DYNAMICAL SYSTEMS

Flow Proof Helps Mathematicians Find Stability in Chaos

By JORDANA CEPELEWICZ

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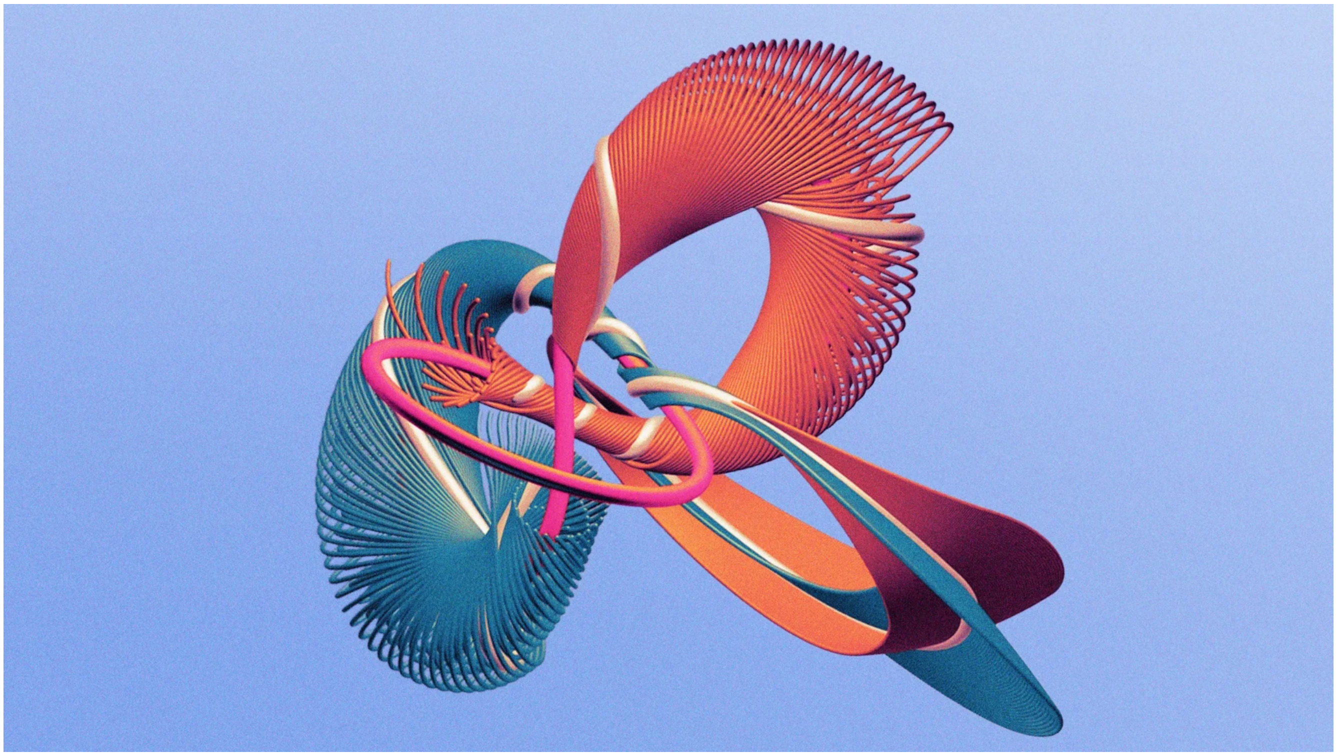
A series of new papers describes how to reconstruct key dynamical systems with relatively little data.



