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Discrete HW 2

1 a) 1. If $A=B$ is true,

- $A \times B = A \times A = B \times B = B \times A$ (transitive prop.)
- Therefore $A \times B = B \times A$

2. If $A \times B = B \times A$ is true

- $A \times B = \{(a,b): a \in A, b \in B\}$ (def of cartesian prod
- $B \times A = \{(b,a): b \in B, a \in A\}$
- If $A \times B = B \times A$,
  $\{(a,b): a \in A, b \in B\} = \{(b,a): b \in B, a \in A\}$
- Thus, $b \in A$ and $a \in B$
- From this, we get $A \subseteq B$ and $B \subseteq A$
- Therefore $A=B$.

Therefore, $A \times B = B \times A$ if and only if $A=B$

b) $|A \triangle B| = |(A-B) \cup (B-A)|$

$|A \triangle B| = |A| + |B| - |A \cap B|$

We must test $|(A-B) \cup (B-A)| = |A| + |B| - |A \cap B|$

Counterexample

Let $A=B$, such that A, B are nonempty.

$|(A-A) \cup (B-B)| = |A| + |A| - |A \cap A|$

$|\emptyset \cup \emptyset| = 2|A| - |A|$

$\emptyset \neq |A|$

$\therefore \quad |A \triangle B| \neq |A| + |B| - |A \cap B|$

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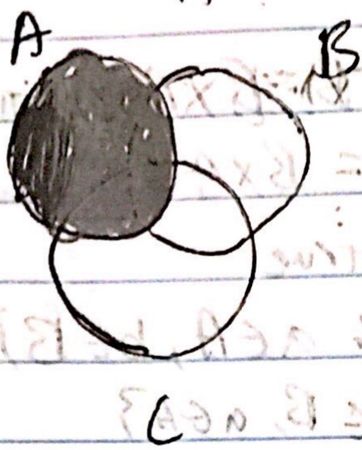
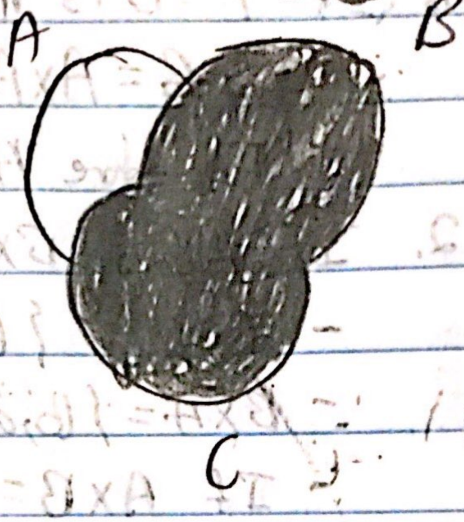
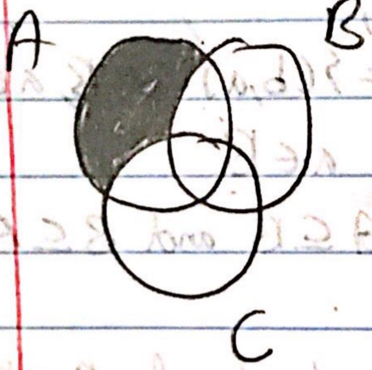
c) A

