Topology of Networks Time-delay Embeddings

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Introduction

Topology and its computational applications have recently received large attention for their capacity to extract robust information from noisy datasets.

We present an approach for detecting recurrent patterns in temporal networks by producing appropriate time-delayed embeddings of the network structure.

Method

- A sequence of network snapshots of length λ is mapped into a high-dimensional metric space.
- Extending previous works [1; 2], we are able to detect the period through persistent cohomology.
- We are then able to reconstruct the period of the recurrent pattern from the cocycle applying techniques introduced in [3].

Time-Delay Embedding

Definition

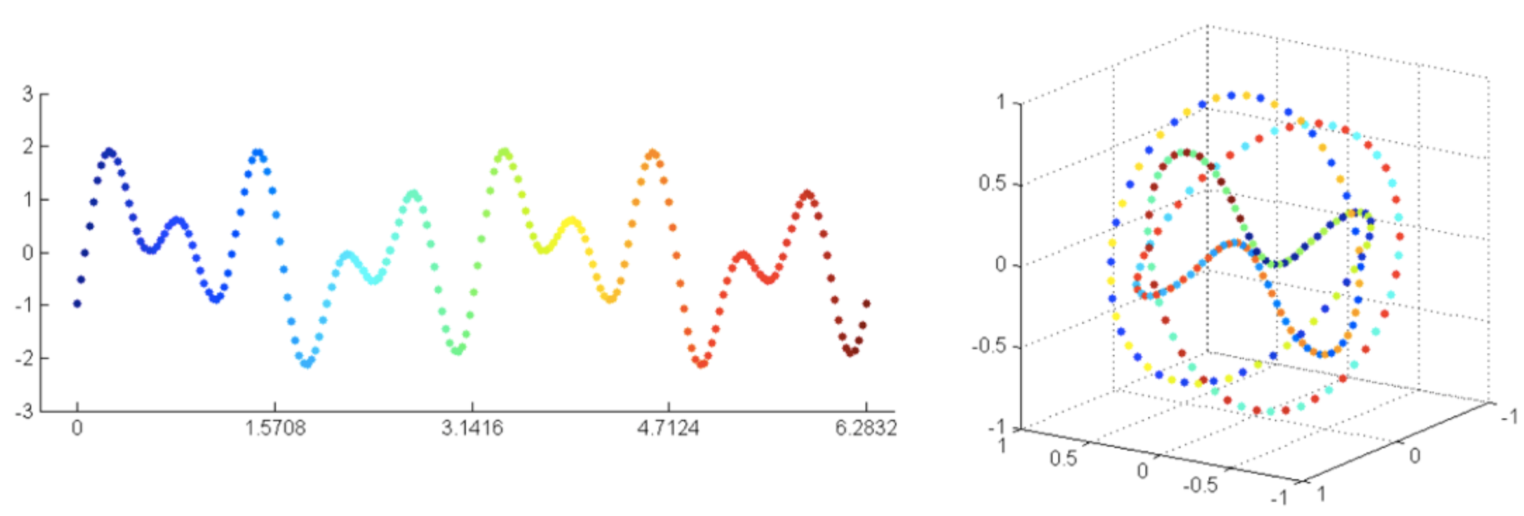
Given a time-series $f: t \to \mathbb{R}$, a time-delay embedding of f is a lift to a time series $\phi: t \to \mathbb{R}^d$ defined by:

$$\phi(t) = (f(t), f(t+\tau), \dots, f(t+(d-1)\tau))$$

Takens Theorem gives conditions under which a smooth attractor can be reconstructed from the observations of a function.

Takens Embedding Theorem (1981)

A smooth attractor can be reconstructed from the observations formed from time delayed values of the scalar measurements.

