B. A. Toledo^{a)}

(Dated: 26 May 2025)

The Lorenz system, a seminal model in chaos theory, is renowned for its complex dynamics and sensitive dependence on initial conditions. This work investigates conserved quantities within the standard Lorenz equations. The system exhibits phase space volume contraction, a characteristic of dissipative systems. Despite the common understanding positing the absence of conserved quantities in its typical chaotic parameter regimes, this study demonstrates that the Lorenz system possesses at least three dynamical constants of motion. The implications of these conserved quantities for the system's chaotic behavior are discussed.

^{a)}Departamento de Física, Facultad de Ciencias, Universidad de Chile, Santiago, Chile.; Electronic mail: btoledoc@uchile.cl

The intricate dance of chaos, famously exemplified by the Lorenz system, is investigated in this study through the lens of its fundamental invariants. This work provides a rigorous demonstration of the existence of three conserved quantities inherent to the standard Lorenz framework, a finding that contrasts with typical expectations for dissipative chaotic systems. This study delves into how the interplay between these specific invariant constraints and the system's inherent dynamical processes gives rise to its complex behavior. The Lorenz system, thus characterized by both complexity and fundamental invariants as demonstrated herein, offers a versatile model for phenomena across diverse fields, from atmospheric science and fluid turbulence to laser physics and biological systems. Appreciating the foundational role of the Lorenz system in nonlinear science necessitates understanding its dynamics within the context of the conserved quantities established by this research.

I. INTRODUCTION

Conserved quantities, or constants of motion, are fundamental concepts in the study of dynamical systems. They represent physical properties, such as energy or momentum, that remain constant as a system evolves. In Hamiltonian mechanics, Noether's theorem elegantly links continuous symmetries of a system's Lagrangian to these conserved quantities, providing profound insights and often simplifying analysis¹. The existence of a sufficient number of independent conserved quantities can render a system integrable, confining its motion to a torus in phase space and leading to regular, predictable behavior. However, many natural and engineered systems, particularly those exhibiting complex or chaotic dynamics, are not conservative. This non-conservative nature is typically associated with phase space volume contraction and the presumed absence of a sufficient number of classical conserved quantities for integrability. One of the most iconic examples of such a system is the Lorenz system, introduced by Edward N. Lorenz in 1963 as a drastically simplified model of atmospheric Rayleigh-Bénard convection². In the physical system of convection, an equilibrium between energy input and output can lead to sustained periodic fluid motions (periodic orbits); depending on the system parameters, these orbits can be stable or unstable, with the latter

often preceding more complex, turbulent-like behavior. The Lorenz model, designed to capture such transitions, is described by a set of three coupled, nonlinear ordinary differential equations:

$$\frac{dx}{dt} = \sigma(y - x),\tag{1}$$

$$\frac{dx}{dt} = \sigma(y - x), \tag{1}$$

$$\frac{dy}{dt} = x(\rho - z) - y, \tag{2}$$

$$\frac{dz}{dt} = xy - \beta z. \tag{3}$$

$$\frac{dz}{dt} = xy - \beta z. (3)$$

Here, x, y, and z are state variables representing the intensity of convective motion and horizontal and vertical temperature variations, respectively. The parameters σ (Prandtl number), ρ (Rayleigh number), and β (a geometric factor) are positive constants. For certain parameter values (e.g., Lorenz's classical values $\sigma = 10$, $\rho = 28$, $\beta = 8/3$), the system exhibits chaotic behavior, characterized by sensitive dependence on initial conditions and the presence of a "strange attractor". It is crucial to distinguish the Lorenz system from Hamiltonian systems. Hamiltonian systems, by definition, conserve energy (if the Hamiltonian is time-independent) and preserve phase space volume, with dynamics governed by symplectic geometry. The Lorenz system, being dissipative, does not fit this framework. Its chaotic behavior arises from an intricate interplay of stretching phase space volumes (leading to sensitivity to initial conditions) and folding them (keeping trajectories within a bounded region), with an overall contraction of volume due to dissipation. Consequently, the standard Lorenz system, particularly in its chaotic regime, is generally understood to lack non-trivial conserved quantities. While there have been attempts to find Lagrangian or Hamiltonian formulations for dissipative systems like Lorenz's, often employing fractional calculus⁴ or embedding within higher-dimensional conservative frameworks⁵, these approaches typically lead to conserved quantities for modified systems and do not straightforwardly apply to the original three-dimensional dissipative equations without careful reinterpretation. This paper challenges this conventional understanding by investigating the existence and nature of conserved quantities within the standard Lorenz equations (1-3). We aim to demonstrate that, despite its dissipative and chaotic nature, the Lorenz system possesses at least three dynamical constants of motion. The analysis will begin by examining the fundamental dynamical properties of the Lorenz system, emphasizing its dissipative character and known symmetries. Subsequently, we will present the derivation of these conserved quantities and explore their implications, commenting on generalized conserved quantities of a similar kind in other contexts. The Lorenz system's significance extends far beyond its meteorological origins, serving as a paradigm for chaotic behavior with profound implications across numerous scientific and engineering disciplines. Its relevance often stems precisely from its deterministic chaos, sensitive dependence on initial conditions, and its dissipative, non-integrable nature. In atmospheric science and fluid dynamics, it remains key for understanding the limits of predictability in weather and as a basic model for thermal convection², with extensions modeling complex geophysical phenomena⁶. Applications in physics include laser instabilities, plasma dynamics, and dynamo theory; in engineering, it has been utilized in nonlinear circuit theory, chaos control, and secure communications^{8,9}. It finds analogies in chemical reactions where transient chaotic phenomena are observed¹⁰, and informs models in biology, medicine¹¹, economics¹², and mathematics, particularly in studies of strange attractors and bifurcations^{13,14}. The broad applicability arises from its qualitative capture of deterministic chaos from simple nonlinear interactions. The very characteristics that traditionally challenge notions of integrability, including the subtleties surrounding its conserved quantities, are what make it a universal prototype. By addressing these points, this paper seeks to provide a comprehensive explanation of how the concept of conserved quantities applies to the Lorenz system, particularly in light of the insights presented herein. We will discuss why this system, with its rich dynamics, remains a cornerstone in the study of nonlinear dynamics and chaos, and how the presence of such invariants could reshape our understanding of its complex behavior.

