

Entropy, purity and gluon cascades at high energies with recombinations and transitions to vacuum

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Abstract

We study one dimensional dipole cascade models in the high-energy limit of QCD. Motivated by data on hadron multiplicities in the LHCb kinematical range, we generalize existing cascade models for splitting and recombination to account also for transitions to the vacuum. This modification allows us to describe the data. Furthermore, we perform both analytical and numerical studies of the cascades and find that the cascade including loop corrections, as well as the one accounting for transitions to the vacuum, can both be related to the Negative Binomial Distribution. In the latter case, however, one must properly normalize the cascade to account for transitions to the vacuum. We also study the scaling properties of the cascades and identify new regimes, which we call "focal" and "parallel." Finally, we investigate the Quantum Information (QI) measures of the cascades, which allow us to highlight their distinctive properties.

1 Introduction

With recent advances in quantum information and quantum computing, high-energy physics has begun to ask questions similar to those asked in the foundations of quantum mechanics [1]. These questions concern manifestations of entanglement that can be observed in various correlations of measured quantities. In high-energy physics, such correlations are studied in processes involving the top quark, Higgs boson decay, α polarization, and multiparticle production, spin correlations in quark-antiquark production and relation of spin and angular momentum [1–10]. Recently, it has been proposed that entanglement can be studied in the process of Deep Inelastic Scattering (DIS) where electron probes a proton with a virtual photon [11]. In isolation, a proton is a coherent, pure

quantum state. Interaction destroys quantum coherence, and moreover, in DIS certain degrees of freedom are effectively integrated out, leading to the formation of a mixed state.

This process has been studied in the high-energy limit of QCD [11] where it is natural to formulate it using Mueller dipole picture [12,13]. The proton wave function is constructed from color dipoles and as one goes to higher and higher energy, more and more dipoles are produced. By tracing out the quarks, keeping the size of the initial dipole fixed, and integrating over the color and transverse sizes of the daughter dipoles, the large- x degrees of freedom become entangled with the low- x degrees of freedom — represented by gluons (or by dipoles in Mueller’s approach). This procedure leads to the emergence of a so-called reduced density matrix characterizing soft gluons *i.e.* low x degrees of freedom [14] and the corresponding partonic entanglement entropy. The entries of the density matrices are given by dipole multiplicities which obey the low x evolution equations [15].

In Ref. [11] it has been conjectured that the partonic entanglement entropy is directly related to the entropy of produced hadrons. This picture has been largely confirmed by recent studies of the DIS data, both fully inclusive and diffractive [16–19], and there are indications that it holds for the pp case as well [6,20]. In the high-energy limit entropy has been found to grow linearly with rapidity [11], which confirms earlier results [21] (see also Ref. [22] for a review) indicating maximal entanglement and signaling conformal properties of QCD at high energies [23]. For other papers addressing the entanglement entropy in high-energy collisions see [24–41].

The proton-proton case is more complex than DIS, because both the projectile and the probe are bound states characterized by strong interactions. Furthermore, the bulk of final states is produced in the central rapidity region where one can not easily define projectile and target. However, one may consider such measurements in which the partonic systems of protons are probed in a phase space region where one proton may be considered as a dilute system of partons, while the other as a dense one. Such processes may be addressed within the framework of Mueller’s one dimensional dipole cascades [13], which are believed to mimic the four dimensional QCD evolution. Different forms of such cascades have been studied in Refs. [11,26,42–50].

Throughout this paper we use the simplest version of the reaction-diffusion process [42], which was shown to follow from the Reggeon field theory with 3-Pomeron and 4-Pomeron couplings [47]. The resulting cascade can be characterized by scaling that emerges from an interplay of the production part and the recombination part [42]. We carry numerical studies of the splitting-recombination cascades and find a new regime of this scaling. We also show that the resulting probability distribution is to a very good approximation a Negative Binomial Distribution (NBD) commonly used to describe multiplicity distributions in hadron production [51]. We show that the cascade multiplicity and the NBD parameter k first rise with increasing energy and then saturate at a fixed, relatively large value. However, the probability distribution never reaches the Poisson limit. Rising k is in flat contradiction with the LHC multiplicity data [52] where k decreases with energy. Increasing k naturally follows from many multiparticle production models such as glittering glasma [53] or string percolation model [54].

To circumvent this discrepancy, we generalize the parton cascade model to include not only dipole recombination but also dipole transition to vacuum. Such transitions appear in the RFT [47], although the probabilistic interpretation is hampered by the negative sign of merging probability [47, 49]. In the generalization proposed here all probabilities are positive. Unfortunately, the behavior of the NBD k parameter is qualitatively not changed, however, the multiplicity does no longer saturate and tends to zero for large y . In this way the system is pulled into a zero particle state, but this limit is achieved with constant k .

We further discuss quantum measures that allow us to characterize properties of the dipole cascades. Those are Krylov complexity, variance, and most importantly entropy and purity. The study follows earlier work [55].

We attempt to use the splitting-recombination cascade to describe hadronic entropy as can be obtained from the LHCb measurement of forward hadron production. However, the recombination is insufficient. One has to add an additional term (discussed above) which effectively models transition of dipoles to states that are not measured or to vacuum. With such modification we achieve description of entropy data. Furthermore, we find its approximate analytical solution which again can be related to the NBD but with a new term corresponding to probability of particles recombining to vacuum.

