A Micro→Macro Bridge for Self-Interacting Dark Matter: SU(2) Skyrme Solitons with Profile-Locked Predictions

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https://github.com/Voxtrium/GR-DM-Interaction-Theory

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Abstract

We present a minimal, calibration–to–prediction framework for self–interacting dark matter (SIDM) based on an SU(2) Skyrme effective field theory (EFT). A single hedgehog soliton profile is solved once and reused everywhere, fixing three dimensionless "shape constants" that lock (i) the mass–size relation, (ii) the per–mass *transfer* cross–section normalization at low velocity, and (iii) a finite–size suppression scale that governs the decline of σ_T/m with velocity. Using two measurements at dwarf scales—the particle mass m and the low–v anchor $(\sigma_T/m)_0$—we algebraically determine the EFT couplings (K_s, e) and then predict $\sigma_T/m(v)$ from dwarfs to clusters with no per–halo retuning. The prediction includes the s–wave effective–range expansion and a profile–derived transfer factor $C_T(k)$; an optional, discrete internal mode with $m_\phi \sim R_*^{-1}$ further suppresses high–v scattering (not used in baseline). On the renormalization side, in an HLS–matched, background–field scheme, the one–loop divergences renormalize the $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$ operators equally along the $a = 1$ line, making $X = eK_s$ marginal at that order (details omitted here). We also record a thin FRW "bridge" that feeds these micro scales into a conserved continuity system via horizon–entropy source terms; it is fully gateable and not used in baseline. The micro sector is unitary and predictive; the macro source is intentionally minimal and testable (and reduces to ΛCDM when gated off).

1 Microphysics: SU(2) Skyrme soliton (locked profile)

Field and Lagrangian. Let $U(x) \in \mathrm{SU}(2)$ and $L_\mu \equiv U^\dagger \partial_\mu U$. We use the Skyrme normalization

$$\mathcal{L} = \frac{F^2}{16}\,\mathrm{Tr}(L_\mu L^\mu) + \frac{1}{32e^2}\,\mathrm{Tr}[L_\mu, L_\nu]^2\,, \qquad K_s \equiv F/2. \tag{1}$$

For bookkeeping one may equivalently write in O(3) unit vector components H with $|H| = 1$:

$$\mathcal{L} = \frac{K_s^2}{2}(\partial H)^2 + \frac{1}{4e^2}(\partial H \times \partial H)^2, \qquad [K_s] = \mathrm{GeV},\ [e] = 1. \tag{2}$$

Hedgehog and dimensionless radius. Use $U(r) = \cos f(r) + i(\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}})\sin f(r)$ with $f(0) = \pi$, $f(\infty) = 0$ and

$$x \equiv eK_s r. \tag{3}$$

Solving the static EOM by shooting yields a unique regular profile $f(x)$ with near–origin series $f(x) = \pi - ax + bx^3 + \cdots$, $b = a^3/[2(5 + 2a^2)]$ (regularity). A representative refined shooting slope is

$$a_{\mathrm{shoot}} = 2.075000000000. \tag{4}$$

This is the dimensionless shooting parameter that defines the profile solution in our conventions.

Canonical EOM and energy density ("x^2 inside ε"). With the convention $x \equiv eK_s r$, the canonical hedgehog EOM and the dimensionless energy density used for all integrals are

$$(x^2 + 2\sin^2 f)\, f_{xx} + 2x f_x + \sin(2f)\left[f_x^2 - 1 - \frac{\sin^2 f}{x^2} \right] = 0, \tag{5}$$

$$\varepsilon(x) = x^2 f_x^2 + 2\sin^2 f + 2\sin^2 f\, f_x^2 + \frac{\sin^4 f}{x^2}. \tag{6}$$

We use the "x^2 inside ε" convention (this preserves invariants under the usual mapping to "x^2 outside"). The dimensionless energy integral is $E = (K_s/e)\, 4\pi \int_0^\infty \varepsilon(x)\, dx$.

Locked shape constants from the solved profile. Define

$$c_m \equiv 4\pi \int_0^\infty \varepsilon\, dx, \qquad c_R \equiv \sqrt{\frac{\int_0^\infty x^2 \varepsilon\, dx}{\int_0^\infty \varepsilon\, dx}}, \qquad c_\sigma \equiv \frac{\pi c_R^2}{c_m}. \tag{7}$$

From the EOM–consistent refined profile we find

$$c_m = 147.565, \qquad c_R = 1.24220, \qquad c_\sigma = 0.032851. \tag{8}$$

What these fix

The micro observables are

$$m = \frac{c_m K_s}{e}, \qquad R_* = \frac{c_R}{eK_s}, \qquad \left(\frac{\sigma_T}{m}\right)_{\rm nat} = \frac{c_\sigma\, e}{K_s^3}, \qquad X \equiv eK_s. \tag{9}$$

Given a dwarf–scale anchor $\big(m, (\sigma_T/m)_0\big)$, convert $(\sigma_T/m)_0$ to natural units and solve

$$K_s^2 = \frac{c_m c_\sigma}{m\, (\sigma_T/m)_{\rm nat}}, \qquad e = \frac{c_m K_s}{m}. \tag{10}$$

