subseteq \mathrm{int}\, B \Rightarrow A \ll B$$

Proof. Take $p \in \mathrm{cl}\, A$. Then $p \in \mathrm{int}\, B \in \tau(\delta)$. By 2.11, $p \not\delta X \setminus \mathrm{int}\, B$ and so $p \ll \mathrm{int}\, B$. By 2.23, there is a $U_p \in \tau(\ll)$ such that $p \ll U_p \ll \mathrm{int}\, B$. Then $\mathrm{cl}\, A \subseteq \bigcup_{p \in \mathrm{cl}\, A} U_p$. Since $\mathrm{cl}\, A$ is compact, there is a finite $F \subseteq \mathrm{cl}\, A$ such that $\mathrm{cl}\, A \subseteq \bigcup_{p \in F} U_p$. By 2.19(2), $\bigcup_{p \in F} U_p \ll \mathrm{int}\, B$. So we have

$$A \subseteq \mathrm{cl}\, A \subseteq \bigcup_{p \in F} U_p \ll \mathrm{int}\, B \subseteq B$$

By 2.18(1), $A \ll B$. □

Theorem 2.33. *Let X be a compact Hausdorff space. There is a unique proximity $\ll$ on X which is compatible with $\tau(X)$. This proximity is defined by, for $A, B \subseteq X$, $A \ll B$ if and only if $\operatorname{cl} A \cap \operatorname{cl}(X \setminus B) = \varnothing$.*

Proof. Since a compact Hausdorff space is normal [9], by 2.31, $\operatorname{cl} A \cap \operatorname{cl}(X \setminus B) = \varnothing$ if and only if A and $X \setminus B$ are completely separated. We showed in 2.30 that δ is then a proximity.

Suppose $A, B \subseteq X$ such that $A \ll B$. By 2.27, A and $X \setminus B$ are completely separated. So $\operatorname{cl} A \cap \operatorname{cl}(X \setminus B) = \varnothing$.

Conversely, suppose $A, B \subseteq X$ such that $\operatorname{cl} A \cap \operatorname{cl}(X \setminus B) = \varnothing$. Then $\operatorname{cl} A \subseteq \operatorname{int} B$. By 2.32, $A \ll B$.

For uniqueness, suppose $\ll_1$ and $\ll_2$ are two proximities on X compatible with $\tau(X)$. Then $(X, \tau(\ll_1))$ and $(X, \tau(\ll_2))$ are both compact Hausdorff and $\tau(\ll_1) = \tau(\ll_2) = \tau(X)$. Let $A, B \subseteq X$. Then $A \ll_1 B$ if and only if $\operatorname{cl} A \cap \operatorname{cl}(X \setminus B)$ if and only if $A \ll_2 B$. □

Remark 2.34. Since $\operatorname{cl} A \cap \operatorname{cl} B = \varnothing$ if and only if $\operatorname{cl} A \subseteq X \setminus \operatorname{int} B$, we may use either to define the unique proximity.

Theorem 2.35. *Let X be a Tychonoff space and Y be a Hausdorff compactification of X. For $A, B \subseteq X$, define $A \not\delta B$ if and only if $\operatorname{cl}_Y A \cap \operatorname{cl}_Y B = \varnothing$. Then δ is a separated proximity on X compatible with $\tau(X)$.*

Proof. First, we will verify that δ is a proximity.

(P1): Trivial.

(P2): Let $A, B, C \subseteq X$. Note that

$$\begin{aligned} &\operatorname{cl}_Y A \cap \operatorname{cl}_Y(B \cup C) \\ &= \operatorname{cl}_Y A \cap (\operatorname{cl}_Y B \cup \operatorname{cl}_Y C) \\ &= (\operatorname{cl}_Y A \cap \operatorname{cl}_Y B) \cup (\operatorname{cl}_Y A \cap \operatorname{cl}_Y C) \end{aligned}$$

Hence $A \not\delta B \cup C$ if and only if $A \not\delta B$ and $A \not\delta C$.

(P3): Suppose $A \subseteq X$. Then $\operatorname{cl}_Y A \cap \operatorname{cl}_Y \varnothing = \varnothing$ and so $A \,\delta\, \varnothing$.

(P4): Suppose $A \not\delta B$. Then $\operatorname{cl}_Y A \cap \operatorname{cl}_Y B = \varnothing$. Since Y is compact Hausdorff, it is normal [9] and thus there is a $U \in \tau(Y)$ such that $\operatorname{cl}_Y A \subseteq U$ and $\operatorname{cl}_Y B \subseteq Y \setminus \operatorname{cl}_Y U$. Hence $\operatorname{cl}_Y A \cap (Y \setminus U) = \varnothing$ and $\operatorname{cl}_Y B \cap \operatorname{cl}_Y U = \varnothing$. Let $C = X \setminus \operatorname{cl}_Y U$. Then $\operatorname{cl}_Y C \subseteq Y \setminus U$. Therefore, $\operatorname{cl}_Y A \cap \operatorname{cl}_Y C = \varnothing$ and so $A \not\delta C$. Also, note that $\operatorname{cl}_Y U = \operatorname{cl}_Y(X \setminus C)$. Hence, $\operatorname{cl}_Y B \cap \operatorname{cl}_Y(X \setminus C) = \varnothing$ and thus $B \not\delta X \setminus C$.

(P5): Suppose $A \not\delta B$. Then $\operatorname{cl}_Y A \cap \operatorname{cl}_Y B = \varnothing$ and thus $A \cap B = \varnothing$.

(P6): Since X is Hausdorff, the proximity is separated by 2.25.

That δ is compatible with $\tau(X)$ follows from 2.30. □

Theorem 2.36 (Smirnov's Compactification Theorem)**.** *A space admits a compatible separated proximity if and only if it is the subspace of a compact Hausdorff space.*

Proof. Let X be a space and δ a separated proximity compatible with $\tau(X)$. By 2.30, $(X, \tau(\delta))$ is Tychonoff. As δ is compatible with $\tau(X)$, X is Tychonoff. So X is the subspace of a compact Hausdorff space, as are all Tychonoff spaces [10, XVII.4.7].

Conversely, let X be the subspace of a compact Hausdorff space. Then X has a Hausdorff compactification [10, XVII.4.7]. By 2.35, there exists a separated proximity compatible with $\tau(X)$. $\square$

Remark 2.37. As a space X is the subspace of a compact Hausdorff space if and only if X has a compactification [10, XVII.4.7], Smirnov's Compactification Theorem can also be stated as, "A space admits a compatible separated proximity if and only if it has a compactification."

Definition 2.38. Let X be a Tychonoff space and Y, Z be Hausdorff compactifications of X. We say Y is **projectively larger** than Z and write $Y \geq Z$ if and only if there is an $f \in \mathrm{C}(Y, Z)$ such that $f|_X = \mathrm{id}_X$.

Theorem 2.39 (Taimanov's Theorem [18])**.** *Let X be a Tychonoff space and Y, Z Hausdorff compactifications of X. Then $Y \geq Z$ if and only if A and B are closed sets in Z and $A \cap B = \varnothing$ implies $\mathrm{cl}_Y(A \cap X) \cap \mathrm{cl}_Y(B \cap X) = \varnothing$.*

Proof. Suppose $Y \geq Z$. Then there is an $f \in \mathrm{C}(Y, Z)$ such that $f(x) = x$ for all $x \in X$. Suppose $A, B \subseteq Z$ are closed and disjoint. Then $f^{\leftarrow}[A]$ and $f^{\leftarrow}[B]$ are closed and disjoint in Y. So $\mathrm{cl}_Y(A \cap X) \cap \mathrm{cl}_Y(B \cap X) \subseteq f^{\leftarrow}[A] \cap f^{\leftarrow}[B] = \varnothing$.

Conversely, suppose $\mathrm{cl}_Y(A \cap X) \cap \mathrm{cl}_Y(B \cap X) = \varnothing$ for all disjoint closed sets $A, B \subseteq Z$. Let $f = \mathrm{id}_X$. Then $f \in \mathrm{C}(X, Z)$ and f can be extended to a function $F \in \mathrm{C}(Y, Z)$ if and only if the filter base $\mathcal{G}_y = \{f[U \cap X] : y \in U \in \tau(Y)\} = \{U \cap X : y \in U \in \tau(Y)\}$ converges for each $y \in Y$ [14, 4.1(l)].

Since Z is compact, $\mathcal{G}_y$ has non-empty adherence [6, V.5.1]. Note that

$$\mathrm{a}\,\mathcal{G}_y = \bigcap_{y \in U \in \tau(Y)} \mathrm{cl}_Z(U \cap X)$$

We will show that $\mathcal{G}_y$ has only one adherent point. Suppose $p, q \in \mathrm{a}\,\mathcal{G}_y$ and $p \neq q$. Since Z is compact Hausdorff and thus Urysohn [9, p.141], there exist $V, W \in \tau(Z)$ such that $\mathrm{cl}_Z V \cap \mathrm{cl}_Z W = \varnothing$, $p \in V$, and $q \in W$. Then, by the hypothesis,

$$\mathrm{cl}_Y(\mathrm{cl}_Z V \cap X) \cap \mathrm{cl}_Y(\mathrm{cl}_Z W \cap X) = \varnothing$$

If $U \in \tau(Y)$ and $y \in U$ then $p \in \operatorname{cl}_Z(U \cap X)$ since $p \in \operatorname{a}\mathcal{G}_y$. Then $V \cap \operatorname{cl}_Z(U \cap X) \neq \varnothing$ and so $V \cap U \cap X \neq \varnothing$. Since this holds for all $U \in \tau(Y)$ such that $y \in U$, it must be that $y \in \operatorname{cl}_Y(V \cap X)$.

By a similar argument using W instead of V, we get that $y \in \operatorname{cl}_Y(W \cap X)$.

But then

$$\begin{aligned} y &\in \operatorname{cl}_Y(V \cap X) \cap \operatorname{cl}_Y(W \cap X) \\ &\subseteq \operatorname{cl}_Y(\operatorname{cl}_Z V \cap X) \cap \operatorname{cl}_Y(\operatorname{cl}_Z W \cap X) \\ &= \varnothing \end{aligned}$$

This is a contradiction. Hence $\operatorname{a}\mathcal{G}_y = \{p\}$ for some $p \in Z$.

Now we will show that $\mathcal{G}_y$ converges to p. Let $V \in \tau(Z)$ such that $p \in V$. To show convergence, we will show that there is an element $G \in \langle \mathcal{G}_y \rangle$ such that $G \subseteq V$. First, note that

$$\{p\} = \bigcap_{U \in \tau(Y), y \in U} \operatorname{cl}_Z(U \cap X) \subseteq V$$

The family $\{Z \setminus \operatorname{cl}_Z(U \cap X) : y \in U \in \tau(X)\} \cup \{V\}$ is an open cover of Z. But Z is compact, so

$$Z = V \cup \bigcup_{i=1}^{n} (Z \setminus \operatorname{cl}_Z(U_i \cap X))$$

Therefore

$$\bigcap_{i=1}^{n} (U_i \cap X) \subseteq \bigcap_{i=1}^{n} \operatorname{cl}_Z(U_i \cap X) \subseteq V$$

and we have shown convergence. Therefore, there is an $F \in \mathrm{C}(Y, Z)$ such that $F|_X = \operatorname{id}_X$ and so $Y \geq Z$. □

Definition 2.40. Let X be a Tychonoff space and Y, Z be Hausdorff compactifications of X. We say Y is **isomorphic** to Z and write $Y \cong Z$ if there is a homeomorphism $f : Y \to Z$ such that $f|_X = \operatorname{id}_X$.

Proposition 2.41. *Let X be a Tychonoff space and Y, Z be Hausdorff compactifications of X. Then $Y \cong Z$ if and only if $Y \geq Z$ and $Z \geq Y$.*

Proof. Suppose $Y \cong Z$. Then there is a homeomorphism $f : Y \to Z$ such that $f|_X = \operatorname{id}_X$. As $f \in \mathrm{C}(Y, Z)$, $Y \geq Z$. Also, $f^{\leftarrow}|_X = \operatorname{id}_X$ and $f^{\leftarrow} \in \mathrm{C}(Z, Y)$. Therefore, $Z \geq Y$.

Conversely, suppose $Y \geq Z$ and $Z \geq Y$. Then there exist $f \in \mathrm{C}(Y, Z)$ and $g \in \mathrm{C}(Z, Y)$ such that $f|_X = \operatorname{id}_X = g|_X$. Hence $(g \circ f) \in \mathrm{C}(Y, Y)$ and $(g \circ f)|_X = \operatorname{id}_X$. Also $\operatorname{id}_Y \in \mathrm{C}(Y, Y)$ and $\operatorname{id}_Y|_X = \operatorname{id}_X$. A continuous extension of a continous function onto a Hausdorff compactification is unique [14, 1.6(d)]. Therefore, $(g \circ f) = \mathrm{id}_Y$. By a similar argument, $(f \circ g) = \mathrm{id}_Z$. So $f = g^{-1}$. Hence, $f^{-1} \in \mathrm{C}(Z, Y)$. Thus, $f : Y \to Z$ is a homeomorphism and $f|_X = \mathrm{id}_X$. So $Y \cong Z$. □

Definition 2.42. Let δ_1 and δ_2 be two proximities defined on a set X. We say δ_1 is **finer** than δ_2 and write $\delta_1 \geq \delta_2$ if and only if for $A, B \subseteq X$, $A \,\delta_1\, B \Rightarrow A \,\delta_2\, B$.

Theorem 2.43. *Let (X, τ) be a Tychonoff space and Y, Z compactifications of X with respective proximities δ_Y and δ_Z. Then,*

$$Y \geq Z \Leftrightarrow \delta_Y \geq \delta_Z$$

Proof. Suppose $\delta_Y \geq \delta_Z$. Let $A, B \subseteq Z$ be $\tau(Z)$-closed such that $A \cap B = \varnothing$. Then $\mathrm{cl}_Z A \cap \mathrm{cl}_Z B = \varnothing$. By 2.33, $A \not\delta_Z B$. By the hypothesis, $A \not\delta_Y B$. Thus $A \cap X \not\delta_Y B \cap X$. Again by 2.33, $\mathrm{cl}_Y(A \cap X) \cap \mathrm{cl}_Y(B \cap X) = \varnothing$. Hence, by 2.39, $Y \geq Z$.

Conversely, suppose $Y \geq Z$ and $A \not\delta_Z B$. Then there is an $f \in \mathrm{C}(Y, Z)$ such that $f|_X = \mathrm{id}_X$. By 2.33, $\mathrm{cl}_Z A \cap \mathrm{cl}_Z B = \varnothing$. Then $f^{\leftarrow}[\mathrm{cl}_Z A] \cap f^{\leftarrow}[\mathrm{cl}_Z B] = \varnothing$. Since $\mathrm{cl}_Y A = \mathrm{cl}_Y f^{\leftarrow}[A] \subseteq f^{\leftarrow}[\mathrm{cl}_Z A]$ and similarly for $\mathrm{cl}_Y B$, we have $\mathrm{cl}_Y A \cap \mathrm{cl}_Y B = \varnothing$. By 2.33, $A \not\delta_Y B$. □

3. Proximity Filters

Definition 3.1. Let $(X, \ll)$ be a proximity space. A **round filter** on X is a filter $\mathcal{F}$ on X such that for all $A \in \mathcal{F}$, there is a $B \in \mathcal{F}$ such that $B \ll A$.