II. THE LORENZ SYSTEM: DYNAMICS AND DISSIPATION

The Lorenz equations (1-3) define a continuous-time dynamical system in a threedimensional phase space. The behavior of this system is critically dependent on the values of the parameters σ , ρ , and β .

A. Dissipative Nature

A key characteristic of the Lorenz system is its dissipative nature. The divergence of the vector field $F = (\sigma(y-x), x(\rho-z) - y, xy - \beta z)$ is given by:

$$\nabla \cdot F = \frac{\partial}{\partial x} (\sigma(y - x)) + \frac{\partial}{\partial y} (x(\rho - z) - y) + \frac{\partial}{\partial z} (xy - \beta z)$$

$$= -\sigma - 1 - \beta. \tag{4}$$

Since $\sigma, \beta > 0$, the divergence is a negative constant: $-(\sigma + 1 + \beta)$. This negative divergence implies that a volume element in phase space contracts exponentially with time, its volume scaling as $e^{-(\sigma+1+\beta)t_{15}}$. This volume contraction is a hallmark of dissipative systems. It means that, as time progresses, trajectories are attracted to a subset of the phase space with zero volume but with non-zero fractal dimension¹⁶.

B. Fixed Points and Stability

The Lorenz system has fixed points where dx/dt = dy/dt = dz/dt = 0. One fixed point is always at the origin (0,0,0), corresponding to no convection. If $\rho > 1$, two additional fixed points, C^+ and C^- , emerge:

$$(x,y,z) = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1). \tag{5}$$

These points represent steady convection. The stability of these fixed points depends on the system parameters. For instance, the origin is stable for $\rho < 1$. For $\rho > 1$, the origin becomes unstable, and C^+ and C^- become stable. However, as ρ increases further, C^+ and C^- also lose stability via a Hopf bifurcation at $\rho_H = \sigma(\sigma + \beta + 3)/(\sigma - \beta - 1)$, provided $\sigma > \beta + 1^{17}$. For Lorenz's parameters, this occurs at $\rho \approx 24.74$.

C. The Strange Attractor

For $\rho > \rho_H$ (e.g., $\rho = 28$ with Lorenz's other parameters), trajectories do not settle to a fixed point or a simple periodic orbit. Instead, they are confined to a complex, bounded region in phase space known as a strange attractor—the Lorenz attractor. This attractor has a fractal structure and is associated with the system's chaotic dynamics. Trajectories on

the attractor exhibit sensitive dependence on initial conditions: nearby trajectories diverge exponentially fast, making long-term prediction impossible despite the deterministic nature of the equations. This "butterfly effect" was one of Lorenz's key discoveries².

D. Symmetries of the Lorenz System

The Lorenz system exhibits a discrete symmetry corresponding to the transformation:

$$(x, y, z) \mapsto (-x, -y, z), \tag{6}$$

which leaves the differential equations invariant. This \mathbb{Z}_2 symmetry implies that the system is equivariant under reflection through the z-axis. As a result, the phase space structure, including fixed points and trajectories, appears symmetrically mirrored across this axis. This symmetry explains the appearance of twin lobes in the Lorenz attractor and the existence of symmetric pairs of nontrivial fixed points, C^+ and C^- . Although this symmetry does not imply the existence of a conserved quantity in the classical sense via Noether's theorem (which applies to continuous symmetries of a Lagrangian), it plays a crucial role in constraining the geometry and bifurcation structure of the system. The \mathbb{Z}_2 symmetry also affects the organization of unstable manifolds and the transition pathways between different regions of the attractor, making it a fundamental structural feature of the Lorenz dynamics.

