

M 408L Practice Exam 1 Solutions

The actual exam will be *much* shorter than this practice exam. This practice exam is mainly here to give you plenty of opportunity to reinforce your calculus skills.

1. **Simple(r) U-subs.** Compute the following integrals:
 - (a) $-\frac{1}{3\pi}$; Use the substitution $u = 3\pi x$ and recall that $\sin(\frac{3}{2}\pi) = -1$.
 - (b) $\frac{1}{12\pi}e^{12\pi x} + C$; Use the substitution $u = 12\pi x$.
 - (c) $\frac{1}{4}$; Use the substitution $u = x - 5$.
 - (d) $\arctan(x - 10) + C$; Use the substitution $u = x - 10$ and recall that $\frac{1}{1+u^2} du = d\arctan(u)$.
2. Answer: π ; You can work out the integral directly even if we don't know the value of c – just treat it like a constant. In this case, the integral will work out to $1 - \cos(c)$. Set this equal to 2 and solve for c to get $c = \pi$ (in fact, the more general solution is $c = (2n + 1)\pi$, where n is any whole number).
3. **Fundamental Theorem of Calculus.** Find the derivative of the following functions:
 - (a) $F'(x) = \sqrt{1 + \cos(x)}$; Just replace t with x .
 - (b) $F'(x)3x^2e^{x^6}$; Replace t with x^3 and multiply by the derivative of x^3 .
 - (c) $F'(x) = -\sec^2(x) \ln(|\tan(x)|)$; Remember to flip the integral first so $\tan(x)$ appears as the top bound of integration. This is why there's a minus sign in front.
 - (d) $F'(x) = 0$; Here, $F(x)$ does not depend on x , so it's just a constant. No matter how scary it looks, the derivative of a constant is zero.
4. Answer: $9 + \sin(2)$; Start by integrating $v(t)$ from 0 to 2, which gives you $4 + \sin(2)$. This is the *change* in position between times 0 and 2. To get the actual position, add this to the initial position, which is 5.
5. **Advanced U-subs.** Compute the following integrals:
 - (a) $\frac{1}{20} \sin(5x^4) + C$; Use the substitution $u = 5x^4$.
 - (b) $3e^{\tan(x)} + C$; Use the substitution $u = \tan(x)$.
 - (c) $\frac{2}{9}(5 + x^3)^{3/2} + C$; Use the substitution $u = 5 + x^3$.
 - (d) $\ln(|\sin(x)|) + C$; start by writing $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and use the substitution $u = \sin(x)$.
 - (e) $-\frac{\cos^{2026}(x)}{2026} + C$; Use the substitution $u = \cos(x)$.

6. Area Between Curves. Find the area of the described regions:

- (a) $\frac{4\sqrt{2}-2}{3} - 2 \ln(2) + 1$; Set up the integral $\int_1^2 (\sqrt{x} - \ln(x)) dx$. By following the hint, you will find that \sqrt{x} is the top function.
- (b) $8 \ln(2) - 4$; Set up the integrals $\int_{-2 \ln 2}^0 (2 - e^{-\frac{1}{2}x}) dx$ and $\int_0^{2 \ln 2} (2 - e^{\frac{1}{2}x}) dx$ and add up the results. We must split the integrals like this since the bottom changes halfway throughout the region. The bounds $2 \ln 2$ and $-2 \ln 2$ are obtained by solving $2 = e^{\frac{1}{2}x}$ and $2 = e^{-\frac{1}{2}x}$ for x .
- (c) 9; This is better thought of as a “right minus left” area problem. By sketching, our right curve is $y = x$, and our left is $y^2 = 4x + 5$. Start by solving these equations for x , then determine the y -values where these two curves meet (this is obtained by solving $y^2 = 4y + 5$ for y). From here, you should have the integral $\int_{-1}^5 \left(y - \frac{y^2 - 5}{4} \right) dy$.
- (d) $\cos(4) - \sin(4) + 2\sqrt{2} + 1$; Following the hint, we need to find the area between $\cos(x)$ and $-\sin(x)$. On the interval $[0, 4]$, $\cos(x)$ and $-\sin(x)$ will be equal at $x = \frac{3\pi}{4}$ and nowhere else. As a result, our top and bottom functions will switch at this point. This lets us set up the integrals $\int_0^{3\pi/4} (\cos(x) + \sin(x)) dx + \int_{3\pi/4}^4 (-\sin(x) - \cos(x)) dx$

7. Volumes. Find the volume of the described solids:

- (a) $\frac{\pi}{2}(e^2 - 3)$; You will need to use washer method since our region will have a “gap” inside. The outer radius is given by e^x and the inner radius is given by 1, so our integral will look like $\pi \int_0^1 (e^x)^2 - 1^2 dx$.
- (b) $\frac{28\pi}{5}$; Another washer problem, but you will need to use dy since the axis of rotation is vertical. In this case, our outer radius is $2 - \frac{y^2}{9}$ and our inner radius is $2 - 1 = 1$, giving us the integral $\pi \int_0^3 \left(2 - \frac{y^2}{9} \right)^2 - 1 dy$
- (c) π ; Since the axis of rotation is vertical, we must use dy . Our solid has no gap, so we can use disk method. In this case, we find our radius by solving $y = e^{x^2}$ for x , which is $x = \sqrt{\ln y}$. From here, set up the integral $\pi \int_1^e (\sqrt{\ln y})^2 dy$.