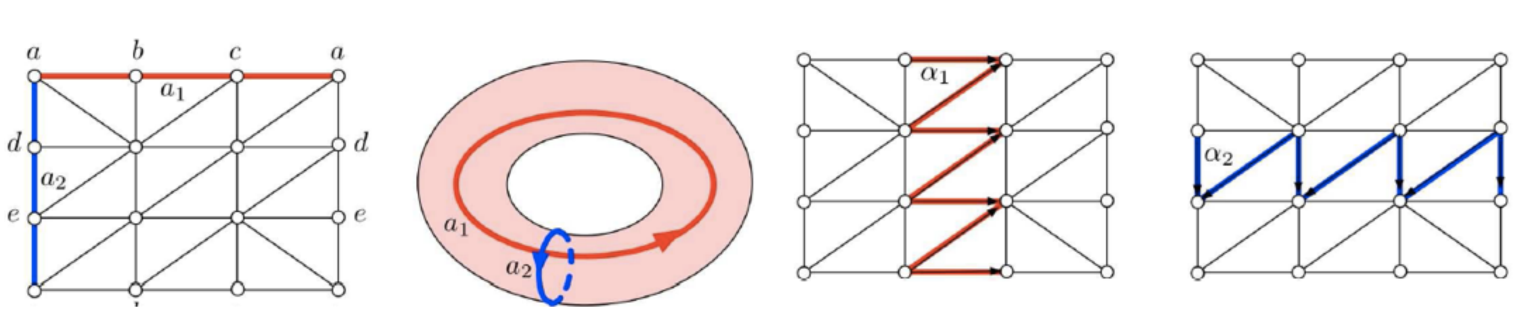


From a periodic time series to its time-delay embedding point cloud.
Figure from Jose A. Perea, and John Harer (2012) [2]

Persistent Cohomology

Homology and Cohomology give some geometrical-topological global information, describing the entire dataset.

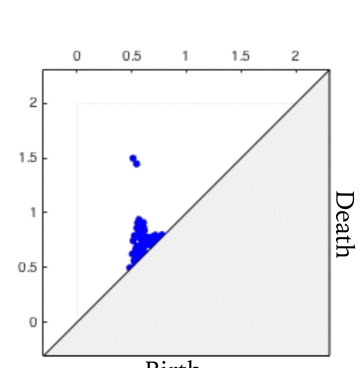
Homology gives information on how many holes of a given dimension there are in a topological space, cohomology can be characterised by saying it assigns quantities to these holes.



The triangulation of a torus highlighting the generators of homology a_1, a_2, and cohomology α_1, α_2. It is important to note the duality between cohomology and homology generators, which is slightly counter-intuitive.
Figure from *Branching and Circular Features in High Dimensional Data* by Wang, Summa, Pascucci and Vejdemo-Johansson

Persistent (co)homology is a way to detect prominent topological features in point clouds. It does so by filtering the complex through a chosen parameter, and then measuring lifespans for each topological feature (holes, voids, tunnels).

The Persistent diagram is a way to represent the persistent (co)homology of a topological space. Each point in the diagram represents a (co)cycle (hole, void, tunnel) and the coordinates of the point are the levels of the threshold at which the (co)cycle first appears and that at which it disappears.



Persistent diagram of the triangulation of a torus

Period Recovering

Morozov, de Silva, Vejdemo-Johansson demonstrated in [3] how to compute $H^1(X;\mathbb{Z})$ persistently and how to yield coordinate functions $X \to S_1$ with circle-valued coordinates.