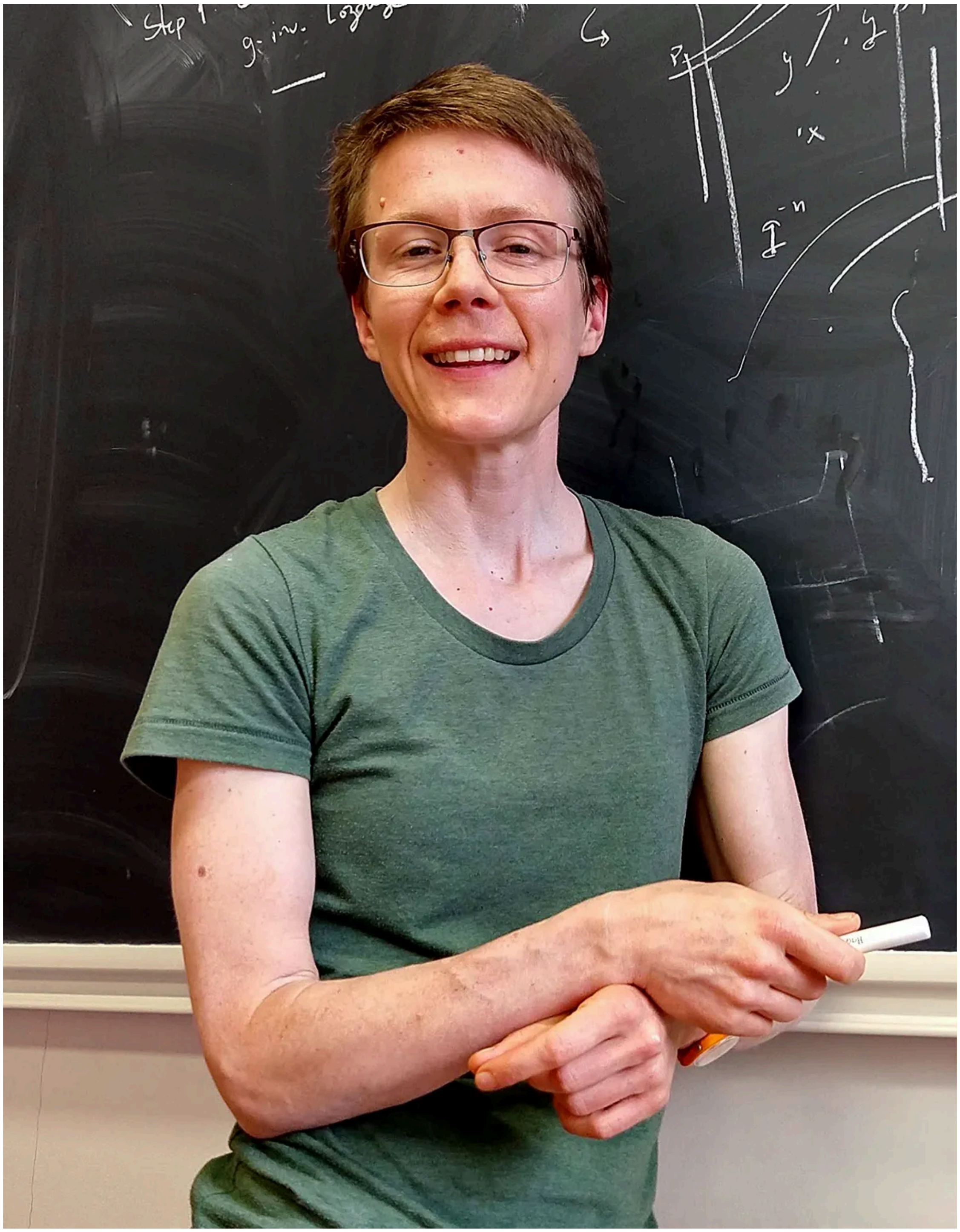
Jos Leys and Étienne Ghys; modified by *Quanta Magazine*

As with so much in mathematics, the proof started with coffee. In September 2019, Kathryn Mann of Cornell University visited Kingston, Ontario, to give a guest lecture at Queen's University. Afterward, she sat down with her host, Thomas Barthelmé, for what was supposed to be a quick cup of coffee. He wanted to get her opinion on a problem he was working on involving mathematical models called dynamical systems, which describe how phenomena as simple as a pendulum's back-and-forth motion or as complex as the weather evolve over time.

Before they knew it, hours had passed. "We were just sitting in this coffee shop, drawing pictures, each of us trying to figure out what the other was trying to say," Mann said. "At the beginning, I was like, this guy makes no sense." But as they learned to speak each other's mathematical languages, both became more optimistic about their chances of finding a solution.

Mann didn't always like mathematics — as a child, she wasn't good at arithmetic — but it was precisely this kind of conversation that ultimately led her to study it. Although initially interested in pursuing a career in philosophy, she realized it wasn't the right fit. For philosophers, "a productive discussion means testing your position against someone else's," she said. "Math is the opposite. You talk to someone, and you're both on the same team from the get-go. If someone's like, 'That doesn't work that way,' you're like, 'Oh, tell me more.' I found that mode of discourse much better."

Barthelmé was interested in particular dynamical systems called Anosov flows, which crop up naturally in many areas of mathematics and act as important toy models. These systems showcase seemingly paradoxical properties all in one place: chaos and stability; rigidity and flexibility; the presence of intrinsic geometric structure amid an underlying topological wilderness.



At first, Kathryn Mann of Cornell University wanted to be a philosopher. But she preferred the way mathematicians share and discuss ideas, and now does research at the intersection of topology, dynamics and group theory.

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Courtesy of Kathryn Mann

Each of those properties arises in dynamical systems that mathematicians have wanted to understand for centuries, from the motion of planets around the sun to the spread of disease through a population. But in such systems, disentangling these different features gets hopelessly complicated. Anosov flows were defined as objects of study in their own right in the 1960s because they exhibit these important behaviors to an extreme degree, making them easier to analyze. "You hope that once you have a perfect picture of this case, you can turn back to the messy real world and approach it with fresh eyes," Mann said.

"In the beginning of the theory of dynamical systems, Anosov flows were like lighthouses telling you the direction where you should go," said Étienne Ghys of the École Normale Supérieure in Lyon, France.

Nevertheless, they have been incredibly difficult to grasp. Though mathematicians have made a lot of progress over the past 60 years, they are still far from achieving the perfect picture Mann spoke of: a classification of all the different kinds of Anosov flows.

