Throughout, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

1 Conditional expectation

Theorem 1.1 (Existence and uniqueness of conditional expectation)**.** *Let $X \in L^1$, and $\mathcal{G} \subseteq \mathcal{F}$. Then there exists a random variable Y such that*

- *Y is $\mathcal{G}$-measurable*
- *$Y \in L^1$, and $\mathbb{E}X\mathbf{1}_A = \mathbb{E}Y\mathbf{1}_A$ for all $A \in \mathcal{G}$.*

Moreover, if Y' is another random variable satisfying these conditions, then $Y' = Y$ almost surely.

We call Y a (version of) the conditional expectation given $\mathcal{G}$.

Proof. (Existence)

Case 1: $X \in L^2$.

Recall that L^2 is a Hilbert space, and that the set of $\mathcal{G}$-measurable random variables is a closed subspace of L^2 (it is closed because the space $L^2(\Omega, \mathcal{G}, \mathbb{P})$ is complete). The projection theorem then gives us the existence and uniqueness of $Y \in L^2 \subseteq L^1$.

Case 2: $X \geq 0 \in L^1$.

Let $X_n = X \wedge n \in L^2$. Then by case 1, we can define $Y_n = \mathbb{E}(X_n \mid \mathcal{G}) \in L^2$. We make the following observation

Lemma 1.0.1. Suppose (X, Y) and (X', Y') are two pairs of random variables satisfying the conditions of the theorem, then $X \geq X'$ implies $Y \geq Y'$ almost surely.

Proof. Let $A = \{Y < Y'\}$. Then $\mathbb{E}Y\mathbf{1}_A = \mathbb{E}X\mathbf{1}_A \geq \mathbb{E}X'\mathbf{1}_A = \mathbb{E}Y'\mathbf{1}_A$, so $\mathbb{E}(Y - Y')\mathbf{1}_A \geq 0$ and $\mathbb{P}(A) = 0$. □

It follows that there is some random variable Y such that $Y_n \uparrow Y$. Clearly Y is $\mathcal{G}$-measurable. For any $A \in \mathcal{G}$, we have

$$\begin{aligned}
\mathbb{E}Y\mathbf{1}_A &= \lim_{n\to\infty} \mathbb{E}Y_n\mathbf{1}_A && \text{(MCV)} \\
&= \lim_{n\to\infty} \mathbb{E}X_n\mathbf{1}_A && \text{(MCV)} \\
&= \mathbb{E}X\mathbf{1}_A
\end{aligned}$$

Case 3: $X \in L^1$.

Write $X = X^+ - X^-$, and apply case 2 to X^+ and X^-.

(Uniqueness) Suppose Y and Y' are two random variables satisfying the conditions of the theorem. The $\{Y > Y'\}$ is in $\mathcal{G}$ so $\mathbb{E}Y\mathbf{1}_{\{Y>Y'\}} = \mathbb{E}Y'\mathbf{1}_{\{Y>Y'\}} \implies \mathbb{E}(Y - Y')\mathbf{1}_{\{Y>Y'\}} = 0 \implies \mathbb{P}(Y > Y') = 0$. Similarly, $\mathbb{P}(Y' > Y) = 0$.

Remark. The above can also be proved using the Radon-Nikodym theorem.

(Proof via Radon-Nikodym) First recall the Radon-Nikodym theorem

Proposition 1.1 (Radon-Nikodym theorem)**.** Let μ, ν be two σ-finite measures on $(\Omega, \mathcal{F})$ such that $\nu \ll \mu$. Then there exists a unique (up to a.e. equivalence) $f \in L^1(\Omega, \mathcal{F}, \mu)$ such that $\nu(A) = \int_A f\, d\mu$ for all $A \in \mathcal{F}$.

Consider the measure on $(\Omega, \mathcal{G})$ given by

$$\mu(A) = \mathbb{E}X\mathbf{1}_A, \quad A \in \mathcal{G}$$

so $\mu \ll \mathbb{P}$. By the Radon-Nikodym theorem, there exists a unique $Y \in L^1(\Omega, \mathcal{G}, \mathbb{P})$ such that $\mu(A) = \int_A Y\, d\mathbb{P}$ for all $A \in \mathcal{G}$.

For general $X \in L^1$, we can write $X = X^+ - X^-$ and apply the above to X^+ and X^-. □

Proposition 1.2 (Equivalent definition for conditional expectation)**.** Let $X, \mathcal{G}$ be as above. Then there exists a random variable Y such that

- Y is $\mathcal{G}$-measurable
- $Y \in L^1$ and $\mathbb{E}XZ = \mathbb{E}YZ$ for all $Z \in L^\infty(\mathcal{G})$

Moreover, $Y = \mathbb{E}(X \mid \mathcal{G})$ almost surely.

Proof. (Existence) Set $Y = \mathbb{E}(X \mid \mathcal{G})$. It is straightforward to see that Y satisfies the conditions of the proposition for simple functions Z. Note that simple functions that are in L^p are dense in L^p for $1 \le p \le \infty$. Let $Z_n \in L^\infty(\mathcal{G})$ be a sequence of simple functions such that $Z_n \to Z$ in L^∞ (in particular, we have almost sure pointwise convergence). Then

$$\begin{aligned}
\mathbb{E}XZ &= \lim_{n\to\infty} \mathbb{E}XZ_n && \text{(DCT)} \\
&= \lim_{n\to\infty} \mathbb{E}YZ_n \\
&= \mathbb{E}YZ && \text{(DCT)}
\end{aligned}$$

□

(Uniqueness) Note that any two random variables satisfying the conditions of the proposition are versions of the conditional expectation given $\mathcal{G}$, which was shown to be unique.

Lemma 1.2.1 (Conditional expectation as a function)**.** Let $X, Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then Y is measurable with respect to $\sigma(X)$ if and only if there exists a Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y(\omega) = f(X(\omega))$ for all $\omega \in \Omega$.

Proposition 1.3 (Properties of conditional expectation)**.** All (in)equality relations below hold almost surely.

1. If $X \geq 0$ a.s., then $\mathbb{E}(X \mid \mathcal{G}) \geq 0$
2. If X and $\mathcal{G}$ are independent, then $\mathbb{E}(X \mid \mathcal{G}) = \mathbb{E}[X]$
3. If $\alpha, \beta \in \mathbb{R}$ and $X_1, X_2 \in L^1$, then
$$\mathbb{E}(\alpha X_1 + \beta X_2 \mid \mathcal{G}) = \alpha \mathbb{E}(X_1 \mid \mathcal{G}) + \beta \mathbb{E}(X_2 \mid \mathcal{G}).$$
4. *Tower property*: If $\mathcal{H} \subseteq \mathcal{G}$, then
$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(X \mid \mathcal{H}) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{H}) \mid \mathcal{G})$$
5. If Z is bounded and $\mathcal{G}$-measurable, then
$$\mathbb{E}(ZX \mid \mathcal{G}) = Z\mathbb{E}(X \mid \mathcal{G}).$$
6. Let $X \in L^1$ and $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$. Assume that $\sigma(X, \mathcal{G})$ is independent of $\mathcal{H}$. Then
$$\mathbb{E}(X \mid \mathcal{G}) = \mathbb{E}(X \mid \sigma(\mathcal{G}, \mathcal{H})).$$

Proof. 1. Follows from the proof of existence and uniqueness of conditional expectation, or just use monotonicity.

2. Let $A \in \mathcal{G}$. Then $\mathbb{E}(\mathbb{E}(X)\mathbf{1}_A) = \mathbb{E}X\mathbb{E}1_A = \mathbb{E}(X1_A)$
3. Use linearity of conditional expectation.
4. For the left equality, let $A \in \mathcal{H}$. Then $\mathbb{E}\left[\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H})\mathbf{1}_A\right] = \mathbb{E}[\mathbb{E}(X \mid \mathcal{G})\mathbf{1}_A] = \mathbb{E}(X\mathbf{1}_A)$. For the right equality, note that $\mathbb{E}(X \mid \mathcal{H})$ is $\mathcal{H}$-measurable
5. Easy if Z is an indicator function. Then use linearity and covergence theorems.
6. Note $\mathbb{E}(X \mid \mathcal{G})$ is $\sigma(\mathcal{G}, \mathcal{H})$-measurable and $\sigma(\mathcal{G}, \mathcal{H})$ is generated by the π-system $\{A \cap B : A \in \mathcal{G}, B \in \mathcal{H}\}$. We show that $\mathbb{E}(X \mid \mathcal{G})$ satisfies the defining property of $\mathbb{E}(X \mid \sigma(\mathcal{G}, \mathcal{H}))$. Let $A \in \mathcal{G}$ and $B \in \mathcal{H}$. Then for any element of the π-system, we have
$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{G})\mathbf{1}_{A\cap B}) = \mathbb{E}[\mathbb{E}(X \mid \mathcal{G})\mathbf{1}_A\mathbf{1}_B] = \mathbb{E}[\mathbb{E}(X\mathbf{1}_A \mid \mathcal{G})\mathbf{1}_B] = \mathbb{E}(\underbrace{X\mathbf{1}_A}_{\in\sigma(\mathcal{G},X)})\mathbb{E}(\mathbf{1}_B) = \mathbb{E}(X\mathbf{1}_{A\cap B})$$
Since finite measures extend uniquely from π-systems, the above holds if $A \cap B$ is replaced by any element of $\sigma(\mathcal{G}, \mathcal{H})$

□

Proposition 1.4 (Properties of conditional expectation)**.** All (in)equality relations below hold almost surely.

