

Emergent classical gravity as stress-tensor deformed field theories

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We introduce a framework in which classical gravity arises from stress-tensor deformations of quantum field theories (QFTs). By expressing the deformed partition function as a gravitational path integral, we propose an equivalent description between a seed QFT coupled to gravity and an effective field theory defined on a deformation-induced metric. This approach, which does not rely on holography or extra dimensions, applies to a variety of gravitational models, including Einstein and Palatini gravity. The deformation parameter acts as a coupling to the emergent geometry, linking stress-energy dynamics to spacetime curvature. The framework provides a calculable, dimension-independent method for studying gravitational theories resulting from QFT deformations.

INTRODUCTION

A longstanding question is whether classical gravity can emerge from microscopic quantum degrees of freedom [1]. Several lines of evidence point in this direction. Thermodynamic and entanglement arguments relate Einstein's equations to equilibrium or near-equilibrium properties of quantum matter [2, 3]. Holographic dualities, most notably AdS/CFT [4, 5], provide concrete realizations of spacetime from large- N QFT data, but rely on specific boundary conditions and asymptotics. Non-holographic approaches that reconstruct geometry directly from QFT therefore offer a broader setting [6, 7].

A particularly promising clue comes from deformations generated by the QFT stress tensor. In two dimensions, the $T\bar{T}$ deformation [8, 9] leads to a solvable flow and admits an interpretation as coupling to (nearly) topological gravity or gravitational dressing [10–13] or random geometry [14]. These results show explicitly that the deformation of QFT by stress-tensor operators can be rephrased as a geometric response. Higher-dimensional stress tensor deformations have been proposed in [15–17], and the generic geometric realization has been demonstrated in [18]. However, beyond $d=2$ and outside special kinematic regimes, there is no general, non-holographic mechanism that derives *classical* gravitational field equations from a purely field-theoretic construction. Related thought experiments and recent proposals suggest that matter dynamics alone may trigger flows toward gravitational behavior [19], but a calculable, dimension-independent framework remains to be formulated.

We propose a framework for studying stress-tensor-induced deformations of a seed QFT, where the deformed partition function is expressed as a path integral over metrics weighted by a gravitational action. This formulation leads to non-local stress tensor deformations of the field theory, which can be identified with local deformations of an effective action defined on a de-

formed metric, allowing us to regard the effective field action with the deformation as a corresponding gravitational action. The framework extends naturally to Palatini gravity [20, 21], suggesting that a broad class of classical gravitational theories can emerge from stress-tensor-driven deformations of QFTs.

ASSUMPTIONS AND FRAMEWORK

The framework is based on relating a deformed QFT to a gravitational theory through the path integral. The gravitational partition function is defined as

$$\mathcal{Z}_{\text{grav}}^{(\lambda)} = \int \mathcal{D}\psi \mathcal{D}g e^{-\hat{S}[g,\psi] - S_{\text{grav}}^{(\lambda)}[g]}, \quad (1)$$

where $\hat{S}[g,\psi]$ denotes the seed theory action and ψ collectively denotes the matter fields.

The metric integral is evaluated in the saddle-point approximation, giving

$$\begin{aligned} \mathcal{Z}_{\text{grav}}^{(\lambda)} &= \int \mathcal{D}\psi \sum_{g^*} \mathcal{Z}_{\text{loops}}^{(\lambda)}[g^*] e^{-\hat{S}[g^*,\psi] - S_{\text{grav}}^{(\lambda)}[g^*]} \\ &\sim \int \mathcal{D}\psi \sum_{\hat{\gamma}} \left(\sum_{\alpha} \mathcal{Z}_{\text{loops}}^{(\lambda)}[g_{\alpha}^*] e^{-S_{\alpha}^{(\lambda)}[\hat{\gamma},\psi]} \right), \end{aligned} \quad (2)$$

where $\{g_{\alpha}^*\}$ denotes the set of all saddle points satisfying $\lim_{\lambda \rightarrow 0} g_{\alpha}^* = \hat{\gamma}$, and $\hat{\gamma}$ represents the background metric on which the deformed field theory is defined. The saddle-point metric $g_{\alpha}^*(\lambda)$ satisfies

$$\frac{\delta}{\delta g_{\mu\nu}} [\hat{S}[g,\psi] + S_{\text{grav}}^{(\lambda)}[g]]|_{g=g_{\alpha}^*} = 0. \quad (3)$$

In this letter, the interpretation (2) is examined at the classical level. The classical deformed field theory action is related to the gravitational action via

$$S_{\alpha}^{(\lambda)}[\hat{\gamma},\psi] = \hat{S}[g_{\alpha}^*,\psi] + S_{\text{grav}}^{(\lambda)}[g_{\alpha}^*], \quad (4)$$

This framework incorporates the random geometric interpretation of two-dimensional TT deformation [13, 14]. The corresponding gravitational action is $S_{\text{grav}}^{(\delta\lambda)} = (1/8\delta\lambda) \int d^2x \sqrt{\hat{\gamma}} \epsilon^{\mu\rho} \epsilon^{\nu\sigma} (g_{\mu\nu} - \hat{\gamma}_{\mu\nu})(g_{\rho\sigma} - \hat{\gamma}_{\rho\sigma})$, where γ will eventually emerge as the background metric. By solving the EOM (3), the unique saddle is given by $g_{\mu\nu}^* = \hat{\gamma}_{\mu\nu} + 2\delta\lambda \epsilon_{\mu\rho} \epsilon_{\nu\sigma} \hat{T}^{\rho\sigma}$. Substituting it into (4), the first-order TT deformed action is obtained. In this letter, the aforementioned framework is employed to investigate more general gravitational actions that involve the Ricci curvature and feature a dynamical metric.

Effective field theory action on a deformed metric

Within the framework above, once the seed theory \hat{S} and gravitational action $S_{\text{grav}}^{(\lambda)}$ are specified, the deformed QFT action $S_{\alpha}^{(\lambda)}$ can, in principle, be obtained perturbatively from Eq. (4). However, determining the metric saddle point $g_{\alpha}^*(\lambda)$ nonperturbatively is generally intractable. Thus, it is useful to define an effective deformed action $S_{\text{EFT}}^{(\lambda)}[\gamma^{(\lambda)}, \psi]$ on a *deformed metric* $\gamma^{(\lambda)}$, which incorporates the saddle g_{α}^* . The metric $\gamma^{(\lambda)}$ is taken as a local function of λ and g_{α}^* (or, more generally, a functional of g_{α}^*). Different choices of $S_{\text{EFT}}^{(\lambda)}$ and $\gamma^{(\lambda)}$ yield the same physical predictions when evaluated on g_{α}^* .

A natural choice is $\gamma^{(\lambda)} \equiv g_{\alpha}^*$, in which case

$$\begin{aligned} S_{\text{EFT}}^{(\lambda)}[g_{\alpha}^*, \psi] &= S_{\alpha}^{(\lambda)}[\hat{\gamma}, \psi] \\ &= \hat{S}[g_{\alpha}^*, \psi] + \int d^d x \mathcal{L}_{\text{st}}^{(\lambda)}(g_{\alpha}^{*\mu\nu}, \hat{T}_{\mu\nu}), \end{aligned} \quad (5)$$

where $\mathcal{L}_{\text{st}}^{(\lambda)}$ is a local function of the seed theory stress tensor obtained by substituting the gravitational EOM into $S_{\text{grav}}^{(\lambda)}$ and eliminating curvature terms.

Taking the total derivative of Eq. (4) with respect to λ and using Eq. (3) yields

$$\frac{d}{d\lambda} S_{\text{EFT}}^{(\lambda)}[g_{\alpha}^*, \psi] \equiv \frac{d}{d\lambda} S_{\alpha}^{(\lambda)}[\hat{\gamma}, \psi] = \partial_{\lambda} S_{\text{grav}}^{(\lambda)}[g_{\alpha}^*]. \quad (6)$$

This formulation provides a direct correspondence between a deformed QFT on the background $\hat{\gamma}$ and a classical gravitational theory, without relying on holography or extra dimensions. The deformation parameter λ appears solely in the gravitational action, and the flow equation (6) relates the action's variation to the underlying gravitational dynamics. For each stress tensor deformation, the corresponding gravitational action is given by relation (4).

LINEARIZED EINSTEIN GRAVITY

As a first application of the framework, the construction is implemented in Einstein gravity to make explicit

how a stress-tensor deformation of the seed QFT reproduces linearized gravitational dynamics. The corresponding gravitational saddle and the non-local stress tensor deformation are explicitly derived.

The starting point is the seed theory coupled to the Einstein–Hilbert action,

$$S = \hat{S}[g, \psi] + \frac{l^2}{2\lambda} \int d^d x \sqrt{g} (R - 2\Lambda). \quad (7)$$

The saddle-point equation for the metric yields the Einstein field equations with the field theory stress tensor as source,

$$R_{\mu\nu}^* - \frac{1}{2} R^* g_{\mu\nu}^* + \Lambda g_{\mu\nu}^* = -\lambda l^{-2} \hat{T}_{\mu\nu}^*, \quad (8)$$

where the subscript α of the metric saddle has been omitted for notational simplicity.