III. CONSERVED QUANTITIES IN THE LORENZ SYSTEM: A CLOSER LOOK

In classical mechanics, a conserved quantity (or first integral) $C(\mathbf{x}, t) \in \mathbb{R}$, where $\mathbf{x} \in \mathbb{R}^n$, is a function whose value remains constant along any trajectory of the system, i.e., dC/dt = 0. If C does not explicitly depend on time, it is a time-independent conserved quantity.

A. Ansatz for the conserved quantities

For the standard Lorenz system with parameters leading to chaos (e.g., $\sigma = 10, \beta = 8/3, \rho = 28$), no general, non-trivial, time-independent, analytic constant of motion was generally understood to exist^{17,18}. The dissipative nature, evidenced by phase space volume contraction, is a strong indicator against the existence of conserved quantities akin to energy

in Hamiltonian systems. In Hamiltonian systems, Liouville's theorem states that phase space volume is preserved, which is fundamentally different from the Lorenz system. However, the results presented below suggest that a remnant fractal set of non-zero dimension might be relevant for understanding how such quantities can persist, although this possibility needs further study in more general systems. The subsequent analysis follows from a modified direct method for finding conserved quantities in dynamical systems. The standard direct method involves the explicit computation of functions $C(\mathbf{x},t)$ whose total time derivatives vanish along solutions of the system, that is, $\frac{dC}{dt} = \nabla C \cdot \dot{\mathbf{x}} + \frac{\partial C}{\partial t} = 0$. This approach does not rely on the existence of an underlying variational principle or symmetries, making it applicable to a broad class of systems, including non-Hamiltonian and dissipative models. Typically, this method begins by assuming a functional form for $C(\mathbf{x},t)$, often polynomial or rational in the state variables, and substituting it into the condition $\frac{dC}{dt} = 0$, leading to a system of partial differential equations for the coefficients of the assumed form. The solutions to this system yield candidate conserved quantities. Symbolic computation tools and algorithmic implementations—such as those based on the Prelle-Singer procedure or Darboux polynomials—can facilitate this process^{19,20}. Although the method may not always succeed in non-integrable systems, it has proven effective in uncovering hidden first integrals in numerous cases, providing valuable insights into the qualitative structure of dynamical systems. In the present work, we will concentrate only on the conserved quantities; the details of the method used to find them will be presented elsewhere. Let us consider the following extended Lorenz system,

$$\frac{dx}{dt} = \sigma(y - x),$$

$$\frac{dy}{dt} = x(\rho - z) - y,$$

$$\frac{dz}{dt} = xy - \beta z,$$

$$\frac{du}{dt} = F_1(x, y, z) u + G_1(x, y, z),$$
(7)

where,

$$F_1(x, y, z) = \sigma - \frac{y + x(z - \rho)}{x - y},$$

$$G_1(x, y, z) = \frac{1}{2\sigma(x - y)} \left[x^2 (z(\beta + 2(z + 1 + \sigma - \rho)) - 2\rho(\sigma + 1) - 2y^2) - x^3y + 2xy((2\beta + 1)z + \rho + 1)\sigma + 1) - 2(\sigma y^2(z + 1) + \beta^2 z^2) \right],$$

from this ansatz, it is assumed that u = u(x, y, z, t) is constructed such that its inclusion does not alter the x, y, z dynamics of the original Lorenz system but facilitates the identification of a conserved quantity.

To illustrate our approach, we focus on a specific Unstable Periodic Orbit (UPO) within an extended Lorenz system that incorporates an auxiliary variable, u. This particular UPO is defined by the initial conditions x(0) = 5.05105452193349, y(0) = 8.93841397633839, z(0) = 12.448299561783278, and u(0) = 0.0, with a period $\tau = 1.5586522282107245$.