1. *Jensen's inequality*: If $c : \mathbb{R} \to \mathbb{R}$ is convex, then
$$\mathbb{E}(c(X) \mid \mathcal{G}) \geq c(\mathbb{E}(X \mid \mathcal{G})).$$
2. *Conditional expectation is a contraction* For $p \geq 1$,
$$\|\mathbb{E}(X \mid \mathcal{G})\|_p \leq \|X\|_p.$$

3. *Monotone convergence theorem* Suppose $X_n \uparrow X$ is a sequence of non-negative random variables. Then

$$\mathbb{E}(X_n \mid \mathcal{G}) \uparrow \mathbb{E}(X \mid \mathcal{G}).$$

4. *Fatou's lemma*: If X_n are non-negative measurable, then

$$\mathbb{E}\left(\liminf_{n\to\infty} X_n \mid \mathcal{G}\right) \leq \liminf_{n\to\infty} \mathbb{E}(X_n \mid \mathcal{G}).$$

5. *Dominated convergence theorem*: If $X_n \to X$ and $Y \in L^1$ such that $Y \geq |X_n|$ for all n, then

$$\mathbb{E}(X_n \mid \mathcal{G}) \to \mathbb{E}(X \mid \mathcal{G}).$$

Proof. 1. Note that a convex function is the supremum of countably many affine functions $c(x) = \sup_{i\in I} a_i x + b_i$. Then

$$\begin{aligned}\mathbb{E}(c(X) \mid \mathcal{G}) &= \mathbb{E}\left(\sup_{i\in I}(a_i X + b_i) \mid \mathcal{G}\right)\\ &\geq \mathbb{E}(a_i X + b_i \mid \mathcal{G}) \quad \forall i \in I && \text{(monotonicity)}\end{aligned}$$

So $\mathbb{E}(c(X) \mid \mathcal{G}) \geq \sup_{i\in I} \mathbb{E}(a_i X + b_i \mid \mathcal{G}) = c(\mathbb{E}(X \mid \mathcal{G}))$.

2. Jensen

3. By monotonicity, $\mathbb{E}(X_n \mid \mathcal{G}) \uparrow Y$ for some Y. By the usual monotone convergence theorem, $\mathbb{E}\mathbb{E}(X_n \mid \mathcal{G}) = \mathbb{E}X_n \to \mathbb{E}Y \leq \mathbb{E}X$ so $Y \in L^1$. Since each of the $\mathbb{E}(X_n \mid \mathcal{G})$ are $\mathcal{G}$-measurable, so is Y. Finally, for any $A \in \mathcal{G}$,

$$\begin{aligned}\mathbb{E}Y\mathbf{1}_A &= \lim_{n\to\infty} \mathbb{E}\mathbb{E}(X_n \mid \mathcal{G})\mathbf{1}_A && \text{(MCV)}\\ &= \lim_{n\to\infty} \mathbb{E}X_n\mathbf{1}_A\\ &= \mathbb{E}X\mathbf{1}_A && \text{(MCV)}\end{aligned}$$

4.

$$\begin{aligned}\mathbb{E}\left(\liminf_{n\to\infty} X_n \mid \mathcal{G}\right) &= \mathbb{E}\left(\lim_{n\to\infty} \underbrace{\inf_{m\geq n} X_m}_{increasing} \mid \mathcal{G}\right)\\ &= \lim_{n\to\infty} \mathbb{E}\left(\inf_{m\geq n} X_m \mid \mathcal{G}\right) && \text{(MCV)}\\ &= \liminf_{n\to\infty} \mathbb{E}\left(\underbrace{\inf_{m\geq n} X_m}_{\leq X_n} \mid \mathcal{G}\right)\\ &\leq \liminf_{n\to\infty} \mathbb{E}(X_n \mid \mathcal{G}) && \text{(monotonicity)}\end{aligned}$$

5. Use Fatou's lemma on $Y + X_n$ and $Y - X_n$.

□

2 Martingales

2.1 Definition and Properties

Definition ((Discrete) stochastic process)**.** A *stochastic process* (in discrete time) is a collection of random variables $(X_n)_{n\in\mathbb{N}}$. A stochastic process is is *integrable* if $X_n \in L^1$ for all n.

Definition (Filtration)**.** A *filtration* is a sequence of σ-algebras $\mathcal{F}_n \subseteq \mathcal{F}$ such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all n. We define $F_\infty = \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$. The *natural filtration* of a stochastic process X is the filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. A stochastic process is *adapted* to a filtration $\mathcal{F}_n$ if X_n is $\mathcal{F}_n$-measurable for all n.

Definition (Martingale)**.** An integrable adapted process $(X_n)_{n\geq 0}$ is a *martingale* if for all $n \geq m$, we have

$$\mathbb{E}(X_n \mid \mathcal{F}_m) = X_m.$$

We say it is a *super-martingale* if

$$\mathbb{E}(X_n \mid \mathcal{F}_m) \leq X_m,$$

and a *sub-martingale* if

$$\mathbb{E}(X_n \mid \mathcal{F}_m) \geq X_m,$$

By the tower property, it is sufficient to check the martingale property for $n = m + 1$.

Remark. The definition can be adapted for any *totally ordered* index set T, such as $\mathbb{R}_+$ or $\mathbb{N}_-$.

Theorem 2.1 (Doob decomposition, non-examinable)**.** *Let X_n be an integrable adapted process. Then there exists a martingale M_n and an integrable predictable process A_n such that $X_n = M_n + A_n$ and $A_0 = 0$, where predictable means that A_n is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$. Moreover, M_n and A_n are unique up to a.s. equivalence.*

Proof. (Existence) Add up the 'known' bits to get A and the 'surprises' to get M. Formally,

$$\begin{aligned} A_n &= A_{n-1} + \mathbb{E}(X_n \mid \mathcal{F}_{n-1}) - X_{n-1} \\ M_n &= M_{n-1} + \underbrace{X_n - \mathbb{E}(X_n \mid \mathcal{F}_{n-1})}_{\text{surprise}} \end{aligned}$$

(Uniqueness) Let $X_n = M_n + A_n = M_n' + A_n'$. Then $M_n - M_n' = A_n' - A_n$ is $\mathcal{F}_{n-1}$-measurable. But $M_n - M_n'$ is a martingale, so $\mathbb{E}(M_n - M_n' \mid \mathcal{F}_{n-1}) = 0$ so $M_n = M_n'$ almost surely. Similarly, $A_n = A_n'$ almost surely. □

Definition (Stopping time)**.** A random variable $T : \Omega \to \mathbb{N} \cup \{\infty\}$ is a *stopping time* if $\{T \leq n\} \in \mathcal{F}_n$ for all n.

In the discrete case, we can equivalently require that $\{T = n\} \in \mathcal{F}_n$ for all n.

Definition (X_T)**.** Let X be a stochastic process and T a stopping time. Then $X_T : \Omega \to \mathbb{R}$ is defined by cases

$$X_T(\omega) = \begin{cases} X_n(\omega) & T(\omega) = n \\ 0 & T(\omega) = \infty \end{cases}$$

Definition (Stopped σ-algebra)**.** Let T be a stopping time. Then the *stopped σ-algebra* is

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n\}.$$

Example. Let $N = \#$ of times a random walk hits -5 before it first hits 10 and T be the first time the random walk hits 10. N is $\mathcal{F}_T$-measurable

Definition (Stopped process)**.** Let X be a stochastic process and T a stopping time. Then the *stopped process* is $X_n^T = X_{T \wedge n}$

Proposition 2.1.

1. If $T, S, (T_n)_{n \geq 0}$ are all stopping times, then

$$T \vee S, T \wedge S, \sup_n T_n, \inf T_n, \limsup T_n, \liminf T_n$$

   are all stopping times.

2. $\mathcal{F}_T$ is a σ-algebra
3. If $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.
4. $X_T \mathbf{1}_{T<\infty}$ is $\mathcal{F}_T$-measurable.
5. If (X_n) is an adapted process, then so is $(X_n^T)_{n \geq 0}$ for any stopping time T.
6. If (X_n) is an integrable process, then so is $(X_n^T)_{n \geq 0}$ for any stopping time T.

Proof.