Definition 3.2. Let $(X, \ll)$ be a proximity space. A maximal round filter on X is called an **end**. We denote the set of all ends on X as $\mathrm{End}(X)$.

Proposition 3.3. *Let $(X, \ll)$ be a proximity space. Every round filter on X is contained in an end.*

Proof. Let $\mathcal{F}$ be a round filter on X. Define $\mathfrak{A} = \{\mathcal{G} : \mathcal{F} \subseteq \mathcal{G}\}$. Note that $\mathfrak{A}$ is partially ordered by set inclusion. Let $\{\mathcal{G}_i\}_{i \in I}$ be a chain in $\mathfrak{A}$. Cleary $\bigcup_{i \in I} \mathcal{G}_i$ is a filter. Let $A \in \bigcup_{i \in I} \mathcal{G}_i$. Then $A \in \mathcal{G}_j$ for some $j \in I$. As $\mathcal{G}_j$ is a round filter, there is a $B \subseteq X$ such that $B \ll A$ and $B \in \mathcal{G}_j$. So $B \in \bigcup_{i \in I} \mathcal{G}_i$. Therefore, $\bigcup_{i \in I} \mathcal{G}_i$ is round. It is also an upper bound on the chain $\{\mathcal{G}_i\}_{i \in I}$. By Zorn's Lemma, $\mathfrak{A}$ has an upper bound $\mathcal{E}$. Clearly $\mathcal{F} \in \mathcal{E}$ and $\mathcal{E}$ is an end. □

Lemma 3.4. *Let $(X, \ll)$ be a proximity space, $\mathcal{F}$ a round filter on X, and $A \subseteq X$ such that $A \cap F \neq \varnothing$ for all $F \in \mathcal{F}$. Then the set*

$$\mathcal{G} = \{C \subseteq X : \exists F \in \mathcal{F}, A \cap F \ll C\}$$

is a round filter.

Proof. First, we verify the three filter axioms.

By way of contradiction, suppose $\varnothing \in \mathcal{G}$. Then there is a $B \in \mathcal{F}$ such that $A \cap B \ll \varnothing$. But then $A \cap B = \varnothing$, a contradiction.

Suppose $B, C \in \mathcal{G}$. Then there are $D, E \in \mathcal{F}$ such that $A \cap D \ll B$ and $A \cap E \ll C$. By 2.19(1), $A \cap D \cap E \ll B \cap C$. But $D \cap E \in \mathcal{F}$ so $B \cap C \in \mathcal{G}$.

Suppose $B \in \mathcal{G}$ and $C \subseteq X$ such that $B \subseteq C$. Then there is an $D \in \mathcal{F}$ such that $A \cap D \ll B \subseteq C$. So $A \cap D \ll C$. Therefore $C \in \mathcal{G}$.

Now, to show $\mathcal{G}$ is round, suppose $B \in \mathcal{G}$. Then there is a $C \in \mathcal{F}$ such that $A \cap C \ll B$. By (Q5), there is a $D \subseteq X$ such that $A \cap C \ll D \ll C$. So $D \in \mathcal{G}$ and $D \ll B$. $\square$

Theorem 3.5. *Let $(X, \ll)$ be a proximity space and $\mathcal{F}$ a round filter on X. Then $\mathcal{F}$ is an end if and only if, for all $A, B \subseteq X$, $A \ll B$ implies $X \setminus A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

Proof. Suppose $\mathcal{F}$ is an end and $A \ll B$. We will proceed in two cases.

Case 1: Suppose $F \cap A \neq \varnothing$ for all $F \in \mathcal{F}$. By 3.4, $\mathcal{G} = \{C \subseteq X : \exists F \in \mathcal{F}, A \cap F \ll C\}$ is a round filter. Let $F \in \mathcal{F}$. Since $\mathcal{F}$ is round, there is a $D \in \mathcal{F}$ such that $D \ll F$. Hence $A \cap D \ll F$. Therefore, $F \in \mathcal{G}$. So $\mathcal{F} \subseteq \mathcal{G}$. But, by the hypothesis, $\mathcal{F}$ is an end, so $\mathcal{F} = \mathcal{G}$. Now, $A \ll B$ implies $A \cap F \ll B$ for all $F \in \mathcal{F}$. Hence $B \in \mathcal{G}$ and thus $B \in \mathcal{F}$.

Case 2: Now suppose there is an $F \in \mathcal{F}$ such that $F \cap A = \varnothing$. Then $F \subseteq X \setminus A$. So $X \setminus A \in \mathcal{F}$.

Conversely, suppose $\mathcal{F}$ is a round filter and $A \ll B$ implies $X \setminus A \in \mathcal{F}$ or $B \in \mathcal{F}$. By way of contradiction, suppose there is a round filter $\mathcal{G}$ such that $\mathcal{F} \subset \mathcal{G}$. Take $G \in \mathcal{G} \setminus \mathcal{F}$. Since $\mathcal{G}$ is round, there is an $H \in \mathcal{G}$ such that $H \ll G$. By the hypothesis, $X \setminus H \in \mathcal{F}$ or $G \in \mathcal{F}$. But $G \notin \mathcal{F}$ by how we chose it, so it must be that $X \setminus H \in \mathcal{F}$. Thus $X \setminus H \in \mathcal{G}$. But also $H \in \mathcal{G}$, so $\varnothing = (X \setminus H) \cap H \in \mathcal{G}$, a contradiction. $\square$

Theorem 3.6. *Let X be a Tychonoff space and $\ll$ a proximity on X compatible with $\tau(X)$. Then for all $p \in X$, $\mathcal{N}(p)$ is an end.*

Proof. It is trivial to show that $\mathcal{N}(p)$ is a filter. To show it is round, take $V \in \mathcal{N}(p)$. Then there is a $U \in \tau(\ll)$ such that $p \in U \subseteq V$. By 2.11, $p \not\delta X \setminus U$. Thus $p \ll U$. By 2.23, there is a $W \in \tau(\ll)$ such that $p \ll W \ll U$. It is clear that $W \in \mathcal{N}(p)$ and thus $\mathcal{N}(p)$ is round.

Now, we will show that $\mathcal{N}(p)$ is an end. Let $A, B \subseteq X$ such that $A \ll B$. Then there is a $C \subseteq X$ such that $A \ll C \ll B$. Either $p \in C$ or $p \in X \setminus C$.

If $p \in C$ then by 2.18(2), $C \subseteq \operatorname{cl} C \ll \operatorname{int} B$. So $x \in \operatorname{int} B$ and thus $B \in \mathcal{N}(p)$.

If $p \in X \setminus C$, then since $A \ll C$, $X \setminus C \ll X \setminus A$. By 2.18(2), $\mathrm{cl}(X \setminus C) \subseteq \mathrm{int}(X \setminus A)$ and so $p \in \mathrm{int}(X \setminus A)$. Therefore, $X \setminus A \in \mathcal{N}(p)$.

By 3.5, $\mathcal{N}(p)$ is an end. □

Definition 3.7. Let $(X, \ll)$ be a proximity space and $\mathcal{F}$ a filter on X. The **round hull** of $\mathcal{F}$ is $\mathrm{rnd}(\mathcal{F}) = \{A \subseteq X : \exists F \in \mathcal{F}, F \ll A\}$.

Proposition 3.8. *Let $(X, \ll)$ be a proximity space and $\mathcal{F}$ a filter on X. Then $\mathrm{rnd}(\mathcal{F})$ is a round filter and $\mathrm{rnd}(\mathcal{F}) \subseteq \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a filter on X. First, we will show that $\mathrm{rnd}(\mathcal{F})$ is a filter.

Suppose, by way of contradiction, that $\varnothing \in \mathrm{rnd}(\mathcal{F})$. Then there is an $F \in \mathcal{F}$ such that $F \ll \varnothing$. But then $\varnothing = F \in \mathcal{F}$, a contradiction.

Let $H, I \in \mathrm{rnd}(\mathcal{F})$. Then there exist $F, G \in \mathcal{F}$ such that $F \ll H$ and $G \ll I$. By 2.19(1), $F \cap G \ll H \cap I$. Since $F \cap G \in \mathcal{F}$ then $H \cap I \in \mathrm{rnd}(\mathcal{F})$.

Let $H \in \mathrm{rnd}(\mathcal{F})$ and $A \subseteq X$ such that $H \subseteq A$. Then there is an $F \in \mathcal{F}$ such that $F \ll H$. So $F \ll A$. Hence $A \in \mathrm{rnd}(\mathcal{F})$.

To show $\mathrm{rnd}(\mathcal{F})$ is round, take $H \in \mathrm{rnd}(\mathcal{F})$. Then there is an $F \in \mathcal{F}$ such that $F \ll H$. So there is an $A \subseteq X$ such that $F \ll A \ll H$. Then $A \in \mathrm{rnd}(\mathcal{F})$ and $A \ll H$. So $\mathrm{rnd}(\mathcal{F})$ is round.

Now, we will show containment. Let $F \in \mathrm{rnd}(\mathcal{F})$. Then there is a $G \in \mathcal{F}$ such that $G \ll F$. So $G \in \mathcal{F}$ and $G \subseteq F$ implying $F \in \mathcal{F}$. □

Theorem 3.9. *Let $(X, \ll)$ be a proximity space and $\mathcal{U}$ an ultrafilter on X. Then $\mathcal{U}$ contains a unique end, namely $\mathrm{rnd}(\mathcal{U})$.*

Proof. Suppose $\mathcal{U}$ is an ultrafilter on X.

First, we will show that $\mathrm{rnd}(\mathcal{U})$ is an end. By 3.8, $\mathrm{rnd}(\mathcal{U})$ is a round filter and $\mathrm{rnd}(\mathcal{U}) \subseteq \mathcal{U}$. We need to show it is maximal. Let $A, B \subseteq X$ such that $A \ll B$. Then there is a $C \subseteq X$ such that $A \ll C \ll B$. If $C \in \mathcal{U}$ then $B \in \mathrm{rnd}(\mathcal{U})$. If $C \notin \mathcal{U}$ then, since $\mathcal{U}$ is an ultrafilter, $X \setminus C \in \mathcal{U}$. Since $X \setminus C \ll X \setminus A$, it follows that $X \setminus A \in \mathrm{rnd}(\mathcal{U})$. By 3.5, $\mathrm{rnd}(\mathcal{U})$ is an end.

To show uniqueness, suppose by way of contradiction that $\mathcal{E}$ is an end on X contained in $\mathcal{U}$ and that there is an $A \in \mathcal{E} \setminus \mathrm{rnd}(\mathcal{U})$. Since $\mathcal{E}$ is round, there is a $B \in \mathcal{E}$ such that $B \ll E$. By 3.5, either $X \setminus B \in \mathrm{rnd}(\mathcal{U})$ or $A \in \mathrm{rnd}(\mathcal{U})$. By our choice of A, the latter case cannot happen. So $X \setminus B \in \mathrm{rnd}(\mathcal{U}) \subseteq \mathcal{U}$ and $B \in \mathcal{E} \subseteq \mathcal{U}$. Thus $\varnothing = (X \setminus B) \cap B \in \mathcal{U}$, a contradiction. □

Definition 3.10. Let $(X, \ll)$ be a proximity space and $A \subseteq X$. Define

$$\mathrm{o}(A) = \{\mathcal{E} \in \mathrm{End}(X) : A \in \mathcal{E}\}$$

Proposition 3.11. *Let $(X, \ll)$ be a proximity space. For all $A, B \subseteq X$,*

(1) $A = \varnothing \Leftrightarrow \mathrm{o}(A) = \varnothing$
(2) $A \subseteq B \Rightarrow \mathrm{o}(A) \subseteq \mathrm{o}(B)$
(3) $\mathrm{o}(A) = \mathrm{o}(\operatorname{int} A)$
(4) $\mathrm{o}(A) \cap \mathrm{o}(B) = \mathrm{o}(A \cap B)$
(5) $\mathrm{o}(A) \cup \mathrm{o}(B) \subseteq \mathrm{o}(A \cup B)$
(6) $A \ll B \Rightarrow \mathrm{End}(X) \setminus \mathrm{o}(B) \subseteq \mathrm{o}(X \setminus A)$

Proof.

(1) Since $\varnothing$ is not in any end, $\mathrm{o}(\varnothing) = \varnothing$.
Suppose $\varnothing \neq A \subseteq X$. Take $a \in A$. By (Q6), there is a $B \subseteq X$ such that $a \ll B \ll A$. Let $\mathcal{F} = \{C \subseteq X : a \in C\}$. It is easy to check that $\mathcal{F}$ is a filter. Also, note that $B \in \mathcal{F}$. Since $B \ll A$ then $A \in \mathrm{rnd}(\mathcal{F})$. By 3.8, $\mathrm{rnd}(\mathcal{F})$ is round and, by 3.3, every round filter is contained in an end. Therefore, there is an end containing A. Thus $\mathrm{o}(A) \neq \varnothing$.
(2) Let $\mathcal{A} \in \mathrm{o}(A)$. Then $A \in \mathcal{A}$. Since $\mathcal{A}$ is an end and $A \subseteq B$ then $B \in \mathcal{A}$. Hence $\mathcal{A} \in \mathrm{o}(A)$.
(3) Since $\operatorname{int} A \subseteq A$, by 3.11(2), $\mathrm{o}(\operatorname{int} A) \subseteq \mathrm{o}(A)$.
For the reverse inclusion, let $\mathcal{E} \in \mathrm{o}(A)$. Then $A \in \mathcal{E}$. Since $\mathcal{E}$ is round, there is a $B \in \mathcal{E}$ such that $B \ll A$. By 2.18(2), $B \subseteq \operatorname{int} A$. Therefore, $\operatorname{int} A \in \mathcal{E}$. So $\mathcal{E} \in \mathrm{o}(\operatorname{int} A)$.
(4) Let $\mathcal{A} \in \mathrm{o}(A \cap B)$. Then $A \cap B \in \mathcal{A}$. Since $\mathcal{A}$ is filter, $A, B \in \mathcal{A}$. So $\mathcal{A} \in \mathrm{o}(A)$ and $\mathcal{A} \in \mathrm{o}(B)$. Hence $\mathcal{A} \in \mathrm{o}(A) \cap \mathrm{o}(B)$.
The converse is the reverse of this argument.
(5) Let $\mathcal{A} \in \mathrm{o}(A) \cup \mathrm{o}(B)$. Then $\mathcal{A} \in \mathrm{o}(A)$ or $\mathcal{A} \in \mathrm{o}(B)$. Thus $A \in \mathcal{A}$ or $B \in \mathcal{A}$. So $A \cup B \in \mathcal{A}$ which implies $\mathcal{A} \in \mathrm{o}(A \cup B)$.
(6) Let $\mathcal{E} \in \mathrm{End}(X) \setminus \mathrm{o}(B)$. Then $B \notin \mathcal{E}$. By 3.5, since $A \ll B$, $X \setminus A \in \mathcal{E}$. Hence $\mathcal{E} \in \mathrm{o}(X \setminus A)$.