Consider the expansion $g = \hat{\gamma} + h$. By employing the standard second-order expansion of the Einstein–Hilbert action [22, 23], and including the gauge-fixing and ghost terms to handle diffeomorphism redundancy [24, 25], the expression for the saddle point h^* in terms of the graviton Green's function is obtained,

$$\begin{aligned} h_{\mu\nu}^*(x) &= -\frac{2\lambda}{l^2} \int d^d y \sqrt{\hat{\gamma}(y)} G_{\mu\nu\rho\sigma}(x, y) \left(\hat{T}^{\rho\sigma}(y) \right. \\ &\quad \left. - \frac{1}{d-2} \hat{T}(y) \hat{\gamma}^{\rho\sigma}(y) \right) + O(\lambda^2), \end{aligned} \quad (9)$$

with G satisfying the equation $[-\hat{\gamma}^{\mu\rho} \hat{\gamma}^{\nu\sigma} \hat{\square} - \hat{R}^{\mu\rho\nu\sigma} - \hat{R}^{\nu\rho\mu\sigma}]_x G_{\rho\sigma\alpha\beta}(x, y) = \delta^{(d)}(x-y) \delta_{\alpha}^{\mu} \delta_{\beta}^{\nu} / \sqrt{\hat{\gamma}}$. Solution (9) is obtained using the perturbation method, which is valid only when $\hat{\gamma}$ satisfies the vacuum Einstein equation. Besides, in the course of our calculation, some total derivative terms involving h have been omitted. This can be realized by introducing suitable boundary terms into the action. The detailed derivations are shown in Supplemental Material A.

Substituting the saddle into the action yields the leading deformation of the QFT,

$$\begin{aligned} S^{(\lambda)} &= \hat{S} + \frac{2\Lambda l^2 \text{Vol}(\mathcal{M})}{\lambda(d-2)} + \frac{\lambda}{2l^2} \int d^d x d^d y \sqrt{\hat{\gamma}(x) \hat{\gamma}(y)} \\ &\quad \times G_{\mu\nu\rho\sigma}(x, y) \hat{T}^{\mu\nu}(x) \left[\hat{T}^{\rho\sigma} - \frac{1}{d-2} \hat{T} \hat{\gamma}^{\rho\sigma} \right](y) \\ &\quad + O(\lambda^2). \end{aligned} \quad (10)$$

Similar non-local deformations were studied in [26]. For a flat background $\hat{\gamma}_{\mu\nu} = \eta_{\mu\nu}$ (with $\Lambda = 0$), the first-order deformation simplifies to

$$S_{[1]}^{(\lambda)} = \frac{\lambda}{2l^2} \int d^d x \hat{T}_{\mu\nu} \frac{1}{-\hat{\square}} \left(\hat{T}^{\mu\nu} - \frac{1}{d-2} \hat{T} \eta^{\mu\nu} \right). \quad (11)$$

Higher-order deformations can be obtained perturbatively, as detailed in Supplemental Material A. On the

fixed background \hat{g} , the stress tensor deformation is complicated and non-local, as the solution to Eq. (8) lacks a compact analytical form. A more practical approach involves separating the metric saddle point from the field theory action and analyzing the effective field theory on the deformed metric.

In the formulation of effective deformed action (5), substituting Eq. (8) into Eq. (7) yields $\mathcal{L}_{\text{st}}^{(\lambda)} = \frac{\sqrt{g^*}}{d-2}(\hat{T}^* + \frac{2l^2\Lambda}{\lambda})$. The corresponding flow equation follows immediately,

$$\partial_\lambda S_{\text{EFT}}^{(\lambda)} = -\frac{2l^2\Lambda \text{Vol}(\mathcal{M})}{(d-2)\lambda^2}. \quad (12)$$

The effective stress tensor is obtained by taking the functional derivative of S_{EFT} with respect to the metric,

$$T_{\text{EFT}}^{(\lambda)\mu\nu} = \hat{T}^{\mu\nu} - \frac{1}{d-2} \left(\hat{T} + \frac{2l^2\Lambda}{\lambda} \right) g^{\mu\nu} - \frac{2}{d-2} \frac{\partial \hat{T}}{\partial g_{\mu\nu}}. \quad (13)$$

which depends on the explicit form of \hat{T} . For a massless free scalar seed, $\hat{\mathcal{L}} = \frac{\sqrt{g}}{2} \nabla^\mu \phi \nabla_\mu \phi$, one has $g_{\mu\nu} \frac{\partial \hat{T}}{\partial g_{\mu\nu}} = \hat{T}$. By taking the trace of Eq. (13), \hat{T} can be rewritten in terms of T_{EFT} . Combining this with Eq. (6) yields the total λ -derivative of the effective field action,

$$\frac{d}{d\lambda} S_{\text{EFT}}^{(\lambda)}[g^*, \psi] = \frac{1}{4\lambda} \int d^d x \sqrt{g^*} \left(T_{\text{EFT}}^{(\lambda)*} + \frac{2(d-4)l^2\Lambda}{(d-2)\lambda} \right). \quad (14)$$

On the deformed metric, the flow equation of the effective field theory action is both compact and local in form. This approach admits a systematic extension to a broad class of gravitational theories.

PALATINI $f(\mathcal{R})$ GRAVITY

In this section, the Palatini $f(\mathcal{R})$ gravity is investigated, corresponding to the field theory deformation induced by the trace of the stress tensor. In the Palatini formalism, the variations of the metric and the connection are independent. The Riemann curvature tensor $\mathcal{R}^\mu{}_{\nu\rho\sigma}$ and the Ricci curvature tensor $\mathcal{R}_{\mu\nu}$ are constructed with the independent connection. The gravitational Lagrangian is formally written as $\mathcal{L}_{\text{grav}}^{(\lambda)} = \frac{\sqrt{g}}{2\lambda} f(\mathcal{R})$, where the Ricci scalar $\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu}$. Varying with respect to $g^{\mu\nu}$ and $\Gamma^\sigma{}_{\mu\nu}$ yields the equations of motion:

$$f'(\mathcal{R}) \mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = -\lambda \hat{T}_{\mu\nu}, \quad (15)$$

$$\bar{\nabla}_\sigma (\sqrt{g} f'(\mathcal{R}) g^{\mu\nu}) = 0.$$

Here $\bar{\nabla}$ denotes the covariant derivative with respect to $\Gamma^\sigma{}_{\mu\nu}$. For a certain f , the trace of the first EOM in (15) provides an algebraic equation between \mathcal{R} and \hat{T} .

The second EOM in (15) can be solved by introducing the following metric [20], $\tilde{g}_{\mu\nu} = (f'(\mathcal{R}))^{\frac{2}{d-2}} g_{\mu\nu}$, and the solution $\Gamma^\sigma{}_{\mu\nu}$ is the Levi-Civita connection of $\tilde{g}_{\mu\nu}$. Subsequently, (15) reduces to Einstein's equation featuring a modified stress tensor [20], which can be solved by applying the perturbation method presented in the previous section. A detailed discussion is provided in Supplementary Material B.

Four-dimensional quadratic Palatini gravity

As an illustrative example, the scenario with four-dimensional spacetime and a quadratic form of the function f is considered,

$$S_{\text{grav}}^{(\lambda)}[g, \Gamma] = \frac{l^2}{2\lambda} \int d^4 x \sqrt{g} \left(-2\Lambda + \mathcal{R} + \alpha l^2 \mathcal{R}^2 \right). \quad (16)$$

With $\Lambda = 0$ and $\alpha = \kappa/(2l^2)$, the action reduces to the Starobinsky model [27]. Taking the trace of Eq. (15) and substituting it into the action yields the effective deformed action:

$$S_{\text{EFT}}^{(\lambda)}[g^*, \psi] = \hat{S}[g^*, \psi] + \int d^4 x \sqrt{g^*} \left[\frac{\alpha\lambda}{2} (\hat{T}^*)^2 + \frac{1+8\alpha l^2\Lambda}{2} \left(\hat{T}^* + \frac{2\Lambda}{\lambda l^{-2}} \right) \right]. \quad (17)$$

The effective stress tensor T_{EFT} depends on the explicit form of \hat{T} . For concreteness, we take a massless scalar seed with $\hat{T}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2$. Varying Eq. (17) with respect to $g_{\mu\nu}$ and inverting expresses \hat{T} in terms of T_{EFT} , from which the flow equation follows:

$$\partial_\lambda S_{\text{EFT}}^{(\lambda)} = \int d^4 x \sqrt{g} \left[-\frac{l^2\Lambda(1+8\alpha l^2\Lambda)}{\lambda^2} + \frac{1}{2\alpha} \left(\frac{1+12\alpha l^2\Lambda - \sqrt{(1+4\alpha l^2\Lambda)^2 - 4\alpha\lambda T_{\text{EFT}}^{(\lambda)}}}{4\lambda} \right)^2 \right]. \quad (18)$$