1. Elementary
2. Elementary
3. Let $A \in \mathcal{F}_S$. For any n, we have $A \cap \{S \leq n\} \in \mathcal{F}_n$ and $A \cap \{T \leq n\} = A \cap \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n$.
4. $X_T \mathbf{1}_{T<\infty} = \sum_{n=1}^{\infty} X_n \mathbf{1}_{\{T=n\}}$ where each of the terms is $\mathcal{F}_T$-measurable.
5. $X_n^T = X_n \mathbf{1}_{\{T \geq n\}} + X_T \mathbf{1}_{\{T<n\}} \mathbf{1}_{\{T<\infty\}}$.
6. $X_n^T = X_n \mathbf{1}_{\{T \geq n\}} + \sum_{k=1}^{n-1} X_k \mathbf{1}_{\{T=k\}}$ so $E|X_n^T| \leq E|X_n| + \sum_{k=1}^{n-1} E|X_k| < \infty$.

$\square$

Theorem 2.2 (Equivalent definitions for super-martingales)**.** *Let $(X_n)_{n \geq 0}$ be an integrable and adapted process. Then the following are equivalent:*

1. *$(X_n)_{n \geq 0}$ is a super-martingale.*

2. *For any bounded stopping time T and any stopping time S,*

$$\mathbb{E}(X_T \mid \mathcal{F}_S) \leq X_{S\wedge T}.$$

3. *(X_n^T) is a super-martingale for any stopping time T.*

4. *For bounded stopping times S, T such that $S \leq T$, we have*

$$\mathbb{E}X_T \leq \mathbb{E}X_S.$$

Proof. – (2) $\Rightarrow$ (1): Let $n \geq m$ and set $T = n, S = m$.

– (2) $\Rightarrow$ (4): Tower rule

– (2) $\Rightarrow$ (3): Let $n \geq m$

$$\mathbb{E}(X_n^T \mid \mathcal{F}_m) = \mathbb{E}(X_{T\wedge n} \mid \mathcal{F}_m) \leq X_{T\wedge m\wedge n} = X_m^T.$$

– (1) $\Rightarrow$ (2) Let $T \leq N$

$$X_T = X_{S\wedge T} + \sum_{k=0}^{N} (X_{k+1} - X_k)\mathbf{1}_{S\leq k<T} \qquad (*)$$

Let $A \in \mathcal{F}_S$.

$$\begin{aligned}\mathbb{E}\left[(X_{k+1} - X_k)\mathbf{1}_{S\leq k<T}\mathbf{1}_A\right] &= \mathbb{E}\left[\mathbb{E}\left[(X_{k+1} - X_k)\underbrace{\mathbf{1}_{S\leq k<T}\mathbf{1}_A}_{\in\mathcal{F}_k} \mid \mathcal{F}_k\right]\right] \\ &= \mathbb{E}\left[\mathbf{1}_{S\leq k<T}\mathbf{1}_A\underbrace{\mathbb{E}\left[(X_{k+1} - X_k) \mid \mathcal{F}_k\right]}_{\leq 0}\right] \\ &\leq 0\end{aligned}$$

so $\mathbb{E}X_T\mathbf{1}_A \leq \mathbb{E}X_{S\wedge T}\mathbf{1}_A$. By Radon-Nikodym, $\mathbb{E}(X_{S\wedge T} - X_T \mid \mathcal{F}_S) \geq 0$. But $X_{S\wedge T}$ is $\mathcal{F}_S$-measurable, so $X_{S\wedge T} - X_T \geq 0$ almost surely.

– (4) $\Rightarrow$ (2) Let $n \geq m$ and $A \in \mathcal{F}_m$. One can check that $T = m\mathbf{1}_A + n\mathbf{1}_{A^c} \leq n$ is a stopping time such that

$$\mathbb{E}((X_n - X_m)\mathbf{1}_A) = \mathbb{E}(X_n - X_T) \leq 0$$

By Radon-Nikodym, $\mathbb{E}(X_m - X_n \mid \mathcal{F}_m) \geq 0$ so $\mathbb{E}(X_n \mid \mathcal{F}_m) \leq X_m$.

– (3) $\Rightarrow$ (1) Let $T = \infty$

□

Proposition 2.2 (Convex transformations of martingales)**.** Let (X_n) be a martingale and $c : \mathbb{R} \to \mathbb{R}$ a convex function. Then $(c(X_n))$ is a sub-martingale.

Proof. Let $S \le T$ be bounded stopping times. Then

$$\begin{aligned}\mathbb{E}(c(X_T) \mid \mathcal{F}_S) &\ge c(\mathbb{E}(X_T \mid \mathcal{F}_S)) && \text{(Jensen)}\\ &= c(X_S)\end{aligned}$$

□

Theorem 2.3 (Optional stopping)**.** *Let $(X_n)_{n\ge 0}$ be a martingale and T a stopping time. Then $E(X_T) = E(X_0)$ if any of the following conditions hold:*

1. *T is almost surely bounded, i.e. there is some N such that $T \le N$ almost surely.*
2. *X has bounded increments, i.e. there is some K such that $|X_{n+1} - X_n| \le K$ for all n almost surely and T is integrable*
3. *There exists an integrable random variable Y such that $|X_n| \le Y$ for all n almost surely and T is finite almost surely, i.e. $\mathbb{P}(T < \infty) = 1$.*

Proof.

1. Use (4) of the previous theorem with $S = 0$, or prove directly.
2. $$\begin{aligned}X_{T\wedge n} &= \sum_{i=1}^{T\wedge n}(X_i - X_{i-1}) + X_0\\ &= \sum_{i=1}^{n}(X_i - X_{i-1})\mathbf{1}_{T\ge i} + X_0\end{aligned}$$

   Note $|X_{T\wedge n}| \le |X_0| + \sum_{i=1}^n |X_i - X_{i-1}|\mathbf{1}_{T\ge i} \le |X_0| + KT \in L^1$. Note that $T \wedge n \to T$ almost surely as $T < \infty$ almost surely. Hence, $X_{T\wedge n} \to X_T$ almost surely and $\mathbb{E}X_{T\wedge n} \to \mathbb{E}X_T$ by dominated convergence theorem. By the previous case, $\mathbb{E}X_{T\wedge n} = \mathbb{E}X_0$ so $\mathbb{E}X_T = \mathbb{E}X_0$.
3. Using the same reasoning, $X_{T\wedge n} \to X_T$ almost surely and $\mathbb{E}|X_{T\wedge n}| \le \mathbb{E}Y$ so dominated convergence holds.

□

2.2 Convergence

Definition (Upcrossing)**.** Let (x_n) be a sequence and (a, b) an interval. An *upcrossing* of (a, b) by (x_n) is a sequence $j, j+1, \dots, k$ such that $x_j \le a$ and $x_k \ge b$. We define

$$\begin{aligned}U_n[a, b, (x_n)] &= \text{number of disjoint upcrossings contained in } \{1, \dots, n\}\\ U[a, b, (x_n)] &= \lim_{n\to\infty} U_n[a, b, (x_n)].\end{aligned}$$

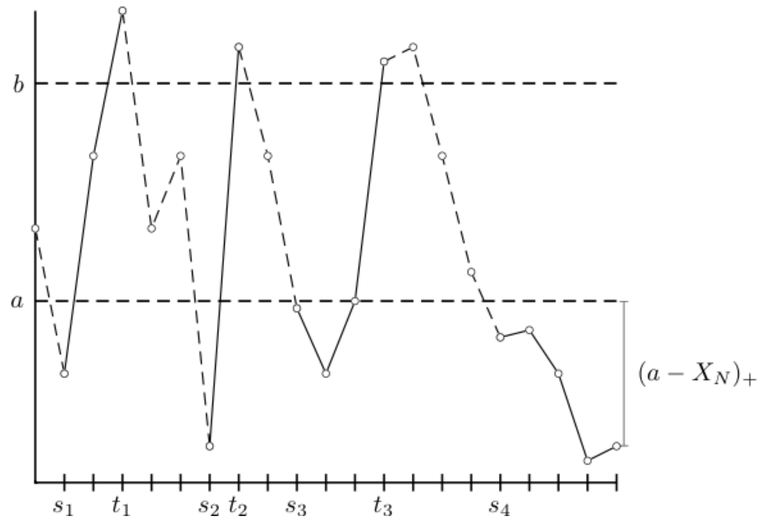


Figure 1: Three upcrossings. Source

The notion of upcrossings is related to the notion of convergence.

Lemma 2.2.1 (Upcrossing and convergence)**.** A sequence (x_n) converges to a limit in the extended real numbers if and only if $U[a, b, (x_n)] < \infty$ for all rationals $a < b$.