The selection of a specific UPO for this analysis does not sacrifice generality. This assertion is grounded in Cvitanović's Periodic Orbit Theory (POT), which posits that chaotic attractors are structured around an infinite ensemble of UPOs. These orbits collectively form the dynamical "skeleton" of chaos²¹. Although individual UPOs are unstable and thus not directly traced by individual long-term trajectories, they fundamentally encode the system's asymptotic behavior. Consequently, POT enables the computation of global observables—such as statistical averages and Lyapunov exponents—via cycle expansions, which are systematically organized sums over these UPOs. Following this theoretical framework, the extended system described by Eqs. (7) (which includes the differential equation for du/dt) was numerically integrated using the standard Lorenz parameters ($\sigma = 10, \beta = 8/3, \rho = 28$) and the specified initial conditions. The outcomes of this integration are presented in Figure 1. This figure specifically illustrates the behavior of the first identified conserved quantity, $C_1(t) \equiv C_1(x, y, z, u)$, where its initial value is denoted by $C_{10} = C_1(0)$. Explicitly, this first conserved quantity is found to be,

$$C_1(x, y, z, u) = xy(z+1) + \frac{1}{2}x^2(z-\rho) - \frac{\beta z^2}{2} + \sigma(x-y)u.$$
 (8)

Calculating \dot{C}_1 yields,

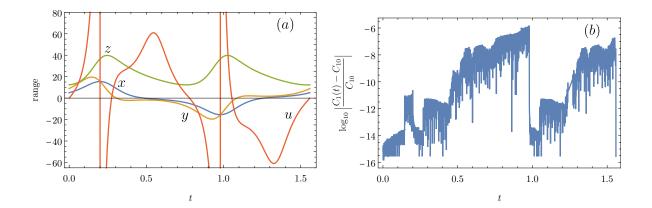


FIG. 1. Extended Lorenz system and its first conserved quantity. In (a), the functions for the UPO considered are shown; in (b), the associated magnitude of the relative error in the conserved quantity estimation is shown on a logarithmic scale.

$$\begin{split} \frac{dC_1}{dt} &= \frac{\partial C_1}{\partial x} \dot{x} + \frac{\partial C_1}{\partial y} \dot{y} + \frac{\partial C_1}{\partial z} \dot{z} + \frac{\partial C_1}{\partial u} \dot{u}, \\ &= \left[\sigma u + x(z - \rho) + y(z + 1) \right] \dot{x} + \left[x(z + 1) - \sigma u \right] \dot{y} + \left[\frac{x^2}{2} + xy - \beta z \right] \dot{z} + \left[\sigma(x - y) \right] \dot{u}, \\ &= \left[\sigma u + x(z - \rho) + y(z + 1) \right] (\sigma(y - x)) + \left[x(z + 1) - \sigma u \right] (x(\rho - z) - y) + \left[\frac{x^2}{2} + xy - \beta z \right] \times \\ &\qquad \qquad (xy - \beta z) + \left[\sigma(x - y) \right] (F_1(x, y, z) u + G_1(x, y, z)), \\ &= \left[\sigma(F_1(x, y, z)(x - y) + x(-\rho - \sigma + z) + \sigma y + y) \right] u + G_1(x, y, z) \sigma(x - y) + \\ &\qquad \qquad + \frac{1}{2} \left(x^2 + 2xy - 2\beta z \right) (xy - \beta z) + \sigma(y - x)(y - \rho x + xz + yz) - x(z + 1)(x(z - \rho) + y), \\ &= 0. \end{split}$$

In the final step, the explicit expressions for F_1 and G_1 were substituted. To express C_1 as $C_1(x, y, z)$, u must be eliminated. This requires solving the equation for du/dt in system (7), whose solution for u(t) is found to be,

$$u(t) = \exp\left(\int_0^t \Phi_1(\tau_1) d\tau_1\right) \int_0^t \exp\left(-\int_0^{\tau_2} \Phi_1(\tau_1) d\tau_1\right) G_1(x(\tau_2), y(\tau_2), z(\tau_2)) d\tau_2,$$

$$\Phi_1(t) = F_1(x(t), y(t), z(t)),$$

and thus, for the first conserved quantity, we obtain,

$$C_{1}(x, y, z) = xy(z+1) + \frac{1}{2}x^{2}(z-\rho) - \frac{\beta z^{2}}{2} + \sigma(x-y) \exp\left(\int_{0}^{t} \Phi_{1}(\tau_{1})d\tau_{1}\right) \times \int_{0}^{t} \exp\left(-\int_{0}^{\tau_{2}} \Phi_{1}(\tau_{1})d\tau_{1}\right) G_{1}(x(\tau_{2}), y(\tau_{2}), z(\tau_{2})) d\tau_{2}.$$
(9)
$$\Phi_{1}(t) = F_{1}(x(t), y(t), z(t)).$$

Now let us extend the Lorenz system in the following manner,

$$\frac{dx}{dt} = \sigma(y - x),$$

$$\frac{dy}{dt} = x(\rho - z) - y,$$

$$\frac{dz}{dt} = xy - \beta z,$$

$$\frac{du}{dt} = F_2(x, y, z) u + G_2(x, y, z),$$
(10)

where,

$$F_2(x, y, z) = \frac{y - x^2 y - \rho(\sigma + 1)x + xz(\beta + \sigma + 1) + \sigma y(\rho - z)}{x(z - \rho) + y},$$

$$G_2(x, y, z) = \frac{1}{x(z - \rho) + y} \left[x^2 \left((z - \rho)(z - \sigma) - y^2 \right) + xy(z(2\beta + 2\sigma + 1) - \sigma(\rho + \sigma + 1)) + \sigma(\sigma + 1)y^2 - \sigma y^2 z - \beta^2 z^2 \right],$$

as in the previous case, let us integrate the system from the same initial conditions and period; the result is shown in Fig. 2.

