Hw6 Discrete Mathematics

Justin Kim (jtk2141)

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Problem 1

(a) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
There are $\frac{n!}{(n-k)!}$ ways to pick k positions from n people. However, since all job positions are same, we have to account for k! repeats, hence why we have to divide $\frac{n!}{(n-k)!}$ by k!.

(b) $k((n-1)P(k-1)) = \frac{k(n-1)!}{(n-k)!}$
There are k ways to pick the mayor's son's position. This leaves us with k-1 positions and n-1 people. Because every position is different, we must be working with permutations. Thus, there are (n-1) P (k-1) different ways to pick k-1 positions from n-1 people. We also have to account for the k different positions the mayor's son can be in, so we multiply that permutation by k.

(c) $((n-1)^{k-1}) + ((n-1)^{k-1}) + ((n-1)^{k-2}) = 2((n-1)^{k-1}) + ((n-1)^{k-2})$
There are 3 possible cases for this: If the mayor's son is vice mayor, that allows n-1 people to fill in k-1 positions, and thus n-1 ways for each of the k-1 positions to be filled. Thus there are $(n-1)^{k-1}$ ways. The same logic applies to the second case in which the mayor's son is head council. Similar logic applies to the 3rd case in which the mayor's son is both head council AND vice mayor, but there will be n-1 people to fill in k-1 positions, so there will be $(n-1)^{k-2}$ ways.

(d) $\binom{n}{2}\binom{n-2}{2}((n-4)P(k-4))$
For the 2 secretary positions, there are $\binom{n}{2}$ ways to fill them. For the 2 adviser positions, there are $\binom{n-2}{2}$ ways to fill them (because the 2 secretary position shave been filled). Now that we have 4 people in the 2 secretary and 2 adviser positions, we have n-4 people and k-4 people. There are n-4 ways for each of the k-4 positions to be filled, which gives us (n-4) P (k-4).

Problem 2

(a) $\frac{10!}{4!4!} = 6300$
There are 10! ways to rearrange 10 different letters, but we also have to conisder the fact that "I" and "S" are repeated 4 times, so we have to divide 10! by the product of 4! and 4!

(b) $\frac{8!}{3!3!} = 1120$
If we take SIP out of the whole word, we have 7 letters, so we have 7! ways to rearrange 7 different letters, but we also have to consider the fact that "I" and "S" are repeated 3 times, so we have to divide 7! by the product of 3! and 3!. There are 8 different ways to place SIP either in front of, behind, or in between the letters of the 7 letter word. Thus we have $(8)(\frac{7!}{3!3!}) = \frac{8!}{3!3!}$.

(c) $\frac{7!}{3!3!} = 140$
If we take SIPM out of the whole word, we have 6 letters, so we have 6! ways to rearrange 6 different letters, but we also have to consider the fact that "I" and "S" are repeated 3 times, so we have to divide 6! by the product of 3! and 3!. There are 7 different ways to place SIP either in front of, behind, or in between the letters of the 7 letter word. Thus we have $(7)(\frac{6!}{3!3!}) = \frac{7!}{3!3!}$.

(d) $(4)(\frac{7!}{3!3!}) = 560$
If we take SIPM out of the whole word, we have 6 letters, so we have 6! ways to rearrange 6 different letters, but we also have to consider the fact that "I" and "S" are repeated 3 times, so we have to divide 6! by the product of 3! and 3!. This gives us $\frac{6!}{3!3!}$. In addition, depending on the position of SIP in the anagram, there could be 1-7 different possible placements of M in the anagram. Thus, we add $1+2+3+4+5+6+7 = 28$ and muliply 28 by $\frac{6!}{3!3!}$ to get the total number of possible anagrams.

Problem 3

(a) $\sum_{i=45}^{99} \binom{99}{i}$.
In order for there to be more 0s than 1s in a 99-bit string, there needs to be at least 45 0s in the 99bit string.

(b) $\frac{208!}{(4!)^{52}}$
Because there are 208 cards, there are 208! ways to order the cards, but there will be $(4)(13) = 52$ repeats of 4 sets of cards. Thus, in order to account for repeats, we do $\frac{208!}{(4!)^{52}}$.

(c) $\frac{\prod_{i=1}^{10} \binom{(4)(i)}{4}}{10!}$
There are 10 ways to make groups of 4 from 40 people. This means that there are $\binom{40}{4}$ ways to pick the first group, $\binom{36}{4}$ to pick the 2nd group, etc. That being said, order is not important so we need to divide by 10!.

4

a)

$$1$$
$$1 \quad 1 \qquad (x+y)^1 \quad 2 \text{ terms}$$
$$1 \quad 2 \quad 1 \qquad (x+y)^2 \quad 3 \text{ terms}$$
$$1 \quad 3 \quad 3 \quad 1 \qquad (x+y)^3 \quad 4 \text{ terms}$$

For $(x+y)^n$ $\exists$ $n+1$ terms in the expansion.

$\therefore$ $(x+y)^{1000}$ has $\boxed{1001 \text{ terms}}$

b) $\binom{14}{5}(2x)^9(3y^2)^5 = (2002 \times 512 \times 243)x^9y^{10}$

$= 249080832x^9y^{10}$

$\boxed{249080832}$

c) Based on pascal's triangle, coefficients will be 1, 5, 10, 10, 5, 1.

$\therefore$ $\left(x^2-\frac{1}{x}\right)^5 = (x^2)^5 + 5(x^2)^4\left(-\frac{1}{x}\right)$

$+ 10(x^2)^3\left(-\frac{1}{x}\right)^2 + 10(x^2)^2\left(-\frac{1}{x}\right)^3 + 5(x^2)\left(-\frac{1}{x}\right)^4$

$+ \left(-\frac{1}{x}\right)^5$

$= \boxed{x^{10} - 5x^7 + 10x^4 - 10x + \frac{5}{x^2} - \frac{1}{x^5}}$

5a

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k!(n-k)!}$$

$$= \frac{(k+n-k)(n-1)!}{k!(n-k)!}$$

$$= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!}$$

$$= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!}$$

$$= \frac{\cancel{k}(n-1)!}{\cancel{k}(k-1)!(n-k)!} + \frac{\cancel{(n-k)}(n-1)!}{k!\cancel{(n-k)}(n-k-1)!}$$

$$= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!}$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k}$$

5b

$$\binom{2n}{n+1} + \binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!(2n-(n+1))!} + \frac{(2n)!}{n!(2n-n)!}$$

$(2n-n)!$

$$= \frac{(2n)!}{(n+1)!(n-1)!} + \frac{(2n)!}{n!n!}$$

$$= \frac{(2n)!(n)(n+1)}{(n+1)!(n+1)!} + \frac{(2n)!(n+1)(n+1)}{(n+1)!(n+1)!}$$

$$= \frac{(2n)!(2n+1)(n+1)}{(n+1)!(n+1)!} = \frac{(n+1)((2n+1)!)}{(n+1)!(n+1)!}$$

$$= \frac{1}{2}\frac{(2n+2)(2n+1)!}{(n+1)!(n+1)!}$$

$$= \frac{(2n+2)!}{(n+1)!(n+1)!}$$

$$= \frac{1}{2}\frac{(2n+2)!}{(n+1)!(2n+2-(n+1))!}$$

$$= \frac{1}{2}\binom{2n+2}{n+1} \quad \text{QED}$$