The total derivative of the effective deformed action is given by

$$\frac{d}{d\lambda} S_{\text{EFT}}^{(\lambda)}[g^*, \psi] = \frac{1}{8\lambda} \int d^4 x \sqrt{g^*} \left[T_{\text{EFT}}^{(\lambda)} + \frac{1+8\alpha l^2\Lambda(1-2\alpha l^2\Lambda)}{2\alpha\lambda} - \frac{(1+4\alpha l^2\Lambda) \sqrt{(1+4\alpha l^2\Lambda)^2 - 4\alpha\lambda T_{\text{EFT}}^{(\lambda)}}}{2\alpha\lambda} \right]. \quad (19)$$

EMERGENT NON-MINIMAL GRAVITY

We proceed to examine a more general action, in which the metric, curvature, and matter fields exhibit non-minimal coupling within the Lagrangian. Formally, this

action [28] can be written as

$$S_{\text{grav}}^{(\lambda)}[g, \psi] = \int d^d x \sqrt{g} \mathcal{A}^{(\lambda)}(g^{\mu\nu}, \mathcal{R}_{\mu\nu}, X_{\mu\nu}^{(i)}, \phi^{(i)}), \quad (20)$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci curvature tensor in the Palatini formalism, and the tensor $X_{\mu\nu}^{(i)} = \nabla_\mu \phi^{(i)} \nabla_\nu \phi^{(i)}$. Variations of (20) with respect to the metric and the independent connection yield

$$\begin{aligned} \frac{\partial \mathcal{A}^{(\lambda)}}{\partial g^{\mu\nu}} - \frac{1}{2} \mathcal{A}^{(\lambda)} g_{\mu\nu} &= 0, \\ \bar{\nabla}_\sigma \left(\sqrt{g} \frac{\partial \mathcal{A}^{(\lambda)}}{\partial \mathcal{R}_{\mu\nu}} \right) &= 0. \end{aligned} \quad (21)$$

Substituting the first equation into the action and eliminating $\mathcal{R}_{\mu\nu}$ yields

$$\begin{aligned} S_{\text{grav}}^{(\lambda)}[g^*, \psi] &= \int d^d x \sqrt{g} \mathcal{A}^{(\lambda)}(g^{\mu\nu}, \mathcal{R}_{\mu\nu}, X_{\mu\nu}^{(i)}, \phi^{(i)}) \Big|_{g=g^*} \\ &= \int d^d x \sqrt{g^*} \mathcal{B}^{(\lambda)}(g^{*\mu\nu}, X_{\mu\nu}^{(i)}, \phi^{(i)}). \end{aligned} \quad (22)$$

Following the prescription in the previous section, the effective deformed action is defined by $S_{\text{EFT}}^{(\lambda)}[g, \psi] = \int d^d x \sqrt{g} \mathcal{B}^{(\lambda)}(g^{\mu\nu}, X_{\mu\nu}^{(i)}, \phi^{(i)})$. The flow equation is

$$\partial_\lambda S_{\text{EFT}}^{(\lambda)} = \int d^d x \sqrt{g} \partial_\lambda \mathcal{B}^{(\lambda)}. \quad (23)$$

Assuming that the theory contains only a single tensor field defined by $X_{\mu\nu} = \sum_{i,j} G_{ij}(\phi) \nabla_\mu \phi^{(i)} \nabla_\nu \phi^{(j)}$. There exist d independent invariants constructed from $X_{\mu\nu}$, which can be expressed as $X_n = X_{\mu_1}^{\mu_n} X_{\mu_2}^{\mu_1} \dots X_{\mu_n}^{\mu_{n-1}} = \text{tr}(X^n)$, for $n = 1, 2, \dots, d$. The Lagrangian $\mathcal{B}^{(\lambda)}$ is a local function of X_n and ϕ . Thus, the effective stress tensor derived from (22) takes the form

$$(T_{\text{EFT}}^{(\lambda)})^\mu_\nu = 2 \sum_{n=1}^d n (X^n)^\mu_\nu \partial_{X_n} \mathcal{B}^{(\lambda)} - \mathcal{B}^{(\lambda)} \delta^\mu_\nu. \quad (24)$$

Suppose that X^μ_ν is diagonalizable, expressed as $X = U \text{diag}(\chi_1, \chi_2, \dots, \chi_d) U^{-1}$. According to (24), the effective stress tensor can be diagonalized by the same matrix U , and its eigenvalues $\{\tau_j\}$ are given by

$$\tau_j^{(\lambda)} = 2\chi_j \partial_{\chi_j} \mathcal{B}^{(\lambda)} - \mathcal{B}^{(\lambda)}, \quad \text{for } j = 1, 2, \dots, d. \quad (25)$$

For a given $\mathcal{B}^{(\lambda)}$, Eq. (25) can be inverted to express χ_j in terms of τ_j ; inserting this into Eq. (23) yields a flow equation written entirely in the effective stress tensor.

Example: Generalized Nambu-Goto action

A notable example is the d -dimensional generalized Nambu-Goto action for a self-interacting scalar field,

$$S_{\text{EFT}}^{(\lambda)} = \int d^d x \sqrt{g} \left[\frac{1 - 2\lambda V - \sqrt{1 - 2\lambda(1 - \lambda V)} \nabla^\mu \phi \nabla_\mu \phi}{\lambda(1 - \lambda V)} \right], \quad (26)$$

where $V = V(\phi)$. Such an effective deformed action satisfies the flow equation [18, 29]

$$\begin{aligned} \partial_\lambda S_{\text{EFT}}^{(\lambda)} &= \int d^d x \sqrt{g} \left[\frac{1}{2d} \text{tr}(T_{\text{EFT}}^2) - \frac{1}{d^2} (\text{tr} T_{\text{EFT}})^2 \right. \\ &\quad \left. - \frac{d-2}{2d^{3/2} \sqrt{d-1}} \text{tr} T_{\text{EFT}} \sqrt{\text{tr}(T_{\text{EFT}}^2) - \frac{1}{d} (\text{tr} T_{\text{EFT}})^2} \right]. \end{aligned} \quad (27)$$

From the perspective of gravity, we propose the following action in the Palatini formalism (for $d \geq 3$),

$$\begin{aligned} \mathcal{A}^{(\lambda)} &= \frac{2(d-1)(1-\lambda V) \nabla^\mu \phi \nabla_\mu \phi}{f(V)} - \frac{(d-2)(1-2\lambda V)}{2\lambda(1-\lambda V)} \\ &\quad + \frac{f(V)}{4\lambda(d-1)} \left(\frac{(d-1)(d-2)}{(1-\lambda V)^2} + \frac{l^2 \mathcal{R}}{(d-2)^2} - \frac{2l\sqrt{\mathcal{R}}}{1-\lambda V} \right), \end{aligned} \quad (28)$$

where the function $f(V) = d(1-\lambda V)(1-2\lambda V) - \sqrt{(1-\lambda V)^2(4-4d+d^2(1-2\lambda V)^2)}$. By utilizing the first EOM in (21), the curvature terms present in (28) are eliminated, thereby recovering the generalized Nambu-Goto action (26). In particular, when $V(\phi) = 0$, the gravitational action (28) reduces to the minimal coupling between a free scalar field and the Palatini $f(\mathcal{R})$ -gravity,

$$\mathcal{A}^{(\lambda)} = (d-1) \nabla^\mu \phi \nabla_\mu \phi + \frac{1}{\lambda(d-1)} \left(\frac{l^2 \mathcal{R}}{2(d-2)^2} - l\sqrt{\mathcal{R}} \right). \quad (29)$$

The gravitational saddle satisfies the Einstein equation with a modified stress tensor, solvable via the perturbation method in Supplemental Material A. More generally, by extending the form of the effective deformed action (26), the gravitational action $\mathcal{A}^{(\lambda)}$ reduces to Einstein or quadratic Palatini action under specific conditions, as detailed in Supplemental Material C.