□

Definition 3.12. Let (X, δ) be a proximity space. For $\mathcal{A}, \mathcal{B} \subseteq \mathrm{End}(X)$, define $\mathcal{A} \not\delta_{\mathrm{E}} \mathcal{B}$ if and only if there exist $A, B \subseteq X$ such that $\mathcal{A} \subseteq \mathrm{o}(A)$, $\mathcal{B} \subseteq \mathrm{o}(B)$, and $A \not\delta B$. As expected, also define $\mathcal{A} \ll_{\mathrm{E}} \mathcal{B}$ if and only if $\mathcal{A} \not\delta_{\mathrm{E}} \mathrm{End}(X) \setminus \mathcal{B}$.

Proposition 3.13. *Let (X, δ) be a proximity space. Then δ_E is a proximity on* $\mathrm{End}(X)$.

Proof. We will verify the proximity axioms.
(P1): Trivial.

(P2): Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \text{End}(X)$ and suppose $\mathcal{A} \not\delta_{\text{E}} \mathcal{B} \cup \mathcal{C}$. Then there are $A, B \subseteq X$ such that $\mathcal{A} \subseteq \text{o}(A)$, $\mathcal{B} \cup \mathcal{C} \subseteq \text{o}(B)$, and $A \not\delta B$. So $\mathcal{B} \subseteq \text{o}(B)$ and $\mathcal{C} \subseteq \text{o}(B)$. Hence $\mathcal{A} \not\delta_{\text{E}} \mathcal{B}$ and $\mathcal{A} \not\delta_{\text{E}} \mathcal{C}$.

Conversely, suppose $\mathcal{A} \not\delta_{\text{E}} \mathcal{B}$ and $\mathcal{A} \not\delta_{\text{E}} \mathcal{C}$. Then there exist $A, B \subseteq X$ such that $\mathcal{A} \subseteq \text{o}(A)$, $\mathcal{B} \subseteq \text{o}(B)$, and $A \not\delta B$. Also, there exist $C, D \subseteq X$ such that $\mathcal{A} \subseteq \text{o}(C)$, $\mathcal{C} \subseteq \text{o}(D)$, and $C \not\delta D$. So $\mathcal{A} \subseteq \text{o}(A) \cap \text{o}(C) = \text{o}(A \cap C)$ and $\mathcal{B} \cup \mathcal{C} \subseteq \text{o}(B) \cup \text{o}(D) \subseteq \text{o}(B \cup D)$. Furthermore, $A \not\delta B$ implies $A \cap C \not\delta B$ and $C \not\delta D$ implies $A \cap C \not\delta D$. Therefore $A \cap C \not\delta B \cup D$. Hence $\mathcal{A} \not\delta_{\text{E}} \mathcal{B} \cup \mathcal{C}$.

(P3): Suppose $\varnothing \not\delta_{\text{E}} \mathcal{B}$. Then there is an $A \subseteq X$ such that $\mathcal{A} \in \text{o}(\varnothing) = \varnothing$, a contradiction.

(P4): Suppose $\mathcal{A} \not\delta_{\text{E}} \mathcal{B}$. Then there exist $A, B \subseteq X$ such that $\mathcal{A} \subseteq \text{o}(A)$, $\mathcal{B} \subseteq \text{o}(B)$, and $A \not\delta B$. Since $A \not\delta B$ then there exists a $C \subseteq X$ such that $A \not\delta C$ and $B \not\delta X \setminus C$.

Let $\mathcal{C} = \text{o}(C)$. Since $\mathcal{A} \subseteq \text{o}(A)$, $\mathcal{C} \subseteq \text{o}(C)$, and $A \not\delta C$, it follows that $\mathcal{A} \not\delta_{\text{E}} \mathcal{C}$. As $B \not\delta X \setminus C$, there exists a $D \subseteq X$ such that $B \not\delta X \setminus D$ and $D \not\delta X \setminus C$. Therefore $D \ll C$. By 3.11(6), $\text{End}(X) \setminus \mathcal{C} = \text{End}(X) \setminus \text{o}(C) \subseteq \text{o}(X \setminus D)$. So we have $\text{End}(X) \setminus \mathcal{C} \subseteq \text{o}(X \setminus D)$, $\mathcal{B} \subseteq \text{o}(B)$, and $B \not\delta X \setminus D$, we have that $\mathcal{B} \not\delta_{\text{E}} \text{End}(X) \setminus \mathcal{C}$.

(P5): Suppose $\mathcal{A} \not\delta_{\text{E}} \mathcal{B}$ Then there exist $A, B \subseteq X$ such that $\mathcal{A} \subseteq \text{o}(A)$, $\mathcal{B} \subseteq \text{o}(B)$, and $A \not\delta B$. Since $A \not\delta B$ then $A \cap B = \varnothing$. Hence $\mathcal{A} \cap \mathcal{B} \subseteq \text{o}(A) \cap \text{o}(B) = \text{o}(A \cap B) = \text{o}(\varnothing) = \varnothing$. □

Remark 3.14. Since $(\text{End}(X), \delta_{\text{E}})$ is a proximity space, δ_{E} induces a topology $\tau(\delta_{\text{E}})$ on $\text{End}(X)$ as defined in 2.10.

Lemma 3.15. *Let (X, δ) be a proximity space, $\mathcal{E} \in \text{End}(X)$, and $\mathcal{A} \subseteq \text{End}(X)$. Then $\mathcal{E} \not\delta_E \mathcal{A}$ if and only if there exists an $A \in \mathcal{E}$ such that $\text{o}(A) \cap \mathcal{A} = \varnothing$.*

Proof. Suppose $\mathcal{E} \not\delta_{\text{E}} \mathcal{A}$. Then there exist $A, B \subseteq X$ such that $\mathcal{E} \in \text{o}(A)$, $\mathcal{A} \subseteq \text{o}(B)$, and $A \not\delta B$. So $A \cap B = \varnothing$ implying that $\text{o}(A) \cap \text{o}(B) = \text{o}(A \cap B) = \text{o}(\varnothing) = \varnothing$. Since $\mathcal{A} \subseteq \text{o}(B)$ we have $\text{o}(A) \cap \mathcal{A} = \varnothing$. Also, $\mathcal{E} \in \text{o}(A)$ implies $A \in \mathcal{E}$.

Conversely, suppose there is an $A \in \mathcal{E}$ such that $\text{o}(A) \cap \mathcal{A} = \varnothing$. Since $\mathcal{E}$ is a round filter, there exist $B, C \in \mathcal{E}$ such that $C \ll B \ll A$. So $C \not\delta X \setminus B$. Also, $\text{o}(A) \cap \mathcal{A} = \varnothing$ and 3.11(6) implies $\mathcal{A} \subseteq \text{End}(X) \setminus \text{o}(A) \subseteq \text{o}(X \setminus B)$. Lastly, $C \in \mathcal{E}$ implies $\mathcal{E} \in \text{o}(C)$. So $\mathcal{E} \not\delta_{\text{E}} \mathcal{A}$. □

Theorem 3.16. *Let $(X, \ll)$ be a proximity space. Then $\mathcal{B} = \{\text{o}(U) : U \in \tau(\ll)\}$ is a base for $\tau(\delta_E)$.*

Proof. First, we show that $\text{o}(U) \in \tau(\delta_{\text{E}})$ for all $U \in \tau(\ll)$. Let $U \in \tau(\ll)$ and take $\mathcal{E} \in \text{o}(U)$. Then $U \in \mathcal{E}$. Note that $\text{o}(U) \cap \text{End}(X) \setminus \text{o}(U) = \varnothing$. So by 3.15, $\mathcal{E} \not\delta_{\text{E}} \text{End}(X) \setminus \text{o}(U)$. So $\mathcal{E} \notin \text{cl}_{\tau(\delta_{\text{E}})}(\text{End}(X) \setminus$

$\mathrm{o}(U)) = \mathrm{End}(X) \setminus \mathrm{int}_{\tau(\delta_{\mathrm{E}})} \mathrm{o}(U)$. Therefore $\mathcal{E} \in \mathrm{int}_{\tau(\delta_{\mathrm{E}})} \mathrm{o}(U)$. Hence $\mathrm{o}(U) \in \tau(\delta_{\mathrm{E}})$.

Let $\mathcal{U} \in \tau(\delta_{\mathrm{E}})$ and take $\mathcal{E} \in \mathcal{U}$. By 2.11, $\mathcal{E} \not\delta_{\mathrm{E}} (\mathrm{End}(X) \setminus \mathcal{U})$. By 3.15, there is an $E \in \mathcal{E}$ such that $\mathrm{o}(E) \cap (\mathrm{End}(X) \setminus \mathcal{U}) = \varnothing$. By 3.11(3), $\mathrm{o}(E) = \mathrm{o}(\mathrm{int}\, E)$. So $\mathcal{E} \in \mathrm{o}(\mathrm{int}\, E) \subseteq \mathcal{U}$ and $\mathrm{o}(\mathrm{int}\, E) \in \mathcal{B}$. $\square$

Proposition 3.17. *Let $(X, \ll)$ be a proximity space. For $A \subseteq X$, $o(X \setminus A) = \mathrm{End}(X) \setminus \mathrm{cl}\, \mathrm{o}(A)$.*

Proof. As $A \cap (X \setminus A) = \varnothing$, by 3.11(1), $\mathrm{o}(A \cap (X \setminus A)) = \varnothing$. By 3.11(4), $\mathrm{o}(A) \cap \mathrm{o}(X \setminus A) = \varnothing$. So $\mathrm{cl}\, \mathrm{o}(A) \cap \mathrm{o}(X \setminus A) = \varnothing$. Therefore, $\mathrm{o}(X \setminus A) \subseteq \mathrm{End}(X) \setminus \mathrm{cl}\, \mathrm{o}(A)$.

For the opposite inclusion, suppose $\mathcal{E} \in \mathrm{End}(X) \setminus \mathrm{cl}\, \mathrm{o}(A)$. As $\mathcal{E} \notin \mathrm{cl}\, \mathrm{o}(A)$, there is a $U \in \tau(\ll_{\mathrm{E}})$ such that $\mathcal{E} \in \mathrm{o}(U)$ and $\mathrm{o}(A) \cap \mathrm{o}(U) = \varnothing$. By 3.11(4), $\mathrm{o}(A \cap U) = \varnothing$. By 3.11(1), $A \cap U = \varnothing$. So $U \subseteq X \setminus A$. By 3.11(2), $\mathrm{o}(U) \subseteq \mathrm{o}(X \setminus A)$. As $\mathcal{E} \in \mathrm{o}(U)$, $\mathcal{E} \in \mathrm{o}(X \setminus A)$. $\square$

Definition 3.18. For a proximity space (X, δ), define $e : X \to \mathrm{End}(X)$ by $e(x) = \mathcal{N}(x)$.

Lemma 3.19. *Let (X, δ) be a proximity space such that $(X, \tau(\delta))$ is Tychonoff. For $A, B \subseteq X$,*

$$A \,\delta\, B \Leftrightarrow e[A] \,\delta_E\, e[B]$$

Proof. Let $A, B \subseteq X$ such that $e[A] \not\delta\, e[B]$. Then there exist $C, D \subseteq X$ such that $e[A] \subseteq \mathrm{o}(C)$, $e[B] \subseteq \mathrm{o}(D)$, and $C \not\delta\, D$. So $C \in e(a)$ and thus $C \in \mathcal{N}(a)$ for all $a \in A$. Also $D \in e(b)$ and thus $D \in \mathcal{N}(b)$ for all $b \in B$. Therefore $a \in C$ for all $a \in A$ and $b \in D$ for all $b \in B$. Also, $A \subseteq C$ and $B \subseteq D$. Since $C \not\delta\, D$ then $A \not\delta\, B$.

Conversely, let $A, B \subseteq X$ such that $A \not\delta\, B$. Then $A \ll X \setminus B$. So there exist $U, V \in \tau(\delta)$ such that $A \ll U \ll V \ll X \setminus B$. Then $A \ll U \ll \mathrm{cl}\, V \subseteq X \setminus B$ and thus $A \subseteq U$ and $B \subseteq X \setminus \mathrm{cl}\, V$. Hence for all $a \in A$, $U \in e(a)$ and for all $b \in B$, $X \setminus \mathrm{cl}\, V \in e(b)$. So $e(a) \in \mathrm{o}(U)$ and $e(b) \in \mathrm{o}(X \setminus \mathrm{cl}\, V)$ for all $a \in A$, $b \in B$. So $e[A] \subseteq \mathrm{o}(U)$ and $e[B] \subseteq \mathrm{o}(X \setminus \mathrm{cl}\, V)$. As $U \ll \mathrm{cl}\, V$, $U \not\delta\, X \setminus \mathrm{cl}\, V$. Therefore $e[A] \not\delta\, e[B]$. $\square$

Lemma 3.20. *Let (X, δ) be a proximity space. For $U \in \tau(\delta)$, $e[U] = \mathrm{o}(U) \cap e[X]$.*

Proof. Let $\mathcal{A} \in e[U]$. Then $\mathcal{A} = \mathcal{N}(u)$ for some $u \in U$. But for all $v \in U$, $U \in \mathcal{N}(v)$. Therefore $U \in \mathcal{A}$ and thus $\mathcal{A} \in \mathrm{o}(U)$. Also, $\mathcal{A} \in e[U] \subseteq e[X]$. So $e[U] \subseteq \mathrm{o}(U) \cap e[X]$.

For the reverse inclusion, let $\mathcal{A} \in \mathrm{o}(U) \cap e[X]$. Then $\mathcal{A} \in \mathrm{o}(U)$ and $\mathcal{A} \in e[X]$. So $U \in \mathcal{A}$ and there is an $x \in X$ such that $\mathcal{A} = \mathcal{N}(x)$. So $U \in \mathcal{N}(x)$ and thus $x \in U$. Therefore $A \in e[U]$. $\square$

Lemma 3.21. *Let (X, δ) be a proximity space. For an ultrafilter $\mathcal{U}$ on* $\mathrm{End}(X)$*, $\mathcal{E} = \{A \subseteq X : e[A] \in \mathrm{rnd}(\mathcal{U})\}$ is an end on X.*

Proof. First we will show that $\mathcal{E}$ is a filter.

Note that $e[\varnothing] = \varnothing$ so $e[\varnothing] \notin \mathrm{rnd}(\mathcal{U})$ since $\mathrm{rnd}(\mathcal{U})$ is a filter. Hence $\varnothing \notin \mathcal{E}$.

Let $A, B \in \mathcal{E}$. Then $e[A], e[B] \in \mathrm{rnd}(\mathcal{U})$. So $e[A] \cap e[B] \in \mathrm{rnd}(\mathcal{U})$. Since e is a homeomorphism onto $e[X]$, $e[A \cap B] = e[A] \cap e[B]$ and so $e[A \cap B] \in \mathrm{rnd}(\mathcal{U})$. Thus, $A \cap B \in \mathcal{E}$.

Let $A \in \mathcal{E}$ and $B \subseteq X$ such that $A \subseteq B$. Then $e[A] \subseteq e[B]$. Since $e[A] \in \mathrm{rnd}(\mathcal{U})$ then $e[B] \in \mathrm{rnd}(\mathcal{U})$. So $B \in \mathcal{E}$.