Now, in a series of recent papers, Barthelmé and Mann, together with Steven Frankel of Washington University in St. Louis, have taken a striking step toward that elusive goal. By translating questions about motion and shape into the language of algebra, they showed that relatively little data is needed to completely and uniquely determine a given Anosov flow. (Their result also holds for related but more general dynamical systems called pseudo-Anosov flows.) Hidden in all that chaos, they found structure.

The mathematicians have already applied their result to address key questions about Anosov flows — namely, how to construct them, and how many of them there could be. "Sometimes results are important because they really provide a new point of view on the subject," said Rafael Potrie, a mathematician at the University of the Republic in Uruguay. "That's what happens here."

A Geometric Perspective

Since the late 19th century, when Henri Poincaré's work in celestial mechanics jump-started the modern theory of dynamical systems, mathematicians have thought about dynamics through the lens of geometry.

Consider a pendulum. Left alone, it hangs vertically, unmoving. But if you lift it and let go, it will swing back and forth. At a given moment, the state of the pendulum can be captured by two pieces of information: its angle from the vertical hanging position and its velocity. As a result, you can represent all possible states of the pendulum as points on a plane, which is known as the state space.

If you start at any one of those points, a differential equation (based on Newton's laws of motion) tells you how the pendulum's angle and velocity will change over time. This motion is captured by a curve, or "trajectory," winding through the state space. If you change the starting angle and velocity of the pendulum, you'll get a different trajectory through the state space.

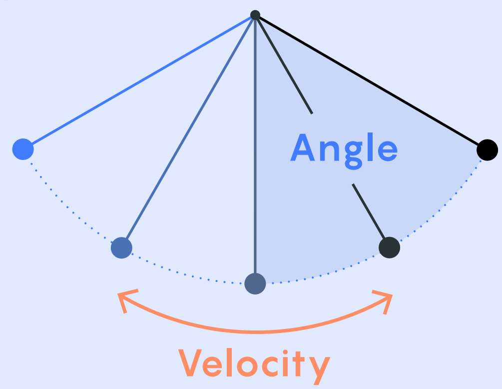
You want to study all such trajectories as a single mathematical object. This geometric way of encoding your dynamical system is called a "flow." Instead of thinking about the pendulum carving out arcs through the air, you can study its behavior by analyzing the flow. In this case, the flow consists of nested ellipses (with each ellipse representing a way the pendulum can oscillate back and forth), as well as curves above and below those ellipses (representing scenarios in which the pendulum spins rapidly like a pinwheel).