Example: $T\bar{T}$ -like deformation in d dimensions

As a significant example, the d -dimensional (root-) $T\bar{T}$ -like deformation introduced in [18] is investigated, and the corresponding gravitational action is constructed. The effective deformed action is presented as follows:

$$\mathcal{B}^{(\lambda)} = \mathcal{B}_0 + \lambda^{1-\Sigma} l^\Delta \prod_{j=1}^d (\chi_j^{\frac{p_j}{2}} - \beta_j^{\frac{p_j}{2}})^{\frac{1}{p_j}}, \quad (30)$$

where $\{\beta_j\}$ is the deformation parameters of the root- $T\bar{T}$ -like operator as discussed in [18], $\{p_j\}$ are the numbers that characterizing the deformation, $\Sigma = \sum_{j=1}^d 1/p_j$, and $\Delta = (2\Sigma + d - 4)d/2$. \mathcal{B}_0 satisfies the differential equation $2\chi_j \partial_{\chi_j} \mathcal{B}_0 - \mathcal{B}_0 = 0$. The solution takes the form

$$\mathcal{B}_0 = C(\phi) \prod_{j=1}^d \chi_j^{\frac{1}{2}} = C(\phi) \frac{\sqrt{\det(X_{\mu\nu})}}{\sqrt{\det(g_{\mu\nu})}}, \quad (31)$$

which is analogous to the action of the Nambu–Goto string [30]. The eigenvalues of the effective stress-energy tensor $\{\tau_j^{(\lambda)}\}$ are obtained by plugging (30) into (25). These eigenvalues can then be substituted into (23) to derive the corresponding flow equation,

$$\partial_\lambda S_{\text{EFT}}^{(\lambda)} = (1 - \Sigma) l^{\frac{\Delta}{1-\Sigma}} b^{\frac{1}{2(1-\Sigma)}} \int d^d x \sqrt{g} \left(\prod_{j=1}^d (\tau_j^{(\lambda)})^{\frac{1}{p_j}} \right)^{\frac{1}{\Sigma-1}}, \quad (32)$$

where $b = \prod_{j=1}^d \beta_j$. In particular, by setting $p_1 = p_2 = \dots = p_d = p$, the deformation operator reduces to the determinant of the effective stress tensor, $\mathcal{O}_{\text{st}} = (\det[(T_{\text{EFT}}^{(\lambda)})^\mu_\nu])^{\frac{1}{d-p}}$ [14, 16].

Next, a class of gravitational actions is constructed that reproduces the effective deformed action given in (30). For simplicity, let the Lagrangian $\mathcal{A}^{(\lambda)}$ to be a local function of the invariants X_n and those built from the Ricci tensor, $\mathcal{R}_n = \text{tr}(\mathcal{R}^n)$. Suppose that \mathcal{R}_ν^μ is diagonalizable and can be expressed as $\mathcal{R} = \tilde{U} \text{diag}(\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_d) \tilde{U}^{-1}$. Under these assumptions, the first EOM in (21) reduces to

$$U^{-1} \tilde{U} \text{diag}(2\tilde{r}_1 \partial_{\tilde{r}_1} \mathcal{A}^{(\lambda)}, \dots, 2\tilde{r}_d \partial_{\tilde{r}_d} \mathcal{A}^{(\lambda)}) (U^{-1} \tilde{U})^{-1} = \text{diag}(\mathcal{A}^{(\lambda)} - 2\chi_1 \partial_{\chi_1} \mathcal{A}^{(\lambda)}, \dots, \mathcal{A}^{(\lambda)} - 2\chi_d \partial_{\chi_d} \mathcal{A}^{(\lambda)}). \quad (33)$$

The two matrices have identical diagonal entries up to a permutation. Relabel the eigenvalues of \mathcal{R}_ν^μ as $\{r_j\}$, which satisfy $2r_j \partial_{r_j} \mathcal{A}^{(\lambda)} = \mathcal{A}^{(\lambda)} - 2\chi_j \partial_{\chi_j} \mathcal{A}^{(\lambda)}$. The explicit form of $\mathcal{A}^{(\lambda)}$ is introduced as follows:

$$\mathcal{A}^{(\lambda)} = \mathcal{B}_0 + \lambda^{1-\Sigma} l^\Delta \prod_{j=1}^d \left(\chi_j^{\frac{p_j}{2}} - (\mathbf{p}_j \chi_j^{\mathbf{q}_j} + \mathbf{s}_j r_j^{\mathbf{q}_j} + \frac{\mathbf{q}_j - 1}{\mathbf{q}_j} \beta_j) \chi_j^{\frac{p_j}{2}} \right)^{\frac{1}{p_j}}, \quad (34)$$

where \mathbf{p}_j , \mathbf{q}_j , and \mathbf{s}_j are arbitrary functions independent of χ and r . The EOM of the Ricci curvature tensor is

$$r_j^{\mathbf{q}_j} = -\frac{\mathbf{p}_j}{\mathbf{s}_j} \chi_j^{\mathbf{q}_j} + \frac{\beta_j}{\mathbf{q}_j \mathbf{s}_j}, \quad (\mathbf{q}_j \neq 1). \quad (35)$$

Substituting it into Eq. (34) recovers the effective deformed action in Eq. (30). As a two-dimensional example, the gravitational action is explicitly formulated as

$$\begin{aligned} S_{\text{grav}}^{(\lambda)}[g, \Gamma, \psi] &= \int d^2 x \sqrt{g} \left\{ \mathcal{B}_0 + \frac{1}{\lambda l^3} \prod_{\theta=\pm} \left[\sqrt{\frac{l}{2}} (X + \theta \sqrt{2X_\nu^\mu X_\mu^\nu - X^2}) \right. \right. \\ &\quad \left. \left. - \sqrt{M^2 l + \sqrt{\mathcal{R} + \theta \sqrt{2\mathcal{R}_\nu^\mu \mathcal{R}_\mu^\nu - \mathcal{R}^2} + \sqrt{X + \theta \sqrt{2X_\nu^\mu X_\mu^\nu - X^2}}} \right] \right\}. \end{aligned} \quad (36)$$

By setting $p_1 = p_2 = 1$, $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{s}_1 = \mathbf{s}_2 = l^{-1}$, $\mathbf{q}_1 = \mathbf{q}_2 = \frac{1}{2}$, and $\beta_1 = \beta_2 = -M^2/\sqrt{2}$, the action (34)

can be identified with (36). By solving the EOM (21), the specific form of the Ricci curvature tensor is obtained,

$$\begin{aligned} \mathcal{R}_\nu^\mu &= (1 + \frac{2\sqrt{2}M^2 l}{\sqrt{\chi_+} + \sqrt{\chi_-}}) X_\nu^\mu + \sqrt{2}M^2 l (\sqrt{2}M^2 l + \sqrt{\chi_+} \\ &\quad + \sqrt{\chi_-} - \frac{X}{\sqrt{\chi_+} + \sqrt{\chi_-}}) \delta_\nu^\mu, \end{aligned} \quad (37)$$

where $\chi_\pm = \frac{1}{2}(X \pm \sqrt{2X_\nu^\mu X_\mu^\nu - X^2})$. By plugging it into the gravitational action and using the definition of the flow equation (23), we have

$$\partial_\lambda S_{\text{EFT}}^{(\lambda)} = \frac{\sqrt{2}}{2M^2 l^2} \int d^2 x \sqrt{g} \left[(T_{\text{EFT}}^{(\lambda)})^\mu_\nu (T_{\text{EFT}}^{(\lambda)})^\nu_\mu - (T_{\text{EFT}}^{(\lambda)})^2 \right], \quad (38)$$

which corresponds to the flow equation of the $T\bar{T}$ deformation [8–10, 19].

CONCLUSION AND OUTLOOK

We present a framework where classical gravity emerges from stress-tensor deformations of quantum field theories. By rewriting the deformed partition function as a gravitational path integral, we establish an equivalence between a seed theory coupled to classical gravity and an effective field theory on a deformation-induced metric. This is different from the metric approach [17, 31], which relates $T\bar{T}$ -like deformations coupled to Einstein gravity to undeformed matter coupled to Ricci-based gravity.

This framework applies to a broad class of gravitational models, including Einstein and Palatini gravity, with the deformation parameter acting as a coupling to the emergent geometry. The method is non-holographic, calculable, and provides a direct link between generic stress-energy deformations and classical gravitational action, without relying on extra dimensions or specific background geometries.

Several directions for future research remain. Quantum corrections should be incorporated to explore the validity of the deformation–gravity correspondence beyond the classical approximation. Extending the framework to include matter with spin, nontrivial topology, or finite temperature would enhance its applicability. Additionally, investigating whether similar deformations can generate other gravitational theories, such as massive or non-local gravity, would test the universality of the approach. A promising direction is to extract holographic properties from the current framework, as recently studied in [32], where gravity emerges as an effective phenomenon from QFT deformed by the stress tensor in the context of holographic renormalization group flow.