Proposition 2.3 (Doob's upcrossing inequality)**.** Let $X = (X_k)$ be a super-martingale and $a < b$. Then

$$(b-a)\mathbb{E}U_n[a, b, X] \leq \mathbb{E}[(X_n - a)^-] \leq \mathbb{E}(|X_n|) + |a|$$

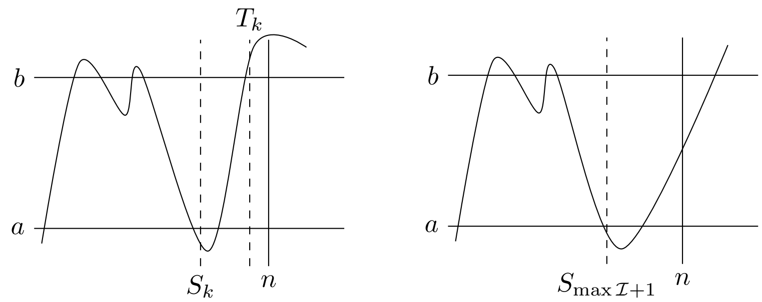
Proof. Assume that X is a super-martingale. We define stopping times S_k, T_k as follows:

- $T_0 = 0$
- $S_{k+1} = \inf\{n : X_n \leq a, n \geq T_n\}$
- $T_{k+1} = \inf\{n : X_n \geq b, n \geq S_{k+1}\}$.

Note that the times alternate, i.e. $S_k \leq T_k \leq S_{k+1} \leq T_{k+1}$

The idea is to sum up the increments of each upcrossing. Consier the set $\mathcal{I} := \{k : S_k < T_k < n\}$ and the sum

$$\underbrace{\sum_{k \in \mathcal{I}} (X_{T_k} - X_{S_k})}_{\geq (b-a)U_n[a,b,X]} + \begin{cases} X_n - X_{S_{\max \mathcal{I}+1}} & \text{if } n > S_{\max \mathcal{I}+1} \\ 0 & \text{otherwise} \end{cases}$$



(a) No extra term

(b) Extra term

Figure 2: Graphs taken from here

Note that each term is of the form $X_U - X_V$ for some *bounded* stopping times $U \geq V$. By the property of super-martingales, $\mathbb{E}(X_U - X_V) \leq 0$ so the sum is negative in expectation.

Hence,

$$(b-a)\mathbb{E}U_n[a,b,X] + \mathbb{E}(\underbrace{\underbrace{(X_n - X_{S_{\max \mathcal{I}+1}})}_{\geq X_n - a \geq -(X_n-a)^-} \mathbf{1}_{n > S_{\max \mathcal{I}+1}}}_{\geq -(X_n-a)^-}) \leq 0$$

$$(b-a)\mathbb{E}U_n[a,b,X] \leq \mathbb{E}((X_n - a)^-) \leq \mathbb{E}(|X_n|) + |a|$$

□

Corollary 1. Under the same assumptions of the previous proposition, by the monotone convergence theorem and noting that $U_n[a,b,X] \uparrow U[a,b,X]$,

$$(b-a)\mathbb{E}U[a,b,X] \leq \sup_n \mathbb{E}[(X_n - a)^-] \leq \sup_n \mathbb{E}(|X_n|) + |a|$$

Proposition 2.4 (Almost sure martingale convergence theorem)**.** Suppose $X = (X_n)_{n\geq 0}$ is a super-martingale that is bounded in L^1, i.e. $\sup_n \mathbb{E}|X_n| < \infty$. Then for any $a < b$, we have $U[a,b,X] < \infty$ almost surely. In particular, there exists an $\mathcal{F}_\infty$-measurable $X_\infty \in L^1$ such that

$$X_n \to X_\infty \text{ a.s. as } n \to \infty.$$

Remark. The intuition is that you cannot make money by betting on a super-martingale (without shorting). For any $a < b$, you can devise a betting strategy where you buy at a and sell at b. If $U[a,b,X] = \infty$, then you make money almost surely.

Proof. Let $a < b$ and n be arbitrary and $\sup_m \mathbb{E}|X_m| = M$. By the previous corollary, So $U[a,b,X] < M + |a| < \infty$ almost surely.

Now consider the set of events where X_n converges

$$A := \cap_{a,b\in\mathbb{Q}}\{U[a,b,X] < \infty\}, \qquad \mathbb{P}(A) = 1$$

and define

$$X_\infty = \begin{cases} \lim X_n & \text{on } A \\ 0 & \text{on } A^c \end{cases}$$

which is $\mathcal{F}_\infty$-measurable. Note that $|X_\infty| = \liminf |X_n|$ almost surely and by Fatou's lemma,

$$\mathbb{E} \liminf |X_n| \le \liminf \mathbb{E}|X_n| \le M < \infty.$$

So $X_\infty \in L^1$. □

Remark. Sone propositions below concern non-negative submartingales. Recall convex transformations, such as $|\cdot|$, turn a martinagle into a sub-martingale.

Proposition 2.5 (Doob's maximal inequality)**.** Let $X = (X_n)_{n\ge 0}$ be a non-negative sub-martingale. Then for any $\lambda > 0$ and writing $X_n^* = \max_{0\le k\le n} X_k$, we have

$$\lambda \mathbb{P}(X_n^* \ge \lambda) \le \mathbb{E} X_n \mathbf{1}_{X_n^* \ge \lambda} \le \mathbb{E} X_n.$$

Proof. Let $T = \inf\{n : X_n \ge \lambda\}$

$$\begin{aligned} \mathbb{E} X_n \ge \mathbb{E} X_{T\wedge n} &= \mathbb{E} X_n \mathbf{1}_{T>n} + \mathbb{E} X_T \mathbf{1}_{T\le n} \\ \mathbb{E} X_n \mathbf{1}_{T\le n} &\ge \lambda \mathbb{P}(T \le n) \\ \mathbb{E} X_n \mathbf{1}_{X_n^* \ge \lambda} &\ge \lambda \mathbb{P}(X_n^* \ge \lambda) \end{aligned}$$

□

Corollary 2. Under the same hypotheses, let $X_n^* \uparrow X^*$. Then

$$\lambda \mathbb{P}(X^* \ge \lambda) \le \sup_{n\ge 0} \mathbb{E} X_n$$

Proposition 2.6 (Doob's L^p inequality)**.** Let $X = (X_n)_{n\ge 0}$ be a martingale or a non-negative sub-martingale. Then for any $p > 1$ and $n \ge 1$, we have

$$\|X_n^*\|_p \le \frac{p}{p-1} \|X_n\|_p.$$

Proof. (magic)

Let $k > 0$ and $T = \inf\{n : X_n \geq k\}$. Then

$$\begin{aligned}
\int |X_n^* \wedge k|^p \, \mathrm{d}\mathbb{P} &= \int \left(\int px^{p-1} \mathbf{1}_{\{0 \leq x \leq |X_n^* \wedge k|\}} \, \mathrm{d}x \right) \mathrm{d}\mathbb{P} \\
&= \int \left(\int px^{p-1} \mathbf{1}_{\{|X_n^*| \geq x\}} \mathbf{1}_{\{0 \leq x \leq k\}} \, \mathrm{d}x \right) \mathrm{d}\mathbb{P} \\
&= \int px^{p-1} \mathbb{P}(|X_n^*| \geq x) \mathbf{1}_{\{0 \leq x \leq k\}} \, \mathrm{d}x \\
&\leq \int px^{p-1} \frac{1}{x} \mathbb{E}|X_n| \mathbf{1}_{\{|X_n^*| \geq x\}} \mathbf{1}_{\{0 \leq x \leq k\}} \, \mathrm{d}x && \text{(Doob's maximal inequality)} \\
&= \int \int px^{p-2} |X_n| \mathbf{1}_{\{|X_n^*| \geq x\}} \mathbf{1}_{\{0 \leq x \leq k\}} \, \mathrm{d}x \, \mathrm{d}\mathbb{P} \\
&= \int \frac{p}{p-1} |X_n| (|X_n^*| \wedge k)^{p-1} \, \mathrm{d}\mathbb{P} \\
&\leq \frac{p}{p-1} \|X_n\|_p \||X_n^*| \wedge k\|_p^{p-1} && \text{(Hölder's inequality)}
\end{aligned}$$

By monotone convergence and taking $k \to \infty$, we get

$$\|X_n^*\|_p^p \leq \frac{p}{p-1} \|X_n\|_p \|X_n^*\|_p^{p-1}$$

and the result follows. □

Corollary 3. Under the same hypotheses, let $X_n^* \uparrow X^*$. Then

$$\|X^*\|_p \leq \frac{p}{p-1} \sup_{n \geq 0} \|X_n\|_p$$

Proposition 2.7 (Equivalent conditions for L^p convergence, $p > 1$)**.** Let $X = (X_n)_{n \geq 0}$ be a martingale, and $p > 1$. Then the following are equivalent:

1. $(X_n)_{n \geq 0}$ is bounded in L^p, i.e. $M = \sup_n \mathbb{E}|X_i|^p < \infty$.
2. $(X_n)_{n \geq 0}$ converges as $n \to \infty$ to a random variable $X_\infty \in L^p$ almost surely and in L^p.
3. There exists a random variable $Z \in L^p$ such that

$$X_n = \mathbb{E}(Z \mid \mathcal{F}_n) \qquad \lim_{n \to \infty} X_n = \mathbb{E}(Z \mid \mathcal{F}_\infty) \text{ a.s.}$$

This gives a bijection between martingales bounded in L^p and $L^p(\mathcal{F}_\infty)$, sending $(X_n)_{n \geq 0} \mapsto X_\infty$.