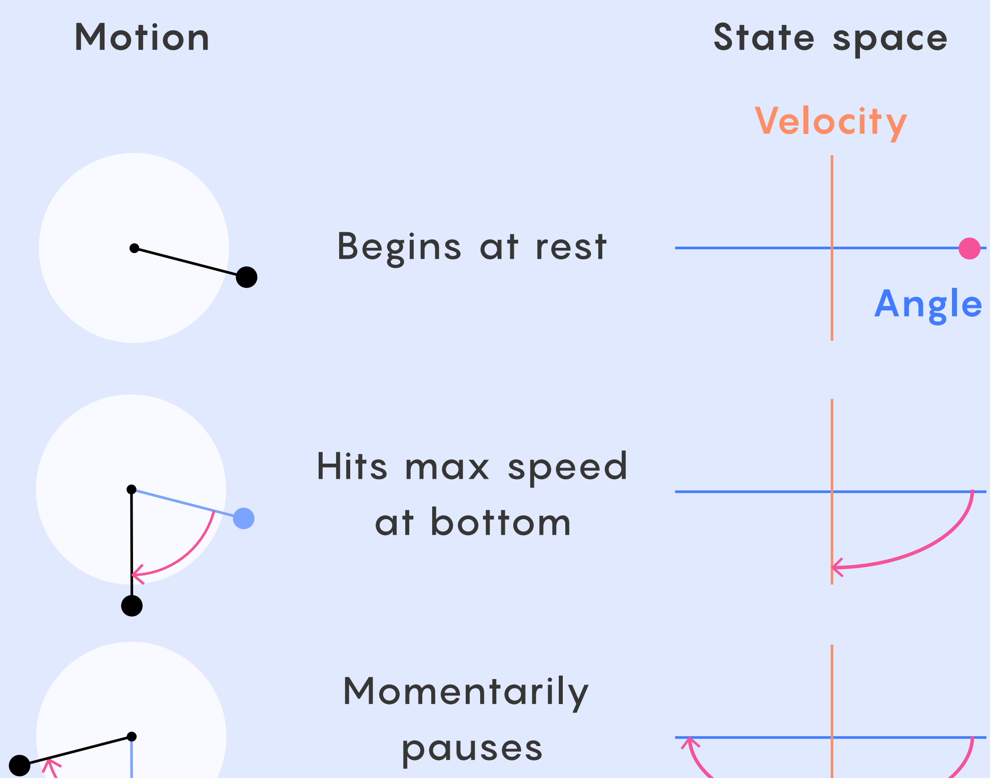
How to Draw a Flow

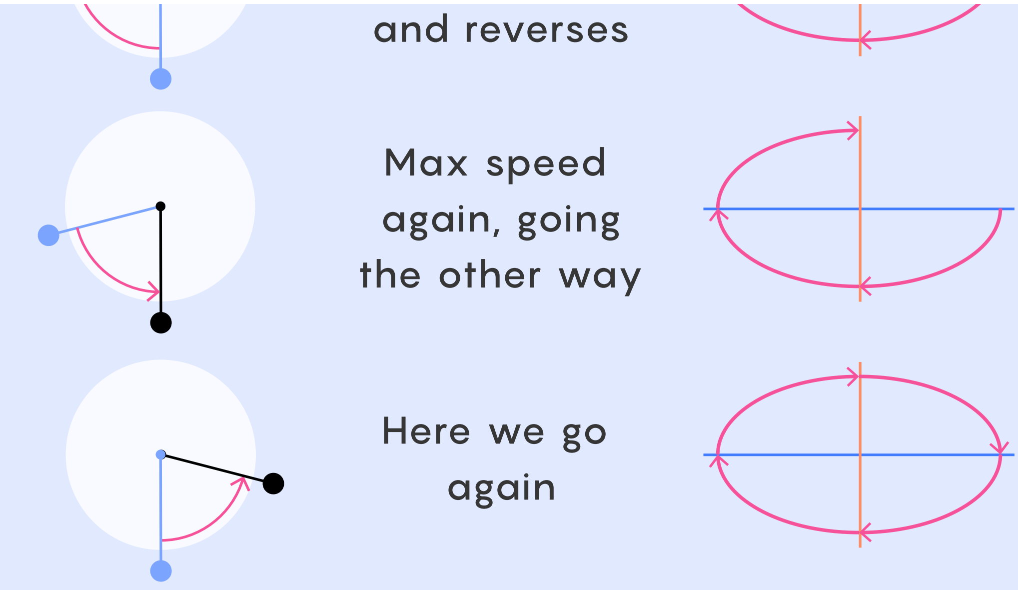
Mathematicians think about dynamical systems (like a pendulum) geometrically in order to understand them.

A SWINGING PENDULUM

A pendulum's motion is defined in terms of angle and velocity. You can map this as a trajectory in a two-dimensional **"state space."**