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[28] Here we assume that the action does not include the Riemann curvature tensor $\mathcal{R}_{\nu\rho\sigma}^{\mu}$. In principle, $\mathcal{A}^{(\mu)}$ should be a local function of all independent invariants composed of $g^{\mu\nu}$, $\mathcal{R}_{\mu\nu}$, and $X_{\mu\nu}^{(i)}$.
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SUPPLEMENTAL MATERIAL

A. Exact deformed action from perturbation method

To determine the exact form of the deformed theory, we compute the metric saddle point g^* and express the deformed action in terms of the background metric $\hat{\gamma}$. Consider the metric perturbation $g = \hat{\gamma} + h$. By employing the techniques in [22, 23], we can expand the Einstein-Hilbert action to the second order of h ,

$$S_{\text{EH}} = l^2 \int d^d x \sqrt{\hat{\gamma}} \left[\hat{R} - 2\Lambda + \left(R^{(1)} + \frac{1}{2}(\hat{R} - 2\Lambda)\hat{\gamma}^{\mu\nu}h_{\mu\nu} \right) + \left(R^{(2)} + \frac{1}{2}R^{(1)}\hat{\gamma}^{\mu\nu}h_{\mu\nu} + \frac{1}{8}(\hat{R} - 2\Lambda)h_{\mu\nu}(\hat{\gamma}^{\mu\nu}\hat{\gamma}^{\rho\sigma} - 2\hat{\gamma}^{\mu\rho}\hat{\gamma}^{\nu\sigma})h_{\rho\sigma} \right) + O(h^3) \right]. \quad (\text{A1})$$

The perturbations of the Ricci scalar curvature are

$$\begin{aligned} R^{(1)} &= \hat{\nabla}^\mu \hat{\nabla}^\nu h_{\mu\nu} - \hat{\square}(\hat{\gamma}^{\mu\nu}h_{\mu\nu}) - \hat{R}^{\mu\nu}h_{\mu\nu}, \\ R^{(2)} &= \hat{R}^{\mu\rho}\hat{\gamma}^{\nu\sigma}h_{\mu\nu}h_{\rho\sigma} - \frac{1}{2}h_{\mu\nu}\hat{\gamma}^{\nu\sigma}\hat{\nabla}^\rho\hat{\nabla}^\mu h_{\rho\sigma} + \frac{1}{4}h_{\mu\nu}\hat{\gamma}^{\mu\rho}\hat{\gamma}^{\nu\sigma}\hat{\square}h_{\rho\sigma} \\ &\quad + \frac{1}{4}h_{\mu\nu}\hat{\gamma}^{\mu\nu}\hat{\gamma}^{\rho\sigma}\hat{\square}h_{\rho\sigma} + (\text{total derivatives}). \end{aligned} \quad (\text{A2})$$

It follows that

$$\begin{aligned} S_{\text{EH}} &= l^2 \int d^d x \sqrt{\hat{\gamma}} \left[\hat{R} - 2\Lambda + \left(\frac{1}{2}(\hat{R} - 2\Lambda)\hat{\gamma}^{\mu\nu}h_{\mu\nu} - \hat{R}^{\mu\nu}h_{\mu\nu} \right) \right. \\ &\quad + \left(\frac{1}{4}h_{\mu\nu}(\hat{\gamma}^{\mu\rho}\hat{\gamma}^{\nu\sigma} - \hat{\gamma}^{\mu\nu}\hat{\gamma}^{\rho\sigma})\hat{\square}h_{\rho\sigma} + \frac{1}{2}h_{\mu\nu}\hat{\gamma}^{\rho\sigma}\hat{\nabla}^\mu\hat{\nabla}^\nu h_{\rho\sigma} - \frac{1}{2}h_{\mu\nu}\hat{\gamma}^{\mu\rho}\hat{\nabla}^\sigma\hat{\nabla}^\nu h_{\rho\sigma} \right. \\ &\quad \left. \left. + h_{\mu\nu}[\hat{R}^{\mu\rho}\hat{\gamma}^{\nu\sigma} - \frac{1}{2}\hat{R}^{\mu\nu}\hat{\gamma}^{\rho\sigma} - \frac{1}{8}(\hat{R} - 2\Lambda)(2\hat{\gamma}^{\mu\rho}\hat{\gamma}^{\nu\sigma} - \hat{\gamma}^{\mu\nu}\hat{\gamma}^{\rho\sigma})]h_{\rho\sigma} \right) + O(h^3) \right]. \end{aligned} \quad (\text{A3})$$

Taking variation of this expansion with respect to $h_{\mu\nu}$, we find the EOM for auxiliary field,

$$\begin{aligned} &\left[\frac{1}{2}(\hat{\gamma}^{\mu\rho}\hat{\gamma}^{\nu\sigma} - \hat{\gamma}^{\mu\nu}\hat{\gamma}^{\rho\sigma})\hat{\square} + \frac{1}{2}(\hat{\gamma}^{\rho\sigma}\hat{\nabla}^\mu\hat{\nabla}^\nu + \hat{\gamma}^{\mu\nu}\hat{\nabla}^\rho\hat{\nabla}^\sigma) - \frac{1}{2}\hat{\gamma}^{\mu\rho}\hat{\nabla}^\sigma\hat{\nabla}^\nu - \frac{1}{2}\hat{\gamma}^{\nu\rho}\hat{\nabla}^\sigma\hat{\nabla}^\mu \right] h_{\rho\sigma} \\ &+ \left[\hat{R}^{\mu\rho}\hat{\gamma}^{\nu\sigma} + \hat{R}^{\nu\rho}\hat{\gamma}^{\mu\sigma} - \frac{1}{2}\hat{R}^{\mu\nu}\hat{\gamma}^{\rho\sigma} - \frac{1}{2}\hat{R}^{\rho\sigma}\hat{\gamma}^{\mu\nu} - \frac{1}{4}(\hat{R} - 2\Lambda)(2\hat{\gamma}^{\mu\rho}\hat{\gamma}^{\nu\sigma} - \hat{\gamma}^{\mu\nu}\hat{\gamma}^{\rho\sigma}) \right] h_{\rho\sigma} \\ &+ \frac{1}{2}(\hat{R} - 2\Lambda)\hat{\gamma}^{\mu\nu} - \hat{R}^{\mu\nu} - l^{-2}\lambda\hat{T}^{\mu\nu} - \tau l^{-2}\lambda h_{\rho\sigma}M^{\mu\nu\rho\sigma} = 0, \end{aligned} \quad (\text{A4})$$

where $M^{\mu\nu\rho\sigma} = \frac{\partial\hat{T}^{\mu\nu}}{\partial\hat{\gamma}_{\rho\sigma}} + \frac{\partial\hat{T}^{\rho\sigma}}{\partial\hat{\gamma}_{\mu\nu}} + \frac{1}{2}\hat{T}^{\mu\nu}\hat{\gamma}^{\rho\sigma} + \frac{1}{2}\hat{T}^{\rho\sigma}\hat{\gamma}^{\mu\nu}$. To ensure that the higher-order terms of the metric perturbation do not affect the leading-order contribution to the stress tensor deformation and to maintain the validity of the perturbation analysis, the saddle point $h_{\mu\nu}^*$ should be proportional to λ (which will be explained in detail in the next subsection). This condition requires that the background metric satisfy the vacuum Einstein field equations,

$$\hat{R}^{\mu\nu} - \frac{1}{2}\hat{R}\hat{\gamma}^{\mu\nu} + \Lambda\hat{\gamma}^{\mu\nu} = 0. \quad (\text{A5})$$

By introducing the trace-reversed variable

$$\tilde{h}_{\rho\sigma} = h_{\rho\sigma} - \frac{1}{2}\hat{\gamma}_{\rho\sigma}h^\alpha_\alpha, \quad (\text{A6})$$

and incorporating the following gauge fixing term and ghost term into the gravitational action [24, 25],

$$\begin{aligned} S_{\text{gauge}} &= -\frac{l^2}{2} \int d^d x \sqrt{\hat{\gamma}} \hat{\gamma}^{\nu\rho}\hat{\nabla}^\mu\tilde{h}_{\mu\nu}\hat{\nabla}^\sigma\tilde{h}_{\rho\sigma}, \\ S_{\text{ghost}} &= -\frac{l^2}{2} \int d^d x \sqrt{\hat{\gamma}} \bar{\eta}_\mu(\hat{\gamma}^{\mu\nu}\hat{\square} + \hat{R}^{\mu\nu})\eta_\nu, \end{aligned} \quad (\text{A7})$$

the above EoM can be simplified as

$$\begin{aligned} & \tau \left[\frac{1}{2} \hat{\square} h^{\mu\nu} - \frac{1}{4} \hat{\gamma}^{\mu\nu} \hat{\square} h_{\alpha}^{\alpha} + \frac{1}{2} (\hat{R}^{\mu\rho\nu\sigma} + \hat{R}^{\nu\rho\mu\sigma}) h_{\rho\sigma} \right] \\ & - l^{-2} \lambda \hat{T}^{\mu\nu} - \tau l^{-2} \lambda h_{\rho\sigma} M^{\mu\nu\rho\sigma} = 0. \end{aligned} \quad (\text{A8})$$

The corresponding solution for auxiliary field takes the form of (9). Plugging it into the gravitational action, we obtain the leading-order contribution to the deformed action,

$$\begin{aligned} S^{(\lambda)}[\hat{\gamma}, \psi] &= \hat{S}[\hat{\gamma}, \psi] + \frac{2\Lambda l^2 \text{Vol}(\mathcal{M})}{\lambda(D-2)} \\ &+ \frac{\lambda l^{-2}}{2} \int d^D x d^D y \sqrt{\hat{\gamma}(x) \hat{\gamma}(y)} G_{\mu\nu\rho\sigma}(x, y) \hat{T}^{\mu\nu}(x) [\hat{T}^{\rho\sigma} - \frac{1}{D-2} \hat{T}_{\alpha}^{\alpha} \hat{\gamma}^{\rho\sigma}](y) \\ &+ O(\lambda^2). \end{aligned} \quad (\text{A9})$$