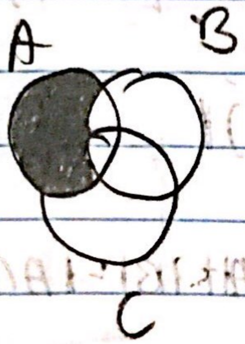


$B \cup C$

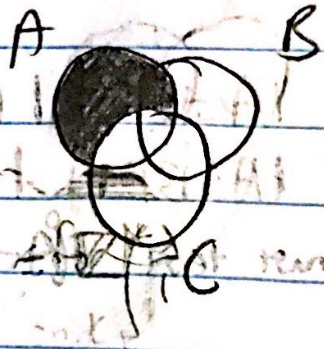




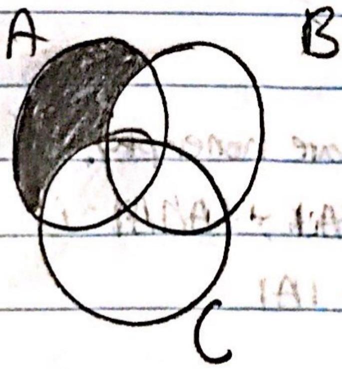
$A-(B \cup C)$



$A-B$

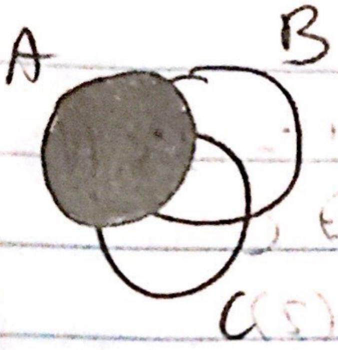


$A-C$

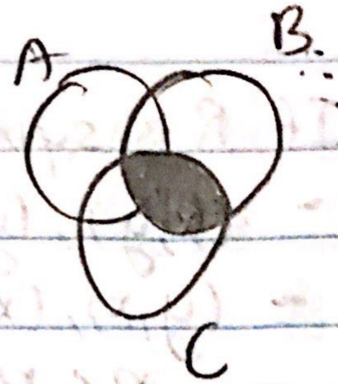


$(A-B) \cap (A-C)$

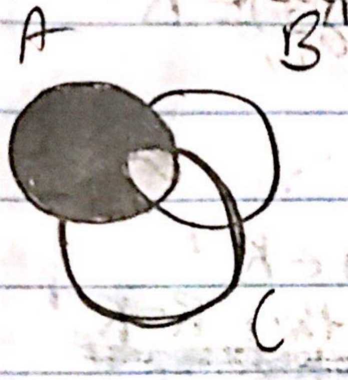
$\therefore A-(B \cup C) = (A-B) \cap (A-C)$



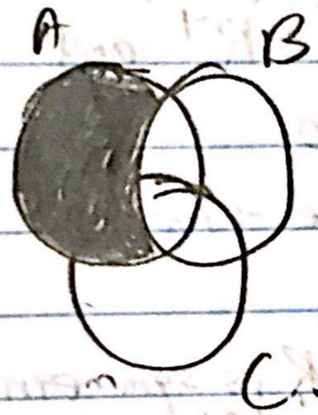
A



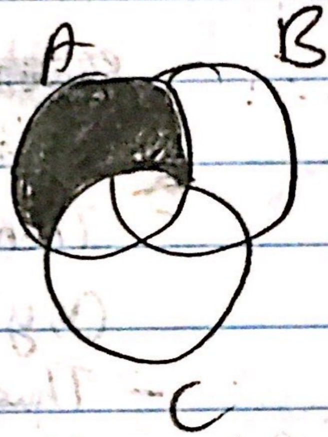
$B \cap C$



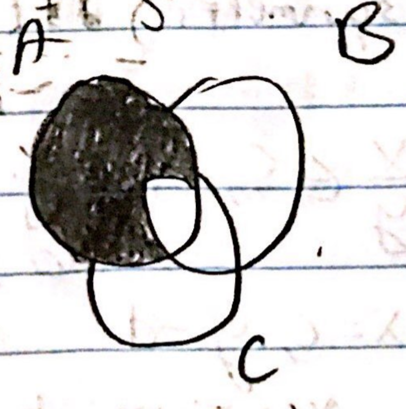
$A - (B \cap C)$



$A - B$



$A - C$



$(A-B) \cup (A-C)$

$\therefore A - (B \cap C) = (A-B) \cup (A-C)$

2 a) 1. If R is symmetric

- $(x,y),(y,x) \in R$
- $(y,x),(x,y) \in R^{-1}$
- $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$
- $\Rightarrow R = R^{-1}$

2. If $R = R^{-1}$

- $(x,y) \in R$, $(y,x) \in R^{-1}$
- Due to the fact that $R = R^{-1}$, $(x,y) \in R^{-1}$ and $(y,x) \in R$ also.
- Therefore we see that $(x,y),(y,x) \in R$.
- Thus, relation R is symmetric

$\therefore$ Relation R is symmetric iff $R = R^{-1}$

b) 1. Reflexivity

- Given fraction $\frac{x}{y} \in S$, for $\left(\frac{x}{y}\right) R \left(\frac{x}{y}\right)$

$x=a$, $y=b$, $x=c$, $y=d$.

- If $ad=bc$, $xy=yx$ based on substitution.
- Thus $\left(\frac{x}{y}\right) R \left(\frac{x}{y}\right)$ exists and makes S reflexive.