The second conserved quantity is then,

$$C_{2}(x, y, z) = xy(z - \sigma) + \frac{\sigma y^{2}}{2} - \frac{\beta z^{2}}{2} + (x(z - \rho) + y) \exp\left(\int_{0}^{t} \Phi_{2}(\tau_{1}) d\tau_{1}\right) \times$$

$$\int_{0}^{t} \exp\left(-\int_{0}^{\tau_{2}} \Phi_{2}(\tau_{1}) d\tau_{1}\right) G_{2}(x(\tau_{2}), y(\tau_{2}), z(\tau_{2})) d\tau_{2}. \tag{11}$$

$$\Phi_{2}(t) = F_{2}(x(t), y(t), z(t)).$$

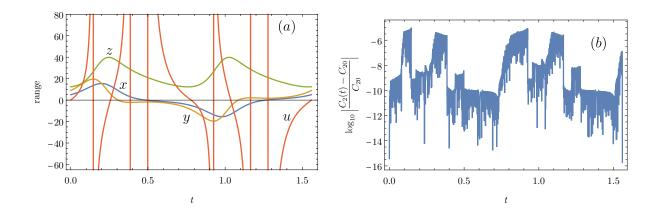


FIG. 2. Extended Lorenz system and its second conserved quantity. In (a), the functions for the UPO considered are shown; in (b), the associated magnitude of the relative error in the conserved quantity estimation is shown on a logarithmic scale.

Finally, the last extension to the Lorenz system we will consider is,

$$\frac{dx}{dt} = \sigma(y - x),$$

$$\frac{dy}{dt} = x(\rho - z) - y,$$

$$\frac{dz}{dt} = xy - \beta z,$$

$$\frac{du}{dt} = F_3(x, y, z) u + G_3(x, y, z),$$
(12)

where,

$$F_3(x, y, z) = \frac{x^2(z - \rho) + xy(\beta + \sigma + 1) - \sigma y^2 - \beta^2 z}{xy - \beta z},$$

$$G_3(x, y, z) = \frac{1}{2(xy - \beta z)} \left[\sigma \left(2x^2(z - \rho) + 2xy + xz(z - 2\rho) - yz(z - 2\rho) \right) - 2\sigma^2(x - y)^2 + 2xy - (\beta + 1)z \right) \left[\sigma \left(2x^2(z - \rho) + 2xy + xz(z - 2\rho) - yz(z - 2\rho) \right) - 2\sigma^2(x - y)^2 + 2xy - (\beta + 1)z \right],$$

as in the previous case, let us integrate the system from the same initial conditions and period; the result is shown in Fig. 3.

The third conserved quantity is then,

$$C_{3}(x,y,z) = \frac{1}{2} x \left(\sigma x - z^{2} + 2\rho z\right) - y(\sigma x + z) + (\beta z - xy) \exp\left(\int_{0}^{t} \Phi_{3}(\tau_{1}) d\tau_{1}\right) \times$$

$$\times \int_{0}^{t} \exp\left(-\int_{0}^{\tau_{2}} \Phi_{3}(\tau_{1}) d\tau_{1}\right) G_{3}(x(\tau_{2}), y(\tau_{2}), z(\tau_{2})) d\tau_{2}.$$

$$\Phi_{3}(t) = F_{3}(x(t), y(t), z(t)).$$
(13)

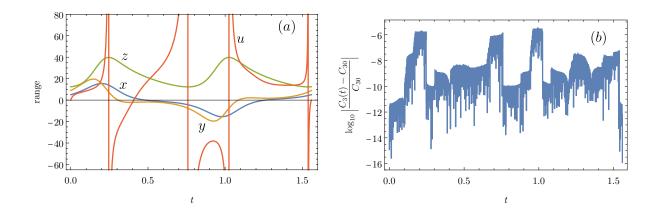


FIG. 3. Extended Lorenz system and its third conserved quantity. In (a), the functions for the UPO considered are shown; in (b), the associated magnitude of the relative error in the conserved quantity estimation is shown on a logarithmic scale.