The paper is organized as follows: in Sect. 2 we formulate the cascade equations and relate the resulting probabilities to geometric and negative binomial distributions. Then in Sect. 3 we study numerical results for the cascades introduced in Sect. 2, identify different evolution regimes, analyze the probability distributions and study the resulting scaling properties. Sect. 4 is devoted to the quantum measures of the multiplicity distributions following from the cascade equations, and in Sect. 5 we apply the formalism developed in the previous sections to the LHCb experimental data on particle multiplicities in the forward rapidity region. Discussion and conclusions are presented in Sect. 6.

2 Dipole evolution equations

A convenient framework to describe multiple particle production in the low x limit is provided by the color dipole picture [12, 13], for a review see [56]. In this approach, which was originally formulated for DIS, the initial dipole is created as a result of the splitting of a virtual photon into a quark-antiquark pair. Then, as the energy increases, more and more dipoles are produced. This cascade can be then used to describe the multiple-particle production. It corresponds to the initial state radiation or, in the Monte Carlo language, to the initial state shower. Starting from the QCD in the large number of colors (N_c) approximation one can derive equation for dipole probability distribution $p_n(y, r)$, which takes the following generic form [12],

$$\frac{\partial p_n(y, \{r\})}{\partial y} = \sum_m K \otimes p_m(y, \{r\}) \quad (1)$$

In this equation, K is the kernel of the Balitsky, Fadin, Kuraev, Lipatov (BFKL) [57, 58] evolution written in two-dimensional position space, and r refers to the size of dipoles in transverse space. This equation can then be used to obtain the BFKL equation in the linear regime and the Balitsky, Kovchegov (BK) equation [59, 60] if multiple scattering and/or recombination are taken into account.

Recently, one dimensional reduction of the equation for dipole multiplicities [15] become attractive due to its simplicity and phenomenological success in describing entropy of produced hadrons (see *e.g.* [11])

$$\partial_y p_n(y) = -\alpha n p_n(y) + \alpha(n-1)p_{n-1}(y). \quad (2)$$

Here, α corresponds to the splitting kernel which in the one-dimensional case is just a number. It directly corresponds to the Pomeron intercept. In the BFKL case $\alpha = 4 \ln 2(N_c \alpha_s / \pi)$, where α_s is a strong coupling constant, here, it has to be either fitted or taken as the BFKL value.

The interpretation of the terms entering Eq. (2) is rather straightforward. The probability of finding n partons (dipoles) in the system is p_n and the total probability that the system transforms into a configuration of $n+1$ dipoles is $\alpha n p_n$. Such a process reduces the probability to find n partons, therefore it enters with a minus sign. Additionally, the system of $n-1$ dipoles can transform into n dipoles by splitting $1 \rightarrow 2$, increasing the probability to find n partons, and the probability is in this case $\alpha(n-1)p_{n-1}$. It enters with the plus sign because it corresponds to the increase in the number of partons (gain). One can explicitly check that the normalization $\sum_n p_n = 1$ is preserved in the course of evolution.

The analytical solution to Eq. (2) is known already for some time (see *e.g.* [61]), as it corresponds to the BFKL evolution where the average multiplicity \bar{n} grows exponentially with y

$$\bar{n}(y) = e^{\alpha y}, \quad (3)$$

which corresponds to the powerlike growth with energy. The probability distribution is also known and is given by the geometric distribution

$$p_n(y) = \frac{1}{\bar{n}} \left(\frac{\bar{n}-1}{\bar{n}} \right)^{n-1}, \quad (4)$$

where $n = 1, 2, \dots$. Note that using the standard notation for the geometric distribution

$$P_m^G(\bar{n}_G) = \frac{1}{\bar{n}_G + 1} \left(\frac{\bar{n}_G}{\bar{n}_G + 1} \right)^m, \quad (5)$$

where $m = 0, 1, 2, \dots$ and \bar{n}_G is the average multiplicity, we have

$$p_n(\bar{n}) = P_{n-1}^G(\bar{n}_G = \bar{n} - 1). \quad (6)$$

where $n = 1, 2, \dots$. Evolution (2) starts from $p_1(0) = 1$, $p_{n>1}(0) = 0$ and never produces a nonzero p_0 , while the geometric distribution (5) includes P_0^G . One can easily check that (6) is properly normalized¹.