Proof. – (1) ⇒ (2) By Jensen, $\mathbb{E}|X_n|^p \geq (\mathbb{E}|X_n|)^p$ so X is bounded in L^1. By the almost sure martingale convergence theorem, there exists an $\mathcal{F}_\infty$-measurable $X_\infty \in L^1$ such that $X_n \to X_\infty$ almost surely. Note also $|X|^* \geq |X_n|$ for all n and $X^* \in L^p$ by Corollary 3. Hence, $X_n \to X_\infty$ in L^p by L^p dominated convergence

– (2) $\Rightarrow$ (3) Let $Z = X_\infty$. Note $X_n \xrightarrow{L^p} X_\infty$ implies that X is bounded in L^p. By Doob's maximal inequality, $|X|^* \in L^p \subseteq L^1$. By conditional dominated convergence,

$$\mathbb{E}(X_\infty \mid \mathcal{F}_n) = \lim_{m\to\infty} \mathbb{E}(X_m \mid \mathcal{F}_n) = X_n$$

– (3) $\Rightarrow$ (1) Conditional expectation is a contraction (Proposition 1.4)

$\square$

Definition (Closed martingale)**.** A martingale in the form $X_n = \mathbb{E}(Z \mid \mathcal{F}_n)$ for some $Z \in L^p$ is called a *martingale closed in* L^p

Definition (Non-examinable, uniform integrability for general measure space)**.** Let $(f_n)_n$ be a family of absolutely integrable functions on some measure space. The family is said to be *uniformly integrable (UI)* if all of the following hold

1. Uniform bound on L^1 norm ($\sup_n \int |f_n| < \infty$)
2. No escape to vertical infinity ($\sup_n \int_{\{|f_n|>\lambda\}} |f_n| \to 0$ as $\lambda \to \infty$)
3. No escape to horizontal infinity (for any $\varepsilon > 0$, there exists a finite measure subset A such that $\sup_n \int_{A^c} |f_n| < \varepsilon$)

Example. A single integrable function is uniformly integrable.

Example. A family of functions which is dominated by some integrable function, i.e. there is $g \in L^1$ such that $|f_i| \le g$ for all $i \in \mathcal{I}$, is uniformly integrable.

Definition (Uniform integrability for random variables)**.** A family of random variables $(X_i)_{i\in\mathcal{I}}$ is *uniformly integrable* if

$$\sup_{i\in\mathcal{I}} \mathbb{E}(|X_i|\mathbf{1}_{|X_i|>\alpha}) \to 0 \text{ as } \alpha \to \infty.$$

Remark. In the finite measure case, functions cannot escape to horizontal infinity and the conditions simplifies just to no escape to vertical infinity.

Proposition 2.8 (Equivalent definition of uniform integrability)**.** A family of random variables $(X_i)_{i\in\mathcal{I}}$ is *uniformly integrable* if and only if both of the following hold

1. It is bounded in L^1 ($\sup_{i\in\mathcal{I}} \mathbb{E}(|X_i|) < \infty$)
2. It is equi-integrable (For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $A \in \mathcal{F}$ with $\mathbb{P}(A) < \delta$, we have

$$\mathbb{E}(|X_i|\mathbf{1}_A) < \varepsilon.$$

for all $i \in \mathcal{I}$.)

Proposition 2.9 (Uniform integrability by domination)**.** Let $\{Y_j : j \in J\}$ be uniformly integrable. Suppose the set $X = \{X_i : i \in I\}$ satisfies for any $i \in I$, there exists $j \in J$ such that $|X_i| \le Y_j$. Then X is uniformly integrable.

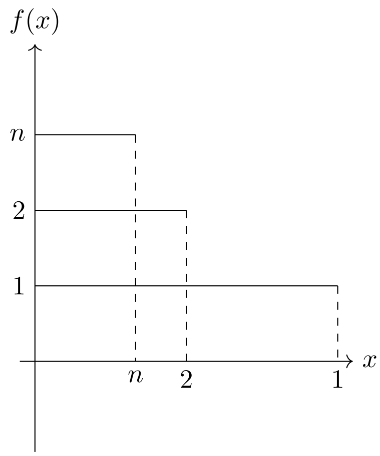


Figure 3: Non uniformly integrable sequence

Theorem 2.4 (Non-examinable, Vitali convergence theorem)**.** *Let $f_1, f_2, \dots$ be a sequence of integrable functions on some measure space and $1 \le p < \infty$. Then $f_n \xrightarrow{L^p} f$ for some measurable f if and only if all of the following hold*

1. *(f_n^p) is uniformly integrable*
2. *$f_n \to f$ in measure*
3. *The sequence cannot escape to horizontal infinity, i.e. for any $\varepsilon > 0$, there exists a finite measure subset A such that $\sup_n \int_{A^c} |f_n|^p < \varepsilon$.*

Remark. In the finite measure case, the third condition is trivially true and almost sure convergence implies convergence in measure, so this implies the L^p dominated convergence theorem.

Proposition 2.10 (Conditional expectations are uniformly integrable)**.** Let $\mathcal{S}$ be a uniformly integrable family of random variables. Then the following set is uniformly integrable

$$\mathcal{S}^* = \{\mathbb{E}(X|\mathcal{G}) \mid X \in \mathcal{S}, \mathcal{G} \text{ is a sub } \sigma\text{-algebra of } \mathcal{F}\}.$$

Proof. Since $\mathcal{S}$ is bounded in L^1, $\mathcal{S}^*$ is bounded in L^1. Let $\varepsilon > 0$. By uniform integrability of $\mathcal{S}$, there exists $\delta > 0$ such that for any $A \in \mathcal{F}$ with $\mathbb{P}(A) < \delta$, we have $\mathbb{E}(|X|\mathbf{1}_A) < \varepsilon$ for all $X \in \mathcal{S}$. Note

$$\mathbb{E}(|\mathbb{E}(X \mid \mathcal{G})|\mathbf{1}_A) \underbrace{\le}_{Jensen} \mathbb{E}(\mathbb{E}(|X| \mid \mathcal{G})\mathbf{1}_A)$$

we wish to show that when A is of the form $\{|X| > \alpha\}$, the right hand side converges to zero as $\alpha \to \infty$. The right hand side becomes $\mathbb{E}|X|\mathbf{1}_{\{|X|>\alpha\}}$

We want to choose α such that $\mathbb{P}(|X| > \alpha) < \delta$. By Markov's inequality,

$$\mathbb{P}(|X| > \alpha) \le \frac{\mathbb{E}|X|}{\alpha}$$

so picking any $\alpha > \frac{\mathbb{E}|X|}{\delta}$ works.

We have shown that for any $\varepsilon > 0$, for any α sufficiently large, $\mathbb{E}(|\mathbb{E}(X \mid \mathcal{G})| \mathbf{1}_{\{|X|>\alpha\}}) \leq \varepsilon$ □

Proposition 2.11 (Equivalent conditions for L^1 convergence)**.** Let $(X_n)_{n\geq 0}$ be a martingale. Then the following are equivalent:

1. $(X_n)_{n\geq 0}$ is uniformly integrable.
2. $(X_n)_{n\geq 0}$ converges to some $X_\infty \in L^1$ almost surely and in L^1.
3. There exists $Z \in L^1$ such that $X_n = \mathbb{E}(Z \mid \mathcal{F}_n)$ almost surely.

Moreover, $X_\infty = \mathbb{E}(Z \mid \mathcal{F}_\infty)$

Proof.

- (1) $\Rightarrow$ (2) X is L^1 bounded and hence converges to some X_∞ almost surely by Proposition 2.4. By Vitali, $X_n \to X_\infty$ in L^1.
- (2) $\Rightarrow$ (3) Let $Z = X_\infty$. Then $X_n = \mathbb{E}(Z \mid \mathcal{F}_n)$ almost surely by Proposition 1.4. (Same as previous proposition) Let $Z = X_\infty$.
$$\|X_n - \mathbb{E}(X_\infty|\mathcal{F}_n)\|_1 = \|\mathbb{E}(X_m - X_\infty \mid \mathcal{F}_n)\|_1 \leq \|X_m - X_\infty\|_1$$
for any $m \geq n$ and the right hand side converges to 0 by L^1 convergence.
- (3) $\Rightarrow$ (1) Conditional expectation is uniformly integrable, see previous example.