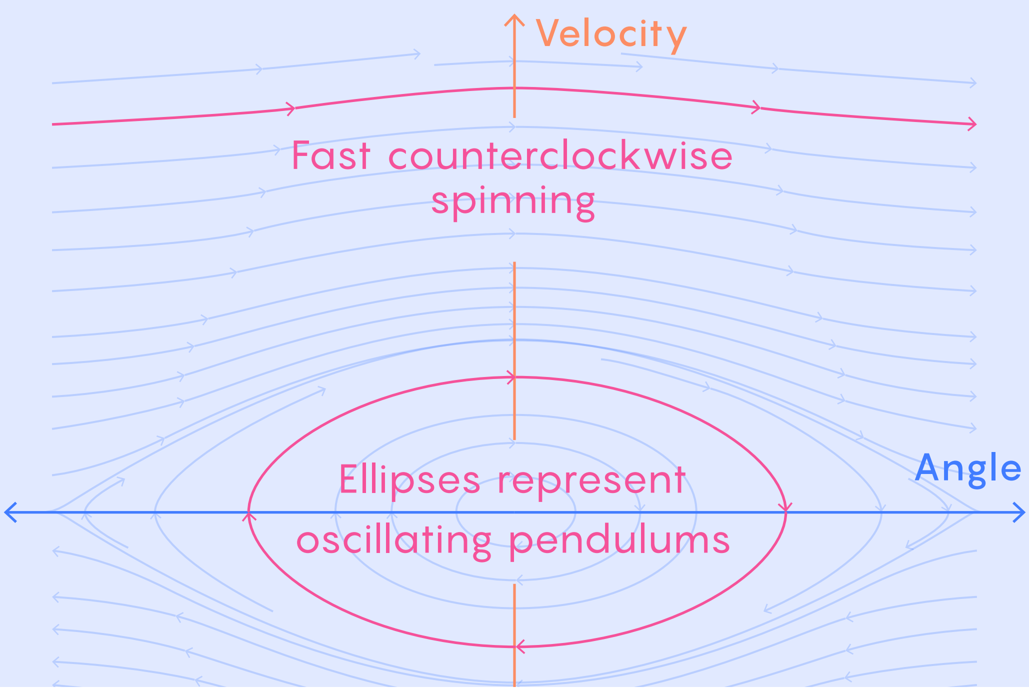


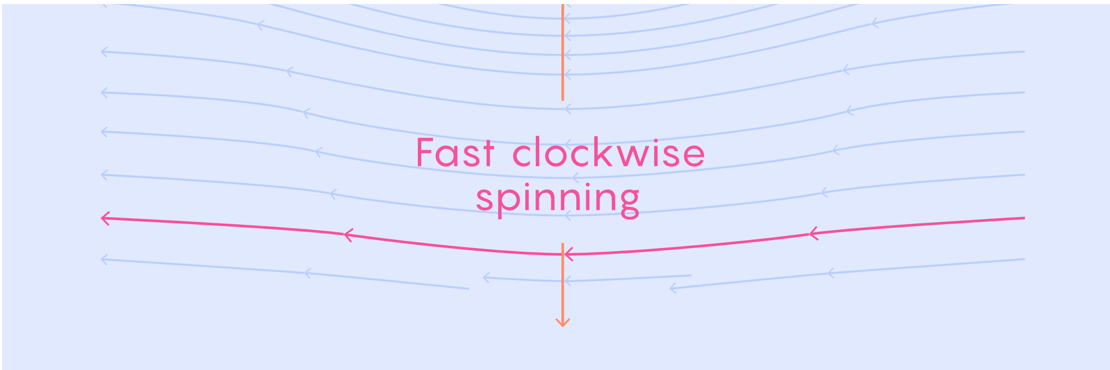


A pendulum without friction would oscillate forever, retracing this trajectory in the state space.

TRAJECTORIES MAP THE FLOW

A “flow” represents all possible motions of the pendulum as infinitely many trajectories.





Merrill Sherman/ *Quanta Magazine*

But flows can get a lot more complicated, involving convoluted, higher-dimensional state spaces. Take 100 particles moving and interacting in space. The flow that captures their behavior is a collection of infinitely many trajectories through a 600-dimensional state space. (To describe just one particle's state, you need six pieces of information: three numbers for its position, and three for velocity. So a description of all 100 particles at once requires 600 numbers.)

To develop tools for studying these more complicated systems, mathematicians needed the right testing grounds — systems simple enough to make sense of, but complicated enough to reflect the properties they were actually interested in.

That's where Anosov flows came into play.

Global Stability, Local Chaos

Even before Poincaré's 19th-century work changed the way dynamical systems were studied, mathematicians were interested in systems where a particle takes the shortest path available: a so-called geodesic. On a plane, particles follow a bunch of straight lines; on the surface of a sphere, they travel along great circles. The topology, or global shape, of the surface affects what these paths look like.

A geodesic flow describes all possible ways a particle can move when not subject to any outside forces. The state space, in general, is four-dimensional: Two of these dimensions correspond to the particle's physical position, while the other two correspond to how fast it's moving. But if you're willing to throw away some information — if, for instance, you don't care how fast the particle is moving but only which way it's facing at any moment in time — you could describe its motion using a three-dimensional state space. Two dimensions describe its position, and the third dimension represents the direction it's facing. That's how mathematicians often think about geodesic flows: as a collection of trajectories through a three-dimensional state space.

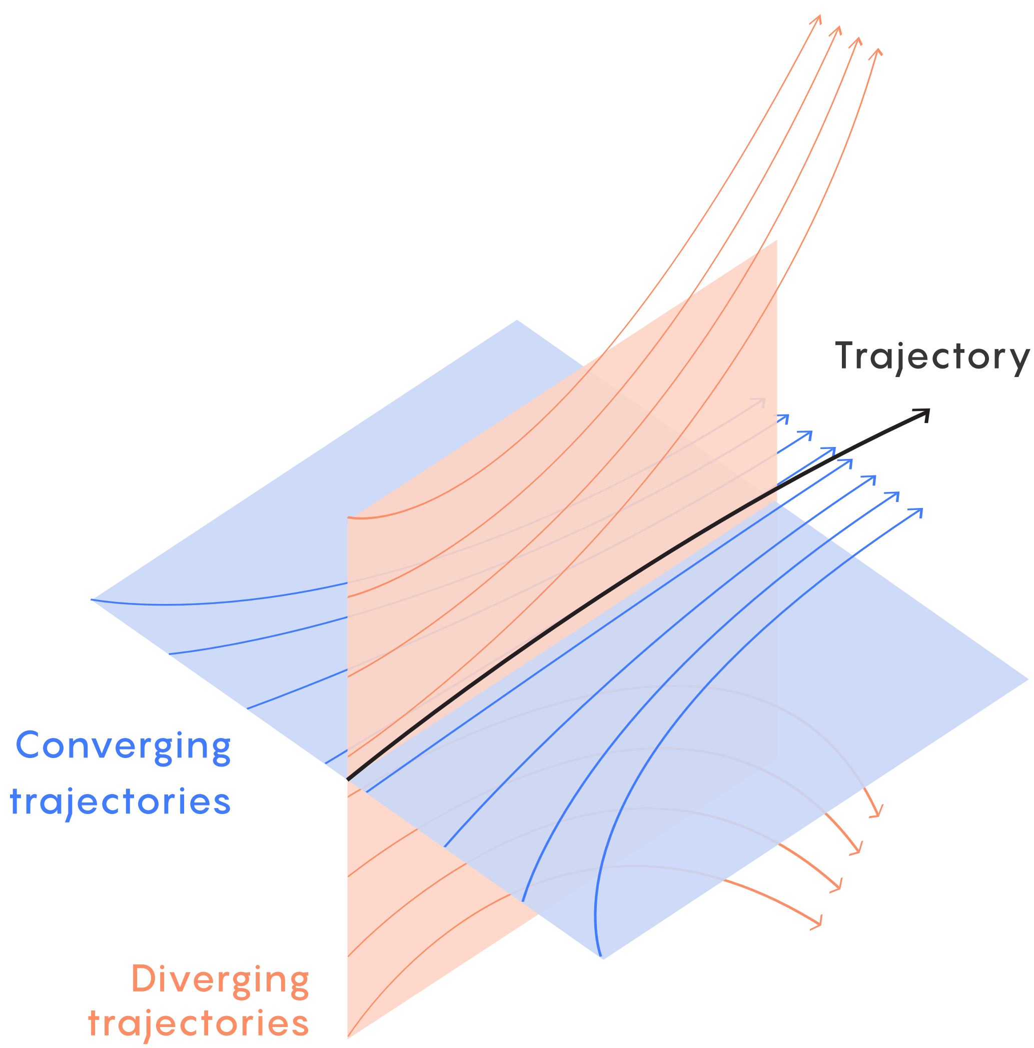
These flows piqued mathematicians' interest for a number of reasons. Since the late 1800s, they have allowed geometers and topologists to better understand the structure of very complicated surfaces — surfaces that are too difficult to study directly, but which become more tractable when you look at how particles move on them. And they're just as crucial for dynamicists, because many mechanical systems in physics can be represented as geodesic flows.