¹See also study of 1 D model in [14] where the lowest multiplicity is in fact p_0

The geometric distribution corresponds to a special limit of the negative binomial distribution (NBD)

$$P_m^{\text{NBD}}(k, \bar{n}_{\text{NBD}}) = \frac{\Gamma(m+k)}{m! \Gamma(k)} \left(\frac{\bar{n}_{\text{NBD}}}{k + \bar{n}_{\text{NBD}}} \right)^m \left(\frac{k}{k + \bar{n}_{\text{NBD}}} \right)^k, \quad (7)$$

where $m = 0, 1, 2, \dots$. Indeed, for $k = 1$ we have

$$P_m^{\text{G}}(\bar{n}_{\text{G}}) = P_m^{\text{NBD}}(1, \bar{n}_{\text{G}}). \quad (8)$$

For $k \rightarrow \infty$

$$P_m^{\text{NBD}}(k \rightarrow \infty, \bar{n}_{\text{NBD}}) = P_m^{\text{Poisson}}(\bar{n}_{\text{NBD}}). \quad (9)$$

The negative binomial distribution is fully determined by two parameters \bar{n}_{NBD} and k . The parameter k is related to the variance [62]

$$\sigma_{\text{NBD}}^2 = \sum_{m=0}^{\infty} m^2 P_m^{\text{NBD}}(k, \bar{n}_{\text{NBD}}) - \bar{n}_{\text{NBD}}^2 \quad (10)$$

in the following way

$$k = \frac{\bar{n}_{\text{NBD}}^2}{\sigma_{\text{NBD}}^2 - \bar{n}_{\text{NBD}}}. \quad (11)$$

In other words, P_m^{NBD} is determined by the first two moments.

As the energy increases, it becomes necessary to take into account the unitarity corrections. In general, this is a complicated problem, leading to nonlinear evolution equations that are nonlocal in transverse dimension [56]. In the 1D scenario, one method is to take into account the terms corresponding to dipole fusion. The resulting equation reads [42]

$$\begin{aligned} \partial_y p_n(y) = & -\alpha n p_n(y) + \alpha(n-1) p_{n-1}(y) \\ & + \beta n(n+1) p_{n+1}(y) - \beta n(n-1) p_n(y) \end{aligned} \quad (12)$$

where the $\beta \simeq \alpha \alpha_s^2$. The equation models contribution corresponding to pomeron loop diagrams [47, 49] and slows down the increase of mean multiplicity.

The interpretation of β terms entering Eq. (12) is straightforward. The system of $n+1$ partons can transform by the annihilation process $2 \rightarrow 1$ into n parton configuration. The probability to find $n+1$ partons is p_{n+1} , and annihilation proceeds by picking one parton out of $n+1$, and the second one out of remaining n partons, which gives an increase of probability of finding n partons by $\beta n(n+1) p_{n+1}$, hence a $+$ sign. In principle there should be a factor $1/2$ in front to avoid double counting, however, since we will treat β as a free parameter, it can be absorbed into β . Analogously, annihilation of two partons out of n reduces the probability, and the emission probability, in this case $\beta n(n-1) p_n$, enters with a $-$ sign. It is easy to show that also in this case $\sum_n p_n = 1$.

This equation has been already studied in the literature where it has been explicitly demonstrated using approximated analytical solution valid at large rapidity [63], and numerical methods valid at any rapidity that the mean multiplicity saturates [55]. We

extend the analysis of [55] and explicitly demonstrate this in the next section, where we will also argue that the probability distribution generated by (12) is very close to the NBD,

$$p_n = P_{n-1}^{\text{NBD}}(k, \bar{n}_{\text{NBD}} = \bar{n} - 1), \quad (13)$$

with k given by (11). It will also turn out that k saturates for large y .

Since experimentally, at least in pp collisions at the LHC [52], k decreases with energy, we have generalized Eq. (12) to the case where two dipoles can annihilate to vacuum, $2 \rightarrow 0$. Such transitions appear in the Reggeon effective field theory [47]. They can also mimic disappearance of particles from the detector or as regions of phase space that are not available in the considered measurement.

To this end we add two more terms proportional to a new parameter γ . By analogy with the previous considerations, we have a loss term where n partons transform into $n - 2$ configuration with probability $-\gamma n(n - 1)p_n$, and a gain term where $n + 2$ partons transform into n parton configuration with probability $+\gamma(n + 1)(n + 2)p_{n+2}$. The resulting equation reads as follows

$$\begin{aligned} \partial_y p_n(y) = & -\alpha n p_n(y) + \alpha(n - 1)p_{n-1}(y) \\ & + \beta n(n + 1)p_{n+1}(y) - \beta n(n - 1)p_n(y) \\ & + \gamma(n + 1)(n + 2)p_{n+2}(y) - \gamma n(n - 1)p_n(y). \end{aligned} \quad (14)$$

One can check that this equation conserves probability. To some extent, similar considerations prompted the authors of the Ref. [18] to generalize the 1 D model to account for measurements in the moving rapidity window. One necessarily had to introduce p_0 that was accounting for probability that particles are not detected in certain region of rapidity. The difference is that in the present paper such term is a part of evolution equation and does not have to be modeled independently.

3 Properties of dipole cascades

In this section, we perform a thorough numerical analysis of Eqs. (12) and (14). It has been argued in Ref. [64] that Eq. (12) has an approximate attractor in a sense that the multiplicity moments of order k scale as $(\alpha/\beta)^k$. We therefore define parameter r

$$\beta = r \alpha. \quad (15)$$

In the case of 1 D Reggeon model of high energy limit of QCD we expect $r \sim \alpha_s^2$. However, in the case of pp scattering α_s can be quite large, and r may be of the order 0.05 or larger. In what follows we vary $\alpha = 0.2 \div 0.8$ and $r = 0.05 \div 0.5$. For smaller r we need better numerical accuracy. Our initial condition will be always $p_1(0) = 1$ and $p_{n>1} = 0$. In each case we check whether the probability distribution generated by the differential equations (2), (12) and (14) are properly normalized.

