Assignment 2

Question 1)

a) 1001101010

b) {2, 4, 6, 7, 9}

Question 2)

a) {∅}

b) {∅, a, {a}, {a, b}, {b}}

c) { (∅, ∅), (∅, {a}), (∅, {b}), (a, ∅), (a, {a}), (a, {b}), ({a, b}, ∅), ({a, b}, {a}), ({a, b}, {b}) }

d) { ∅, {∅}, {a}, {{a, b}}, {∅, a}, {∅, {a, b}}, {a, {a, b}, {∅, a, {a, b}} }

Question 3)

a) $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

Left Hand Side

$= x \in \overline{A \cap B \cap C} \rightarrow x \notin A \cap B \cap C$ (Definition of Complement)

$= \neg\ (x \in A \land x \in B \land x \in C)$ (Definition of Intersection)

$= \neg\ (x \in A) \lor \neg\ (x \in B) \lor \neg\ (x \in C)$ (De Morgan's Law)

$= x \notin A \lor x \notin B \lor x \notin C$ (Definition of $\notin$)

$= x \in \overline{A} \lor x \notin \overline{B} \lor x \notin \overline{C}$ (Definition of complement)

$= x \in \overline{A} \cup \overline{B} \cup \overline{C}$ (Definition of union of sets)

$\therefore$ Equals to RHS

b)

A B C	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1 1 1	0	0	0	0	0
1 1 0	1	0	0	1	1
1 0 1	1	0	1	0	1
1 0 0	1	0	1	1	1
0 1 1	1	1	0	0	1
0 1 0	1	1	0	1	1
0 0 1	1	1	1	0	1
0 0 0	1	1	1	1	1

Question 4)

a) Transitive, Antisymmetric.

b) Reflexive, Symmetric, Transitive

c) Reflexive, Symmetric

Question 5)

a) The number of nonzero entries in the matrix representing R^{-1} is k.

Since R^{-1} is the inverse of R, (b,a) ∈ R^{-1} when (a,b) ∈ R and (b,a) ∉ R^{-1} when (a,b) ∉ R.

Therefore, the matrix representing R^{-1} would have the same number of nonzero entries as the matrix representing R.

Hence number of nonzero entries is k.

b) The number of nonzero entries in the matrix representing $\overline{R}$ is $n^2 - k$.

Since $\overline{R}$ is the complement of R. (a,b) ∈ $\overline{R}$ when (a,b) ∉ R and (a,b) ∉ $\overline{R}$ when (a,b)∈ R.

Therefore, since there are k entries in the matrix representing R that is nonzero, k will be the number of entries in the matrix $\overline{R}$ that are zero.

And since the relation R is on the set A with n elements, the dimensions of the matrix representing R would be n x n. Therefore, the number of entries in the matrix representing R is n^2.

Therefore, number of nonzero entries in the matrix representing $\overline{R}$ is $n^2 - k$.

Question 6)

a) For a relation to be an equivalence relation, it has to be reflexive, symmetric and transitive.

Reflexive: If ((a,b),(a,b)) ∈ R for every (a,b) ∈ A

Since a + b = b + a (commutative property of addition)

$((a,b),(a,b)) \in R$

Therefore, R is reflexive.

Symmetric: $((a,b),(c,d)) \in R \rightarrow ((c,d), (a,b)) \in R$

$(a,b)R(c,d) \Leftrightarrow a + d = b + c$

$d + a = c + b$

$c + b = d + a$

$c + b = d + a \rightarrow ((c,d),(a,b)) \in R$

Therefore, R is symmetric.

Transitive: $((a,b),(c,d)) \in R \wedge ((c,d),(e,f)) \in R \rightarrow ((a,b),(e,f)) \in R$

$(a,b)R(c,d) \Leftrightarrow a + d = b + c$

$(c,d)R(e,f) \Leftrightarrow c + f = d + e$

Rearranging the equations

$a = b + c - d$

$f = d + e - c$

$a + f = (b + c - d) + (d + e - c)$

$= b + e$

$a + f = b + e \rightarrow ((a,b),(e,f)) \in R$

Therefore, R is transitive.