Firstly, it is noted that the conserved quantities $C_k(x,y,z)$ for k=1,2 respect the \mathbb{Z}_2 symmetry; that is, they are left invariant under this transformation. However, this is not the case for $C_3(x,y,z)$, which transforms into a distinct conserved quantity $\bar{C}_3(x,y,z) = C_3(-x,-y,z)$, different from $C_3(x,y,z)$ itself. On the other hand, the inclusion of integral terms is a significant feature. This term implies that the value of $C_k(x,y,z)$ is not solely determined by the instantaneous state of the system (x(t),y(t),z(t)), but also depends on the entire history of its trajectory from an initial time (taken as $t_0=0$ in the explicit integral forms) to the current time t. Such a construction suggests a form of non-locality in time. This characteristic contrasts with classical conserved quantities derived from Noether's theorem, which are typically local functions of the system's generalized coordinates and velocities (or fields and their derivatives) at a single instant of time¹⁸. The existence of such history-dependent conserved quantities for the Lorenz system warrants further investigation into their physical and mathematical implications.

DISCUSSION AND FUTURE DIRECTIONS

The Lorenz system, a cornerstone of deterministic chaos and nonlinear dynamics, continues to yield profound insights into the structure of complex systems. Contrary to the prevailing view that its chaotic regime is devoid of conserved quantities, this work establishes the existence of *three nontrivial conserved quantities* within the standard Lorenz equations.

This finding holds throughout the chaotic attractor, where each periodic orbit—whether stable or unstable—is associated with specific constant values for each of these conserved quantities $C_k(x,y,z)$. For period-1 orbits intersecting a Poincaré section, the constant value of such a conserved quantity along the orbit can be conceptually likened to a characteristic parameter, analogous to how a radius defines a circumference. This topological equivalence can be illustrated: consider the geometric center of the closed curve formed by the orbit. One can then place a unit sphere at this center, project the curve onto the sphere's surface (its 'shadow'), and subsequently project this shadow onto the sphere's equator. Since this composite transformation is invertible, such an equivalence is implied. A key aspect of these conserved quantities, as formulated via the proposed ansatz, is their non-local character in time, imparted by an integral component. This feature marks a conceptual departure from classical conserved quantities. Typically, conserved quantities arising from continuous symmetries via Noether's first theorem are local functions of the system's state variables and their time derivatives at a single point in time¹⁸. This classical interpretation, as established in foundational works²², posits that conserved quantities do not qualitatively depend on the system's history, being expressed locally in terms of dynamic variables and their derivatives at a given instant. Standard texts on classical mechanics¹ and field theory²³ corroborate this, deriving conserved quantities like energy, momentum, and angular momentum as functions of the system's instantaneous state. The insights derived from foundational analyses reinforce that while standard Noether charges are defined instantaneously, more generalized theoretical frameworks or specific physical contexts can indeed accommodate conserved quantities defined through integrals over time or space. Such integral-defined quantities inherently encapsulate information about the system's accumulated history or its global properties, rather than being solely dependent on its state at a specific moment. This notion of history-dependent or non-local conserved quantities is not without precedent. For instance, extensions to Noether's theorem address scenarios where the action is invariant only up to a total derivative. In such cases, while the resulting conserved current often remains local, the allowance for invariance up to a surface term represents a significant generalization²⁴. Furthermore, in physical systems characterized by memory or retardations, often described by integro-differential equations, it is common for physical quantities to emerge whose evolution involves time integrals. If such systems can be derived from a variational principle involving a non-local Lagrangian (e.g., one that depends on inte-

grals of dynamic variables over past times), then any conserved quantity derived through a Noether-like procedure could inherit this non-locality and, consequently, a dependence on history^{25–27}. Although the Lorenz system is a dissipative, chaotic system—distinct from the conservative, integrable systems where action-angle variables (defined by integrals over closed paths in phase space ^{15,18}) are most commonly applied—the mathematical structure of a conserved quantity involving an integral over time, such as the proposed $C_k(x,y,z)$ discussed herein, suggests a broader class of invariants than those strictly local in time. The integral term effectively imbues such a quantity with a "memory" of the path traversed by the system through its state space. The implications of rigorously establishing these nonlocal conserved quantities for the Lorenz system are significant, as they could provide new analytical tools for constraining the system's long-term dynamics and offer a deeper understanding of the structure of its strange attractor. Future research must focus on the precise conditions under which these integrals converge and maintain their time-invariance along the system's trajectories (beyond period-1 orbits). Furthermore, it would be compelling to investigate whether these quantities are linked to any underlying, perhaps non-obvious, symmetries or whether they can offer insights into the effective dimensionality or emergent organizational principles within the chaotic flow. Connections to information-theoretic measures that accumulate over time or to concepts from non-equilibrium statistical mechanics might also prove to be fruitful avenues for exploration^{3,15}. This discovery fundamentally alters the conventional understanding of the Lorenz system, necessitating a re-examination of its properties, including its potential integrability, underlying symmetries, and long-term behavior. The presence of these conserved quantities prompts a re-evaluation of the notion of "conservation" in the context of chaotic dynamics. Future theoretical work should investigate the origin and mathematical structure of these invariants to determine if they arise from hidden symmetries, geometric constraints, or deeper algebraic frameworks. Such insights could pave the way for a more unified understanding of conservation laws in dissipative systems. The well-documented \mathbb{Z}_2 symmetry of the Lorenz system, when considered alongside these newly identified conserved quantities, suggests the possibility of symmetrybased reductions or alternative interpretations of its dynamics. These could, in turn, lead to effective simplifications or new perspectives, facilitating both analytical and numerical study. From a computational standpoint, the confirmation of these conserved quantities, supported by numerical results (e.g., Figs. 1-3), offers a crucial validation target for emerg-