3.1 Cascade β

Let us first consider Eq. (12) which we will call the β -cascade, in contrast to the α -cascade of Eq. (2) and the γ -cascade of Eq. (14). In Fig. 1 we plot first 10 probabilities for $\alpha = 0.5$, and $r = 0.1$ and 0.5 . We see that for small r probabilities cross in the vicinity of some rapidity, which we call y_{cross} , and then rearrange. On the other hand, for larger r 's probabilities do not cross and are almost parallel for large values of y . In what follows, we will refer to these two regimes as *focal* and *parallel*, respectively. The physical region of α and β couplings relevant to QCD evolution is in the focal regime.

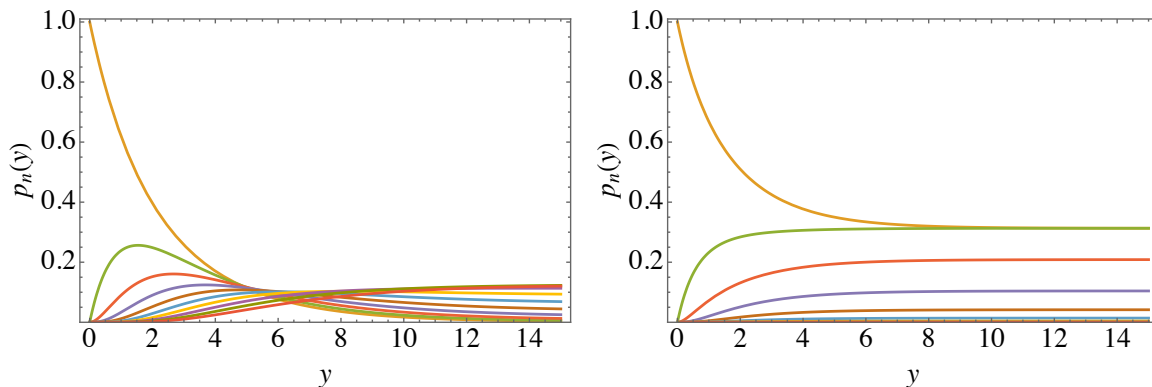


Figure 1: Probabilities $p_n(y)$ ($n = 1, \dots, 10$) for the β branching (12) for $r = 0.1$ (left) and $r = 0.5$ (right), and $\alpha = 0.5$. One can see that for small r probabilities cross at $y_{\text{cross}} \simeq 6.5$, whereas for larger r they reach asymptotic values without crossing.

Next, we study the first two moments of the probability distribution generated by Eq. (12). In Fig. 2 we show the average multiplicity $\bar{n}(y)$ for $\alpha = 0.2, 0.5$ and 0.8 and different values of r . For comparison we also show $\bar{n}(y)$ for $\beta = 0$, *i.e.* for the α -cascade without recombination. One can see that for small y multiplicity rises parallel to the case $\beta = 0$ and then saturates at the value, which depends only on r . For $r \leq 0.2$ asymptotic value of the multiplicity $\bar{n}(y \rightarrow \infty) \simeq 1/r$. For larger r multiplicities scale to one common value, which is, however, larger than $1/r$. As is clear from Fig. 2, for fixed r multiplicity saturates faster for larger α .

As is clear from Fig. 2 the saturation value of \bar{n} fulfills the attractor scaling of Ref. [64] $\bar{n}_{\text{sat}} = 1/r$ but only in the focal regime. We illustrate this in the left panel Fig. 3 where we show asymptotic values of $r\bar{n}$ as functions of r .

Next, we analyze variance $\sigma^2(y)$ for the same set of parameters as for \bar{n} above. We see that in the focal regime (left panel of Fig. 4) $\sigma^2(y)$ has a maximum which shifts to the left with increasing α , and saturates for large rapidities. For $r < 0.2$ the asymptotic value is $1/r$, *i.e.* $\sigma_{\text{sat}}^2 = \bar{n}_{\text{sat}} = 1/r$. However, as we will show below, this does not mean that the distribution is Poissonian. For larger values of r , in the parallel regime, the dependence on r deviates from $1/r$. This is illustrated in the right panel of Fig. 3.

While the probability distribution for the α cascade is determined by the geometric distribution (6) with the mean value of $\bar{n}_G = \bar{n} - 1$, it is reasonable to ask if the probability