□

Lemma 2.11.1 (Stopped UI process)**.** Let X be a uniformly integrable martingale and T be any stopping time. Then the following statements about the stopped process X^T hold:

1. $X_{T\wedge n} = \mathbb{E}(X_\infty|\mathcal{F}_{T\wedge n})$
2. X^T is uniformly integrable
3. $X^T_n \to X_T$ in L^1 and almost surely

Proof. Since X is UI, we use the fact that the martingale can be represented as a conditional expectation of some $X_\infty \in L^1$. Then

$$\begin{aligned} X_{T\wedge n} &= \mathbb{E}(X_n|\mathcal{F}_{T\wedge n}) \\ &= \mathbb{E}(\mathbb{E}(X_\infty \mid \mathcal{F}_n) \mid \mathcal{F}_{T\wedge n}) \\ &= \mathbb{E}(X_\infty|\mathcal{F}_{T\wedge n}) \end{aligned}$$

By Proposition 2.10, conditional expectations are uniformly integrable. By Proposition 2.11, $X^T_n \to X^T_\infty$ in L^1 and almost surely for some $X^T_\infty \in L^1$. By considering different values of T, one can see that $X^T_\infty = X_T$ almost surely. □

Proposition 2.12 (Optional stopping for arbitrary stopping times)**.** If $(X_n)_{n\geq 0}$ is a uniformly integrable martingale, and S, T are arbitrary stopping times, then $\mathbb{E}(X_T \mid \mathcal{F}_S) = X_{S\wedge T}$. In particular $\mathbb{E}X_T = X_0$.

Note that we are now allowing arbitrary stopping times, so T may be infinite with non-zero probability. Hence we define

$$X_T = \sum_{n=0}^{\infty} X_n \mathbf{1}_{T=n} + X_\infty \mathbf{1}_{T=\infty}.$$

Proof. We have proven the result for bounded stopping times. For the stopped process $X^T = (X_{T\wedge n})_{n\geq 0}$, we have $\mathbb{E}(X_n^T \mid \mathcal{F}_S) = X_{S\wedge T\wedge n}$. What we would like to do is it take the limit as $n \to \infty$.

From Lemma 2.11.1,

$$\|\mathbb{E}(X_{T\wedge n} - X_T | \mathcal{F}_S)\|_1 \leq \|X_{T\wedge n} - X_T\|_1 \to 0$$

so $X_{S\wedge T\wedge n} \xrightarrow{L^1} \mathbb{E}(X_T \mid \mathcal{F}_S)$.

Lemma 2.11.1 also says $X_{S\wedge T\wedge n} \xrightarrow{L^1} X_{S\wedge T}$ so $X_{S\wedge T} = \mathbb{E}(X_T \mid \mathcal{F}_S)$ almost surely □

2.3 Applications of martingales

Definition (Backwards filtration)**.** A *backwards filtration* on a measurable space $(E, \mathcal{E})$ is a sequence of σ-algebras $\hat{\mathcal{F}}_n \subseteq \mathcal{E}$ such that $\hat{F}_{n+1} \subseteq \hat{F}_n$. We define

$$\hat{\mathcal{F}}_\infty = \bigcap_{n\geq 0} \hat{\mathcal{F}}_n.$$

Definition (Backwards martingale)**.** An adapted process $(X_n)_{n\geq 0}$ is a *backwards martingale* with respect to a backwards filtration $(\hat{\mathcal{F}}_n)_{n\geq 0}$ if all of the following hold

1. $X_0 \in L^1$
2. $\mathbb{E}(X_n | \hat{\mathcal{F}}_{n+1}) = X_{n+1}$

Proposition 2.13 (Equivalent definition for backwards martingale)**.** $(X_n)_{n\geq 0}$ is a backwards martingale if and only if there exists some $Y \in L^1$ such that $X_n = \mathbb{E}(Y | \hat{\mathcal{F}}_n)$ almost surely.

Proof.

- ($\implies$) Let $Y = X_0$.
- ($\impliedby$) Tower property and conditional expectation being a contraction

□

Remark. Let $I = \mathbb{N}$ or $\mathbb{R}_+$ and $(X_t)_{t\in I}$ be a backwards martingale with respect to a backwards filtration $(\hat{\mathcal{F}}_t)_{t\in I}$ and $s \in I$. Then $(X_{s-t})_{0\leq t\leq s}$ is a martingale with respect to the filtration $(\mathcal{F}_{s-t})_{0\leq t\leq s}$.

Proposition 2.14 (Backwards martingale convergece)**.** Let $(X_n)_{n\geq 0}$ be a backwards martingale with $X_0 \in L^p$ for some $p \in [0, \infty)$ and $X_\infty = \mathbb{E}(X_0|\hat{\mathcal{F}}_\infty)$. Then $X_n \to X_\infty$ in L^p and almost surely.

Proof. Proof is basically the same as the forwards case. Note that the martingale $(X_{s-t})_{0\leq t\leq s}$ has the same number of upcrossings on $[a, b]$ as $(-X_{s-t})_{0\leq t\leq s}$ on $[-b, -a]$. By noting that $\|X_t\|_1 \leq \|X_0\|_1$ for all t, the margingale is bounded in L_1 and as before we have $X_t \to X_\infty$ almost surely to some X_∞ which is also in L^1 by Fatou's lemma.

Using similar arguments, we see that $X_\infty \in L^p$.

Next, show $X_\infty = \mathbb{E}(X_0|\hat{\mathcal{F}}_\infty)$ almost surely. Pick any $A \in \hat{\mathcal{F}}_\infty$. Then $A \in \hat{\mathcal{F}}_n$ for all n. So

$$\begin{aligned}
\mathbb{E}(X_\infty \mathbf{1}_A) &= \lim_{n\to\infty} \mathbb{E}(X_n \mathbf{1}_A) && \text{(Vitali convergence theorem)} \\
&= \lim_{n\to\infty} \mathbb{E}(\mathbb{E}(X_0|\hat{\mathcal{F}}_n)\mathbf{1}_A) \\
&= \lim_{n\to\infty} \mathbb{E}(X_0 \mathbf{1}_A) \\
&= \mathbb{E}(X_0 \mathbf{1}_A)
\end{aligned}$$

Finally, prove $X_n \to X_\infty$ in L^p. Note that $(|X_n - X_\infty|^p)_{n\geq 0}$ is UI by 2.9 since

$$|X_n - X_\infty|^p = |\mathbb{E}(X_0 - X_\infty|\hat{\mathcal{F}}_n)|^p \leq \mathbb{E}(|X_0 - X_\infty|^p|\hat{\mathcal{F}}_n)$$

and the set of variables on the RHS is UI. By Vitali, $X_n \to X_\infty$ in L^p. $\square$

3 Continuous time stochastic processes

Uncountability bad.

Definition (Continuous time stochastic process)**.** A *continuous time stochastic process* is a family of random variables $(X_t)_{t\geq 0}$ (or $(X_t)_{t\in[a,b]}$).

We can think of a continuous time stochastic process as a function

$$(\omega, t) \mapsto X_t(\omega)$$

In the discrete case, this function is $(\mathcal{F} \otimes \mathcal{P}(\mathbb{N}))$-measurable. This is not true in the continuous case, i.e. the function is not necessarily $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

Worse still, the first hitting time of a measurable set A need not be a stopping time. Let $T_a = \inf\{t \geq 0 : X_t \in A\}$ so

$$\{T_A \leq t\} = \bigcup_{0\leq s\leq t} \{X_s \in A\} \notin \mathcal{F}_t$$

since the union is uncountable.

To remedy this, we reduce to the countable case by considering continuous or more generally cadlag processes, which are determined by any dense subset of $\mathbb{R}_+$, e.g. $\mathbb{Q}_+$.

Definition (Cadlag)**.** Let (M, d) be a metric space and $E \subseteq \mathbb{R}$ A function $f : E \to M$ is *cadlag* if it is right continuous and has left limits, i.e. for all $t \geq 0$,

$$\lim_{s \to t^+} f(s) = f(t) \qquad \lim_{s \to t^-} f(s) \text{ exists.}$$

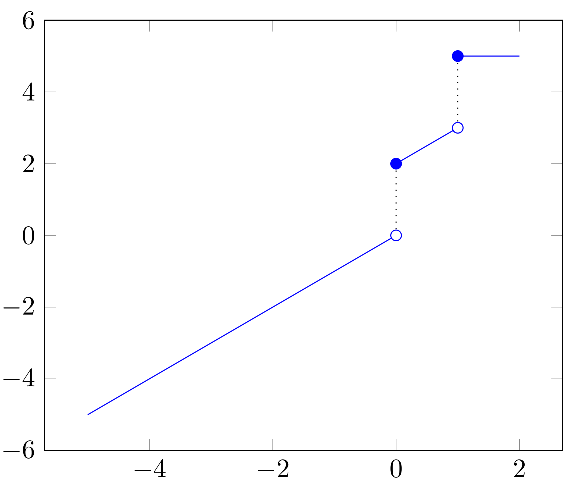


Figure 4: Cadlag function

Definition (Continuous/cadlag processes)**.** We say a stochastic process is *continuous* (resp. cadlag) if the set of $\omega \in \Omega$ such the map $t \mapsto X_t(\omega)$ is continuous (resp. cadlag) has probability 1.

Definition (Continuous time filtration)**.** A *continuous-time filtration* is a family of σ-algebras $(\mathcal{F}_t)_{t \geq 0}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ if $s \leq t$. Define $\mathcal{F}_\infty = \sigma(\mathcal{F}_t : t \geq 0)$.