ing methodologies. These include symbolic regression, data-driven discovery of dynamical laws, and machine learning algorithms designed to uncover hidden invariants from timeseries data²⁸⁻³⁰. Such approaches could further illuminate the internal structure of the Lorenz system and potentially be generalized to other systems exhibiting similar dynamics. Moreover, the implications of this finding extend beyond a single system. Many physical and engineered systems are modeled as high-dimensional or coupled ensembles of Lorenztype oscillators. The role of these conserved quantities in such networks—particularly in the emergence of collective behavior, synchronization, and pattern formation—remains an open and compelling area for investigation. For instance, physically extended Lorenz systems, which incorporate additional variables to better approximate fluid dynamics, may preserve or reflect these conserved structures in complex ways³¹. Despite this progress, significant analytical challenges remain. Understanding how these conserved quantities interact with the global attractor geometry and what constraints they impose on invariant measures and bifurcation structures represents a deep and largely unexplored mathematical frontier. Classical open problems—such as the full resolution of Smale's 14th problem³² or the rigorous classification of attractor properties across parameter regimes—may now be approached with renewed tools and motivation. Ultimately, the Lorenz system demonstrates that intricate and seemingly erratic behavior can emerge from deceptively simple equations. The revelation that this behavior can coexist with the internal constraints of conserved quantities reshapes our understanding of chaos itself. This aligns with the broader understanding that while the classical formulation of Noether's theorem implies history-independent conserved quantities, generalizations or more sophisticated physical contexts readily admit conserved quantities defined via temporal or spatial integrals, which inherently reflect the system's "history" or global properties. This suggests that even within dissipative and turbulent regimes, order and structure may persist.

IV. CONCLUSION

The Lorenz system, long considered a cornerstone in the study of nonlinear dynamics, has traditionally been characterized by the absence of classical conserved quantities within its chaotic operational regime. This view, closely associated with its dissipative nature and the resulting contraction of phase space volume, has supported the prevailing interpretation of the system as fundamentally non-integrable and inherently unpredictable in the long term. However, the identification of three nontrivial conserved quantities in the present work necessitates a significant revision of this perspective. These invariants demonstrate that, contrary to conventional wisdom, constraining structures can coexist with dissipation and chaos. While the Lorenz system remains non-Hamiltonian and exhibits strong nonlinear and dissipative behavior, the existence of conserved quantities reconfigures our understanding of its internal dynamics and the mechanisms underlying the formation of its strange attractor. The system's well-known rotational (\mathbb{Z}_2) symmetry may now be seen in the context of a richer structure that gives rise to these invariants, suggesting a deeper geometric or algebraic Rather than diminishing the Lorenz system's importance as a model of chaotic behavior, this newfound structure enhances its value as a paradigm for studying the interplay between order and chaos. It illustrates that complex dynamics can arise not only from the absence of constraints but also from their subtle and non-obvious presence. The Lorenz system continues to inform an impressive spectrum of scientific domains—from atmospheric modeling and fluid mechanics to laser physics, chemistry, biology, engineering, and pure mathematics—offering fresh insights into the behavior of complex systems. In light of these findings, it stands as a testament to the evolving nature of our understanding of dynamical systems and the potential for conserved structures even in contexts where they were previously thought to be absent.

ACKNOWLEDGMENTS

The author conducted this research independently at the Departamento de Física, Facultad de Ciencias, Universidad de Chile, without external funding.

DATA AVAILABILITY STATEMENT

Numerical calculations were performed using Mathematica version 12.0.

REFERENCES

¹H. Goldstein, C. P. Poole, and J. L. Safko, *Classical Mechanics*, 3rd ed. (Addison Wesley, 2002).