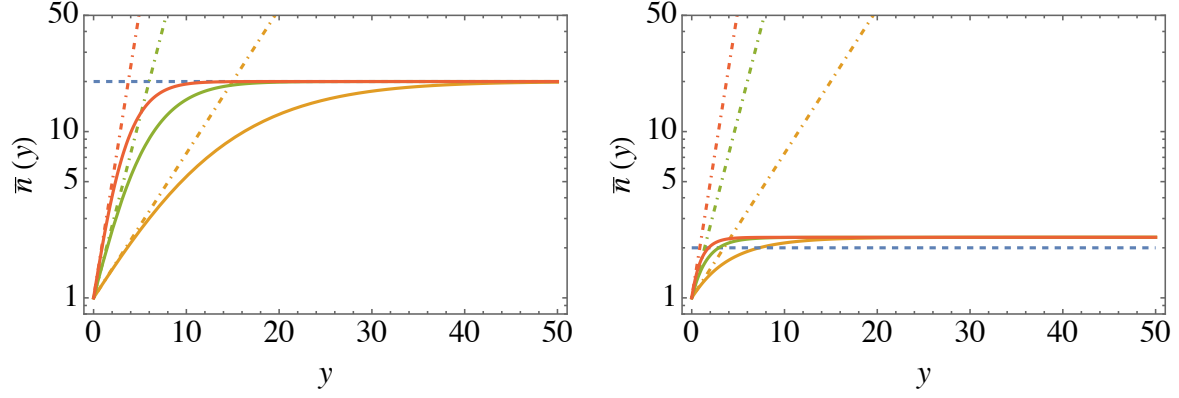


Figure 2: Mean multiplicities of the β cascade (solid lines) as functions of y for $\alpha = 0.2$ (orange), $\alpha = 0.5$ (green) and $\alpha = 0.8$ (red). Values of r : 0.05 (left panel) and 0.5 (right panel). For comparison dashed-dotted line corresponds to $\beta = 0$. Dashed blue line: $1/r$.

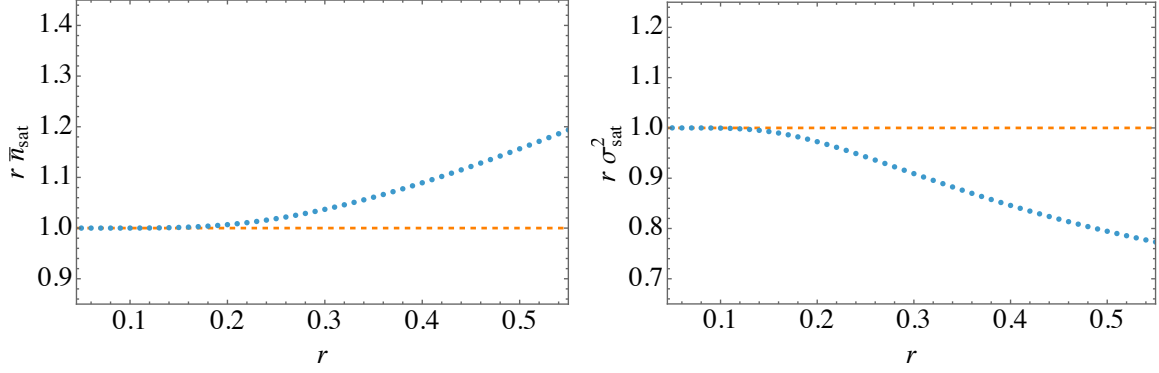


Figure 3: Asymptotic values of $r \bar{n}_{\text{sat}}$ and $r \sigma_{\text{sat}}^2$ as functions of r . Dashed orange lines correspond to $1/r$ dependence discussed in the text.

distributions given by generalized Eq.(12), that accounts for merging, can be described by the negative binomial distribution (7), such that it tends to (6) for $\beta \rightarrow 0$. So we want to check if

$$p_n(y) \stackrel{?}{=} p_n^{\text{NBD}}(y) \equiv P_{n-1}^{\text{NBD}}(k(y), \bar{n}(y) - 1), \quad (16)$$

which for $k = 1$ indeed reduces to (6).

In order to fully determine the NBD we need to know parameter $k(y)$. One can easily convince oneself that the variance of p_n is equal to σ_{NBD}^2 . Indeed

$$\sigma^2 = \sum_{n=1} n^2 p_n - \bar{n}^2 \quad (17)$$

$$\begin{aligned} &= \sum_{m=0} m^2 P_m^{\text{NBD}}(k, \bar{n} - 1) - (\bar{n} - 1)^2 \\ &= \sigma_{\text{NBD}}^2. \end{aligned} \quad (18)$$

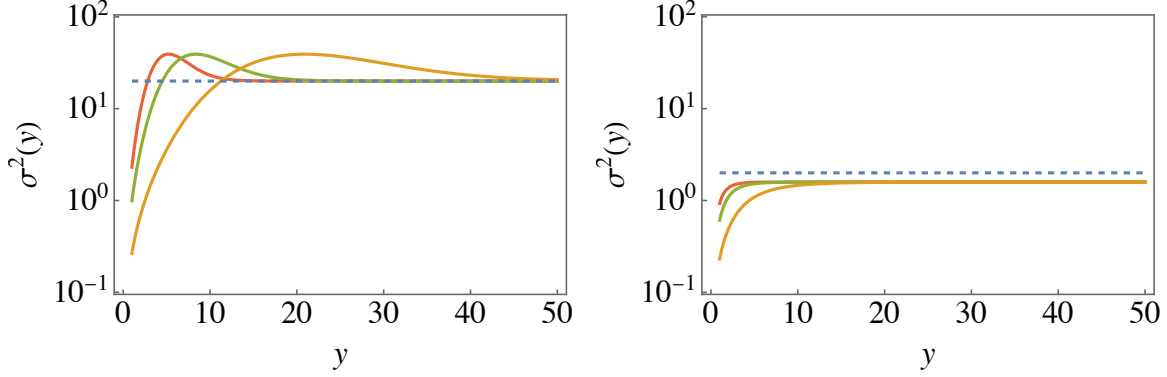


Figure 4: Variance of the β cascade (solid lines) as a function of y for $\alpha = 0.2$ (orange), $\alpha = 0.5$ (green) and $\alpha = 0.8$ (red). Values of r : 0.05 (left panel) and 0.5 (right panel). Dashed blue line: $1/r$.