Proposition 3.1. Let $(X_t)_{t \geq 0}$ be a cadlag adapted process and S, T stopping times. Then

1. $S \wedge T$ is a stopping time.
2. If $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.
3. $X_T \mathbf{1}_{T < \infty}$ is $\mathcal{F}_T$-measurable.
4. $(X_t^T)_{t \geq 0} = (X_{T \wedge t})_{t \geq 0}$ is adapted.

To show 3., it is useful to have the following characterisation of $\mathcal{F}_T$ measurability.

Lemma 3.1.1 (Characterisation of $\mathcal{F}_T$-measurability)**.** A random variable Z is $\mathcal{F}_T$-measurable iff $Z\mathbf{1}_{\{T \leq t\}}$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Proof. $\Longrightarrow$ is just by definition. For $\Longleftarrow$,

1. It is true by working with the definition for $Z = c\mathbf{1}_A$, $A \in \mathcal{F}$
2. It is true for simple functions Z since measurability is preserved under linear combinations
3. It is true for any measurable Z since it is a pointwise limit of simple functions and measurability is preserved under limits

□

Corollary 4. $X_T\mathbf{1}_{T=t}$ is $\mathcal{F}_T$-measurable.

Proof. Let $s \geq 0$. Then $X_T\mathbf{1}_{T=t}\mathbf{1}_{T\leq s} = X_t\mathbf{1}_{s\geq t}$ which is $\mathcal{F}_s$-measurable by considering cases. □

Proposition 3.2. Let $(X_t)_{t\geq 0}$ be a cadlag adapted process and S, T stopping times. Then

1. $S \wedge T$ is a stopping time.
2. If $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.
3. $X_T\mathbf{1}_{T<\infty}$ is $\mathcal{F}_T$-measurable.
4. $(X_t^T)_{t\geq 0} = (X_{T\wedge t})_{t\geq 0}$ is adapted.

Proof. 1 and 2 are same as the discrete case. For 3, we have

$$X_T\mathbf{1}_{T\leq t} = X_T\mathbf{1}_{\{T<t\}} + X_T\mathbf{1}_{\{T=t\}}.$$

The term on the right is $\mathcal{F}_t$-measurable by Corollary 4. Let $(T_n)_{n\geq 0} \subseteq \mathbb{Q}$ be your favourite rational sequence of stopping times with $T_n \downarrow T$. Since the process is right continuous, X_T can be approximated from above, i.e. $X_T\mathbf{1}_{\{T<t\}} = \lim_{n\to\infty} X_{T_n\wedge t}\mathbf{1}_{T<t}$. Moreover,

$$X_{T_n\wedge t} = \sum_{q\in\mathbb{Q}, q\leq t} \underbrace{X_{T_n}\mathbf{1}_{T_n=q}}_{\in\mathcal{F}_{T_n}\subseteq\mathcal{F}_t} + \underbrace{X_t\mathbf{1}_{T_n>t}}_{\in\mathcal{F}_t}$$

is $\mathcal{F}_t$-measurable. It follows that $X_T\mathbf{1}_{\{T<t\}}$ is $\mathcal{F}_t$-measurable and the result follows by Lemma 3.1.1.

For 4, $X_{T\wedge t} \in \mathcal{F}_{T\wedge t} \subseteq \mathcal{F}_t$. □

Definition (Hitting time)**.** Let $A \in \mathcal{B}(\mathbb{R})$. Then the hitting time of A is

$$T_A = \inf\{t \geq 0 : X_t \in A\}$$

Lemma 3.2.1 (Process at hitting time of a closed set)**.** Let $A \in \mathcal{B}(\mathbb{R})$ be closed and X be a *continuous* adapted process. then $X_{T_A} \in A$

Proof. There is a sequence $t_n \downarrow T_A$ such that $X_{t_n} \in A$ for all n. Since A is closed and $X_{t_n} \to X_{T_A}$, we have $X_{T_A} \in A$. □

Proposition 3.3 (Hitting time of closed set is stopping time)**.** Let $A \in \mathcal{B}(\mathbb{R})$ be closed and X be a *continuous* adapted process. Then T_A is a stopping time if $T_A < \infty$.

Proof. Let $t \geq 0$. Then

$$\begin{aligned}
\{T_A \leq t\} &= \{\text{at least one } 0 \leq s \leq t \text{ such that } X_s \in A\} && (X_{T_A} \in A) \\
&= \{\inf_{s \leq t} d(X_s, A) = 0\} && (s \to d(X_s, A) \text{ is a.s. continuous}) \\
&= \{\inf_{s \leq t, s \in \mathbb{Q}} d(X_s, A) = 0\} && (X \text{ is continuous})
\end{aligned}$$

□

Definition (Right continuous filtration)**.** Given a continuous filtration $(\mathcal{F}_t)_{t \geq 0}$, we define

$$\mathcal{F}_t^+ = \bigcap_{s > t} \mathcal{F}_s \supseteq \mathcal{F}_t.$$

We say $(\mathcal{F}_t)_{t \geq 0}$ is *right continuous* if $\mathcal{F}_t = \mathcal{F}_t^+$.

Proposition 3.4 (Hitting time of open set is stopping time for right continuous filtrations)**.** Let $(X_t)_{t \geq 0}$ be an adapted process (to $(\mathcal{F}_t)_{t \geq 0}$) that is cadlag, and let A be an open set. Then T_A is a stopping time with respect to $\mathcal{F}_t^+$.

Proof.

$$\{T_A < t\} = \bigcup_{s < t, s \in \mathbb{Q}} \{X_s \in A\} \in \mathcal{F}_t$$

$$\{T_A \leq t\} = \bigcap_{n \geq 1} \{T_A < t + \frac{1}{n}\} \in \mathcal{F}_t^+$$

□

Before proving continuous time analogues of previous theorems, recall the following properties of cadlag functions and processes.

Proposition 3.5 (Properties of Cadlag/continuous functions)**.**

1. Let D be a dense subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be a cadlag function. Then for any $x \in \mathbb{R} \cup \{\pm\infty\}$,

$$\lim_{y \to x} f(y) = z \iff \lim_{y \to x, y \in D} f(y) = z.$$

2. Let D be a dense subset of $\mathbb{R}_+$ and $(X_t)_{t \geq 0}, (Y_t)_{t \geq 0}$ be cadlag processes. Suppose almost surely

$$X_t = Y_t \text{ for all } t \in D.$$

then almost surely $X_t = Y_t$ for all $t \geq 0$.

3. Let $f : X \to Y$ be a continuous function on a metric space and $E \subseteq X$ be dense. Then $f(E)$ is dense in $f(X)$. Analgously (I *think* the following is true), let $f : X \to Y$ with $X, Y \subseteq \mathbb{R}$ be a cadlag function and $E \subseteq X$ with E being dense in X and X being open. Then $f(E)$ is dense in $f(X)$.

Proposition 3.6 (Almost sure martingale convergence theorem, continuous version)**.** Suppose $X = (X_t)_{t\geq 0}$ is a super-martingale that is bounded in L^1. Then there exists an $\mathcal{F}_\infty$-measurable $X_\infty \in L^1$ such that

$$X_n \to X_\infty \text{ a.s. as } n \to \infty.$$

Proof. Note that $X' = (X_t)'_{t\in\mathbb{Q}_+}$ is a super-martingale that is bounded in L^1. By the discrete time almost sure martingale convergence theorem, there exists an $\mathcal{F}_\infty$-measurable $X'_\infty \in L^1$ such that $X'_q \to X'_\infty$ almost surely as $q \to \infty$. By the previous proposition, $X_t \to X_\infty$ almost surely as $t \to \infty$. □

Proposition 3.7 (Doob's maximal inequality, continuous version)**.** Let $X = (X_t)_{t\geq 0}$ be a non-negative cadlag sub-martingale. Then for any $\lambda > 0$ and writing $X_t^* = \sup_{0\leq s\leq t} X_t$, we have

$$\lambda \mathbb{P}(X_t^* \geq \lambda) \leq \mathbb{E}X_t$$

Proof. Note that $X_t^* = \sup_{0\leq s\leq t, s\in\mathbb{Q}_+} X_s$. the result follows from the discrete time analogue. □

Proposition 3.8 (Doob's L^p inequality, continuous version)**.** Let $(X_t)_{t\geq 0}$ be as above. Then

$$\|X_t^*\|_p \leq \frac{p}{p-1}\|X_t\|_p.$$

Proof. The proof of the discrete time version can be applied directly after using the continuous version of Doob's maximal inequality. □

Proposition 3.9 (Optional stopping theorem for UI martingales, continuous version)**.** Let $(X_t)_{t\geq 0}$ be a UI martingale and S, T be any stopping times. Then

$$\mathbb{E}(X_T \mid \mathcal{F}_S) = X_{S\wedge T} \text{ a.s.}$$

Proof. Let $T_n = \frac{1}{2^n}\lceil 2^n T\rceil$ and $S_n = \frac{1}{2^n}\lceil 2^n S\rceil$ so $T_n \downarrow T$ and $S_n \downarrow S$ as $n \to \infty$. It follows that $X_{T_n} \to X_T$ and $X_{S_n} \to X_S$ almost surely. By the discrete time optional stopping theorem, $\mathbb{E}(X_{T_n} \mid \mathcal{F}_{S_n}) = X_{S_n\wedge T_n}$ almost surely. Let $A \in \mathcal{F}_S \subseteq \mathcal{F}_{S_n}$. Then

$$\mathbb{E}X_{T_n}\mathbf{1}_A = \mathbb{E}X_{S_n\wedge T_n}\mathbf{1}_A$$

By the discrete time version again, $X_{T_n} = \mathbb{E}(X_\infty \mid \mathcal{F}_{T_n})$ almost surely so $(X_{T_n})_{n\geq 0}$ is UI and similarly so is $(X_{S_n\wedge T_n})_{n\geq 0}$. By Vitali convergence, take $n \to \infty$ in the above equation to get

$$\mathbb{E}(X_T \mid \mathcal{F}_S) = X_{S\wedge T} \text{ a.s.}$$

□

The previous discussion focused on cadlag processes. The following proposition shows that this is not too harsh a restriction.