8. Integration by Parts. Compute the following integrals:

- (a) $\frac{1}{\pi} - \frac{4}{\pi^3}$; Use integration by parts twice. For the first round, $u = x^2$ and $dv = \sin(\pi x)$. For the second round, $u = x$ and $dv = \cos(\pi x)$.

- (b) $-\frac{2e^{3x} \cos(2x)}{13} + \frac{3e^{3x} \sin(2x)}{13} + C$; You will need multiple rounds of integration by parts. Each time, use $dv = e^{3x}$, and u will be a trigonometric function. After a couple applications, you will cycle back to the original integral; move it to the other side and solve for the original integral.
- (c) $x \ln^2(x) - 2(x \ln(x) - x) + C$; You don't have too many options for what u and dv are. In this case, setting $u = x$ and $dv = \ln(x)$ will work.
- (d) $-\frac{5}{16e^4} + \frac{1}{16}$; Use $u = \ln(x)$ and $dv = x^{-5}$.

9. Advanced Trig Integrals. Compute the following integrals:

- (a) $\frac{1}{8} \left(x - \frac{1}{4} \sin(4x) \right) + C$; The trig identity $\sin(2x) = 2 \sin(x) \cos(x)$ and the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$ may be useful.
- (b) $\frac{1}{3} \left(-\frac{\cos^3(3x)}{3} + \frac{\cos^5(3x)}{5} \right) + C$; Pull out a $\sin^2(3x)$ and use the Pythagorean identity. From here, u -sub with $u = \cos(3x)$.
- (c) $-\frac{1}{2} \cos^4(x) + C$; Start by using the double angle formula $\sin(2x) = 2 \sin(x) \cos(x)$, then u -sub with $u = \cos(x)$.
- (d) $\frac{\tan^6(x)}{6} + \frac{\tan^8(x)}{8} + C$; you can either factor out $\sec^2(x)$ and apply Pythagorean identity to the remaining $\sec^2(x)$, or pull out $\sec(x) \tan(x)$ and apply Pythagorean identity to the remaining $\tan^4(x)$.
- (e) $\frac{1}{2} \left(-\frac{1}{7} \cos(7x) + \cos(x) \right) + C$; Use the identity $\cos(A) \sin(B) = \frac{\sin(A+B) + \sin(A-B)}{2}$ with $A = 4x$ and $B = 3x$.

10. Trig Substitution. Compute the following integrals:

- (a) $\frac{1}{2} \left(\arcsin(x) - \frac{1}{2} \sin(2 \arcsin(x)) \right) + C$; Use the substitution $x = \sin(\theta)$.
- (b) $-2 \operatorname{arcsec}\left(\frac{5}{2}x\right) + \sqrt{25x^2 - 4} + C$; Use the substitution $x = \frac{2}{5} \sec(\theta)$ to force the square root to contain $\sqrt{\sec^2(\theta) - 1}$.
- (c) $-\frac{\sqrt{1+x^2}}{x} + C$; Use the substitution $x = \tan(\theta)$.
- (d) $\frac{-9\sqrt{-x^2+9} - 2x^2\sqrt{9-x^2}}{243x^3} + C$; Use the substitution $x = 3 \sin(\theta)$.

11. Freestyle. Compute the following integrals by any means necessary. Try to use the easiest method(s) available!

- (a) 0; the function we're integrating is odd (\cos is even, \sin is odd, and x^{100} is even).

- (b) 75; take advantage of the fact that $|x|$ is even to rewrite the problem as $2 \int_0^5 3x \, dx$.
- (c) $\frac{x^2}{2} - \frac{1}{2} \ln|x^2 + 1| + C$; technically, you can use trig sub with $x = \tan(\theta)$, but polynomial long division makes this much easier. Using long division, we will find that $\frac{x^3}{1+x^2} = x - \frac{x}{x^2+1}$. The second term can be integrated with u -sub.
- (d) $x - \arctan(x) + C$; Again, trig-sub is possible here, but the easier approach is to notice that $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$. The antiderivative of the second term is simply \arctan .
- (e) $\frac{1}{2} (e^{x^2} x^2 - e^{x^2}) + C$; This problem requires a non-obvious integration by parts. Following the hint, we should set $u = x^2$ and $dv = xe^{x^2}$. You will only need to apply IBP once since $\int vdu$ can be solved with a single substitution.
- (f) $\frac{5}{3} \arctan(x^3) + C$; A non-obvious u -sub. Taking $u = x^3$ is the best choice since it allows us to cancel the x^2 in the numerator, leaving us with $\frac{5}{3} \int \frac{1}{1+u^2} \, du$.