Hence we arrive at the formula for k

$$k = \frac{(\bar{n} - 1)^2}{\sigma^2 - (\bar{n} - 1)} \quad (19)$$

given in terms of \bar{n} and variance of the probability distribution p_n of the β -cascade (12). One can see from Eq. (19) that when $\sigma^2 - \bar{n} = 0$, which would normally correspond to the Poisson distribution, the true distribution is still of the negative binomial type with large but finite $k < \infty$. However, numerically it may be undistinguishable from the Poisson distribution with $k = \infty$. This is a consequence of the fact that the β -cascade never fills p_0 , *i.e.* a state without any particles.

To verify the conjecture (16) we have evolved Eq. (12) for $\alpha = 0.5$ and $r = 0.05$ (focal regime), and $r = 0.5$ (parallel regime). The results are shown in Fig. 5 as blue circles. For each case we have computed the corresponding \bar{n} and k , and compared $p_n(y)$ and $p_n^{\text{NBD}}(y)$ shown in Fig. 5 as orange squares. Looking at Fig. 5 one concludes that the NBD is almost indistinguishable from the numerical multiplicity distribution of the β -cascade both in the focal and parallel regimes. We have checked that the same behavior is observed for other values of the cascade parameters. To test how well NBD describes the β -cascade we have also computed the third moment. The difference between the true value and the NBD value is indeed very small.

An important and interesting question is the energy dependence of the NBD parameter k (19). From our previous discussion on saturation of \bar{n} and σ^2 , see Fig. 3, we expect that k should saturate for large y at $k_{\text{sat}} = (1/r - 1)^2$, at least in the focal region. To this end in Fig. 6 we plot parameter k as a function of y for different couplings α and two values of r . We see that for small r (focal regime) parameter k saturates at k_{sat} defined above, whereas for larger r (parallel regime) the saturation still takes place but at a different value. Saturation is faster for larger α . The r dependence of k_{sat} is shown in Fig. 7. Interestingly, independently of α and r the asymptotic value of k is always finite, and therefore the probability distribution does not have a Poissonian limit for large y .

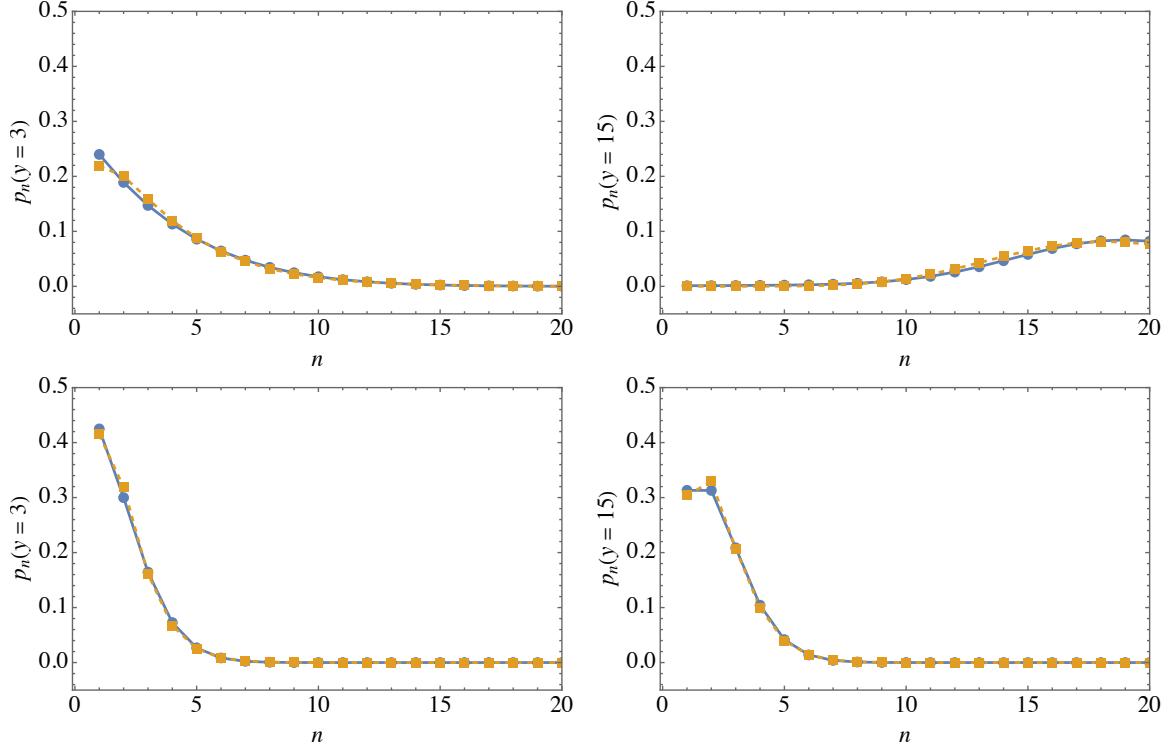


Figure 5: Probability distributions $p_n(y)$ for the β -cascade (blue circles) with $\alpha = 0.5$ and the corresponding NBD (orange squares) for selected values of $y = 3$ and 15 . In the upper row we have plotted cascade for $r = 0.05$, whereas the lower row corresponds to $r = 0.5$. Lines are drawn to guide the eyes.