- ²E. N. Lorenz, "Deterministic nonperiodic flow," Journal of the Atmospheric Sciences **20**, 130–141 (1963).
- ³S. H. Strogatz, Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering, 2nd ed. (Westview Press, 2015).
- ⁴V. E. Tarasov, "Fractional dynamical systems," in *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media* (Springer Berlin Heidelberg, Berlin, Heidelberg, 2010) pp. 293–313.
- ⁵J. Cresson, "Fractional embedding of differential operators and lagrangian systems," Journal of Mathematical Physics **48**, 033504 (2007).
- ⁶T. N. Palmer, "Extended-range atmospheric prediction and the lorenz model," Bulletin of the American Meteorological Society **74**, 49–66 (1993).
- ⁷H. Haken, "Analogy between higher instabilities in fluids and lasers," Physics Letters A **53**, 77–78 (1975).
- ⁸L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," Phys. Rev. Lett. **64**, 821–824 (1990).
- ⁹L. Liu, S. Miao, and S. Liu, "A novel image encryption algorithm based on lorenz system and dynamic s-box," Cryptography **6**, 53 (2022).
- ¹⁰S. K. Scott, B. Peng, A. S. Tomlin, and K. Showalter, "Transient chaos in a closed chemical system," The Journal of Chemical Physics **94**, 1134–1140 (1991).
- ¹¹R. Schwartz, Biological Modeling and Simulation: A Survey of Practical Methodologies (MIT Press, Cambridge, MA, 2008).
- ¹²B. Mandelbrot, "The variation of certain speculative prices," The Journal of Business **36**, 394–419 (1963).
- ¹³É. Ghys, "Knots and dynamics," in *Proceedings of the International Congress of Mathematicians*, Vol. 1 (European Mathematical Society, Madrid, 2007) pp. 247–277.
- ¹⁴J. S. Birman and R. F. Williams, "Knotted periodic orbits in dynamical systems—i: Lorenz's equations," Topology **22**, 47–82 (1983).
- ¹⁵E. Ott, Chaos in Dynamical Systems, 2nd ed. (Cambridge University Press, 2002).
- ¹⁶T. Alberti, D. Faranda, V. Lucarini, R. V. Donner, B. Dubrulle, and F. Daviaud, "Scale dependence of fractal dimension in deterministic and stochastic lorenz-63 systems," Chaos: An Interdisciplinary Journal of Nonlinear Science 33, 023144 (2023).

- ¹⁷C. Sparrow, The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors (Springer-Verlag, 1982).
- ¹⁸M. Tabor, Chaos and Integrability in Nonlinear Dynamics: An Introduction (Wiley, 1989).
- ¹⁹V. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, "On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations," Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **461**, 2451–2477 (2005).
- ²⁰G. Bluman and S. Anco, Symmetry and Integration Methods for Differential Equations, Applied Mathematical Sciences No. v. 154 (Springer, 2002).
- ²¹P. Cvitanović *et al.*, *Chaos: Classical and Quantum* (Niels Bohr Institute, 2005) online version available at https://chaosbook.org.
- ²²E. Noether, "Invariante variationsprobleme," Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 235–257 (1918), english translation: Transport Theory and Statistical Physics, 1(3), 186–207 (1971).
- ²³M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley, 1995).
- ²⁴P. J. Olver, Applications of Lie Groups to Differential Equations, 2nd ed. (Springer-Verlag, 1993).
- ²⁵F. Riewe, "Nonconservative lagrangian and hamiltonian mechanics," Physical Review E **53**, 1890–1899 (1996).
- ²⁶Z. E. Musielak, "General conditions for the existence of non-standard lagrangians and their physical interpretation," Chaos, Solitons & Fractals **38**, 954–962 (2008).
- ²⁷B. D. Vujanović and S. E. Jones, *Variational Methods in Nonconservative Phenomena* (Academic Press, 1988).
- ²⁸D. Angelis, F. Sofos, and T. E. Karakasidis, "Artificial intelligence in physical sciences: Symbolic regression trends and perspectives," Archives of Computational Methods in Engineering **30**, 3845–3865 (2023).
- ²⁹M. Schmidt and H. Lipson, "Distilling free-form natural laws from experimental data," Science **324**, 81–85 (2009).
- ³⁰W. L. Cava, P. Orzechowski, B. Burlacu, F. O. de Francca, M. Virgolin, Y. Jin, M. Kommenda, and J. H. Moore, "Contemporary symbolic regression methods and their relative performance," Advances in neural information processing systems **2021 DB1**, 1–16 (2021).

- ³¹V. Krishnamurthy, J. L. Kinter III, and J. Shukla, "A physically extended lorenz system," Journal of the Atmospheric Sciences **56**, 29–43 (1999).
- $^{32}\mathrm{W}.$ Tucker, "A rigorous ode solver and smale's 14th problem," Foundations of Computational Mathematics **2**, 53–117 (2002).