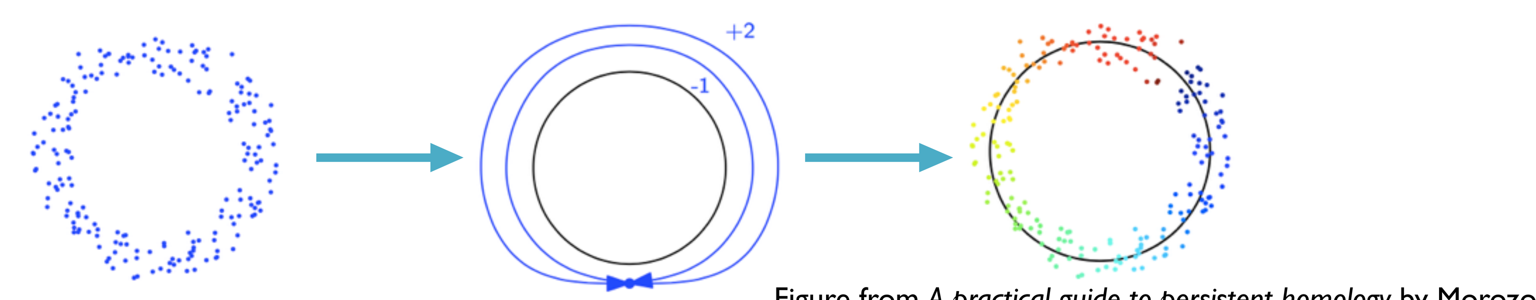


Figure from *A practical guide to persistent homology* by Morozov

1. Compute persistent cohomology classes
2. Turn each representative cocycle z∗ into a map, $X \to S_1$
3. Smooth that map (minimize variation across edges), staying within the same cohomology/ homotopy class (done projecting onto the harmonic space).

How to apply this to analysing periodic systems?

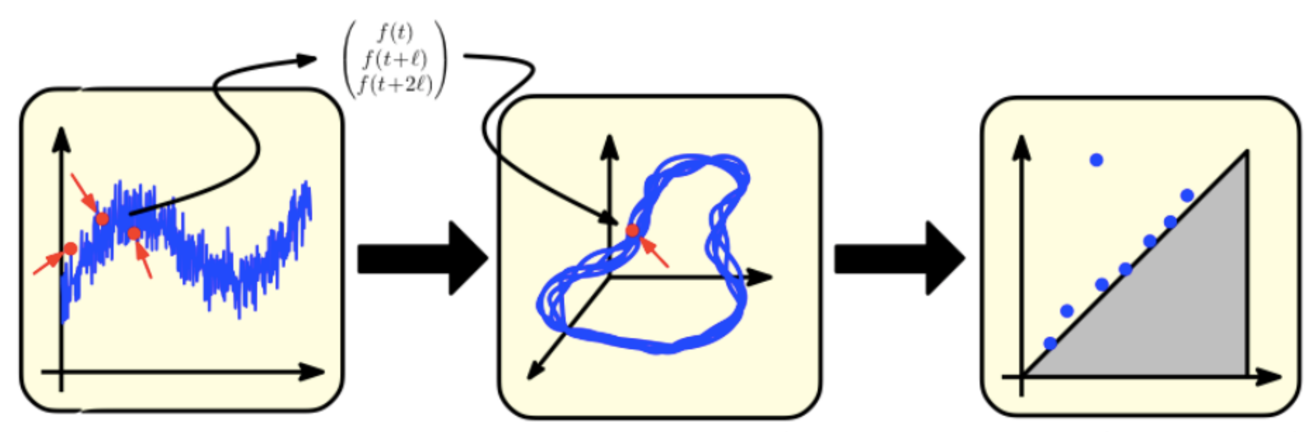


Figure from Vejdemo-Johansson, Skraba, and de Silva in [1]

- Sufficiently well described periodic or recurrent signals will describe circles in the observation space.
- Circles can be detected by persistent cohomology as cocycles, of which circular coordinates can be computed.
- Unrolling a circular coordinate, followed by linear regression, measures periodicity.

Results

A temporal network is seen as a multi varied time series.