To show $\mathcal{E}$ is round, take $A \in \mathcal{E}$. Then $e[A] \in \mathrm{rnd}(\mathcal{U})$. So there is a $\mathcal{B} \in \mathrm{rnd}(\mathcal{U})$ such that $\mathcal{B} \ll_{\mathrm{E}} e[A]$. By 3.19, $e^{\leftarrow}[\mathcal{B}] \ll e^{\leftarrow}[e[A]]$ and as $e^{\leftarrow}[e[A]] = A$, $e^{\leftarrow}[\mathcal{B}] \ll A$. Now, note that, since $\mathcal{B} \subseteq e[A]$, $\mathcal{B} \subseteq e[X]$. Hence $e[e^{\leftarrow}[\mathcal{B}]] = \mathcal{B}$. As $\mathcal{B} \in \mathrm{rnd}(\mathcal{U})$, $e[e^{\leftarrow}[\mathcal{B}]] \in \mathrm{rnd}(\mathcal{U})$. Therefore $e^{\leftarrow}[\mathcal{B}] \in \mathcal{E}$ and $e^{\leftarrow}[\mathcal{B}] \ll A$. So $\mathcal{E}$ is round.

Finally, we must show that $\mathcal{E}$ is an end. Let $A, B \subseteq X$ such that $A \ll B$ and $X \setminus A \notin \mathcal{E}$. By 3.5, it is enough to show that $B \in \mathcal{E}$. Since $A \ll B$ then $X \setminus B \ll X \setminus A$. So by 3.19, $e[X \setminus B] \ll_{\mathrm{E}} e[X \setminus A]$. Since $X \setminus A \notin \mathcal{E}$ then $e[X \setminus A] \notin \mathrm{rnd}(\mathcal{U})$. But, $\mathrm{rnd}(\mathcal{U})$ is an end, so $\mathrm{End}(X) \setminus e[X \setminus B] \in \mathrm{rnd}(\mathcal{U})$. Therefore $e^{\leftarrow}[\mathrm{End}(X) \setminus e[X \setminus B]] \in \mathcal{E}$. So as $e^{\leftarrow}[\mathrm{End}(X) \setminus e[X \setminus B]] = X \setminus (X \setminus B) = B$, we have that $B \in \mathcal{E}$. ☐

Theorem 3.22. *Let (X, δ) be a proximity space. Then $(\mathrm{End}(X), \tau(\delta_E))$ is compact Hausdorff.*

Proof. First we will show that $(\mathrm{End}(X), \tau(\delta_{\mathrm{E}}))$ is Hausdorff.

Let $\mathcal{E}, \mathcal{F} \in \mathrm{End}(X)$ such that $\mathcal{E} \neq \mathcal{F}$. Let $B \in \mathcal{E} \setminus \mathcal{F}$. Then there is an $A \in \mathcal{E}$ such that $A \ll B$. Since $B \notin \mathcal{F}$ and $\mathcal{F}$ is an end, then by 3.5, $X \setminus A \in \mathcal{F}$. So $A \in \mathcal{E}$ and $X \setminus A \in \mathcal{F}$. Also, $\mathrm{o}(A) \cap \mathrm{o}(X \setminus A) = \mathrm{o}(\varnothing) = \varnothing$.

A space is compact if and only if every ultrafilter converges [8]. Let $\mathcal{U}$ be an ultrafilter on $\mathrm{End}(X)$. We will show that $\mathcal{U}$ converges to $\mathcal{E} = \{A \subseteq X : e[A] \in \mathrm{rnd}(\mathcal{U})\}$ which is an end by 3.21. Let $A \subseteq X$ such that $\mathcal{E} \in \mathrm{o}(A)$. Then it is enough to show that $\mathrm{o}(A) \in \mathcal{U}$. Since $\mathcal{E} \in \mathrm{o}(A)$ then $A \in \mathcal{E}$. Since $\mathcal{E}$ is round, there is a $B \in \mathcal{E}$ such that $B \ll A$. Then there is a $U \in \tau(\delta)$ such that $B \ll U \ll A$. As $B \subseteq U$ and $B \in \mathcal{E}$ then $U \in \mathcal{E}$. Therefore, $e[U] \in \mathrm{rnd}(\mathcal{U})$. By 3.20, $e[U] \subseteq \mathrm{o}(U)$ and thus $\mathrm{o}(U) \in \mathrm{rnd}(\mathcal{U})$. Since $U \subseteq A$, by 3.11(2), $\mathrm{o}(U) \subseteq \mathrm{o}(A)$. Hence $\mathrm{o}(A) \in \mathrm{rnd}(\mathcal{U}) \subseteq \mathcal{U}$. ☐

Lemma 3.23. *Let $(X, \ll)$ be a proximity space such that $(X, \tau(\ll))$ is Tychonoff. Then e is a homeomorphism onto its image.*

Proof. To show e is continuous, let $x \in X$. Then $e(x) = \mathcal{N}(x)$. By 3.16, there is a $U \in \tau(\ll)$ such that $\mathcal{N}(x) \in \mathrm{o}(U)$. By 3.20, $e[U] \subseteq \mathrm{o}(U)$. So e is continuous.

To show that e is injective, let $a, b \in X$ such that $a \neq b$. Since $(X, \tau(\delta))$ is Hausdorff and thus δ is separated, $a \not\delta\, b$. By 3.19, $e(a) \not\delta\, e(b)$. By (P5), $e(a) \neq e(b)$.

Since the property of being Hausdorff is hereditary [12, p.67], $e[X]$ is Hausdorff. As e is a continuous bijection from a compact space to a Hausdorff space, e is a homeomorphism [7, p.316]. $\square$

Lemma 3.24. *Let (X, δ) be a proximity space. Then $e[X]$ is dense in $(\mathrm{End}(X), \tau(\delta_E))$.*

Proof. By way of contradiction, suppose $\mathcal{A} \in \mathrm{End}(X) \setminus \mathrm{cl}_{\tau(\delta_{\mathrm{E}})}\, e[X]$. Then there exist $A, B \subseteq X$ such that $\mathcal{A} \subseteq \mathrm{o}(A)$, $e[X] \subseteq \mathrm{o}(B)$, and $A \not\delta\, B$. Since $e[X] \subseteq \mathrm{o}(B)$ then $e(x) \in \mathrm{o}(B)$ for all $x \in X$. Hence $B \in e(x)$ for all $x \in X$. So $B = X$. Since $A \not\delta\, B$ then it must be that $A = \varnothing$. As $\mathcal{A} \subseteq \mathrm{o}(\varnothing)$, $\mathcal{A} = \varnothing$. But $\mathcal{A} \in \mathrm{End}(X)$, a contradiction. $\square$

Theorem 3.25. *Let (X, δ) be a proximity space such that $(X, \tau(\delta))$ is Tychonoff. Then $(\mathrm{End}(X), \delta_E)$ is a Hausdorff compactification of (X, δ).*

Proof. This follows immediately from 3.22, 3.23, and 3.24. $\square$

Corollary 3.26. *Let (X, δ) be a proximity space such that $(X, \tau(\delta))$ is Tychonoff. For $A, B \subseteq X$,*

$$A \,\delta\, B \Leftrightarrow \mathrm{cl}_{\tau(\delta_E)}\, e[A] \cap \mathrm{cl}_{\tau(\delta_E)}\, e[B] = \varnothing$$

Proof. Let $A, B \subseteq X$. By 3.19, $A \,\delta\, B$ if and only if $e[A] \;\delta_{\mathrm{E}}\; e[B]$. By 3.22, $(\mathrm{End}(X), \tau(\delta_{\mathrm{E}}))$ is compact Hausdorff so, by 2.33, $e[A] \;\delta_{\mathrm{E}}\; e[B]$ if and only if $\mathrm{cl}_{\tau(\delta_{\mathrm{E}})}\, e[A] \cap \mathrm{cl}_{\tau(\delta_{\mathrm{E}})}\, e[B] = \varnothing$. $\square$

Theorem 3.27 (Smirnov's Theorem [16])**.** *Let X be Tychonoff, $\mathcal{D}$ the collection of all proximities on X compatible with $\tau(X)$, and $\mathcal{K}$ the collection of all Hausdorff compactifications of X. Then*

$$\nu : \mathcal{D} \to \mathcal{K} : \delta \mapsto (\mathrm{End}(X), \tau(\delta_E))$$

is a bijection.

Proof. First, we will show ν is injective. Let $Y, Z \in \mathcal{K}$ such that $Y \cong Z$ and let δ_Y and δ_Z be proximities on Y and Z respectively. By 2.41, $Y \geq Z$ and $Z \geq Y$. By 2.43, $\delta_Y \geq \delta_Z$ and $\delta_Z \geq \delta_Y$. Hence $\delta_Y = \delta_Z$. Therefore, ν is injective.

To show ν is surjective, take $Y \in \mathcal{K}$. As Y is compact Hausdorff, by 2.33, there is a unique proximity δ_Y on Y compatible with $\tau(Y)$ and is defined by, for $A, B \subseteq Y$, $A \,\delta_Y\, B$ if and only if $\mathrm{cl}_Y A \cap \mathrm{cl}_Y B = \varnothing$. Define δ by, for $A, B \subseteq X$, $A \,\delta\, B$ if and only if $A \,\delta_Y\, B$. By 2.13, δ is a proximity on X.

Let $\mathcal{A}, \mathcal{B} \subseteq \mathrm{End}(X)$ such that $\mathcal{A}, \mathcal{B}$ are disjoint and closed under the topology $\tau(\delta_{\mathrm{E}})$. Then $\mathcal{A} \not\delta_{\mathrm{E}} \mathcal{B}$. By 3.19, $e^{\leftarrow}[\mathcal{A}] \not\delta\, e^{\leftarrow}[\mathcal{B}]$. Thus, $e^{\leftarrow}[\mathcal{A}] \not\delta_Y\, e^{\leftarrow}[\mathcal{B}]$. So $\mathrm{cl}_Y e^{\leftarrow}[\mathcal{A}] \cap \mathrm{cl}_Y e^{\leftarrow}[\mathcal{B}] = \varnothing$. By 2.39, $Y \geq \mathrm{End}(X)$.

Now, let $A, B \subseteq Y$ such that A, B are disjoint closed. Then $A \not\delta_Y B$. Then $A \cap X \not\delta\, B \cap X$. By 3.19, $e[A \cap X] \not\delta_{\mathrm{E}}\, e[B \cap X]$. So $\mathrm{cl}_{\mathrm{End}(X)} e[A \cap X] \cap \mathrm{cl}_{\mathrm{End}(X)} e[B \cap X] = \varnothing$. By 2.39, $\mathrm{End}(X) \geq Y$.

By 2.41, $Y = \mathrm{End}(X)$; therefore, ν is surjective. □

4. De Vries Algebras

Definition 4.1. A **de Vries algebra** is a Boolean algebra $(B, \leq)$ with a binary relation $\ll$ on B such that for all $a, b, c, d \in B$,

(D1) $0 \ll 0$
(D2) $a \ll b \Rightarrow a \leq b$
(D3) $a \ll b \Rightarrow -b \ll -a$
(D4) $a \ll b \wedge c \Leftrightarrow a \ll b$ and $a \ll c$
(D5) $a \ll b \Rightarrow \exists c \in B$ s.t. $a \ll c \ll b$
(D6) $0 \neq a \in B \Rightarrow \exists b \in B$ s.t. $b \neq 0$ and $b \ll a$

Henceforth, when we say B is a de Vries algebra, we will assume $\leq$ is the partial order of the underlying Boolean algebra and $\ll$ is the endowed de Vries algebra relation on B.

Proposition 4.2. *Let B be a de Vries algebra. For all $a \in B$,*

$$a = \bigvee_{b \ll a} b$$

Proof. By (D2), $b \ll a$ implies $b \leq a$. It is immediate that $\bigvee_{b \ll a} b \leq a$.

Suppose, by way of contradiction, that $\bigvee_{b \ll a} b < a$. Then $a - \bigvee_{b \ll a} b > 0$. By (D6), there is a $c \in B$ such that $0 < c \ll \bigvee_{b \ll a} b$. So $c \leq \bigvee_{b \ll a} b$. Also, $c \leq a - \bigvee_{b \ll a} b$. Therefore,

$$c \leq \bigvee_{b \ll a} b \wedge (a - \bigvee_{b \ll a} b) = 0$$

This is a contradiction. □

Example 4.3. Let $(B, \leq)$ be a complete Boolean algebra. Then $(B, \leq)$ is a de Vries algebra.

Theorem 4.4. *Let (X, τ) be a space. Then $(\mathcal{RO}(X), \leq)$ is a complete Boolean algebra with the following operations. For $U, V \in \mathcal{RO}(X)$,*

- $1 = X$

- $U \wedge V = U \cap V$
- $U \vee V = \operatorname{int}\operatorname{cl}(U \cup V)$
- $-U = X \setminus \operatorname{cl} U$

If $\{U_i\}_{i\in A}$ is an infinite collection of regular open sets of X, then

- $\bigwedge_{i\in A} U_i = \operatorname{int} \bigcap_{i\in A} U_i$
- $\bigvee_{i\in A} U_i = \operatorname{int}\operatorname{cl} \bigcup_{i\in A} U_i$

The partial order $\leq$ on $(\mathcal{RO}(X), \leq)$ is set inclusion.

Proof. [13, 7.21 & 7.22]. □

Example 4.5. Let (X, τ) be a compact Hausdorff space. For $U, V \in \mathcal{RO}(X)$, define $U \ll V$ if and only if $\operatorname{cl} U \subseteq V$. Then $(\mathcal{RO}(X), \ll)$ is a de Vries algebra.

Example 4.6. Let $(X, \ll)$ be a proximity space. Then $(\mathcal{P}(X), \ll)$ is a de Vries algebra if and only if $(X, \tau(\ll))$ is discrete.

Proof. It is well known that $(\mathcal{P}(X), \ll)$ is a complete Boolean algebra with the partial order as subset inclusion.

Whether $(X, \tau(\ll))$ is discrete or not, (D1) through (D5) are clearly satisfied. It is enough to check (D6).

Suppose X is discrete and $A \subseteq X$ such that $A \neq \varnothing$. Then there is an $a \in A$. Since $A \in \tau(\ll)$ then $a \not\delta X \setminus A$. Hence $\{x\} \ll A$ and thus (D6) is satisfied.

Conversely, suppose X is not discrete. Then there is a limit point x in X. Note that $\{x\}$ is not open. So $X \setminus \{x\}$ is not closed. Hence it must be that $x \,\delta\, X \setminus \{x\}$. Thus $\{x\} \not\ll \{x\}$. So if there is a set $A \subseteq X$ such that $A \ll \{x\}$ then $A = \varnothing$. Therefore, (D6) is not satisfied. □

Example 4.7. Let $(X, \ll)$ be a proximity space. Then $(\mathcal{RO}(X), \ll)$ is a de Vries algebra.

Proof. It is enough to check (D6). Let $\varnothing \neq U \in \mathcal{RO}(X)$. Take $x \in U$. Since U is open, $x \ll U$. By 2.24, there is a $V \in \mathcal{RO}(X)$ such that $x \ll V \ll U$. □

Remark 4.8. By 4.6, it is clear that a de Vries algebra and a proximity space $(X, \ll)$ are similar structures. As a result, they have many of the same properties.