By the dawn of the 20th century, the French mathematician Jacques Hadamard had started to investigate geodesic flows on "negatively curved" surfaces — that is, surfaces that look like a saddle at any given point. (This is perhaps impossible to visualize, but mathematically useful.) He found that these kinds of geodesic flows are always chaotic: If you tweak a particle's initial position by a tiny amount, it will end up on a vastly different trajectory, meaning that you can't predict the system's long-term behavior. (The geodesic flow on the surface of a sphere, on the other hand, does not have this property.)

"In some ways, these systems … are maximally chaotic," said Andy Hammerlindl of Monash University in Australia.

In the 1960s, building on Hadamard's work, the Russian mathematician Dmitri Anosov observed that if you slightly adjust the equation that defines the geodesic flow, all the trajectories shift just a bit: You can wiggle your original flow into the new one without changing its overall structure. Under typical circumstances, such structural stability is by no means guaranteed — but it is for these kinds of geodesic flows. "The tag line is 'global stability, local chaos,'" Mann said. "In dynamics, you're really interested in this confluence of stability and chaos." The two coexist in many dynamical systems, striking a subtle and crucial balance that mathematicians have been trying to disentangle since Poincaré's work on our solar system.

Anosov discovered that both chaos and stability arise automatically in a geodesic flow because its trajectories converge and diverge like lines drawn on a piece of taffy as it is squeezed together and stretched out.



At any given point on an Anosov flow, trajectories converge in one direction (shown in blue) and diverge in the other (shown in red). This converging and diverging behavior occurs at every point in the Anosov flow, giving rise to many other properties of interest.

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Merrill Sherman/*Quanta Magazine;* source: Thomas Barthelmé

Anosov generalized the concept of a geodesic flow on a negatively curved surface by writing down particular mathematical conditions for such taffy-like behavior. These generalized flows now bear his name. Any geodesic flow on a negatively curved surface is one. But Anosov thought there might be a whole litany of dynamical systems that act this way.