For small r parameter k saturates at k_{sat} that is growing with $r \rightarrow 0$.

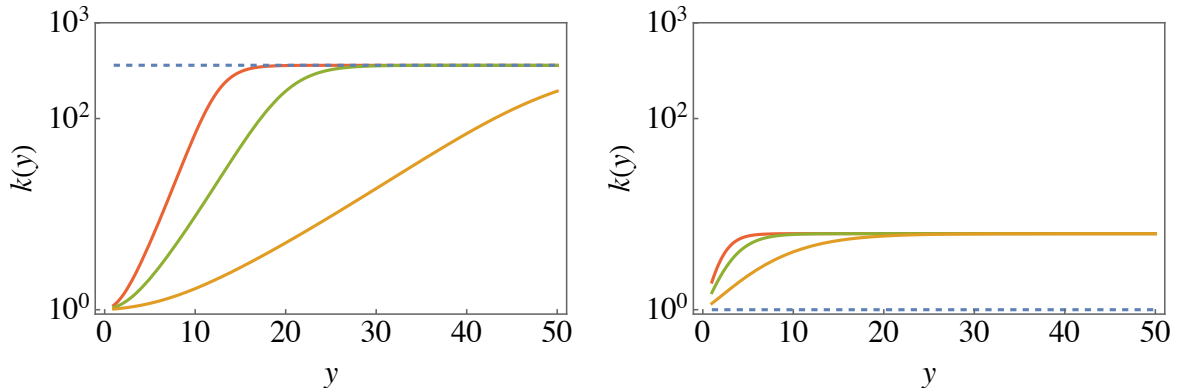


Figure 6: Parameter k of the NBD for the β -cascade (solid lines) as a function of y for $\alpha = 0.2$ (orange), $\alpha = 0.5$ (green) and $\alpha = 0.8$ (red). Values of r : 0.05 (left) and 0.5 (right). Dashed blue line corresponds to the saturation value $k_{\text{sat}} = (1/r - 1)^2$.

Let us summarize our findings for the β -cascade. The cascade shows a scaling behavior

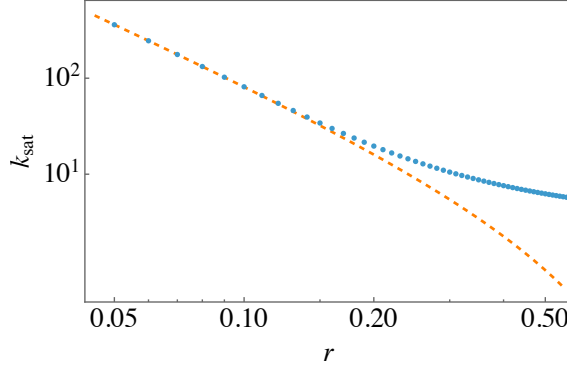


Figure 7: Asymptotic values of k_{sat} a function of r . Dashed orange line corresponds to $(1/r - 1)^2$.

for large y : average multiplicity, variance and the NBD parameter k saturate for large y at a value which depends only on r . Saturation is achieved faster for larger α . In the focal regime ($r < 0.2$) the scaling behavior is very simple

$$\begin{aligned}\bar{n} &\rightarrow \bar{n}_{\text{sat}} = \frac{1}{r}, \\ \sigma^2 &\rightarrow \sigma_{\text{sat}}^2 = \frac{1}{r}, \\ k &\rightarrow k_{\text{sat}} = \left(\frac{1}{r} - 1\right)^2.\end{aligned}\tag{20}$$

One should note that the above scaling laws have to break at $r \rightarrow 0$. We know that for the BFKL-like cascade ($r = 0$) \bar{n} does not saturate and grows with y to infinity (3), which is consistent with the $1/r$ behavior. However, in the case of geometric distribution (4) the variance $\sigma_G^2 = \bar{n}_G(\bar{n}_G + 1) = \bar{n}(\bar{n} - 1)$ does not scale as $1/r$ but rather as $1/r^2$ and, as a consequence, $k = 1$. In contrast, for the β -cascade k tends to ∞ for small r , Fig. 7, therefore, the transition from the NBD to the geometric distribution with $k = 1$ is discontinuous.

3.2 Cascade γ

Now we turn to Eq. (14) describing the cascade with recombination and with additional γ -term (γ -cascade), which we parametrize as

$$\gamma = s\beta.\tag{21}$$

The γ -term is responsible for the disappearance of particles during evolution. An immediate consequence of this process is an increase of p_0 with y , which was identically zero in the case of cascade (12). Moreover, the γ term is *defocusing* the probability distribution. This is shown in Fig. 8 where the γ term with $s = 0.5$ and 1.5 is added to the β -cascade shown in the left panel of Fig. 1.