A simpler case is that the background metric is flat, $\hat{\gamma}_{\mu\nu} = \eta_{\mu\nu}$. The Green's function

$$G_{\mu\nu\alpha\beta}(x, y) = \eta_{\mu\alpha} \eta_{\nu\beta} G_{(-\hat{\square})}(x, y). \quad (\text{A10})$$

Then, the leading order deformation can be simplified as

$$S_{[1]}^{(\lambda)}[\hat{\gamma}, \psi] = \frac{\lambda l^{-2}}{2} \int d^D x \sqrt{\hat{\gamma}} \hat{T}_{\mu\nu} \frac{1}{-\hat{\square}} (\hat{T}^{\mu\nu} - \frac{1}{D-2} \hat{T}_{\alpha}^{\alpha} \eta^{\mu\nu}). \quad (\text{A11})$$

The Einstein-Hilbert action contains higher-order corrections of $h_{\mu\nu}$. In principle, for the perturbation method to be valid, we must demonstrate that the higher-order corrections of $h_{\mu\nu}$ do not affect the leading-order form of the stress tensor deformation. The gravitational action can be formally written as

$$\begin{aligned} S &= \hat{S}[\hat{\gamma} + h, \psi] + \frac{1}{2\lambda} S_{\text{grav}}[\hat{\gamma} + h] \\ &= \hat{S}[\hat{\gamma}, \psi] - \frac{1}{2} \int d^d x \sqrt{\hat{\gamma}} h_{\mu\nu} \hat{T}^{\mu\nu} - \frac{1}{2} \sum_{k=1}^{\infty} \int d^d x h_{\mu\nu} \left(\prod_{i=1}^k h_{\mu_i \nu_i} \right) \frac{\partial^k}{\prod_{i=1}^k \partial \hat{\gamma}_{\mu_i \nu_i}} (\sqrt{\hat{\gamma}} \hat{T}^{\mu\nu}) \\ &+ \frac{l^2}{2\lambda} \sum_{k=0}^{\infty} \int d^D x \sqrt{\hat{\gamma}} \mathcal{F}^{(k)}, \end{aligned} \quad (\text{A12})$$

where

$$\begin{aligned} \mathcal{F}^{(k)} &= \mathcal{F}^{(k)}(\hat{\gamma}_{\mu\nu}, h_{\mu\nu}, \hat{\nabla}_{\rho} h_{\mu\nu}, \hat{\nabla}_{\rho} \hat{\nabla}_{\sigma} h_{\mu\nu}) \sim h^k, \\ [\mathcal{F}^{(k)}] &= [L]^{-2}. \end{aligned} \quad (\text{A13})$$

The EOM for auxiliary field is

$$\begin{aligned} & -\lambda l^{-2} \left[\hat{T}^{\mu\nu} + \sum_{k=1}^{\infty} \tau^k \left(\prod_{i=1}^k h_{\mu_i \nu_i} \right) \frac{1}{\sqrt{\hat{\gamma}}} \left(\frac{\partial^k}{\prod_{i=1}^k \partial \hat{\gamma}_{\mu_i \nu_i}} (\sqrt{\hat{\gamma}} \hat{T}^{\mu\nu}) + k \frac{\partial^k}{\partial \hat{\gamma}_{\mu\nu} \prod_{i=2}^k \partial \hat{\gamma}_{\mu_i \nu_i}} (\sqrt{\hat{\gamma}} \hat{T}^{\mu_1 \nu_1}) \right) \right] \\ & + \sum_{k=0}^{\infty} \tau^k \tilde{\mathcal{F}}^{(k)\mu\nu} = 0, \end{aligned} \quad (\text{A14})$$

where

$$\tilde{\mathcal{F}}^{(k)\mu\nu}(x) = \frac{1}{\sqrt{\hat{\gamma}(x)}} \frac{\delta}{\delta h_{\mu\nu}(x)} \int d^d x' \sqrt{\hat{\gamma}} \mathcal{F}^{(k+1)} \sim h^k. \quad (\text{A15})$$

The first two coefficients have been calculated in the previous section,

$$\begin{aligned} \tilde{\mathcal{F}}^{(0)\mu\nu} &= \frac{1}{2} (\hat{R} - 2\Lambda) \hat{\gamma}^{\mu\nu} - \hat{R}^{\mu\nu}, \\ \tilde{\mathcal{F}}^{(1)\mu\nu} &= \frac{1}{2} \hat{\square} h^{\mu\nu} - \frac{1}{4} \hat{\gamma}^{\mu\nu} \hat{\square} h_{\alpha}^{\alpha} + \frac{1}{2} \left[\hat{R}^{\mu\rho\nu\sigma} + \hat{R}^{\nu\rho\mu\sigma} + \hat{R}^{\mu\rho} \hat{\gamma}^{\nu\sigma} + \hat{R}^{\nu\rho} \hat{\gamma}^{\mu\sigma} \right. \\ &\quad \left. - \hat{R}^{\mu\nu} \hat{\gamma}^{\rho\sigma} - \hat{R}^{\rho\sigma} \hat{\gamma}^{\mu\nu} - \frac{1}{2} (\hat{R} - 2\Lambda) (2\hat{\gamma}^{\mu\rho} \hat{\gamma}^{\nu\sigma} - \hat{\gamma}^{\mu\nu} \hat{\gamma}^{\rho\sigma}) \right] h_{\rho\sigma}. \end{aligned} \quad (\text{A16})$$

Suppose that the solution can be written as a power series in λ ,

$$h_{\mu\nu}^* = \sum_{k=-\infty}^{\infty} \lambda^k h_{[k]\mu\nu}^*. \quad (\text{A17})$$

It is clear that when $\tilde{\mathcal{F}}_{\mu\nu}^{(0)} = 0$, there exists a solution of the auxiliary field that satisfies

$$h_{[k]\mu\nu}^* = 0, \text{ for } k \leq 0. \quad (\text{A18})$$

Such a saddle point corresponds to the seed theory with a non-local stress tensor deformation, and the form of the deformation can be solved order by order using perturbation method. Below, we proceed to calculate the deformation up to the second order in λ . One can easily find that

$$\begin{aligned} \tilde{\mathcal{F}}^{(2)\mu\nu} = & \frac{1}{8} \left(4\hat{\nabla}_\rho h^{\mu\nu} \hat{\nabla}_\sigma h^{\rho\sigma} - 8\hat{\nabla}_\rho h^{\mu\rho} \hat{\nabla}_\sigma h^{\nu\sigma} - 2h^{\mu\nu} \hat{\nabla}_\sigma \hat{\nabla}_\rho h^{\rho\sigma} + 8h^{\nu\rho} \hat{\nabla}_\sigma \hat{\nabla}_\rho h^{\mu\sigma} + 4h^{\rho\sigma} \hat{\nabla}_\sigma \hat{\nabla}_\rho h^{\mu\nu} + 2h^{\mu\nu} \hat{\square} h_\rho^\rho \right. \\ & - 8h^{\nu\rho} \hat{\square} h_\rho^\mu + 2h_\rho^\rho \hat{\square} h^{\mu\nu} - 8h^{\mu\rho} (\hat{\nabla}_\rho \hat{\nabla}_\sigma h^{\nu\sigma} + \hat{\nabla}_\sigma \hat{\nabla}_\rho h^{\nu\sigma} - \hat{\square} h_\rho^\nu - \hat{\nabla}_\sigma \hat{\nabla}^\nu h_\rho^\sigma) - 4h_\rho^\rho \hat{\nabla}_\sigma \hat{\nabla}^\nu h^{\mu\sigma} - \hat{\gamma}^{\mu\nu} \hat{\nabla}_\sigma h_\delta^\delta \hat{\nabla}^\sigma h_\rho^\rho \\ & + 8\hat{\nabla}_\sigma h_\rho^\nu \hat{\nabla}^\sigma h^{\mu\rho} - 4\hat{\gamma}^{\mu\nu} \hat{\nabla}_\rho h^{\rho\sigma} \hat{\nabla}_\delta h_\sigma^\delta + 4\hat{\gamma}^{\mu\nu} \hat{\nabla}_\sigma h_\rho^\rho \hat{\nabla}_\delta h_\sigma^\delta + 2\hat{\gamma}^{\mu\nu} h_\rho^\rho \hat{\nabla}_\delta \hat{\nabla}_\sigma h^{\sigma\delta} - 2\hat{\gamma}^{\mu\nu} h_\rho^\rho \hat{\square} h_\sigma^\sigma - 2\hat{\gamma}^{\mu\nu} \hat{\nabla}_\sigma h_{\rho\delta} \hat{\nabla}^\delta h^{\rho\sigma} \\ & + 3\hat{\gamma}^{\mu\nu} \hat{\nabla}_\delta h_{\rho\sigma} \hat{\nabla}^\delta h^{\rho\sigma} + 4\hat{\nabla}_\rho h^{\nu\rho} \hat{\nabla}^\mu h_\sigma^\sigma + 4h_\rho^\rho \hat{\nabla}^\mu \hat{\nabla}_\sigma h^{\nu\sigma} - 4h^{\rho\sigma} \hat{\nabla}^\mu \hat{\nabla}^\nu h_{\rho\sigma} - 8\hat{\nabla}^\sigma h^{\mu\rho} \hat{\nabla}^\nu h_{\rho\sigma} + 4\hat{\nabla}^\mu h^{\rho\sigma} \hat{\nabla}^\nu h_{\rho\sigma} \\ & \left. + 4\hat{\nabla}_\rho h^{\mu\rho} \hat{\nabla}^\nu h_\sigma^\sigma - 2\hat{\nabla}^\mu h_\rho^\rho \hat{\nabla}^\nu h_\sigma^\sigma - 2\hat{\nabla}_\rho h_\sigma^\sigma (\hat{\nabla}^\rho h^{\mu\nu} - 2\hat{\nabla}^\nu h^{\mu\rho}) - 8\hat{\nabla}_\sigma h_\rho^\sigma \hat{\nabla}^\nu h^{\mu\rho} - 4h_\rho^\rho \hat{\nabla}^\nu \hat{\nabla}_\sigma h^{\mu\sigma} + 2h_\rho^\rho \hat{\nabla}^\nu \hat{\nabla}^\mu h_\sigma^\sigma \right). \end{aligned} \quad (\text{A19})$$