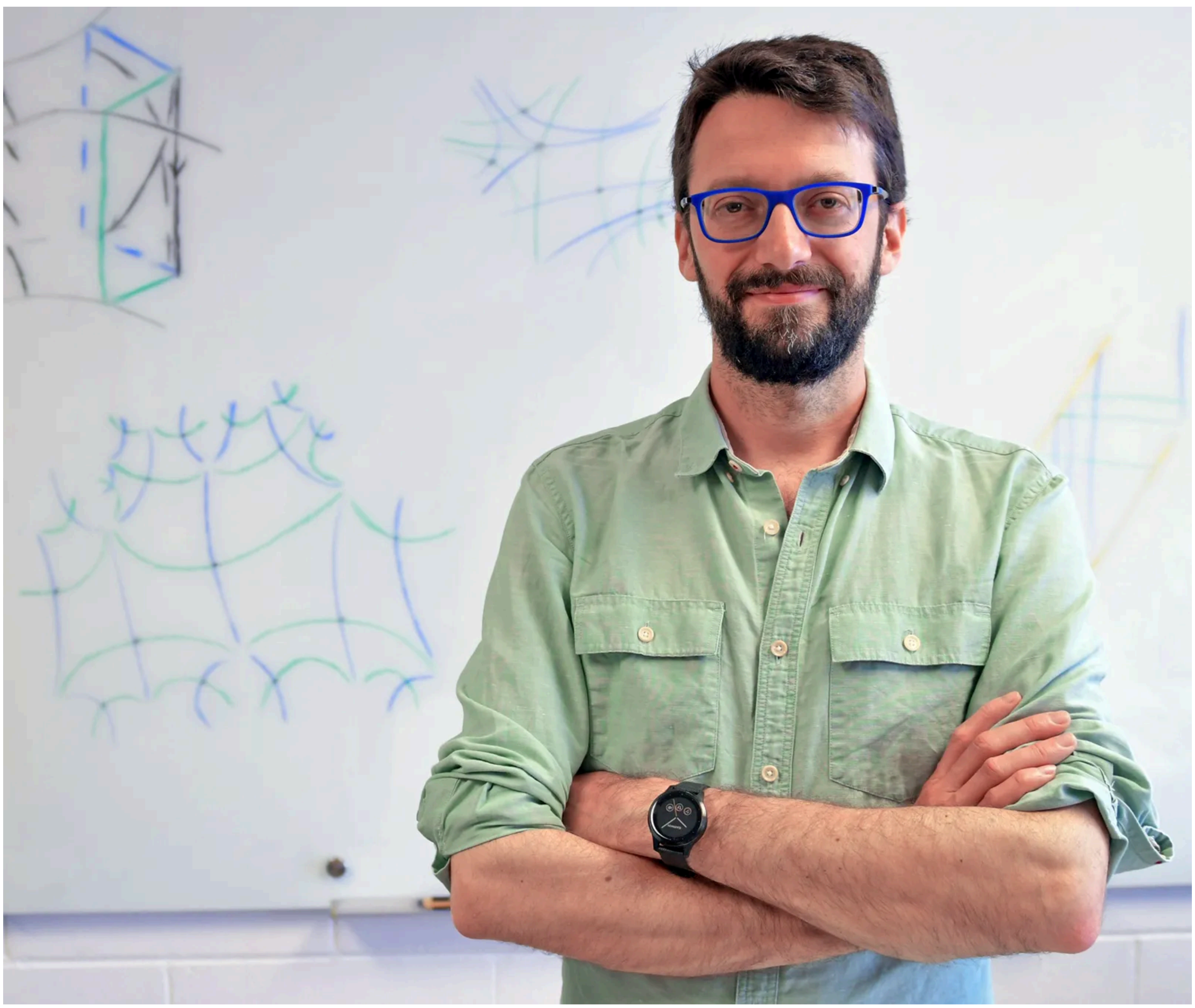
Although this definition might seem oddly specific, such stretching and compressing can be found in many dynamical systems. But it's not as obvious or ubiquitous in those contexts.

In an Anosov flow, the taffy-like behavior appears everywhere — making it an extreme case and therefore a particularly good model system for the development of new tools and insights.

Just as scientists might try to learn about gene expression in a fruit fly before moving on to humans, mathematicians have proved results about topological, statistical and other properties in Anosov flows and then extended that work to other dynamical systems. For example, in the 1970s, mathematicians used what they knew about Anosov flows (and related systems) to formulate a conjecture about what kinds of flows can exhibit structural stability. In the 1990s, Shuhei Hayashi of the University of Tokyo proved that it was true.

Shortcut to Identification

Anosov could only come up with one other family of systems that fit his criteria. But since then, mathematicians have uncovered a sprawling zoo of examples. (Most of these are flows in a three-dimensional state space. Higher-dimensional Anosov flows remain poorly understood.)



When Thomas Barthelmé and his colleagues showed that you can reconstruct an Anosov flow with relatively little data, he was surprised they'd managed to get their proof to work. "I think people would have guessed that it shouldn't be true, because it's fairly weak information," he said.

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Andrew Carroll/Queens University

Mathematicians want to understand Anosov flows completely: to find all examples of them, to analyze their structure, to assess which spaces can support them and which cannot, and to determine how many different flows can live in a given space.

But to do any of that, they first need to get the basics down — which includes finding a good way to characterize a given Anosov flow.

That's what Barthelmé, Frankel and Mann have now done.

Imagine an Anosov flow as a complicated tangle of infinitely many trajectories, which together fill up a three-dimensional state space like yarn. This state space is what's known as a manifold. If you zoom in on any part of it, it will look like regular three-dimensional space, but globally, it can have a very complicated structure, full of holes and other strange features.

Some of the trajectories in the manifold loop back on themselves, representing how a particle might eventually return to its initial state (occupying the same position on a surface and pointed in the same direction). Most trajectories, however, never return to the same point: Instead of forming a closed trajectory, they meander indefinitely, a forever unspooling thread.

The three mathematicians proved that for most Anosov flows (as well as pseudo-Anosov flows), knowing just the closed, or "periodic," trajectories allows you to completely determine the entire system. "You're not losing a lot by going to this simpler thing," Frankel said. "It's actually capturing all the information." There are some exceptions, but in those cases, the trio showed that you need just one additional piece of information to characterize the flow.

