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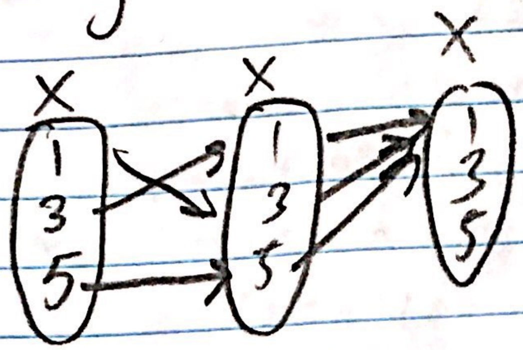
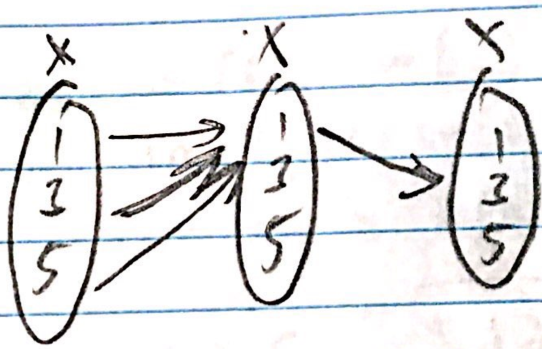
1 a. $g \circ f$



$f \circ g$



b. f, f^{-1}

c. f, f^{-1}

d. $im(f) = \{3, 1, 5\}$

$im(g) = \{1\}$

$im(f^{-1}) = \{3, 1, 5\}$

$im(f \circ g) = \{3\}$

$im(g \circ f) = \{1\}$

2 a

- $y \in Y \rightarrow y = f(x)$ for some distinct $x \in X$ due to injective nature of f.
- Therefore $f^{-1}(y)$ has distinct x $(x = f^{-1}(y))$
- Thus, $f^{-1}(f(x)) = x$ for all $x \in X$.

b. $im(f_1) = im(f_2) \cap im(f_3) \rightarrow$
$(im(f_2) \cap im(f_3) \subseteq im(f_1)) \wedge$
$(im(f_1) \subseteq im(f_2) \cap im(f_3))$

- $im(f_1) \subseteq im(f_2) \cap im(f_3)$ proof
- $\forall y \in im(f_1)$
- $y = f_1(x)$ such that $x \in A_1 \cap A_2$
- if $x \in A_1, A_2$ then $y_1 = f_2(x)$ such that $y_1 \in im(f_2) \wedge y_2 \in f_3(x)$ such that $y_2 \in im(f_3)$
- if $f_1, f_2, f_3 \subseteq f$ then $f_1(x) = f_2(x) = f_3(x)$ because of vertical line test
- Therefore, $y \in im(f_2) \wedge y \in im(f_3)$ which means $y \in im(f_2) \cap im(f_3)$
- Thus, $im(f_1) \subseteq im(f_2) \cap im(f_3)$

- $im(f_2) \cap im(f_3) \subseteq im(f_1)$ proof
  - if $\exists y \in im(f_2) \cap im(f_3)$ then $y \in im(f_2) \wedge y \in im(f_3)$
  - if $y \in im(f_2), im(f_3)$ then $y = f_2(x_1), f_3(x_2)$ for some
  - $x_1 \in A_1$ and $x_2 \in A_2$
  - $f_2(x_1) = y$ and $f_3(x_2) = y$
  - Since f is one to one, $f_1, f_2, f_3 \subseteq f$ such that $x_1 = x_2$
  - From this, we get $x_1 \in (A_1 \cap A_2)$ because $x_1 \in A_1, A_2$
  - Therefore, $f_1(x_1) \in im(f_1)$
  - The definition of a function denotes that any element of the domain cannot map over to more than one value in the image.
  - Therefore $f_1, f_2, f_3 \subseteq f$ so $f_1(x_1) = f_2(x_1) = f_3(x_1)$
  - $\therefore im(f_2) \cap im(f_3) \subseteq im(f_1)$

- Because $im(f_2) \cap im(f_3) \subseteq im(f_1)$ and $im(f_1) \subseteq im(f_2) \cap im(f_3)$ $im(f_1) = im(f_2) \cap im(f_3)$

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3. a. Counterexample: $f(x) = \frac{1}{x}$, $g(x) = x^2$

$(g \circ f)(z) = \left(\frac{1}{z}\right)^2$ $\quad z = \left(\frac{1}{y}\right)^2$

$(g \circ f)^{-1}(z) = \pm\sqrt{z}$ $\quad zy^2 = 1$

$y = \pm\sqrt{z}$

$(f^{-1})(x) = \frac{1}{x}$

$(g^{-1})(x) = \pm\sqrt{x}$

$(f^{-1} \circ g^{-1})(z) = \frac{1}{\pm\sqrt{z}}$

$\frac{1}{\pm\sqrt{z}} \neq \pm\sqrt{z} \quad \therefore (g \circ f)^{-1}(z) \neq (f^{-1} \circ g^{-1})(z)$

b. Prove $(g \circ f)^{-1}$ exists

- g and f are bijective
- thus $g \circ f$ must be bijective (bijective meare one-to-one and onto)
- If $g \circ f$ is bijective, $g \circ f$ must have an inverse
- Therefore $g \circ f^{-1}$ exists

Assume:

Prove $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$: - $x \in X, y \in Y, z \in Z$

- $\exists (y, z) \in f : Y \to Z$ and f is bijective
- $\exists (x, y) \in g : X \to Y$ and g is bijective.
- $\exists (z, y) \in f^{-1} : Z \to Y$ b/c def. of inverse
- $\exists (y, x) \in g^{-1} : Y \to X$ b/c def of inverse.
- $\exists (z, x) \in f^{-1} \circ g^{-1} : Z \to X$
- $\exists (x, z) \in g \circ f : X \to Z$
- $\exists (z, x) \in (g \circ f)^{-1} : Z \to X$

$(z, x) \in f^{-1} \circ g^{-1} : Z \to X = (z, x) \in (g \circ f)^{-1} : Z \to X$

$\therefore (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

4. - $f : X \to Y \wedge |X| = |Y|$

1. f is injective $\to$ f is surjective
   - If f is injective and $|X| = |Y|$ for any $y \in Y$, there exists on $x \in X$ such that $f(x) = y$
   - $\therefore$ f is surjective

2. f is surjective $\to$ f is injective.
   - If f is surjective and $|X| = |Y|$, then $\forall y \in Y$, there exists one distinct $x \in X$ such that $f(x) = y$
   - Therefore, if $f(x_1) = f(x_2)$ $x_1 = x_2$ $\therefore$ f is injective.

$\therefore$ f is injective $\longleftrightarrow$ f is surjective
given $f: X \to Y \wedge |X| = |Y|$

5. - Assume $x, y \in N$ where $|N| = 7$ and all elements of N are distinct
- Possible remainders for any 2 distinct integers are 0, 1 or 9, 2 or 8, 3 or 7, 4 or 6, 5
- If $|N| = 7$ and 6 possible combinations of remainders there must always exist $x, y \in N$ such that $x+y$ or $x-y$ is a multiple of 10.