Proposition 4.9. *Let B be a de Vries algebra. For all $a, b, c, d \in B$,*

$$a \leq b \ll c \leq d \Rightarrow a \ll d$$

Proof. The proof is analogous to 2.18(1). □

Proposition 4.10. *Let B be a de Vries algebra and $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$ finite families of elements of B such that $a_i \ll b_i$ for $i = 1, \ldots, n$.*

(1) $\bigwedge_{i=1}^n a_i \ll \bigwedge_{i=1}^n b_i$
(2) $\bigvee_{i=1}^n a_i \ll \bigvee_{i=1}^n b_i$

Proof. The proof is analogous to 2.19. □

Definition 4.11. Let B be a de Vries algebra.

- A **round filter** on B is a filter $\mathcal{F}$ on B such that for all $a \in \mathcal{F}$, there is a $b \in \mathcal{F}$ such that $b \ll a$.
- A maximal round filter on B is called an **end**. We denote the set of all ends on B as $\mathrm{End}(B)$.
- The **round hull** of a filter $\mathcal{F}$ on B is $\mathrm{rnd}(\mathcal{F}) = \{a \in B : \exists b \in \mathcal{F}, b \ll a\}$.

Proposition 4.12. *Let B be a de Vries algebra.*

(1) Every round filter on B is contained in an end.
(2) A round filter $\mathcal{F}$ on B is an end if and only if for all $a, b \in B$, $a \ll b$ implies $-a \in \mathcal{F}$ or $b \in \mathcal{F}$.
(3) For a filter $\mathcal{F}$ on B, $\mathrm{rnd}(\mathcal{F})$ is a round filter and $\mathrm{rnd}(\mathcal{F}) \subseteq \mathcal{F}$.

Proof. The proofs are analogous to 3.3, 3.5, and 3.8 respectively. □

Definition 4.13. For $a \in B$, define $\mathrm{o}(a) = \{\mathcal{E} \in \mathrm{End}(B) : a \in \mathcal{E}\}$

Proposition 4.14. *Let B be a de Vries algebra. For all $a, b \in B$,*

(1) $a = 0 \Leftrightarrow \mathrm{o}(a) = \varnothing$
(2) $a \leq b \Rightarrow \mathrm{o}(a) \subseteq \mathrm{o}(b)$
(3) $\mathrm{o}(a) \cap \mathrm{o}(b) = \mathrm{o}(a \wedge b)$
(4) $\mathrm{o}(a) \cup \mathrm{o}(b) \subseteq \mathrm{o}(a \vee b)$
(5) $a \ll b \Rightarrow \mathrm{End}(B) \setminus \mathrm{o}(b) \subseteq \mathrm{o}(-a)$
(6) $\{\mathrm{o}(a) : a \in B\}$ *is a base for a compact Hausdorff topology on* $\mathrm{End}(B)$.
(7) $\mathrm{o}(-a) = \mathrm{End}(B) \setminus \mathrm{cl}(\mathrm{o}(a))$

Proof. The proofs are analogous to 3.11(1); 3.11(2); 3.11(5); 3.11(6); 3.16 and 3.22; and 3.17 respectively. □

Definition 4.15. Let A, B be de Vries algebras. A function $f : A \to B$ is a **de Vries morphism** if for all $a, b \in A$,

(M1) $f(0) = 0$
(M2) $f(a \wedge b) = f(a) \wedge f(b)$
(M3) $a \ll b \Rightarrow -f(-a) \ll f(b)$
(M4) $f(a) = \bigvee_{b \ll a} f(b)$

Proposition 4.16. *Let A, B be de Vries algebras and $f : A \to B$ a de Vries morphism. For all $a, b \in A$,*

(1) $f(1) = 1$

(2) $f(a) \leq -f(-a)$
(3) $a \ll b \Rightarrow f(a) \ll f(b)$
(4) $a \leq b \Rightarrow f(a) \leq f(b)$

Proof.

(1) By (D1), $0 \ll 0$. By (M3), $-f(-0) \ll f(0)$. So $-f(1) \ll f(0)$. By (M1), $-f(1) \ll 0$ and by (D3), $-0 \ll f(1)$. Hence $1 \ll f(1)$. By (D2), $1 \leq f(1)$ and thus $f(1) = 1$.
(2) Note that $0 = f(0) = f(a \wedge -a)$. By (M2), $f(a \wedge -a) = f(a) \wedge f(-a)$. Hence $f(a) \leq -f(-a)$.
(3) Suppose $a \ll b$. By (M1), $f(0) = 0$. But $f(0) = f(a \wedge -a) = f(a) \wedge f(-a)$ by (M2). Therefore, $f(a) \leq -f(-a)$. Since $a \ll b$, by (M3), $-f(-a) \ll f(b)$. So we have $f(a) \leq -f(-a) \ll f(b)$. By 4.9, $f(a) \ll f(b)$.
(4) Suppose $a \leq b$. Then $a \wedge b = a$. So $f(a \wedge b) = f(a)$. By (M2), $f(a) \wedge f(b) = f(a)$. So $f(a) \leq f(b)$.

□

Remark 4.17. Let $(A, \ll)$, $(B, \ll)$ be de Vries algebras and $f : A \to B$ a de Vries morphism. In general, $f(-a) \neq -f(a)$. Therefore, f may not be a Boolean algebra homomorphism.

Remark 4.18. Let A, B, C be de Vries algebras and $f : A \to B$, $g : B \to C$ de Vries morphisms. If we consider morphism composition as normal function composition, it is clear that $g \circ f$ satisfies $(M1)$ through $(M3)$. It is not necessarily true, however, that $g \circ f$ satisfies $(M4)$.

Definition 4.19. Let A, B, C be de Vries algebras and $f : A \to B$, $g : B \to C$ de Vries morphisms. Define

$$(g * f)(a) = \bigvee_{b \ll a} (g \circ f)(b)$$

Proposition 4.20. *Let A, B, C be de Vries algebras and $f : A \to B$, $g : B \to C$ de Vries morphisms. Then $g * f : A \to C$ is a de Vries morphism.*

Proof. We will verify the de Vries morphism axioms.

(M1): Note that $(g * f)(0) = \bigvee_{a \ll 0} (g \circ f)(a)$. But $a \ll 0$ implies $a = 0$. So $(g * f)(0) = (g \circ f)(0)$ and $(g \circ f)(0) = g(f(0)) = g(0) = 0$ by (M1).

(M2): Let $a,b\in A$. Then $(g*f)(a\wedge b)=\bigvee_{c\ll a\wedge b}(g\circ f)(c)$. By (D4), $c\ll a\wedge b$ if and only if $c\ll a$ and $c\ll b$. So

$$\begin{aligned}\bigvee_{c\ll a\wedge b}(g\circ f)(c) &= \bigvee_{c\ll a}(g\circ f)(c)\wedge\bigvee_{c\ll b}(g\circ f)(c)\\ &= (g*f)(a)\wedge(g*f)(b)\end{aligned}$$

Therefore, $(g*f)(a\wedge b)=(g*f)(a)\wedge(g*f)(b)$.

(M3): Suppose $a\ll b$. By (D5), there exist $c,d\in A$ such that $a\ll c\ll d\ll b$. By (D3), $-d\ll -c\ll -a$. By 4.16(3), $(g\circ f)(-d)\ll (g\circ f)(-c)$. Since $-c\ll -a$, $(g\circ f)(-d)\ll\bigvee_{e\ll -a}(g\circ f)(e)$. By (D3), $-\bigvee_{e\ll -a}(g\circ f)(e)\ll (g\circ f)(d)$. Since $d\ll b$, $-\bigvee_{e\ll -a}(g\circ f)(e)\ll \bigvee_{e\ll b}(g\circ f)(e)$. Hence $-(g*f)(-a)\ll(g*f)(b)$.

(M4): Let $a\in A$. By (D5), for all $b\in A$ such that $b\ll a$, there is a $c\in A$ such that $b\ll c\ll a$. Therefore, $\bigvee_{b\ll a}(g\circ f)(b)\le\bigvee_{c\ll a}\bigvee_{b\ll c}(g\circ f)(b)$. Hence $(g*f)(a)\le\bigvee_{c\ll a}(g*f)(c)$.

For inequality in the other direction, note that if $b\ll a$ and $c\ll b$ then $c\ll a$. Therefore, $\bigvee_{c\ll b}(g\circ f)(c)\le\bigvee_{c\ll a}(g\circ f)(c)$. Hence $(g*f)(b)\le(g*f)(a)$ for all $b\ll a$. So $\bigvee_{b\ll a}(g*f)(b)\le(g*f)(a)$. $\square$

Proposition 4.21. *Let A,B,C,D be de Vries algebras and $f:A\to B$, $g:B\to C$, $h:C\to D$ de Vries morphisms. For $a,b\in A$,*

(1) $(g*f)(a)\le(g\circ f)(a)$
(2) $a\ll b\Rightarrow(g\circ f)(a)\le(g*f)(b)$
(3) $a\ll b\Rightarrow((h*g)\circ f)(a)\le(h\circ(g*f))(b)$
(4) $a\ll b\Rightarrow(h\circ(g*f))(a)\le((h*g)*f)(b)$

Proof.

(1) If $a=0$, it is clear from (M1) and 4.20 that $(g*f)(a)=0$ and $(g\circ f)(a)=0$.
Suppose $a\neq 0$. By (D6), there is a $c\in A$ such that $c\ll a$. By 4.16(3), $f(c)\ll f(a)$ and thus $(g\circ f)(c)\ll(g\circ f)(a)$. By (D2), $(g\circ f)(c)\le(g\circ f)(a)$. So $(g*f)(a)=\bigvee_{c\ll a}(g\circ f)(c)\le(g\circ f)(a)$.

(2) Suppose $a\ll b$. By (D5), there is a $c\in A$ such that $a\ll c\ll b$. By (D2), $a\le c$. Applying 4.16(4) twice, $(g\circ f)(a)\le(g\circ f)(c)$. Hence $(g\circ f)(a)\le\bigvee_{d\ll b}(g\circ f)(d)=(g*f)(b)$.

(3) Suppose $a\ll b$. By 4.21(1),

$$\begin{aligned}((h*g)\circ f)(a) &= (h*g)(f(a))\\ &\le (h\circ g)(f(a))\\ &= (h\circ(g\circ f))(a)\end{aligned}$$

Since $a\ll b$, by 4.21(2), $(g\circ f)(a)\le(g*f)(b)$. By 4.16(4), $(h\circ(g\circ f))(a)\le(h\circ(g*f))(b)$.

(4) Suppose $a \ll b$. By (D5), there is a $c \in A$ such that $a \ll c \ll b$. By 4.21(1), $(g * f)(a) \leq (g \circ f)(a)$. By 4.16(4),

$$\begin{aligned}(h \circ (g * f))(a) &\leq (h \circ (g \circ f))(a)\\ &= (h \circ g)(f(a))\end{aligned}$$

Since $a \ll c$ then $f(a) \ll f(c)$ by 4.16(3). So by 4.21(2),

$$\begin{aligned}(h \circ g)(f(a)) &\leq (h * g)(f(c))\\ &= ((h * g) \circ f)(c)\end{aligned}$$

As $c \ll b$, applying 4.21(2) again,

$$((h * g) \circ f)(c) \leq ((h * g) * f)(b)$$

Therefore, we have $(h \circ (g * f))(a) \leq ((h * g) * f)(b)$.

$\square$

Theorem 4.22. *The class of de Vries algebras with de Vries morphisms and composition $*$ as defined in 4.19 is a category. We will denote this category as* **DeV**.

Proof. We only need to verify two things: that the composition of morphisms is associative and that each object has an identity morphism.

First, we will show associativity of composition. Let A, B, C, D be de Vries algebras and $f : A \to B, g : B \to C, h : C \to D$ be de Vries morphisms. We will show that $h * (g * f) = (h * g) * f$.

For $a = 0$, it is clear from 4.20 and (M1) that $(h * (g * f))(a) = 0$ and $((h * g) * f)(a) = 0$.

Suppose $a \neq 0$.

First we will show that $(h * (g * f))(a) \leq ((h * g) * f)(a)$.

By 4.21(4), for all $b \in B$ such that $b \ll a$,

$$(h \circ (g * f))(b) \leq ((h * g) * f)(a)$$

So we have

$$\begin{aligned}(h * (g * f))(a) &= \bigvee_{b \ll a} (h \circ (g * f))(b)\\ &\leq ((h * g) * f)(a)\end{aligned}$$

Now we will show that $((h * g) * f)(a) \leq (h * (g * f))(a)$.

By (D6), there is a $b \in A$ such that $0 \neq b \ll a$. By (D5), there is a $c \in A$ such that $b \ll c \ll a$. By 4.21(3),

$$((h * g) \circ f)(b) \leq (h \circ (g * f))(c)$$

Since $c \ll a$,

$$\begin{aligned}(h \circ (g * f))(c) &\leq \bigvee_{d \ll a} (h \circ (g * f))(d) \\ &= (h * (g * f))(a)\end{aligned}$$

So we have, for all $b \in A$ such that $b \ll a$,

$$((h * g) \circ f)(b) \leq (h * (g * f))(a)$$

Therefore

$$\begin{aligned}((h * g) * f)(a) &= \bigvee_{b \ll a} ((h * g) \circ f)(b) \\ &\leq (h * (g * f))(a)\end{aligned}$$

Now, let A, B be de Vries algebras and $f \in \mathrm{M}(A, B)$. Define $1_A : A \to A$ as $1_A = \mathrm{id}_A$. It is trivial to show that axioms (M1), (M2), and (M3) are satisfied. That (M4) is satisfied follows directly from 4.2. Therefore, 1_B is indeed a de Vries morphism. Also, it is clear that $1_A \circ f = f = f \circ 1_B$. Therefore, 1_A is an identity morphism.

For any $a \in A$,

$$\begin{aligned}(f * 1_A)(a) &= \bigvee_{b \ll a} (f \circ 1_A)(b) \\ &= \bigvee_{b \ll a} f(b)\end{aligned}$$

It follows from (M4) that

$$(f * 1_A)(a) = f(a)$$

Similarly,

$$\begin{aligned}(1_B * f)(a) &= \bigvee_{b \ll a} (1_B \circ f)(b) \\ &= \bigvee_{b \ll a} f(b) \\ &= f(a)\end{aligned}$$

Therefore, 1_A is an identity morphism. □

5. De Vries Duality

Definition 5.1. Define $\Phi : \mathbf{CPT_2} \to \mathbf{DeV}$ by

$$\Phi(X) = (\mathcal{RO}(X), \ll)$$

where $\ll$ is the unique proximity on X given in 2.34 and defined by $U \ll V$ if and only if $\operatorname{cl} U \subseteq V$. For $X, Y \in \operatorname{ob} \mathbf{CPT_2}$ and a morphism $f \in \mathrm{M}(X, Y)$, define $\Phi(f) : \mathcal{RO}(Y) \to \mathcal{RO}(X)$ by

$$\Phi(f)(U) = \operatorname{int} \operatorname{cl} f^{\leftarrow}[U]$$

The corresponding diagram is

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi"'] & Y \arrow[d, "\Phi"] \\ (\mathcal{RO}(X), \ll) & (\mathcal{RO}(Y), \ll) \arrow[l, "\Phi(f)"] \end{tikzcd}$$

Remark 5.2. By 4.5, $(\mathcal{RO}(X), \ll)$ is a de Vries algebra. So the contravariant functor Φ is well-defined.