Hence R is an equivalence relation.

b) [2,1] = {(a+1,a) | a ∈ Z^+ }

Question 7)

For a relation to be a partial order, it must be reflexive, antisymmetric and transitive

a) The relation represented by the digraph is not a partial order relation. This is because the relation fails to meet the transitive property. For a relation to be transitive ∀a∀b∀c ((a,b) ∈ R ∧ (b,c) ∈ R → (a,c) ∈ R). However in the digraph, (a,b) ∈ R ∧ (b,d) ∈ R but (a,d) ∉ R. Therefore, fails the transitive property. Hence it is not a partial order relation.

b) The relation represented by the digraph is also not a partial order relation. As it fails to meet the transitive property. For a relation to be transitive ∀a∀b∀c ((a,b) ∈ R ∧ (b,c) ∈ R → (a,c) ∈ R). However, in the digraph (c,d) ∈ R ∧ (d,b) ∈ R but (c,b) ∉ R. Therefore, fails the transitive property. Hence it is not a partial order relation.

Question 8)

a) If a string is a palindrome, first half of the string is equal to the second half of the string. And a bit string has only two cases, 1 and 0.

Two cases: n is even, or n is odd.

No. of permutations when

n is even = $\left(\underbrace{2 * 2 * 2 * 2 * 2 * 2 * \ldots.}_{n/2}\right)$ * 1

$= 2^{\frac{n}{2}}$

n is odd = $\left(\underbrace{2 * 2 * 2 * 2 * 2 * 2 * \ldots.}_{(n-1)/2}\right)$ * 2 * 1

$= 2^{\frac{n-1}{2}}$*2

$= 2^{\frac{n+1}{2}}$

b) Only have to consider first 4 bits of the string due to being a palindrome.

Due to having to contain 2 consecutive 0s, can therefore be grouped as 1.

So only have to find permutations of first 3 bits.

However, creates an overlap of repeated numbers, therefore have to take 4.

No of permutations = $3 * 2 * 2$ - 4

= 8

c) **First case:** All zeros

1 way.

Second case: Having only one 1 in the first 4 bits

10000001
01000010
00100100

00011000

Therefore 4 ways

Third case: Having two 1s in the first 4 bits

11000011
01100110
00111100 (Not accepted as contains 3 consecutive 1s)
10100101
01011010
10011001

Therefore 5 ways

Fourth case: Having three 1s in the first 4 bits

10111101 (Not Accepted as contains 3 consecutive 1s)
11011011

Therefore only 1 way.

Adding all 4 cases, gives us **11** ways to form a bit string palindrome without 3 consecutive 1s.

Question 9)

a) $= C(15,8) * \left(\frac{P(8,8)}{8}\right) * \left(\frac{P(7,7)}{7}\right)$

$= C(15,8) * 7! * 6!$

$= 2.3351328 * 10^{10}$ ways

b) Case 1: They sit on the 8-seat table

$C(13,6) * \left(\frac{P(7,7)}{7}\right) * 2! * \left(\frac{P(7,7)}{7}\right)$

Case 2: They sit on the 7-seat table

$C(13,5) * \left(\frac{P(6,6)}{6}\right) * 2! \ * \left(\frac{P(8,8)}{8}\right)$

Ans = $C(13,6) * \left(\frac{P(7,7)}{7}\right) * 2! * \left(\frac{P(7,7)}{7}\right) + \ C(13,5) * \left(\frac{P(6,6)}{6}\right) * 2! * \left(\frac{P(8,8)}{8}\right)$

= 3335904000 ways

c) **Total numbers of ways:**

= $2.335 * 10^{10}$ ways

Number of ways two people are sitting on the same table:

Case 1: Sitting on the 8-seat table

$C(13,6) * \left(\frac{P(8,8)}{8}\right) * \left(\frac{P(7,7)}{7}\right)$

Case 2: Sitting on the 7-seat table

$C(13,5) * \left(\frac{P(7,7)}{7}\right) * \left(\frac{P(8,8)}{8}\right)$

Number of ways = $C(13,6) * \left(\frac{P(8,8)}{8}\right) * \left(\frac{P(7,7)}{7}\right) + C(13,5) * \left(\frac{P(7,7)}{7}\right) * \left(\frac{P(8,8)}{8}\right)$

= $1.08972864 * 10^{10}$

Number of ways two people are not sitting on the same table:

= total number of ways - number of ways they are sitting on the same table

= $2.3351328 * 10^{10} \ - 1.08972864 * 10^{10}$

= $1.24540416 * 10^{10}$ ways

ANS = $\mathbf{1.24540416 * 10^{10}}$ **ways**

Question 10)

a) **Number of possible characters to choose** = 26 + 26 + 10 + 6

= 68

Number of ways for each password Lengths

8 characters = 68^8

9 characters = 68^9

10 characters = 68^{10}

11 characters = 68^{11}

12 characters: 68^{12}

Total number of ways = $68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12}$

= $9.920671339 * 10^{21}$ ways

Each possible password takes 1 nano second, therefore it will take

$9.920671339 * 10^{21}$ nanoseconds.

b) **Number of possible characters to choose if not including special character**

= 26 + 26 + 10

= 62

Number of ways containing no special characters

= $62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12}$

= $3.279156378 * 10^{21}$ ways

Therefore, number of ways for password containing at least 1 special character:

(Total number of ways) - (Number of ways containing no special character)

$= 9.920671339 * 10^{21} - 3.279156378 * 10^{21}$

$= 6.641514961 * 10^{21}$ ways.

c)

Inclusion-exclusion principle for set of 4.

$|A \cap B \cap C \cap D| = |A| + |B| + |C| + |D| - (|A \cap B| + |A \cap C| + |A \cap D| +$

$|B \cap C| + |B \cap D| + |C \cap D|) + (|A \cap B \cap C| + |A \cap B \cap D| +$

$|A \cap C \cap D| + |B \cap C \cap D|) - |\mathrm{A} \cup \mathrm{B} \cup \mathrm{C} \cup \mathrm{D}|$

A = contains at least one lower case

B = contains at least one upper case

C = contains at least one digit

D = contains at least one special character

$\mathrm{A} = 9.920671339 * 10^{21} - 3.086433436 * 10^{19}$

$= 9.889807005 * 10^{21}$

$\mathrm{B} = 9.889807005 * 10^{21}$

$\mathrm{C} = 8.446020983 * 10^{21}$

$\mathrm{D} = 6.641514961 * 10^{21}$

$|A \cap B| = 9.889807005 * 10^{21} + 9.889807005 * 10^{21} - 9.920671339 * 10^{21} -$

$3.002396888 * 10^{14}$

$= 9.858942971 * 10^{21}$

$$|A \cap C| = 9.889807005 * 10^{21} + 8.446020983 * 10^{21} - 9.920671339 * 10^{21} - 1.190112485 * 10^{18}$$

$$= 8.416346761 * 10^{21}$$

$$|A \cap D| = 9.889807005 * 10^{21} + 6.641514961 * 10^{21} - 9.915797575 * 10^{21}$$

$$= 6.615524391 * 10^{21}$$

$$|B \cap C| = 8.416346761 * 10^{21}$$

$$|B \cap D| = 6.615524391 * 10^{21}$$

$$|C \cap D| = 8.446020983 * 10^{21} + 6.641514961 * 10^{21} - 9.920671339 * 10^{21} - 3.985412605 * 10^{20}$$

$$= 4.768323345 * 10^{21}$$

$$|A \cap B \cap C| = 9.920671339 * 10^{21} - 2611802880 - (9.889807005 * 10^{21} + 9.889807005 * 10^{21} + 8.446020983 * 10^{21}) + 9.858942971 * 10^{21} + 8.416346761 * 10^{21} + 8.416346761 * 10^{21}$$