Symmetry

- if $ad=bc$, $\left(\frac{a}{b}, \frac{c}{d}\right) \in R$

- If $ad = bc$, then $cb = da$ and therefore $\left(\frac{c}{d}, \frac{a}{b}\right) \in R$

- $\left(\frac{a}{b}\right) R \left(\frac{c}{d}\right) \longrightarrow \left(\frac{c}{d}\right) R \left(\frac{a}{b}\right)$ $\therefore$

S is symmetric

Transitivity

- given $\frac{x}{y} \in S$, if $\left(\frac{a}{b}\right) R \left(\frac{c}{d}\right)$ and $\left(\frac{c}{d}\right) R \left(\frac{x}{y}\right)$, $ad = bc$ and $cy = dx$

- If $ad = bc$ and $cy = dx$ then
$$\frac{ad}{b} = c = \frac{dx}{y} \quad \text{and} \quad \frac{bc}{a} = d = \frac{cy}{x}$$

- From this, we get $ady = bdx$ and $bcx = acy$

- $\frac{a\cancel{d}y}{\cancel{d}} = \frac{b\cancel{d}x}{\cancel{d}}$ $\quad ay = bx$

- $\frac{b\cancel{c}x}{\cancel{c}} = \frac{a\cancel{c}y}{\cancel{c}}$ $\quad bx = ay$

- $\left(\frac{a}{b}\right) R \left(\frac{c}{d}\right)$ and $\left(\frac{c}{d}\right) R \left(\frac{x}{y}\right)$ implies $\left(\frac{a}{b}\right) R \left(\frac{x}{y}\right)$ S is transitive.

ii Equivalence class:

- $[\frac{2}{3}] = \{x \in S : xR\frac{2}{3}\}$
- $xR(\frac{2}{3})$ implies $c=2, d=3$
 such that there is an integer a and integer b in which $2a = 3b$

$$\frac{a}{b} = \frac{2}{3}$$

= If $\frac{a}{b} = \frac{2}{3}$, then equivalence

class must be subset of S ~~in which~~ that contains fractions equivalent to $\frac{2}{3}$

$(\text{i.e. } \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots)$

- $[\frac{2}{3}] = \{x \in S \mid x = \frac{2a}{3a}, a \in \mathbb{Z}\}$

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3. a).

base case: $2^{2(0)} - 1 = 2^0 - 1 = 0$

$3|0$

$2^{2(1)} - 1 = 2^2 - 1 = 3$

$3|3$

hypothesis:

$P(n) = 2^{2n} - 1$ is divisible by 3

$P(n+1) = 2^{2(n+1)} - 1$

$= 2^{2n+2} - 1$

$= 4(2^{2n}) - 1$

$= (3+1)(2^{2n}) - 1$

$= \underbrace{(3)(2^{2n})}_{\text{divisible by 3}} + \underbrace{2^{2n} - 1}_{\text{divisible by 3.}}$

~~$\therefore$ $3|2^{2n}$ $3|2^{2n}-1$ by induction~~

$\therefore$ $3|2^{2n} - 1$ $\forall n \in \mathbb{Z}^*$ by induction

b) base case

$S_2 = (1 - \frac{1}{2}) = \frac{1}{2}$

$S_3 = (1 - \frac{1}{2})(1 - \frac{1}{3}) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

hypothesis: $P(n) = (1 - \frac{1}{2})(1 - \frac{1}{3}) \cdots (1 - \frac{1}{n}) = \frac{1}{n}$

$P(n+1) = \left(1 - \frac{1}{n+1}\right)(P(n))$

$= \left(\frac{n+1}{n+1} - \frac{1}{n+1}\right)\left(\frac{1}{n}\right) = \left(\frac{\cancel{n}}{n+1}\right)\left(\frac{1}{\cancel{n}}\right) = \frac{1}{n+1}$

Thus, $P(n+1) = \left(1-\frac{1}{2}\right)(1-3)\cdots\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$

$\therefore$ by induction $S_n = (1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{n}) = \frac{1}{n}$

4a) $\exists n \in \mathbb{Z}^*$

Let's assume n, $3 \nmid (2^{2n}-1)$

<u>Base case:</u>

$P(n): 2^{2n}-1$

$P(0): 2^{2(0)}-1 = 1-1=0$

$3 \mid 0$

$P(0)$ is true.

If $P(n-1)$ is true then $3 \mid (2^{2(n-1)}-1)$

- $\exists a,\ 3a = 2^{2(n-1)}-1$
- $\exists a,\ 3a = 2^{2n-2}-1$
- $\exists a,\ 4(3a) = \left(\frac{2^{2n}}{4} - \frac{4}{4}\right)4$
- $\exists a,\ 4(3a) = 2^{2n}-4$
- $\exists a,\ 4(3a) = 2^{2n}-(3+1)$
- $\exists a,\ 4(3a)+3 = 2^{2n}-1$
- $\exists a,\ 3(4a+1) = 2^{2n}-1$, which contradicts our initial assumption $\exists n,\ 3 \nmid (2^{2n}-1)$.

$\therefore \forall n \in \mathbb{Z}^*,\ 3 \mid (2^{2n}-1)$

b) Assume $\exists\, n \geq 2,\ S_n = (1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{n}) \neq \frac{1}{n}$

$P(n) = (1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{n})$

Base case: $S_2 = (1-\frac{1}{2}) = \frac{1}{2}$ ✓ $\therefore P(2)$ is true.