Proposition 5.3. *Let U and V be open sets in the space X. Then*

$$\operatorname{int} \operatorname{cl}(U \cap V) = \operatorname{int} \operatorname{cl} U \cap \operatorname{int} \operatorname{cl} V$$

Proof. Note that

$$\begin{aligned} \operatorname{int} \operatorname{cl}(U \cap V) &\subseteq \operatorname{int}(\operatorname{cl} U \cap \operatorname{cl} V) \\ &= \operatorname{int} \operatorname{cl} U \cap \operatorname{int} \operatorname{cl} V \end{aligned}$$

For the other inclusion, note that

$$\begin{aligned} \operatorname{cl}(U \cap V) &= \operatorname{cl}(U \cap \operatorname{cl} V) \\ &= \operatorname{cl}(U \cap \operatorname{int} \operatorname{cl} V) \\ &= \operatorname{cl}(\operatorname{cl} U \cap \operatorname{int} \operatorname{cl} V) \\ &= \operatorname{cl}(\operatorname{int} \operatorname{cl} U \cap \operatorname{int} \operatorname{cl} V) \end{aligned}$$

As $\operatorname{int} \operatorname{cl} U \cap \operatorname{int} \operatorname{cl} V$ is open,

$$\begin{aligned} \operatorname{int} \operatorname{cl} U \cap \operatorname{int} \operatorname{cl} V &\subseteq \operatorname{int} \operatorname{cl}(\operatorname{int} \operatorname{cl} U \cap \operatorname{int} \operatorname{cl} V) \\ &= \operatorname{int} \operatorname{cl}(U \cap V) \end{aligned}$$

□

Theorem 5.4. *Let $X, Y \in \operatorname{ob} \mathbf{CPT_2}$ and $f \in \mathrm{M}(X, Y)$. Then $\Phi(f)$ is a de Vries morphism from $\Phi(Y)$ to $\Phi(X)$.*

Proof. (M1): $\Phi(f)(\varnothing) = \operatorname{int} \operatorname{cl} f^{\leftarrow}[\varnothing] = \operatorname{int} \operatorname{cl} \varnothing = \varnothing$

(M2): Let $U, V \in \mathcal{RO}(Y)$. Since $f \in \mathrm{C}(X, Y)$, $f^{\leftarrow}[U]$ and $f^{\leftarrow}[V]$ are open. Therefore, by 5.3,

$$\begin{aligned}\Phi(f)(U \cap V) &= \operatorname{int} \operatorname{cl} f^{\leftarrow}[U \cap V] \\ &= \operatorname{int} \operatorname{cl}(f^{\leftarrow}[U] \cap f^{\leftarrow}[V]) \\ &= \operatorname{int} \operatorname{cl} f^{\leftarrow}[U] \cap \operatorname{int} \operatorname{cl} f^{\leftarrow}[V] \\ &= \Phi(f)(U) \cap \Phi(f)(V)\end{aligned}$$

(M3): Let $U, V \in \mathcal{RO}(Y)$ such that $U \ll V$. Then $\operatorname{cl} U \subseteq V$. So $f^{\leftarrow}[\operatorname{cl} U] \subseteq f^{\leftarrow}[V]$. As $f^{\leftarrow}[\operatorname{cl} U] = X \setminus f^{\leftarrow}[Y \setminus \operatorname{cl} U]$, we have

$$X \setminus f^{\leftarrow}[Y \setminus \operatorname{cl} U] \subseteq f^{\leftarrow}[V]$$

Now, $f^{\leftarrow}[V]$ is open, so $f^{\leftarrow}[V] \subseteq \operatorname{int} \operatorname{cl} f^{\leftarrow}[V]$. Likewise, $f^{\leftarrow}[Y \setminus \operatorname{cl} U] \subseteq \operatorname{int} \operatorname{cl} f^{\leftarrow}[Y \setminus \operatorname{cl} U]$ and thus $X \setminus \operatorname{int} \operatorname{cl} f^{\leftarrow}[Y \setminus \operatorname{cl} U] \subseteq X \setminus f^{\leftarrow}[Y \setminus \operatorname{cl} U]$. So,

$$X \setminus \operatorname{int} \operatorname{cl} f^{\leftarrow}[Y \setminus \operatorname{cl} U] \subseteq \operatorname{int} \operatorname{cl} f^{\leftarrow}[V]$$

Therefore,

$$\operatorname{cl}(X \setminus \operatorname{cl} \operatorname{int} \operatorname{cl} f^{\leftarrow}[Y \setminus \operatorname{cl} U]) \subseteq \operatorname{int} \operatorname{cl} f^{\leftarrow}[V]$$

This is exactly

$$-\Phi(f)(-U) \ll \Phi(f)(V)$$

(M4): Let $U \in \mathcal{RO}(Y)$. First, we will show that $\Phi(f)(U) \subseteq \bigvee_{V \ll U} \Phi(f)(V)$.

Let $V \in \mathcal{RO}(Y)$ such that $V \ll U$. Then $\operatorname{cl} V \subseteq U$. So,

$$\begin{aligned}\operatorname{cl} f^{\leftarrow}[V] &\subseteq f^{\leftarrow}[\operatorname{cl} V] \\ &\subseteq f^{\leftarrow}[U] \\ &\subseteq \operatorname{cl} f^{\leftarrow}[U]\end{aligned}$$

Therefore, $\operatorname{int} \operatorname{cl} f^{\leftarrow}[V] \subseteq \operatorname{int} \operatorname{cl} f^{\leftarrow}[U]$. So $\Phi(f)(V) \subseteq \Phi(f)(U)$ for all $V \ll U$. Hence $\bigvee_{V \ll U} \Phi(f)(V) \subseteq \Phi(f)(U)$.

For reverse inclusion, let $V \in \mathcal{RO}(Y)$ such that $V \ll U$. Then $\operatorname{cl} V \subseteq U$. Let $p \in U$. Since X is Tychonoff, there is a $T \in \mathcal{RO}(X)$ such that $p \in T \subseteq \operatorname{cl}(T) \subseteq U$. Since $\operatorname{cl}(T) \subseteq U$, $T \ll U$. Therefore, $\operatorname{int} \operatorname{cl} f^{\leftarrow}[T] \subseteq \Phi(f)(T) \subseteq \bigvee_{V \ll U} \Phi(f)(V)$. As $f^{\leftarrow}[T]$ is open, $f^{\leftarrow}[T] \subseteq \operatorname{int} \operatorname{cl} f^{\leftarrow}[T]$. So $f^{\leftarrow}[T] \subseteq \bigvee_{V \ll U} \Phi(f)(V)$. But $f^{\leftarrow}(p) \in f^{\leftarrow}[T]$; therefore, $f^{\leftarrow}(p) \in \bigvee_{V \ll U} \Phi(f)(V)$ for all $p \in U$. So $f^{\leftarrow}[U] \subseteq \bigvee_{V \ll U} \Phi(f)(V)$. Note that $\bigvee_{V \ll U} \Phi(f)(V) \in \mathcal{RO}(X)$. Thus $\Phi(f)(U) = \operatorname{int} \operatorname{cl} f^{\leftarrow}[U] \subseteq \bigvee_{V \ll U} \Phi(f)(V)$. □

Theorem 5.5. Φ *is a contravariant functor from* $\mathbf{CPT_2}$ *to* $\mathbf{DeV}$.

Proof. Let $X \in \operatorname{ob}\mathbf{CPT_2}$. Note that $1_X = \mathrm{id}_X$ and for any $B \in \operatorname{ob}\mathbf{DeV}$, $1_B = \mathrm{id}_B$. Let $U \in \mathcal{RO}(X)$. Then

$$\begin{aligned}\Phi(1_X)(U) &= \operatorname{int}\operatorname{cl} U\\ &= U\\ &= 1_{\Phi(X)}(U)\end{aligned}$$

Now let $X, Y, Z \in \operatorname{ob}\mathbf{CPT_2}$, $f \in \mathrm{M}(X, Y)$, and $g \in \mathrm{M}(Y, Z)$. We will show that $\Phi(g \circ f) = \Phi(f) * \Phi(g)$.

Let $V \in \mathcal{RO}(Z)$. If $U \in \mathcal{RO}(Z)$ such that $U \ll V$ then $\operatorname{cl} U \subseteq V$. As $U \subseteq \operatorname{cl} U \subseteq V$, $\operatorname{cl} g^{\leftarrow}[U] \subseteq g^{\leftarrow}[\operatorname{cl} U] \subseteq g^{\leftarrow}[V]$. Hence $\operatorname{int}\operatorname{cl} g^{\leftarrow}[U] \subseteq g^{\leftarrow}[V]$. So we have $\operatorname{int}\operatorname{cl} f^{\leftarrow}[\operatorname{int}\operatorname{cl} g^{\leftarrow}[U]] \subseteq \operatorname{int}\operatorname{cl} f^{\leftarrow}[g^{\leftarrow}[V]]$. Thus, for $U \ll V$,

$$\begin{aligned}(\Phi(f) \circ \Phi(g))[U] &= \operatorname{int}\operatorname{cl} f^{\leftarrow}[\operatorname{int}\operatorname{cl} g^{\leftarrow}[U]]\\ &\subseteq \operatorname{int}\operatorname{cl} f^{\leftarrow}[g^{\leftarrow}[V]]\\ &= \Phi(g \circ f)[V]\end{aligned}$$

Therefore,

$$\begin{aligned}(\Phi(f) * \Phi(g))[U] &= \bigvee_{U \ll V} (\Phi(f) \circ \Phi(g))[U]\\ &\subseteq \Phi(g \circ f)[V]\end{aligned}$$

For the reverse inclusion, let $U \in \mathcal{RO}(Z)$. First we will show that

$$f^{\leftarrow}[g^{\leftarrow}[U]] = \bigcup_{V \ll U} f^{\leftarrow}[g^{\leftarrow}[V]]$$

One inclusion is trivial. For the other, let $u \in f^{\leftarrow}[g^{\leftarrow}[U]]$. Then $(g \circ f)(u) \in U$. As X is Tychonoff, there is a $V \in \mathcal{RO}(X)$ such that

$$(g \circ f)(u) \in V \subseteq \operatorname{cl} V \subseteq U$$

So $u \in f^{\leftarrow}[g^{\leftarrow}[V]]$ and $V \ll U$. Thus $u \in \bigcup_{V \ll U} f^{\leftarrow}[g^{\leftarrow}[V]]$ giving the desired equality.

Now, as $g^{\leftarrow}[U]$ is open, $g^{\leftarrow}[U] \subseteq \operatorname{int}\operatorname{cl} g^{\leftarrow}[U]$ and since $\operatorname{int}\operatorname{cl} g^{\leftarrow}[U]$ is open,

$$f^{\leftarrow}[g^{\leftarrow}[U]] \subseteq \operatorname{int}\operatorname{cl} f^{\leftarrow}[\operatorname{int}\operatorname{cl} g^{\leftarrow}[U]]$$

So

$$\begin{aligned}
\operatorname{int}\operatorname{cl} f^{\leftarrow}[g^{\leftarrow}[U]] &= \operatorname{int}\operatorname{cl} \bigcup_{V \ll U} f^{\leftarrow}[g^{\leftarrow}[V]] \\
&\subseteq \operatorname{int}\operatorname{cl} \bigcup_{V \ll U} \operatorname{int}\operatorname{cl} f^{\leftarrow}[\operatorname{int}\operatorname{cl} g^{\leftarrow}[V]] \\
&= \bigvee_{V \ll U} \operatorname{int}\operatorname{cl} f^{\leftarrow}[\operatorname{int}\operatorname{cl} g^{\leftarrow}[V]]
\end{aligned}$$

Hence $\Phi(g \circ f) \subseteq \Phi(f) * \Phi(f)$.

□

Theorem 5.6. *Let $(B, \ll)$ be a de Vries algebra. For all $a \in B$, $\mathrm{o}(a) \in \mathcal{RO}(\mathrm{End}(B))$.*

Proof. By 4.14(7), $\mathrm{o}(a) = \mathrm{End}(B) \setminus \operatorname{cl}\mathrm{o}(-a)$. Therefore

$$\begin{aligned}
\operatorname{int}\operatorname{cl}\mathrm{o}(a) &= \operatorname{int}\operatorname{cl}\mathrm{End}(B) \setminus \operatorname{cl}\mathrm{o}(-a) \\
&= \mathrm{End}(B) \setminus \operatorname{cl}\operatorname{int}\operatorname{cl}\mathrm{o}(-a)
\end{aligned}$$

Since $\mathrm{o}(-a)$ is open, we have

$$\begin{aligned}
\operatorname{int}\operatorname{cl}\mathrm{o}(a) &= \mathrm{End}(B) \setminus \operatorname{cl}\operatorname{int}\operatorname{cl}\operatorname{int}\mathrm{o}(-a) \\
&= \mathrm{End}(B) \setminus \operatorname{cl}\operatorname{int}\mathrm{o}(-a) \\
&= \mathrm{End}(B) \setminus \operatorname{cl}\mathrm{o}(-a) \\
&= \mathrm{o}(a)
\end{aligned}$$

Therefore, $\mathrm{o}(a) \in \mathcal{RO}(\mathrm{End}(B))$. □

Theorem 5.7. *Let $(B, \ll)$ be a de Vries algebra. For all $U \in \mathcal{RO}(\mathrm{End}(B))$, $U = \mathrm{o}(a)$ for some $a \in B$.*

Proof. Note that, since U is open, we can write $U = \bigcup_{c \in C} \mathrm{o}(c)$ for some $C \subseteq B$. By 4.14(4), $\bigcup_{c \in C} \mathrm{o}(c) \subseteq \mathrm{o}(\bigvee_{c \in C} c)$. Therefore, $U \subseteq \mathrm{o}(\bigvee_{c \in C} c)$. Letting $a = \bigvee_{c \in C} c$ gives $U \subseteq \mathrm{o}(a)$.

Now, since $\mathrm{End}(B) \setminus \operatorname{cl} U$ is open, we can write $\mathrm{End}(B) \setminus \operatorname{cl} U = \bigcup_{d \in D} \mathrm{o}(d)$ for some $D \subseteq B$.

Take $c \in C$ and $d \in D$. Clearly $\mathrm{o}(c) \subseteq U$ and $\mathrm{o}(d) \subseteq \mathrm{End}(B) \setminus \operatorname{cl} U$. So $\mathrm{o}(c) \cap \mathrm{o}(d) = \varnothing$. By 4.14(3), $\mathrm{o}(c \wedge d) = \varnothing$. By 4.14(1), $c \wedge d = 0$. Hence $c \leq -d$. Since this holds for all $c \in C$ and $a = \bigvee_{c \in C} c$, $a \leq -d$ for all $d \in C$. So $d \leq -a$ and, by 4.14(2), $\mathrm{o}(d) \subseteq \mathrm{o}(-a)$ for all $d \in C$. Since $\mathrm{End}(B) \setminus \operatorname{cl} U = \bigcup_{d \in D} \mathrm{o}(d)$, $\mathrm{End}(B) \setminus \operatorname{cl} U \subseteq \mathrm{o}(-a)$. So $\mathrm{End}(B) \setminus o(-a) \subseteq \operatorname{cl} U$.