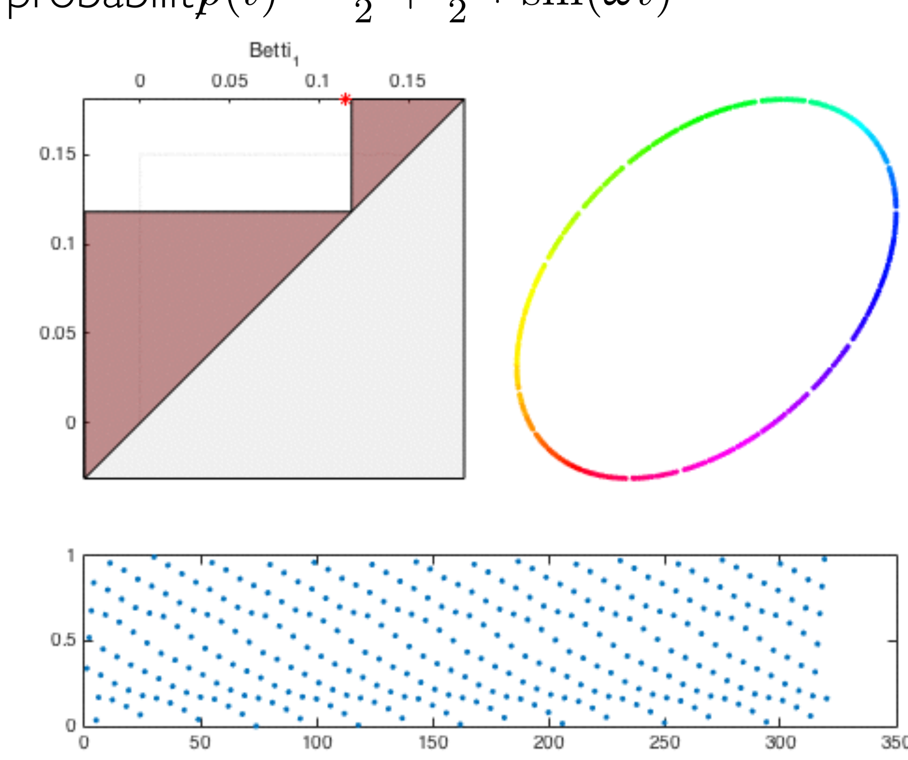
A sequence of network snapshots is embedded into a high-dimensional metric space as $\phi(t)$

$$\phi(t) = \; (f_1(t), f_2(t), \dots, f_N(t), f_1(t+\alpha), f_2(t+\alpha), \dots, f_N(t+\alpha), \dots, f_1(t+\alpha(d-1)), f_2(t+\alpha(d-1)), \dots, f_N(t+\alpha(d-1)))$$

In order to detect the circles in the high-dimensional embedding space we construct a simplicial complex approximating the topology of the embedded point cloud (eg. Vietoris Rips), and then create a persistence diagram for 1-dimensional cohomology.

Example

As a temporal network toy model we used an Erdos-Renyi random graph $ER(N, p(t))$ with time-dependant probability $p(t) = \frac{1}{2} + \frac{1}{2} * \sin(\omega t)$