The saddle of the auxiliary field to the second order in λ takes the form

$$\begin{aligned} \tau h_{\mu\nu}^*(x) = & -2\lambda t_{\mu\nu}(x) + 4\lambda^2 l^{-2} \int d^d y \sqrt{\hat{\gamma}(y)} G_{\mu\nu\eta\xi}(x, y) \left(\delta_\alpha^\eta \delta_\beta^\xi - \frac{1}{D-2} \hat{\gamma}^{\eta\xi}(y) \hat{\gamma}_{\alpha\beta}(y) \right) \left[M^{\alpha\beta\rho\sigma} t_{\rho\sigma} \right. \\ & + \frac{l^2}{4} \left(4\hat{\nabla}_\rho t^{\alpha\beta} \hat{\nabla}_\sigma t^{\rho\sigma} - 8\hat{\nabla}_\rho t^{\alpha\rho} \hat{\nabla}_\sigma t^{\beta\sigma} - 2t^{\alpha\beta} \hat{\nabla}_\sigma \hat{\nabla}_\rho t^{\rho\sigma} + 8t^{\beta\rho} \hat{\nabla}_\sigma \hat{\nabla}_\rho t^{\alpha\sigma} + 4t^{\rho\sigma} \hat{\nabla}_\sigma \hat{\nabla}_\rho t^{\alpha\beta} + 2t^{\alpha\beta} \hat{\square} t_\rho^\rho \right. \\ & - 8t^{\beta\rho} \hat{\square} t_\rho^\alpha + 2t_\rho^\rho \hat{\square} t^{\alpha\beta} - 8t^{\alpha\rho} (\hat{\nabla}_\rho \hat{\nabla}_\sigma t^{\beta\sigma} + \hat{\nabla}_\sigma \hat{\nabla}_\rho t^{\beta\sigma} - \hat{\square} t_\rho^\beta - \hat{\nabla}_\sigma \hat{\nabla}^\beta t_\rho^\sigma) - 4t_\rho^\rho \hat{\nabla}_\sigma \hat{\nabla}^\beta t^{\alpha\sigma} - \hat{\gamma}^{\alpha\beta} \hat{\nabla}_\sigma t_\delta^\delta \hat{\nabla}^\sigma t_\rho^\rho \\ & + 8\hat{\nabla}_\sigma t_\rho^\beta \hat{\nabla}^\sigma t^{\alpha\rho} - 4\hat{\gamma}^{\alpha\beta} \hat{\nabla}_\rho t^{\rho\sigma} \hat{\nabla}_\delta t_\sigma^\delta + 4\hat{\gamma}^{\alpha\beta} \hat{\nabla}_\sigma t_\rho^\rho \hat{\nabla}_\delta t_\sigma^\delta + 2\hat{\gamma}^{\alpha\beta} t_\rho^\rho \hat{\nabla}_\delta \hat{\nabla}_\sigma t^{\sigma\delta} - 2\hat{\gamma}^{\alpha\beta} t_\rho^\rho \hat{\square} t_\sigma^\sigma - 2\hat{\gamma}^{\alpha\beta} \hat{\nabla}_\sigma t_{\rho\delta} \hat{\nabla}^\delta t^{\rho\sigma} \\ & + 3\hat{\gamma}^{\alpha\beta} \hat{\nabla}_\delta t_{\rho\sigma} \hat{\nabla}^\delta t^{\rho\sigma} + 4\hat{\nabla}_\rho t^{\beta\rho} \hat{\nabla}^\alpha t_\sigma^\sigma + 4t_\rho^\rho \hat{\nabla}^\alpha \hat{\nabla}_\sigma t^{\beta\sigma} - 4t^{\rho\sigma} \hat{\nabla}^\alpha \hat{\nabla}^\beta t_{\rho\sigma} - 8\hat{\nabla}^\sigma t^{\alpha\rho} \hat{\nabla}^\beta t_{\rho\sigma} + 4\hat{\nabla}^\alpha t^{\rho\sigma} \hat{\nabla}^\beta t_{\rho\sigma} \\ & + 4\hat{\nabla}_\rho t^{\alpha\rho} \hat{\nabla}^\beta t_\sigma^\sigma - 2\hat{\nabla}^\alpha t_\rho^\rho \hat{\nabla}^\beta t_\sigma^\sigma - 2\hat{\nabla}_\rho t_\sigma^\sigma (\hat{\nabla}^\rho t^{\alpha\beta} - 2\hat{\nabla}^\beta t^{\alpha\rho}) - 8\hat{\nabla}_\sigma t_\rho^\sigma \hat{\nabla}^\beta t^{\alpha\rho} - 4t_\rho^\rho \hat{\nabla}^\beta \hat{\nabla}_\sigma t^{\alpha\sigma} + 2t_\rho^\rho \hat{\nabla}^\beta \hat{\nabla}^\alpha t_\sigma^\sigma \left. \right] (y) \\ & + O(\lambda^3), \end{aligned} \quad (\text{A20})$$

where

$$t_{\mu\nu}(x) = \frac{1}{l^2} \int d^d x' \sqrt{\hat{\gamma}(x')} G_{\mu\nu\rho\sigma}(x, x') \left[\hat{T}^{\rho\sigma} - \frac{1}{D-2} \hat{T}_\alpha^\alpha \hat{\gamma}^{\rho\sigma} \right] (x'). \quad (\text{A21})$$