Therefore

$$\begin{aligned} \mathrm{o}(a) &= \mathrm{End}(B) \setminus \mathrm{cl}\,\mathrm{o}(-a) \\ &= \mathrm{int}(\mathrm{End}(B) \setminus \mathrm{o}(-a)) \\ &\subseteq \mathrm{int}\,\mathrm{cl}\,U \\ &= U \end{aligned}$$

□

Theorem 5.8. *Let B be a de Vries algebra. Then $B \to \mathcal{RO}(\mathrm{End}(B)) : b \mapsto \mathrm{o}(b)$ is a bijection.*

Proof. By 5.7, the mapping is surjective.

To show the mapping is injective, let $a, b \in B$ such that $a \neq b$. Then $-a \wedge b \neq 0$. By (D6), there is a $c \neq 0$ such that $c \ll -a \wedge b$. Let $\mathcal{F} = \{d \in B : c \ll d\}$.

First, we will show that $\mathcal{F}$ is a filter. Clearly $0 \notin \mathcal{F}$. Suppose $a, b \in \mathcal{F}$. Then $c \ll a$ and $c \ll b$. By (D4), $c \ll a \wedge b$ and hence $a \wedge b \in \mathcal{F}$. Finally, suppose $a \in \mathcal{F}$ and $b \in B$ such that $a \leq b$. Then since $c \ll a \leq b$, by 4.9, $c \ll b$; therefore, $b \in \mathcal{F}$. So $\mathcal{F}$ is a filter.

To show that $\mathcal{F}$ is round, let $a \in \mathcal{F}$. Then $c \ll a$. So there is a $b \in B$ such that $c \ll b \ll a$. So $b \in \mathcal{F}$ and $b \ll a$. Thus $\mathcal{F}$ is round.

This round filter $\mathcal{F}$ is contained in an end, call it $\mathcal{E}$.

Now, note that $-a \wedge b \in \mathcal{F} \subseteq \mathcal{E}$. Hence $-a \in \mathcal{E}$ and $b \in \mathcal{E}$. Since $-a \in \mathcal{E}$, $a \notin \mathcal{E}$. So $\mathcal{E} \in \mathrm{o}(b) \setminus \mathrm{o}(a)$. Therefore, $\mathrm{o}(a) \neq \mathrm{o}(b)$. □

Definition 5.9. Define $\Psi : \mathbf{DeV} \to \mathbf{CPT_2}$ by

$$\Psi(B) = \mathrm{End}(B)$$

For $A, B \in \mathrm{ob}\,\mathbf{DeV}$ and a morphism $f \in \mathrm{M}(A, B)$, define $\Psi(f) : \mathrm{End}(B) \to \mathrm{End}(A)$ by

$$\Psi(f)(\mathcal{E}) = \{a \in A : \exists b \in A \text{ s.t. } b \ll a \text{ and } f(b) \in \mathcal{E}\}$$

The corresponding diagram is

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Psi"'] & B \arrow[d, "\Psi"] \\ \mathrm{End}(A) & \mathrm{End}(B) \arrow[l, "\Psi(f)"] \end{tikzcd}$$

Theorem 5.10. *Let A, B be de Vries algebras and $f : A \to B$ a de Vries morphism. For $\mathcal{E} \in \mathrm{End}(B)$, $\Psi(f)(\mathcal{E}) \in \mathrm{End}(A)$ and thus $\Psi(f)$ is well-defined.*

Proof. By way of contradiction, suppose $0 \in \Psi(f)(\mathcal{E})$. Then there is a $b \in A$ such that $b \ll 0$ and $f(b) \in \mathcal{E}$. But $b \ll 0$ implies $b \leq 0$ and so $b = 0$. Hence $f(0) \in \mathcal{E}$. But by (M1), $f(0) = 0$, a contradiction.

Let $a, b \in \Psi(f)(\mathcal{E})$. Then there exist $c, d \in A$ such that $c \ll a$, $d \ll b$, $f(c) \in \mathcal{E}$, and $f(d) \in \mathcal{E}$. By 4.10(1), $c \wedge d \ll a \wedge b$. Also, by (M2), $f(c \wedge d) = f(c) \wedge f(d)$ and $f(c) \wedge f(d) \in \mathcal{E}$. Hence $a \wedge b \in \Psi(f)(\mathcal{E})$.

Let $a \in \Psi(f)(\mathcal{E})$ and $b \in A$ such that $a \leq b$. Then there is a $c \in A$ such that $c \ll a$ and $f(c) \in \mathcal{E}$. But $c \ll a \leq b$ implies, by 4.9, that $c \ll b$. Hence $b \in \Psi(f)(\mathcal{E})$.

To show $\Psi(f)(\mathcal{E})$ is an end, let $a, b \in A$ such that $a \ll b$. By (D5), there exist $c, d \in A$ such that $a \ll c \ll d \ll b$. As $c \ll d$, by (M3), $-f(-c) \ll f(d)$. But $\mathcal{E}$ is an end, so either $f(-c) \in \mathcal{E}$ or $f(d) \in \mathcal{E}$. If $f(-c) \in \mathcal{E}$ then since $-c \ll -a$, by the definition of $\Psi(f)(\mathcal{E})$, $-a \in \Psi(f)(\mathcal{E})$. On the other hand, if $f(d) \in \mathcal{E}$ then since $d \ll b$ then $b \in \Psi(f)(\mathcal{E})$. So either $-a \in \Psi(f)(\mathcal{E})$ or $b \in \Psi(f)(\mathcal{E})$. Hence $\Psi(f)(\mathcal{E})$ is an end. $\square$

Theorem 5.11. *For $A, B \in \operatorname{ob}\mathbf{DeV}$ and $f \in \mathrm{M}(A, B)$, $\Psi(f)$ is a continuous function and thus $\Psi(f)$ is a morphism in $\mathbf{CPT_2}$ from $\Psi(B)$ to $\Psi(A)$.*

Proof. Let $\mathcal{E} \in \mathrm{End}(B)$. Then let $a \in A$ such that $\Psi(f)(\mathcal{E}) \in \mathrm{o}(a)$. We will show there is a $d \in B$ such that $\Psi(f)(\mathcal{E}) \in \Psi(f)[\mathrm{o}(d)] \subseteq \mathrm{o}(a)$.

As $\Psi(f)(\mathcal{E}) \in \mathrm{o}(a)$, $a \in \Psi(f)(\mathcal{E})$. So there is a $0 \neq b \in \mathcal{A}$ such that $b \ll a$ and $f(b) \in \mathcal{E}$. Then $\mathcal{E} \in \mathrm{o}(f(b))$ and hence $\Psi(f)(\mathcal{E}) \in \Psi(f)[\mathrm{o}(f(b))]$.

To conclude, we will show that $\Psi(f)[\mathrm{o}(f(b))] \subseteq \mathrm{o}(a)$. Let $\mathcal{F} \in \mathrm{o}(f(b))$. Then $f(b) \in \mathcal{F}$. But, $b \ll a$ and thus $f(b) \leq f(a)$. As $f(b) \in \mathcal{F}$, $a \in \Psi(f)(\mathcal{F})$. Hence $\Psi(f)(\mathcal{F}) \in \mathrm{o}(a)$. $\square$

Theorem 5.12. *For $B \in \operatorname{ob}\mathbf{DeV}$, $\Psi(1_B) = 1_{F(B)}$.*

Proof. Let $\mathcal{E} \in \mathrm{End}(B)$. As $1_{\Psi(B)} = \mathrm{id}_{\Psi(B)}$, $1_{\Psi(B)}(\mathcal{E}) = \mathcal{E}$.

Also, $1_B = \mathrm{id}_B$. So,

$$\begin{aligned}\Psi(1_B)(\mathcal{E}) &= \{a \in B : \exists b \in B \text{ s.t. } b \ll a \text{ and } 1_B(b) \in \mathcal{E}\} \\ &= \{a \in B : \exists b \in B \text{ s.t. } b \ll a \text{ and } b \in \mathcal{E}\}\end{aligned}$$

To finish, we will prove that $\Psi(1_B)(\mathcal{E}) = \mathcal{E}$.

Let $a \in \Psi(1_B)(\mathcal{E})$. Then there is a $b \in B$ such that $b \ll a$ and $b \in \mathcal{E}$. So $b \leq a$ and thus $a \in \mathcal{E}$.

Let $a \in \mathcal{E}$. Since $\mathcal{E}$ is an end and hence round, there is a $b \in \mathcal{E}$ such that $b \ll a$. Hence $a \in \Psi(1_B)(\mathcal{E})$. $\square$

Theorem 5.13. *Ψ is a contravariant functor from $\mathbf{DeV}$ to $\mathbf{CPT_2}$.*

Proof. Let $A, B, C \in \operatorname{ob}\mathbf{DeV}$, $f \in \mathrm{M}(A, B)$, and $g \in \mathrm{M}(B, C)$. All that is left to show is that $\Psi(g * f) = \Psi(f) \circ \Psi(g)$.

Let $\mathcal{E} \in \operatorname{End}(A)$ and note that

$$\begin{aligned}(\Psi(f) \circ \Psi(g))(\mathcal{E}) &= \{a \in A : \exists b \in A \text{ s.t. } b \ll a \text{ and } f(b) \in \\ &\qquad \{c \in B : \exists d \in B \text{ s.t. } d \ll c \text{ and } g(d) \in \mathcal{E}\}\} \\ &= \{a \in A : \exists b \in A \text{ s.t. } b \ll a \text{ and} \\ &\qquad \exists d \in B \text{ s.t. } d \ll f(b) \text{ and } g(d) \in \mathcal{E}\}\end{aligned}$$

Let $a \in \Psi(g * f)(\mathcal{E})$. Then there is a $b \in A$ such that $b \ll a$ and $\Psi(g*f)(b) \in \mathcal{E}$. As $b \ll a$, there is a $c \in A$ such that $b \ll c \ll a$. Hence $f(b) \ll f(c)$. By 4.21(1), $(g * f)(b) \leq (g \circ f)(b)$. Since $(g * f)(b) \in \mathcal{E}$ then $(g \circ f)(b) \in \mathcal{E}$. Therefore, $a \in (\Psi(f) \circ \Psi(g))(\mathcal{E})$.

Now, let $a \in (\Psi(f) \circ \Psi(g))(\mathcal{E})$. Then there is a $b \in A$ such that $b \ll a$ and there is a $d \in B$ such that $d \ll f(b)$ and $g(d) \in \mathcal{E}$. As $b \ll a$, there is a $c \in A$ such that $b \ll c \ll a$. By 4.16(3), $d \ll f(b) \ll f(c)$ and $g(d) \ll (g \circ f)(b) \ll (g \circ f)(c)$. So $g(d) \ll \bigvee_{e \ll c}(g \circ f)(e) = (g * f)(c)$. Since $g(d) \in \mathcal{E}$, $(g * f)(c) \in \mathcal{E}$. This fact along with $c \ll a$ yields $a \in \Psi(g * f)(\mathcal{E})$. $\square$

Proposition 5.14. *Let X be a compact Hausdorff space. For any $x, y \in X$, there exist $U, V \in \mathcal{RO}(X)$ such that $x \in U$, $y \in V$, and $\operatorname{cl} U \cap \operatorname{cl} V = \varnothing$.*

Proof. Let $x, y \in X$. Since X is Hausdorff, there exist $U, V \in \mathcal{RO}(X)$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$. Then $U \cap \operatorname{cl} V = \varnothing$. As $x \in U$ and X is Tychonoff, there is a $W \in \mathcal{RO}(X)$ such that $x \in W \subseteq \operatorname{cl} W \subseteq U$. Therefore, $\operatorname{cl} W \cap \operatorname{cl} V = \varnothing$. $\square$

Lemma 5.15. *Let X be compact Hausdorff and $\mathcal{E} \in \operatorname{End}(\mathcal{RO}(X), \ll)$. Then there is an $x \in X$ such that $\{x\} = \mathrm{a}\,\mathcal{E}$ and, furthermore, $\mathcal{E}$ converges to x.*

Proof. Let $\mathcal{E} \in \operatorname{End}(\mathcal{RO}(X), \ll)$. As X is compact, $\mathrm{a}\,\mathcal{E} \neq \varnothing$ [6, V.5.1]. By way of contradiction, let $x, y \in \mathrm{a}\,\mathcal{E}$ such that $x \neq y$. Then $x, y \in \operatorname{cl}(E)$ for all $E \in \mathcal{E}$. By 5.14, there exist $U, V \in \mathcal{RO}(X)$ such that $x \in U$, $y \in V$, and $\operatorname{cl} U \cap \operatorname{cl} V = \varnothing$. Hence $\operatorname{cl} U \subseteq \operatorname{int}(X \setminus V)$. So $U \ll \operatorname{int}(X \setminus V)$. As $\mathcal{E}$ is an end, either $X \setminus \operatorname{cl} U \in \mathcal{E}$ or $\operatorname{int}(X \setminus V) \in \mathcal{E}$. Suppose $X \setminus \operatorname{cl} U \in \mathcal{E}$. Note that $\operatorname{cl}(X \setminus \operatorname{cl} U) = X \setminus \operatorname{int}\operatorname{cl} U = X \setminus U$. As $x \in U$, $x \notin \operatorname{cl}(X \setminus \operatorname{cl} U)$. Therefore, $x \notin \mathrm{a}\,\mathcal{E}$, a contradiction. On the other hand, suppose $\operatorname{int}(X \setminus V) \in \mathcal{E}$. Note that $\operatorname{cl}\operatorname{int}(X \setminus V) = X \setminus \operatorname{int}\operatorname{cl} V = X \setminus V$. As $y \in V$, $y \notin \operatorname{cl}\operatorname{int}(X \setminus V)$, a contradiction.

So take $x \in \mathrm{a}\,\mathcal{E}$. We will show that $\mathcal{E}$ converges to x. Let $U \in \mathcal{RO}(X)$ such that $x \in U$. Then $x \ll U$. So there is a $V \in \mathcal{RO}(X)$

such that $x \ll V \ll U$. Hence $X \setminus \operatorname{cl} V \in \mathcal{E}$ or $U \in \mathcal{E}$. If $X \setminus \operatorname{cl} V \in \mathcal{E}$ then $x \in \operatorname{cl}(X \setminus \operatorname{cl} V) = X \setminus V$. But $x \in V$, a contradiction. Therefore, $U \in \mathcal{E}$. □

Theorem 5.16. *For $X \in \operatorname{ob} \mathbf{CPT_2}$, $(\Psi \circ \Phi)(X)$ is homeomorphic to X.*

Proof. Define $f : X \to \operatorname{End}(\mathcal{RO}(X), \ll)$ by $f(x) = \mathcal{N}(x) \cap \mathcal{RO}(X)$.