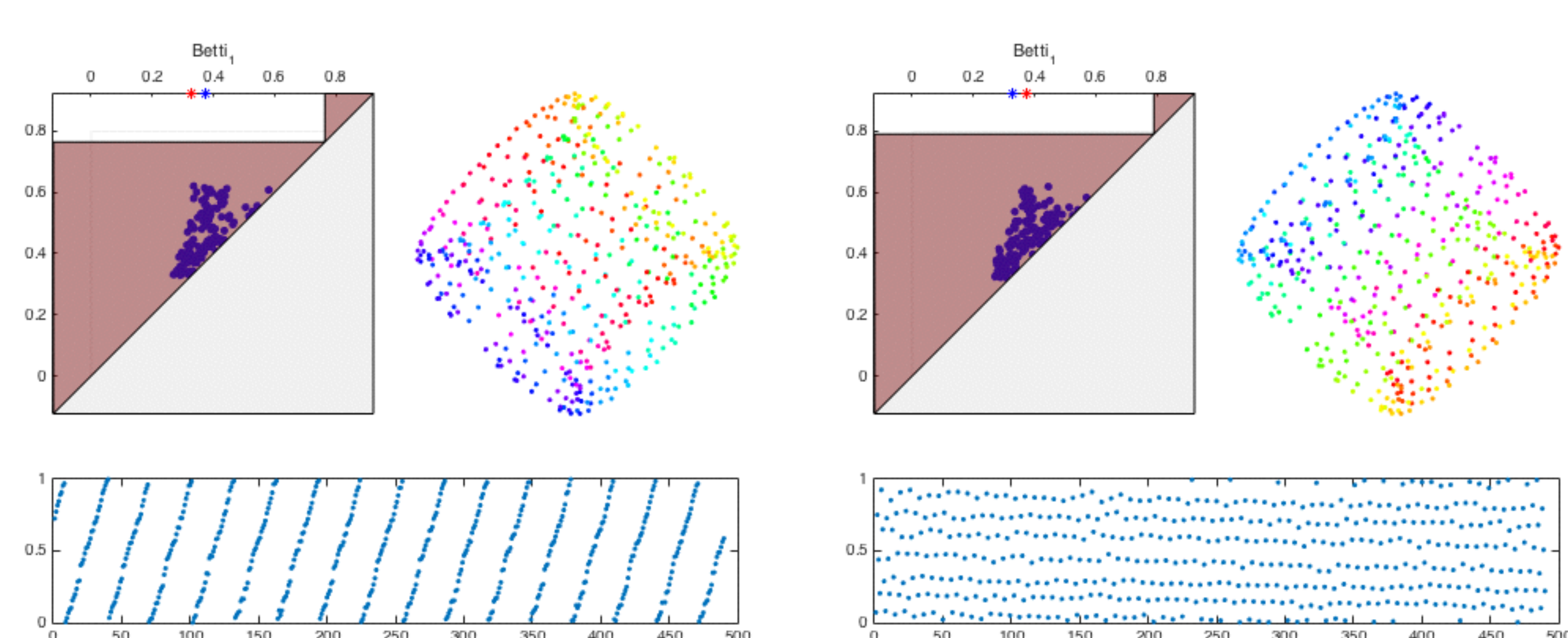


(a) The persistent diagram of a time-delay embedding of a with delay=5 and dimension 6; (b) the first two coordinates of the data, plotted with colours to indicate the circular coordinate; (c) the circular coordinate plotted against the time.

For this model with 100 nodes, $\omega = 1$ with an embedding with time step 20 and dimension 5.
We were able to successfully recover the period $T = \frac{1}{0.1592} = 6.2814$

We then studied a temporal model with two connected components $ER(N, p_1(t))$ $ER(N, p_2(t))$ with different frequencies $\omega_1 = 1$, $\omega_2 = 150$ and probability

$$p_1(t) = \frac{1}{2} + \frac{1}{2} * \sin(\omega_1 t) \quad p_2(t) = \frac{1}{2} + \frac{1}{2} * \cos(\omega_2 t)$$



Conclusions

Our method can successfully detect recurrent structures in temporal network through the persistent diagram.

Outlook

Recover all the periods of the network, and identify the recurrent structure.
Study the circular coordinates of the cycles detected with non-costant frequencies.
Applications to real-world networks.

References

[1] Vejdemo-Johansson, Skraba, and de Silva. Topological Analysis of Recurrent Systems. In *NIPS 2012 Workshop on Algebraic Topology and Machine Learning, December 8th, Lake Tahoe, Nevada* (pp. 1-5)

[2] Perea, and Harer. Sliding windows and persistence: An application of topological methods to signal analysis. In NIPS Workshop on Algebraic Topology and Machine Learning (2012)

[3] De Silva, Morozov, and Vejdemo-Johansson. Persistent cohomology and circular coordinates. *Discrete & Computational Geometry*, *45*(4), (2011). 737-759.

[4] Patania, Vaccarino, Nazem, Jadanza and Petri. Cohomological method for period recovering in temporal networks. *In preparation*

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