For $m = 6.283$ GeV and $(\sigma_T/m)_0 = 0.10\ \mathrm{cm^2\,g^{-1}}$, the calibration gives

$$K_s = 0.04104958\ \mathrm{GeV}, \quad e = 0.96410663, \quad X = 0.039576176\ \mathrm{GeV}, \tag{11}$$

$$R_* = 31.38757\ \mathrm{GeV^{-1}} \simeq 6.1936 \times 10^{-13}\ \mathrm{cm}, \quad \mu = \frac{m}{2} = 3.1415\ \mathrm{GeV}. \tag{12}$$

A useful profile scale is

$$v_{R_*} \equiv \frac{1}{\mu R_*} = 0.01015\, c \approx 3.04 \times 10^3\ \mathrm{km\,s^{-1}}, \tag{13}$$

which matches the turnover seen in $C_T(k)$ (e.g. $C_T \approx 0.5$ near $v \simeq 3000\ \mathrm{km\,s^{-1}}$ in our calibration run). The low–v scattering length fixed by the anchor is $a = \sqrt{m(\sigma_T/m)_{\rm nat}/(4\pi)} = 1.5130465 \times 10^1\ \mathrm{GeV^{-1}}$ (here a is the physical scattering length).

An optional threshold scale is $m_\phi \sim R_*^{-1} \approx 3.186 \times 10^{-2}$ GeV.

2 Predictive scattering: $\sigma_T/m(v)$ from one curve

Effective range (s–wave) and transfer factor. Let $k = \mu v$ (with v in units of c), and write $k \cot \delta_0(k) = -1/a + (r_e/2)k^2$, where a is the physical scattering length (not to be confused with the near–origin shooting slope a_{shoot}). We set $r_e = \xi R_*$ with $\xi = \mathcal{O}(1)$ (e.g. $\xi = 2/3$). The per–mass transfer cross–section is

$$\frac{\sigma_T}{m}(v) = \left(\frac{\sigma_T}{m}\right)_0 \frac{1}{\left(1 - \frac{1}{2} a r_e k^2\right)^2 + (ak)^2} \times C_T(k), \tag{14}$$

where

$$C_T(k) = \frac{1}{4\pi} \int d\Omega \, (1 - \cos\theta) \, |F_{\text{prof}}(q)|^2, \qquad q = 2k \sin(\theta/2), \tag{15}$$

and the profile–derived form factor is

$$F_{\text{prof}}(q) = \frac{\displaystyle\int_0^\infty dx \; \varepsilon(x) \; j_0\big((q/X)\,x\big)}{\displaystyle\int_0^\infty dx \; \varepsilon(x)}, \qquad j_0(z) = \frac{\sin z}{z}. \tag{16}$$

For a single internal mode, multiply by $S_\phi(q) = 1/(1 + q^2/m_\phi^2)$ with $m_\phi \sim R_*^{-1}$. This discrete option is not used in our baseline predictions.

What is fixed vs. what is chosen. Once $\big(m, (\sigma_T/m)_0\big)$ are set at dwarfs, (K_s, e, R_*) are fixed and the shape of $\sigma_T/m(v)$ follows from the universal profile. The only residual choice is $\xi = \mathcal{O}(1)$. If stronger high–v suppression is demanded by data, m_ϕ is a single, transparent knob.

3 Renormalization hygiene (sketch; details on request)

In an HLS–matched scheme with background–field gauge, one–loop divergences renormalize $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$ equally along $a = 1$, so $X = eK_s$ is marginal at that order; we keep details off this note.

4 Macro bridge (optional; not used here)

We have a thin FRW "bridge" that injects micro scales into continuity equations via a horizon–entropy source. It is fully gateable (set to zero) so the micro result stands alone; details omitted here.

5 What is already done; what is next

Done: (i) locked hedgehog profile and constants (c_m, c_R, c_σ); (ii) algebraic calibration (K_s, e) to $\big(m, (\sigma_T/m)_0\big)$; (iii) $\sigma_T/m(v)$ with effective range + profile transfer $C_T(k)$ and optional m_ϕ (off in baseline); (iv) one–loop equality along $a = 1$ (sketch; details on request).

Next: (1) full phase–shift computation (beyond ER) to fix $(a, r_e, \dots)$ directly from the soliton potential; (2) irreducible two–loop check; (3) astrophysical comparison: dwarfs$\rightarrow$clusters with no per–halo knobs; (4) empirical calibration of the thin FRW bridge if used (else set it to zero and the micro result stands on its own).

6 Predictions and falsifiability

- Velocity trend: one curve anchored at dwarfs predicts a finite–size decline toward clusters; v_{R_*} is fixed by the calibrated R_* and μ.
- Optional threshold: an internal mode with $m_\phi \sim R_*^{-1}$ provides additional high–v suppression if demanded by data (not used in baseline).
- No per–halo retuning: after dwarf anchoring, the same parameters apply to all systems (predictivity).
- RG stability (one loop, sketch): $X = eK_s$ marginal $\Rightarrow$ the calibrated mass/size do not undergo a leading runaway.
- Macro bridge: optional and gateable; not used in baseline.

v (km/s)	1000	2000	3000
σ_T/m (baseline; anchored at 0.10)	~ 0.09	~ 0.07	~ 0.05