If $P(k-1)$ is true, then $(1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{k-1}) = \frac{1}{k-1}$

$$(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{k-1}\right)\left(1-\frac{1}{k}\right) = \left(\frac{1}{k-1}\right)\left(1-\frac{1}{k}\right)$$

$$= \left(\frac{1}{k-1}\right)\left(\frac{k-1}{k}\right)$$

$$= \frac{1}{k}$$

Thus contradicting our initial assumption

$\therefore\ \forall n \geq 2,\ S_n = \frac{1}{n}$

5 a)

Base case: If $n=1 \rightarrow 2^0 = 1$

hypothesis: $\forall n \in \mathbb{Z}^+,\ n = c_k 2^k + c_{k-1} 2^{k-1} \ldots + c_1 2^1 + c_0 2^0$

where $k \in \mathbb{N},\ c_k \in \{0,1\}$

Induction.

If $P(k+1)$ is true then, any positive integer m can be written as $c_{k+1} 2^{(k+1)} + c_k 2^k \ldots c_1 2^1 + c_0 2^0$

See next pages

Case 1 $c_{k+1} = 0$

$$m = c_{k+1} 2^{k+1} + c_k 2^k + c_{k-1} 2^{k-1} + \dots + c_1 2^1 + c_0 2^0$$

$$n + c_{k+1} 2^{k+1} = c_{k+1} 2^{k+1} + c_k 2^k + c_{k-1} 2^{k-1} + \dots + c_1 2^1 + c_0 2^0$$

$n + 0$

$$n + (0)(2^{k+1}) = (0)(2^{k+1}) + c_k 2^k + c_{k-1} 2^{k-1} + \dots c_1 2^1 + c_0 2^0$$

$$n = c_k 2^k + c_{k-1} 2^{k-1} \dots + c_1 2^1 + c_0 2^0$$

$$n = n \checkmark$$

Case 2 $c_{k+1} = 1$

$$n + c_{k+1} 2^{k+1} = c_{k+1} 2^{k+1} + c_k 2^k + c_{k-1} 2^{k-1} + \dots + c_1 2^1 + c_0 2^0$$

$$n + (1)(2^{k+1}) = (1)(2^{k+1}) + c_k 2^k + c_{k-1} 2^{k-1} + \dots + c_1 2^1 + c_0 2^0$$

$$n + 2^{k+1} = 2^{k+1} + c_k 2^k \dots c_1 2^1 + c_0 2^0$$

$$n + 2(2^k) = 2(2^k + c_k 2^{k-1} + c_{k-1} 2^{k-2} + \dots c_1 2^0 + c_0 2^{-1})$$

$$n = 2(\cancel{2^k} + c_k 2^{k-1} + c_{k-1} 2^{k-2} \dots + c_1 2^0 + c_0 2^{-1} - \cancel{2^k})$$

$$n = 2(c_k 2^{k-1} + c_{k-1} 2^{k-2} \dots + c_1 2^0 + c_0 2^{-1})$$

$$n = c_k 2^k + c_{k-1} 2^{k-1} \dots + c_1 2^1 + c_0 2^0$$

$$n = n \checkmark$$

$\therefore \forall n \in \mathbb{Z}^+, n = c_k 2^k + c_{k-1} 2^{k-1} \dots + c_1 2^1 + c_0 2^0$

where $k \in \mathbb{N}$, $c_k \in \{0, 1\}$

b) Any product of 2 or more integers is odd.

base case:

Let $a_1 \equiv 1 \pmod 2$ b/c a_1, a_2 are odd

Let $a_2 \equiv 1 \pmod 2$

$a_1 a_2 \equiv 1^2 \pmod 2$

$a_1 a_2 \equiv 1 \pmod 2$

hypothesis: $a_1 a_2 \cdots a_n \equiv 1^n \pmod 2$ such that $a_1, a_2 \cdots a_{n+1}$ are all odd numbers $1 \pmod 2$

$a_1 a_2 \cdots a_{n+1} \equiv 1^{n+1} \pmod 2$

$\underbrace{a_1 a_2 \cdots a_n}_{1^n \pmod 2} \underbrace{a_{n+1}}_{1 \pmod 2} \equiv 1^n \pmod 2 \cdot 1 \pmod 2$

$1^n \pmod 2 \cdot 1 \pmod 2$

$\therefore$ 1 raised to any power is 1

$a_1 a_2 \cdots a_{n+1} \equiv 1 \pmod 2$

$\therefore$ product of 2 or more odd integers is odd.

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