$$= 8.386672839 * 10^{21}$$

$$|A \cap B \cap D| = 9.920671339 * 10^{21} - 1.1111 * 10^{12} - (9.889807005 * 10^{21} + 9.889807005 * 10^{21} + 6.641514961 * 10^{21}) + 9.858942971 * 10^{21} + 6.615524391 * 10^{21} + 6.615524391 * 10^{21}$$

$$= 6.58953412 * 10^{21}$$

$$|A \cap C \cap D| = 9.920671339 * 10^{21} - 9.924610658 * 10^{16} - (9.889807005 * 10^{21} + 8.446020983 * 10^{21} + 6.641514961 * 10^{21}) + 8.416346761 * 10^{21} + 6.615524391 * 10^{21} + 4.768323345 * 10^{21}$$

$$= 4.743423641 * 10^{21}$$

$$|B \cap C \cap D| = 4.743423641 * 10^{21}$$

$|A \cap B \cap C \cap D|$

$$= 9.889807005 * 10^{21} + 9.889807005 * 10^{21} + 8.446020983 * 10^{21}$$

$$+\ 6.641514961 * 10^{21}$$

$$-(9.858942971 * 10^{21} + \ 8.416346761 * 10^{21} + \ 6.615524391$$

$$* 10^{21} + \ 8.416346761 * 10^{21} + \ 6.615524391 * 10^{21}$$

$$+\ 4.768323345 * 10^{21})$$

$$+\ (8.386672839 * 10^{21} + 6.58953412 * 10^{21} + \ 4.743423641$$

$$* 10^{21} + \ 4.743423641 * 10^{21}) - \ 9.920671339 \ * \ 10^{21}$$

$$= \ 4.718524236 * 10^{21}\ ways$$

$\boldsymbol{ANS = \ 4.718524236 * 10^{21}\ ways}$

Question 11)

a) **D = Have Covid-19**

P = Tests positive

$P(D) = 0.0001$

$P(P|D) = 0.99$

$P(P|\overline{D}) = 0.002$

$$P(D|P) = \frac{P(D) * P(P|D)}{P(D) * P(P|D) + P(P|\overline{D}) * P(\overline{D})}$$

$$= \frac{0.0001 * 0.99}{0.0001 * 0.99 + 0.002 * 0.9999}$$

$$= 0.04717$$

$= 4.717\%$

b) $P(\bar{P}|D) = 1 - 0.99 = 0.01$

$P(\bar{P}|\bar{D}) = 1 - 0.002 = 0.998$

$$P(D|\bar{P}) = \frac{P(D) * P(\bar{P}|D)}{P(D) * P(\bar{P}|D) + P(\bar{P}|\bar{D}) * P(\bar{D})}$$

$$= \frac{0.0001 * 0.01}{0.0001 * 0.01 + 0.998 * 0.9999}$$

$= \quad 1.0021 * 10^{-6}$

$= \quad 0.0001\%$

Question 12)

a)

$P(send\ 0) = \frac{2}{3}$

$P(send\ 1) = \frac{1}{3}$

$P(received\ 0\ |send\ 1) = \ 0.1$

$P(received\ 0\ |send\ 0) = 0.8$

$P(receive\ 0) = P(correctly\ received\ 0) * P(incorrectly\ received\ 0)$

$= P(received\ 0|\ send\ 0) * P(send\ 0) + P(received\ 0\ |send\ 1)\ *\ P(send\ 1)$

$= \quad (0.8 * \frac{2}{3}) + (0.1 * \frac{1}{3})$

$= \quad 0.567$

$= \quad 56.7\%$

b)

$P(send\ 0\ |received\ 0)$

$$= \frac{P(send\ 0) * P(received\ 0\ |send\ 0)}{P(send\ 0) * P(received\ 0\ |send\ 0) + P(received\ 0\ |\ send\ 1) * P(send\ 1)}$$

$$= \frac{\left(\frac{2}{3}\right) * 0.8}{\left(\frac{2}{3}\right) * 0.8 + 0.1 * \frac{1}{3}}$$

$= 0.941$

$= 94.1\%$