The corresponding stress tensor deformation is

$$\begin{aligned} & S_{[2]}^{(\lambda)}[\hat{\gamma}, \psi] \\ = & -\frac{\lambda^2 l^2}{2} \int d^d x \sqrt{\hat{\gamma}} \left[\frac{2}{l^2} t_{\mu\nu} t_{\rho\sigma} M^{\mu\nu\rho\sigma} + (t_\rho^\rho)^2 (\hat{\nabla}_\nu \hat{\nabla}_\mu t^{\mu\nu} - \hat{\square} t_\mu^\mu) - 2t^{\mu\nu} (3\hat{\nabla}_\mu t^{\rho\sigma} \hat{\nabla}_\nu t_{\rho\sigma} - \hat{\nabla}_\mu t_\rho^\rho \hat{\nabla}_\nu t_\sigma^\sigma + 4\hat{\nabla}_\nu t_\sigma^\sigma \hat{\nabla}_\mu t_\rho^\rho \right. \\ & + 4\hat{\nabla}_\nu t_\mu^\rho \hat{\nabla}_\rho t_\sigma^\sigma + 4t_\mu^\rho \hat{\nabla}_\rho \hat{\nabla}_\nu t_\sigma^\sigma - 4t_\mu^\rho \hat{\nabla}_\rho \hat{\nabla}_\sigma t_\nu^\sigma - 2\hat{\nabla}_\rho t_\sigma^\sigma \hat{\nabla}^\rho t_{\mu\nu} - 4\hat{\nabla}_\rho t_\mu^\rho \hat{\nabla}_\sigma t_\nu^\sigma - 8\hat{\nabla}_\nu t_\mu^\rho \hat{\nabla}_\sigma t_\rho^\sigma + 4\hat{\nabla}^\rho t_{\mu\nu} \hat{\nabla}_\sigma t_\rho^\sigma - 4t^{\rho\sigma} \hat{\nabla}_\sigma \hat{\nabla}_\rho t_{\mu\nu} \\ & + 4t^{\rho\sigma} \hat{\nabla}_\sigma \hat{\nabla}_\rho t_{\mu\nu} - 4t_\mu^\rho \hat{\nabla}_\sigma \hat{\nabla}_\rho t_\nu^\sigma + t_{\mu\nu} \hat{\nabla}_\sigma \hat{\nabla}_\rho t^{\rho\sigma} + 4t_\mu^\rho \hat{\square} t_{\nu\rho} - t_{\mu\nu} \hat{\square} t_\rho^\rho - 4\hat{\nabla}_\nu t_{\rho\sigma} \hat{\nabla}^\sigma t_\mu^\rho - 2\hat{\nabla}_\rho t_{\nu\sigma} \hat{\nabla}^\sigma t_\mu^\rho + 6\hat{\nabla}_\sigma t_{\nu\rho} \hat{\nabla}^\sigma t_\mu^\rho) \\ & - t_\mu^\mu (\hat{\nabla}_\rho t_\sigma^\sigma \hat{\nabla}^\rho t_\nu^\nu + 4\hat{\nabla}_\nu t^{\nu\rho} \hat{\nabla}_\sigma t_\rho^\sigma - 4\hat{\nabla}^\rho t_\nu^\nu \hat{\nabla}_\sigma t_\rho^\sigma - 4t^{\nu\rho} (\hat{\nabla}_\rho \hat{\nabla}_\nu t_\sigma^\sigma - \hat{\nabla}_\rho \hat{\nabla}_\sigma t_\nu^\nu - \hat{\nabla}_\sigma \hat{\nabla}_\rho t_\nu^\nu + \hat{\square} t_{\nu\rho})) + 2\hat{\nabla}_\rho t_{\nu\sigma} \hat{\nabla}^\sigma t^{\nu\rho} \\ & \left. - 3\hat{\nabla}_\sigma t_{\nu\rho} \hat{\nabla}^\sigma t^{\nu\rho} \right]. \end{aligned} \quad (\text{A22})$$

B. $f(\mathcal{R})$ gravity in Palatini formalism

In this Supplemental Material, we briefly review some basics of Palatini formalism of $f(\mathcal{R})$ gravity. The EOMs have been shown in (15). Taking the trace of the first EOM yields

$$f'(\mathcal{R})\mathcal{R} - \frac{d}{2}f(\mathcal{R}) = -\lambda\hat{T}. \quad (\text{A23})$$

For a certain f , this equation is an algebraic equation in \mathcal{R} . The solution can be formally written as $\mathcal{R} = \mathcal{R}(\hat{T}(g^*))$. Plugging it into the gravitational action, we have

$$S_{\text{EFT}}^{(\lambda)}[g^*, \psi] = \hat{S}[g^*, \psi] + \frac{1}{2\lambda} \int d^d x \sqrt{g^*} f(\mathcal{R}(\hat{T}(g^*))). \quad (\text{A24})$$

To obtain the specific form of the deformed action, we ought to find the exact solution g^* . By introducing the following metric conformal to $g_{\mu\nu}$ [20],

$$\tilde{g}_{\mu\nu} = (f'(\mathcal{R}))^{\frac{2}{d-2}} g_{\mu\nu}, \quad (\text{A25})$$

the second EOM in (15) can be rewritten as

$$\bar{\nabla}_\sigma(\sqrt{\tilde{g}} \tilde{g}^{\mu\nu}) = 0. \quad (\text{A26})$$

This equation is the definition of the Levi-Civita connection of $\tilde{g}_{\mu\nu}$, which gives

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\tilde{g}^{\sigma\lambda}(\partial_\mu\tilde{g}_{\nu\lambda} + \partial_\nu\tilde{g}_{\mu\lambda} - \partial_\lambda\tilde{g}_{\mu\nu}). \quad (\text{A27})$$

The Ricci curvature tensor after the conformal transformation is

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{f'(\mathcal{R})}(\nabla_\mu\nabla_\nu + \frac{1}{d-2}g_{\mu\nu}\square)f'(\mathcal{R}) + \frac{d-1}{d-2}\frac{1}{f'(\mathcal{R})^2}\nabla_\mu f'(\mathcal{R})\nabla_\nu f'(\mathcal{R}). \quad (\text{A28})$$

Plugging this expression into the first EOM in (15), we obtain Einstein's field equation with a modified stress tensor,

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \\ &= -\frac{\lambda}{f'}\hat{T}_{\mu\nu} - \frac{1}{2}(\mathcal{R} - \frac{f}{f'})g_{\mu\nu} + \frac{1}{f'}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\square)f' - \frac{d-1}{d-2}\frac{1}{f'^2}(\nabla_\mu f'\nabla_\nu f' - \frac{1}{2}g_{\mu\nu}\nabla^\rho f'\nabla_\rho f'), \end{aligned} \quad (\text{A29})$$

which can be solved using the perturbation method in Supplemental Material A.

C. Emergence of gravity from generalized Nambu-Goto action

Inspired by the deformed field theory action (26), we can more generally express the action in the following form,

$$S_{\text{EFT}}^{(\lambda)}[g, \phi] = \int d^d x \sqrt{g} \left[\frac{1 - \lambda A^{(\lambda)} - \sqrt{1 - \lambda B^{(\lambda)} - 2\lambda C^{(\lambda)}(\nabla^\mu \phi \nabla_\mu \phi)^{2q}}}{\lambda C^{(\lambda)}} \right]. \quad (\text{A30})$$

The corresponding gravitational action is constructed as follows,

$$\mathcal{A}^{(\lambda)} = \frac{2q(d-2q)(\nabla^\mu \phi \nabla_\mu \phi)^{2q}}{F^{(\lambda)}(A, B)} - \frac{F^{(\lambda)}(A, B)}{4\lambda q^2(d-2q)} \left(\mathcal{R}^q - \frac{qC^{(\lambda)}}{d-4q}\mathcal{R}^{2q} - \frac{(d-2q)(d-4q)[F^{(\lambda)}(A, B) - 4q^2(1 - \lambda A^{(\lambda)})]}{4qC^{(\lambda)}} \right), \quad (\text{A31})$$

where $F^{(\lambda)}(A, B) = dq(1 - \lambda A^{(\lambda)}) - |q|\sqrt{(d-4q)^2 - \lambda d^2 A^{(\lambda)}(2 - \lambda A^{(\lambda)}) + 8\lambda q(d-2q)B^{(\lambda)}}$. Here we set $l = 1$ for simplicity. By using the first EOM in (21), we find

$$\mathcal{R}^q = \frac{(d-2q)(d-4q)[F^{(\lambda)}(A, B) - 4q^2\sqrt{1 - \lambda B^{(\lambda)} - 2\lambda C^{(\lambda)}(\nabla^\mu \phi \nabla_\mu \phi)^{2q}}]}{2qC^{(\lambda)}F^{(\lambda)}(A, B)}. \quad (\text{A32})$$

Plugging it into (A31), we recover the effective deformed action (A30). Some of the gravitational actions discussed in the text can be derived from (A31). If we set $q = 1/2$, $A^{(\lambda)} = 2V$, $B^{(\lambda)} = 0$, and $C^{(\lambda)} = (1 - \lambda V)$, then (A31) reduces to the gravitational action (28), which corresponds to the Nambu-Goto action. If we set $q = 1$, and $B^{(\lambda)} = A^{(\lambda)}(2 - \lambda A^{(\lambda)})$, then (A31) reduces to

$$\mathcal{A}^{(\lambda)} = \frac{(d-2q)(\nabla^\mu \phi \nabla_\mu \phi)^2}{2q(1 - \lambda A^{(\lambda)})} - \frac{1 - \lambda A^{(\lambda)}}{\lambda(d-2q)} \left(\mathcal{R} - \frac{qC^{(\lambda)}}{d-4q} \mathcal{R}^2 \right), \quad (\text{A33})$$

which corresponds to the quadratic Palatini $f(\mathcal{R})$ gravity with $\Lambda = 0$, and $\alpha = -\frac{qC^{(\lambda)}}{d-4q}$. Suppose that $A^{(\lambda)}$ is a local function of $C^{(\lambda)}$ and has the Taylor expansion

$$A(C) = A_0 + A_1 C + A_2 C^2 + \dots \quad (\text{A34})$$

By taking the limit $C \rightarrow 0$, we obtain the Einstein's gravity,

$$\mathcal{A}^{(\lambda)} = \frac{(d-2q)(\nabla^\mu \phi \nabla_\mu \phi)^2}{2q(1 - \lambda A_0)} - \frac{1 - \lambda A_0}{\lambda(d-2q)} \mathcal{R}. \quad (\text{A35})$$

The effective deformed action (A30) will remain finite in this limit if we impose the condition $1 - \lambda A_0 > 0$.