First, we must show that $f(x) \in \operatorname{End}(\mathcal{RO}(X), \ll)$ for all $x \in X$. Let $x \in X$. Clearly, $\varnothing \notin f(x)$. Let $U, V \in \mathcal{N}(x) \cap \mathcal{RO}(X)$. Then $U \cap V \in \mathcal{N}(x)$ and, by 5.3, $U \cap V \in \mathcal{RO}(X)$. Now, let $U \in \mathcal{N}(x) \cap \mathcal{RO}(X)$ and $V \in \mathcal{RO}(X)$ such that $U \ll V$. Then $\operatorname{cl} U \subseteq V$. As $x \in U$, $x \in V$. So $V \in \mathcal{N}(x) \cap \mathcal{RO}(X)$ and hence $f(x)$ is a filter. Let $U \in \mathcal{N}(x) \cap \mathcal{RO}(X)$. Then $x \in U$. As X is compact Hausdorff, there is a $V \in \mathcal{RO}(X)$ such that $x \in V \ll U$. So $f(x)$ is round. By 3.6, $\mathcal{N}(x)$ is an end on X and thus $\mathcal{N}(x) \cap \mathcal{RO}(X)$ is an end on $\mathcal{RO}(X)$.

To show f is injective, let $x, y \in X$ such that $x \neq y$. By 5.14, there exist $U, V \in \mathcal{RO}(X)$ such that $x \in U$ and $v \in V$. Therefore, $U \in (\mathcal{N}(x) \cap \mathcal{RO}(X)) \setminus (\mathcal{N}(y) \cap \mathcal{RO}(X))$. Hence $f(x) \neq f(y)$.

To show f is surjective, let $\mathcal{E} \in \operatorname{End}(\mathcal{RO}(X), \ll)$. By 5.15, $\operatorname{a} \mathcal{E} = \{x\}$ for some $x \in X$ and, furthermore, $x \in E$ for all $E \in \mathcal{E}$. Hence $\mathcal{E} \subseteq \mathcal{N}(x)$. As $\mathcal{N}(x)$ is an end, it must be that $\mathcal{E} = \mathcal{N}(x)$. So $\mathcal{E} = f(x)$.

To show f is continuous, we will first show that $f^{\leftarrow}[\operatorname{o}(U)] = U$ for all $U \in \mathcal{RO}(X)$. Let $x \in f^{\leftarrow}[\operatorname{o}(U)]$. Then $f(x) \in \operatorname{o}(U)$ and thus $U \in f(x)$. So $U \in \mathcal{N}(x)$ and hence $x \in U$. The opposite inclusion is the reverse of this argument. As $\{\operatorname{o}(U) : U \in \mathcal{RO}(X)\}$ is a base for a topology on $\operatorname{End}(\mathcal{RO}(X), \ll)$, this shows f is continuous.

As f is a continuous bijection from a compact space to a Hausdorff space, f is a homeomorphism [7]. □

Theorem 5.17. *For $B \in \operatorname{ob} \mathbf{DeV}$, $(\Phi \circ \Psi)(B)$ is algebra isomorphic to B.*

Proof. We will show that the mapping $B \to \mathcal{RO}(\operatorname{End}(B), \ll) : b \mapsto \operatorname{o}(b)$ is an algebra isomorphism.

By 4.14(1), $\operatorname{o}(0) = \varnothing$, and clearly, $\operatorname{o}(1) = \operatorname{End}(B)$. So, in the Boolean algebra $(\mathcal{RO}(\operatorname{End}(B)), \ll)$, $\operatorname{o}(0) = 0$ and $\operatorname{o}(1) = 1$.

Let $a, b \in B$. By 4.14(3), $\operatorname{o}(a \wedge b) = \operatorname{o}(a) \wedge \operatorname{o}(b)$. By 4.14(7), $\operatorname{End}(B) \setminus \operatorname{cl} \operatorname{o}(a) = \operatorname{o}(-a)$. But in the Boolean algebra $(\mathcal{RO}(\operatorname{End}(B)), \ll)$, $\operatorname{End}(B) \setminus \operatorname{cl} \operatorname{o}(a) = -\operatorname{o}(a)$. So $-\operatorname{o}(a) = \operatorname{o}(-a)$ and thus, using De Morgan's law, $\operatorname{o}(a \vee b) = \operatorname{o}(a) \vee \operatorname{o}(b)$.

By 5.8, the mapping is bijective and thus an isomorphism. □

Lemma 5.18. *Let $A, B \in \operatorname{ob}\mathbf{DeV}$ and $f \in \mathrm{M}(A,B)$. For $a \in A$,*

$$\mathrm{o}(f(a)) = \operatorname{int}\operatorname{cl}\bigcup_{b \ll a}\mathrm{o}(f(b))$$

Proof. Let $U = \bigcup_{b \ll a}\mathrm{o}(f(b))$. Clearly, $U \subseteq \mathrm{o}(f(a))$. As $\mathrm{o}(f(a)) \in \mathcal{RO}(\operatorname{End}(B))$, $U \subseteq \operatorname{int}\operatorname{cl}\mathrm{o}(f(a))$. For the reverse inclusion, note that $\operatorname{End}(B) \setminus \operatorname{cl} U \in \mathcal{RO}(\operatorname{End}(B))$. By 5.8, there is a $c \in B$ such that $\operatorname{End}(B) \setminus \operatorname{cl} U = \mathrm{o}(c)$. So $\operatorname{int}\operatorname{cl} U = \operatorname{End}(B) \setminus \operatorname{cl}\mathrm{o}(c)$. Also, $U \cap \mathrm{o}(c) = \varnothing$ and thus, for $d \ll a$, $\mathrm{o}(c) \cap \mathrm{o}(f(d)) = \varnothing$. Therefore, $\mathrm{o}(c \wedge f(d)) = \varnothing$ and thus $c \wedge f(d) = 0$. Hence $f(d) \leq -c$. As this holds for all $d \ll a$, we have

$$f(a) = \bigvee_{d \ll a} f(d) \leq -c$$

and thus

$$\begin{aligned}\mathrm{o}(f(a)) &\subseteq \mathrm{o}(-c)\\ &= \operatorname{End}(B) \setminus \operatorname{cl}\mathrm{o}(c)\\ &= \operatorname{int}\operatorname{cl} U\end{aligned}$$

□

Lemma 5.19. *Let $A, B \in \operatorname{ob}\mathbf{DeV}$ and $f \in \mathrm{M}(A,B)$. For $a \in A$,*

$$(\Phi \circ \Psi)(f)(\mathrm{o}(a)) = \mathrm{o}(f(a))$$

Proof. First, note that $(\Phi \circ \Psi)(f)(\mathrm{o}(a)) = \operatorname{int}\operatorname{cl}\Psi(f)^{\leftarrow}[\mathrm{o}(a)]$.

Suppose $b \in A$ such that $b \ll a$. Take $\mathcal{E} \in \mathrm{o}(f(b))$. Then $f(b) \in \mathcal{E}$. As $b \ll a$, $a \in \Psi(f)(\mathcal{E})$. So $\Psi(f)(\mathcal{E}) \in \mathrm{o}(a)$ and thus $\mathcal{E} \in \Psi(f)^{\leftarrow}(\mathrm{o}(a))$. Hence $\mathrm{o}(f(b)) \subseteq \Psi(f)^{\leftarrow}[\mathrm{o}(a)]$. Since this holds for all $b \ll a$, $\bigcup_{b \ll a}\mathrm{o}(f(b)) \subseteq \Psi(f)^{\leftarrow}[\mathrm{o}(a)]$. By 5.18,

$$\begin{aligned}\mathrm{o}(f(a)) &= \operatorname{int}\operatorname{cl}\bigcup_{b \ll a}\mathrm{o}(f(b))\\ &\subseteq \operatorname{int}\operatorname{cl}\Psi(f)^{\leftarrow}[\mathrm{o}(a)]\end{aligned}$$

For the reverse inclusion, let $\mathcal{E} \in \Psi(f)^{\leftarrow}[\mathrm{o}(a)]$. Then $\Psi(f)(\mathcal{E}) \in \mathrm{o}(a)$ and thus $a \in \Psi(f)(\mathcal{E})$. So there is a $b \in A$ such that $b \ll a$ and $f(b) \in \mathcal{E}$. Hence $f(b) \leq f(a)$ and thus $f(a) \in \mathcal{E}$. Therefore, $\mathcal{E} \in \mathrm{o}(f(a))$. So we have $\Psi(f)^{\leftarrow}[\mathrm{o}(a)] \subseteq \mathrm{o}(f(a))$. As $\mathrm{o}(f(a)) \in \mathcal{RO}(\operatorname{End}(B))$, $\operatorname{int}\operatorname{cl}\Psi(f)^{\leftarrow}[\mathrm{o}(a)] \subseteq \mathrm{o}(f(a))$. □

Theorem 5.20. *For $B \in \operatorname{ob}\mathbf{DeV}$, define $\eta(B) : \mathcal{RO}(\operatorname{End}(B), \ll) \to B$ by $\eta(B)(\mathrm{o}(b)) = b$. Then $\eta : \Phi \circ \Psi \to 1_{\mathbf{DeV}}$ is a natural isomorphism.*

Proof. Note that, by 5.8, $\eta(B)$ is a bijection for any $B \in \operatorname{ob}\mathbf{DeV}$ and so η is well-defined.

For $A, B \in \operatorname{ob}\mathbf{DeV}$ and $f \in \mathrm{M}(A, B)$, we have the following diagram.

$$\begin{tikzcd}
(\Phi \circ \Psi)(A) \arrow[r, "\eta(A)"] \arrow[d, "(\Phi\circ\Psi)(f)"'] & A \arrow[d, "f"] \\
(\Phi \circ \Psi)(B) \arrow[r, "\eta(B)"'] & B
\end{tikzcd}$$

We want to show this diagram commutes. First,

$$(1_{\mathbf{DeV}}(f) \circ \eta(A))(\mathrm{o}(a)) = 1_{\mathbf{DeV}}(f)(a) = f(a)$$

Also, by 5.19,

$$(\eta(B) \circ (\Phi \circ \Psi)(f))(\mathrm{o}(a)) = \eta(B)(\mathrm{o}(f(a))) = f(a)$$

So the diagram commutes and thus η is a natural transformation.

By 5.17, $\eta(B)$ is an isomorphism for all $B \in \operatorname{ob}\mathbf{DeV}$. Therefore, η is a natural isomorphism. □

Lemma 5.21. *Let $X, Y \in \operatorname{ob}\mathbf{CPT_2}$ and $f \in \mathrm{M}(X, Y)$. For $x \in X$,*

$$(\Psi \circ \Phi)(f)(\mathcal{N}(x)) = \mathcal{N}(f(x)) \cap \mathcal{RO}(Y)$$

Proof. Note that

$$(\Psi \circ \Phi)(f)(\mathcal{N}(x)) = \{U \in \mathcal{RO}(Y) : \exists V \in \mathcal{RO}(Y) \text{ s.t. } \operatorname{cl} V \subseteq U \\ \text{and } \operatorname{int}\operatorname{cl} f^{\leftarrow}[V] \in \mathcal{N}(x)\}$$

Let $U \in (\Psi \circ \Phi)(f)(\mathcal{N}(x))$. Then $U \in \mathcal{RO}(Y)$ and there is a $V \in \mathcal{RO}(Y)$ such that $\operatorname{cl} V \subseteq U$ and $\operatorname{int}\operatorname{cl} f^{\leftarrow}[V] \in \mathcal{N}(x)$. So $x \in \operatorname{int}\operatorname{cl} f^{\leftarrow}[V]$. As $\operatorname{cl} V \subseteq U$, $\operatorname{cl} f^{\leftarrow}[V] \subseteq f^{\leftarrow}[U]$. Hence $\operatorname{int}\operatorname{cl} f^{\leftarrow}[V] \subseteq \operatorname{int} f^{\leftarrow}[U] = f^{\leftarrow}[U]$. So $x \in f^{\leftarrow}[U]$ and thus $f(x) \in U$. As U is open, $U \in \mathcal{N}(f(x))$.

For the reverse inclusion, let $U \in \mathcal{N}(f(x)) \cap \mathcal{RO}(Y)$. Then $U \in \mathcal{RO}(Y)$ and $f(x) \in U$. As Y is compact Hausdorff, there is a $V \in \mathcal{RO}(Y)$ such that $\operatorname{cl} V \subseteq U$ and $V \in \mathcal{N}(f(x))$. So $f(x) \in V$ and thus $x \in f^{\leftarrow}[V]$. As f is continuous and V is open, $f^{\leftarrow}[V] \subseteq \operatorname{int}\operatorname{cl} f^{\leftarrow}[V]$. Hence $x \in \operatorname{int}\operatorname{cl} f^{\leftarrow}[V]$ and so $\operatorname{int}\operatorname{cl} f^{\leftarrow}[V] \in \mathcal{N}(x)$. Therefore, $U \in (\Psi \circ \Phi)(f)(\mathcal{N}(x))$. □

Theorem 5.22. *For $X \in \operatorname{ob}\mathbf{CPT_2}$, define $\zeta(X) : X \to \operatorname{End}((\mathcal{RO}(X), \ll))$ by $\zeta(X)(x) = \mathcal{N}(x) \cap \mathcal{RO}(X)$. Then $\zeta : 1_{\mathbf{CPT_2}} \to \Psi \circ \Phi$ is a natural isomorphism.*

Proof. For $X, Y \in \operatorname{ob}\mathbf{CPT_2}$ and $f \in \mathrm{M}(X, Y)$, we have the following diagram.

$$\begin{tikzcd} X \arrow[r, "\zeta(X)"] \arrow[d, "f"'] & (\Psi \circ \Phi)(X) \arrow[d, "(\Psi\circ\Phi)(f)"] \\ Y \arrow[r, "\zeta(Y)"'] & (\Psi \circ \Phi)(Y) \end{tikzcd}$$

We want to show this diagram commutes. Note that

$$(\zeta(Y) \circ f)(x) = \zeta(Y)(f(x)) = \mathcal{N}(f(x)) \cap \mathcal{RO}(Y)$$

Also, by 5.21,

$$(\Psi \circ \Phi)(f) \circ \zeta(X))(x) = (\Psi \circ \Phi)(f)(\mathcal{N}(x)) = \mathcal{N}(f(x)) \cap \mathcal{RO}(Y)$$

So the diagram commutes and thus ζ is a natural transformation.

By 5.16, $\zeta(X)$ is a homeomorphism for all $X \in \operatorname{ob} \mathbf{CPT_2}$. Therefore, ζ is a natural isomorphism. □

Theorem 5.23 (De Vries Duality Theorem [3])**.** *The categories* $\mathbf{CPT_2}$ *and* $\mathbf{DeV}$ *are dually equivalent.*

Proof. By 5.5, Φ is a contravariant functor from $\mathbf{CPT_2}$ to $\mathbf{DeV}$ and by 5.13, Ψ is a contravariant functor from $\mathbf{DeV}$ to $\mathbf{CPT_2}$. By 5.20 and 5.22, $\eta : \Phi \circ \Psi \to 1_{\mathbf{DeV}}$ and $\zeta : 1_{\mathbf{CPT_2}} \to \Psi \circ \Phi$ are natural isomorphisms. Therefore, $\mathbf{CPT_2}$ and $\mathbf{DeV}$ are dually equivalent. □

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