Definition (Version)**.** Let X and Y be processes defined on the same probability space. We say X is a *version* of Y if for any t in the index set, $X_t = Y_t$ almost surely, i.e.

$$\forall t, \mathbb{P}(X_t = Y_t) = 1.$$

Definition (Usual conditions)**.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Let $\mathcal{N}$ be the set of all $\mathcal{F}$-measurable sets of measure zero. Define the filtration

$$\tilde{\mathcal{F}}_t = \sigma(\mathcal{F}_t \cup \mathcal{N}).$$

We say $(\mathcal{F}_t)_{t\geq 0}$ satisfies the *usual conditions* if

$$\tilde{\mathcal{F}}_t = \mathcal{F}_t \text{ for all } t \geq 0.$$

Lemma 3.9.1 (Upcrossing and convergence)**.** Let $f : \mathbb{Q}_+ \to \mathbb{R}$, $a < b$, $a, b \in \mathbb{Q}_+$, $I \subseteq \mathbb{Q}_+$ be bounded. Suppose all of the following statements are true

1. f is bounded on I
2. Number of upcrossings of the interval $[a, b]$ by f on I is finite, i.e. $U(a, b, I, f) < \infty$, where

$$U(a, b, I, f) = \sup\{n \geq 0 : \exists 0 \leq s_1 < t_1 < \cdots < s_n < t_n \in I \text{ such that } f(s_i) < a, f(t_i) > b, 1 \leq i \leq n\}$$

Then, for any $t \in \mathbb{Q}_+$,

$$\lim_{s\downarrow t} f(s), \lim_{s\uparrow t} f(s) \text{ exist and are finite.}$$

Remark. If I satisfies the conditions of the lemma, then any $J \subseteq I$ also satisfies the conditions of the lemma.

Proof. Consider any rational sequence $t_n \downarrow t$. By 2.2.1, $f(t_n)$ converges. Since f is bounded on I, the limit is finite. Let $s_n \downarrow t$ be any other rational sequence. Combine the two sequences to get a sequence $r_n \downarrow t$. Then $f(r_n)$ converges to the same limit as $f(t_n)$ and hence $f(s_n)$ converges to the same limit. Hence, $\lim_{s\downarrow t} f(s)$ exists and is finite. The proof for $\lim_{s\uparrow t} f(s)$ is similar. □

Proposition 3.10 (Martingale regularisation theorem)**.** Let $(X_t)_{t\geq 0}$ be a martingale wrt a filtration $(\mathcal{F}_t)_{t\geq 0}$. Then there exists a cadlag process $\tilde{X}$ such that which is a martingale with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and satisfies

$$\mathbb{P}(X_t = \mathbb{E}(\tilde{X}_t \mid \mathcal{F}_t)) = 1 \text{ for all } t \geq 0.$$

Proof. The candidate for $\tilde{X}_t$ is $\lim_{s\downarrow t, s\in\mathbb{Q}_+} X_s$ on a set of probability 1. The outline of the proof is as follows:

1. That the limit exists and is finite on a suitable set of probability 1 using the previous lemma.
2. For any t, $X_t = \mathbb{E}(\tilde{X}_t \mid \mathcal{F}_t)$ almost surely.
3. $\tilde{X}$ is a martingale
4. $\tilde{X}$ is cadlag

Step 1

Let a, b, I be as in the previous lemma and $\sup I < K < \infty$. By the corollary to Doob's maximal inequality 2, for any $\lambda > 0$,

$$\lambda \mathbb{P}(\sup_{s \in I} |X_s| \geq \lambda) \leq \sup_{s \in I} \mathbb{E}|X_s| \leq \mathbb{E}|X_K|$$

so $\mathbb{P}(\sup_{s \in I} |X_s| \geq \lambda) \to 0$ as $\lambda \to \infty$ so $\mathbb{P}(\sup_{s \in I} |X_s| < \infty) = 1$.

The following analogue to the corollary of the upcrossing inequality 1 can be proven

$$(b-a)\mathbb{E}U[a, b, I, X] \leq \sup_n \mathbb{E}(|X_n|) + |a| \leq \mathbb{E}|X_K| + |a| < \infty.$$

by taking finite subsets $J \subseteq I$ and taking the limit as $J \uparrow I$. It follows that $U[a, b, I, X] < \infty$ almost surely.

Hence, for any a, b, I as in the previous lemma, we can construct a set $\Omega_{a,b,I}$ satisfying the conditions of the previous lemma. Define $I_M = \mathbb{Q}_+ \cap [0, M]$ and $\Omega_0 := \bigcap_{M \in \mathbb{N}} \bigcap_{a,b \in \mathbb{Q}, a<b} \Omega_{a,b,I_M}$. Then $\mathbb{P}(\Omega_0) = 1$.

We show that on Ω_0, the conditions of the lemma hold. For any a', b', I', pick M' so $I' \subseteq I_{M'}$. Then $\Omega_{a',b',I'} \subseteq \Omega_{a',b',I_{M'}} \subseteq \Omega_0$.

Now, define

$$\tilde{X}_t = \begin{cases} \lim_{s \downarrow t, s \in \mathbb{Q}_+} X_s & \text{on } \Omega_0 \\ 0 & \text{on } \Omega_0^c \end{cases}$$

Observe that $\tilde{X}_t$ is $\tilde{\mathcal{F}}_t$-measurable

Step 2

Let $t_n \downarrow t$, $t_n \in \mathbb{Q}_+$. Then $(X_{t_n})_{n \geq 0}$ is a backwards martingale with respect to the backwards filtration $(\mathcal{F}_{t_n})_{n \geq 0}$. By the backwards martingale convergence theorem 2.14, we have $X_{t_n} \xrightarrow{L^1} \tilde{X}_t$ and hence $\underbrace{\mathbb{E}(X_{t_n} \mid \mathcal{F}_t)}_{=X_t} \xrightarrow{L^1} \mathbb{E}(\tilde{X}_t \mid \mathcal{F}_t)$ and hence $X_t = \mathbb{E}(\tilde{X}_t \mid \mathcal{F}_t)$ almost surely.

Step 3

Let $s < t$, $s_n \downarrow s$, $s_n \in \mathbb{Q}_+$ and $s_0 < t$. By the backwards martingale convergence theorem again, $\underbrace{\mathbb{E}(X_t \mid \mathcal{F}_{s_n})}_{=X_{s_n}} \to \mathbb{E}(X_t \mid \cap_{n \geq 0} \mathcal{F}_{s_n}) = \mathbb{E}(X_t \mid \mathcal{F}_s^+)$ almost surely. Recall that by definition $X_{s_n} \to \tilde{X}_s$ almost surely. Hence $\tilde{X}_s = \mathbb{E}(\tilde{X}_t \mid \mathcal{F}_s^+)$ almost surely.

To conclude that $\tilde{X}$ is a martingale, we use the following result: for any random variable X, σ-algebra $\mathcal{G}$ and collection of measure zero sets $\mathcal{N}$,

$$\mathbb{E}(X \mid \sigma(\mathcal{G} \cup \mathcal{N})) = \mathbb{E}(X \mid \mathcal{G}).$$

which can probably be proven using π-λ theorem.

It follows that $\tilde{X}$ is a martingale.

Step 4

Only show right continuity and leave left limit as an exercise. Suppose not. Then there exists sequence $t_n \downarrow t$ and $\varepsilon > 0$ such that $|\tilde{X}_{t_n} - \tilde{X}_t| > \varepsilon$ for all n. We can pick a rational sequence $t'_n \downarrow t$ with $t'_n \geq t_n$ and approximate $\tilde{X}_{t_n}$ by $X_{t'_n}$ such that $|X_{t'_n} - \tilde{X}_{t_n}| < \frac{\varepsilon}{2}$ for all n. Then $|X_{t'_n} - \tilde{X}_t| \geq \frac{\varepsilon}{2}$ for all n. But $X_{t'_n} \to \tilde{X}_t$ almost surely, which is a contradiction. $\square$