The work provides a way to tell whether two different flows are equivalent — that is, whether there is a particular mathematical way to transform each trajectory in one flow into a trajectory in the other. You can't manually check this, trajectory by trajectory; you need a shortcut, a way to identify the flow with less information.

To get this shortcut, Barthelmé, Frankel and Mann turned to a key tool from algebra that topologists often make use of: the fundamental group.

A Translation to Algebra

The fundamental group is effectively a list of loops on the manifold (and all their combinations) that encode information about the manifold's shape. Consider the surface of a doughnut, or torus — a two-dimensional manifold. You can construct one loop by starting at a point on the torus, passing through the hole, and coming back around to where you started. If instead of going through the hole, you go around it, you've formed a second loop. The fundamental group of the torus is the set of these two loops and all of their combinations (for instance, you can imagine circling the hole twice, or passing through the hole once and then circling it, and so on). The fundamental group of a 3D manifold gets more complicated.

Every periodic trajectory in a given Anosov flow corresponds to a class of loops represented in the fundamental group. According to Barthelmé, Frankel and Mann, for most Anosov (and pseudo-Anosov) flows, knowing this subset is enough to allow you to reconstruct the entire flow. You don't even need to know whether certain trajectories intertwine, or how many copies of a given periodic trajectory there are. From the periodic data alone, you can build up your flow — first to get one- and two-dimensional objects that encode various aspects of the flow, and finally to get the three-dimensional flow itself.

Steven Frankel of Washington University in St. Louis was drawn to Anosov flows because they allow him to test out new approaches to complicated dynamical systems.

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Joe Angeles

"This was a surprise to me," Barthelmé said. "I think people would have guessed that it shouldn't be true, because it's fairly weak information."

That's what makes the work so intriguing. "If you pick a trajectory at random, it will densely fill up the space. And periodic trajectories don't do that. They come back to where they started, and don't see most of the space," said Amie Wilkinson of the University of Chicago. And yet if you take those periodic trajectories together, you can understand the full structure of the flow. "That's the beauty of the result."

There are still infinitely many periodic trajectories in an Anosov flow. But that infinite number is "countable" — a smaller infinity than the "uncountable" total number of trajectories. It's similar to how there are infinitely many fractions between the numbers 1 and 2, but many more irrational numbers like $\sqrt{2}$ in that interval. As a result, reducing a flow to just its periodic trajectories can be useful for assessing whether two systems are equivalent.

Barthelmé, Frankel and Mann also found exceptions to their rule: In some cases, it's possible for two flows to be different while having the same periodic trajectories. But these exceptions turned out to have a very particular structure, and the mathematicians were able to determine that only one more piece of information was needed to characterize them.

Since finishing their proof late last year, Barthelmé and Mann have teamed up with Sergio Fenley, a mathematician at Florida State University who has done a lot of research on Anosov flows, to fully characterize these exceptions. In a new paper that they have not yet posted online, they cataloged the situations that give rise to these more complicated flows. In doing so, they not only relied on the previous result, but also ended up constructing new flows that share this one intriguing property: that they can't be fully characterized by their periodic data. "This is amazing," Potrie said. "It's like with astronomy: Sometimes you're studying the orbits of planets, and by understanding them better, you detect that there should be some planet there that you didn't know about. And that tells you to point your telescope so that you see it. I think that this happened in this work."

"It actually gives a very clean way to close up this subject," Fenley said. "There's kind of a bad object that can happen, but the bad object, it turns out, is not so bad."

Counts and Classifications

Several mathematicians, including Potrie and Fenley, have already used Barthelmé, Frankel and Mann's result to complete other proofs about related dynamical systems.

Barthelmé and Mann have also used their work to make headway on one of the biggest open questions about Anosov flows: Is it possible for a given 3D manifold to support an infinite number of different Anosov flows?

Just as the contours of a riverbank affect the possible ways the water in a river can flow, the structure of a manifold affects what sorts of dynamical flows are possible. (Whitewater rapids don't show up in broad, flat planes.)

It was already known that many 3D manifolds cannot serve as a state space for any Anosov flows at all. And a few years ago, Mann and Jonathan Bowden, a mathematician at the University of Regensburg in Germany, proved that for any finite number of your choosing — 15,000 different flows, or 15 million, or 15 billion — you can find a 3D manifold that has at least that many different flows. (Another group showed this independently.)

But it's still not known if you can find a manifold with infinitely many different Anosov flows. Barthelmé and Mann proved a special case by combining their new work with other recent results in an area called contact geometry.

"There are definitely things brewing at this interface," said Boris Hasselblatt of Tufts University. "It's new and exciting."

This will all hopefully help with the long-term goal of classifying Anosov flows (including higher-dimensional ones). But it also provides new directions for research and gives mathematicians a better understanding of the relationship between topology and dynamics. According to Potrie, it will be intriguing to study this periodic data, with its corresponding group structure, in its own right. "There are plenty of questions that open up, just because they defined this object, this extraction," he said. "Now we need to understand it."