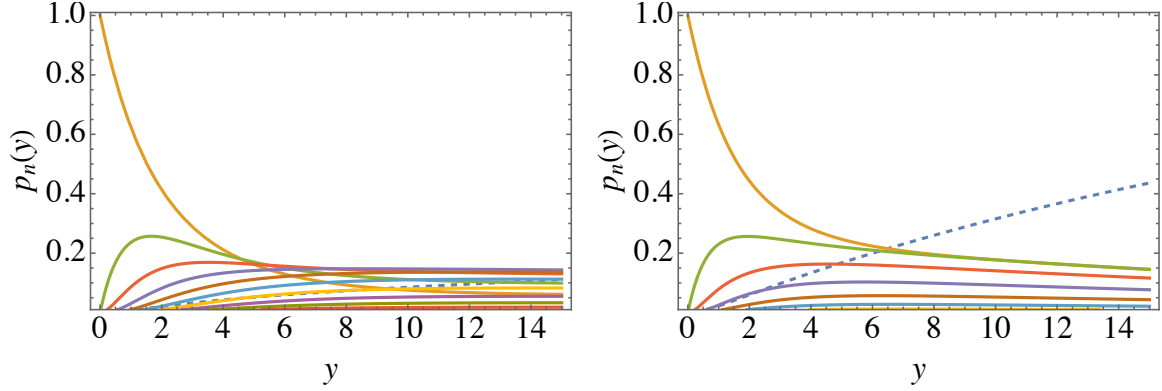


Figure 8: Probabilities $p_n(y)$ ($n = 0, \dots, 10$) for the γ branching (14) for $\alpha = 0.5$, $r = 0.1$ and $s = 0.5$ (left), and $s = 1.5$ (right). Probability p_0 is shown as a blue dashed line. The corresponding β -cascade is shown in the left panel of Fig. 1.

In Fig. 9 we show the mean value and variance for the γ -cascade with $\alpha = 0.5$ and $r = 0.1$ for different values of s . For comparison \bar{n} and σ^2 for the corresponding β -cascade are shown in blue (upper curves) together with the asymptotic values shown by dashed lines. Mean multiplicity for the α -cascade is shown by dashed-dotted line. We see that both average multiplicity and variance do not saturate and start decreasing for large y . The decrease is faster for larger s . This pattern is general and does not depend on α and r . For larger s the decrease of \bar{n} and σ^2 is getting stronger.

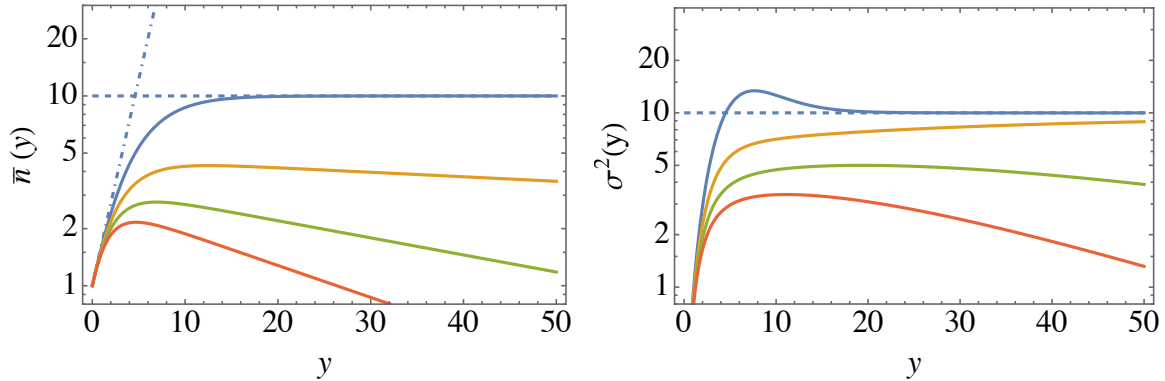


Figure 9: Mean multiplicity (left) and variance (right) for the γ branching (14) for $\alpha = 0.5$, $r = 0.1$ and $s = 0, 0.5, 1, 1.5$ (from top to bottom). Horizontal dashed lines show saturation values for the corresponding β -casccade. In the left panel mean value for the corresponding α -cascade is shown with a dashed-dotted line.

In the case of γ -cascade the identification (16) breaks down. However, it is conceivable that after subtracting p_0 , the remainder does still behave like an NBD. Subtracting p_0 from a normalized γ -cascade distribution makes the rest unnormalized. To correct for

this, we define a new distribution

$$p_n \rightarrow p_{\gamma n} = \frac{p_n}{1 - p_0}, \quad n = 1, 2, \dots, \quad (22)$$

where p_n 's are numerical solutions of Eq. (14). Note that the normalized $p_{\gamma n}$ distribution has a different mean than p_n previously denoted by \bar{n} (note that $n = 0$ does not contribute to the mean value)

$$\bar{n}_\gamma = \frac{\bar{n}}{1 - p_0}. \quad (23)$$

Analogously, the variance reads

$$\sigma_\gamma^2 = \sum_{n=1} n^2 p_{\gamma n} - \bar{n}_\gamma^2 = \frac{1}{(1 - p_0)^2} \left((1 - p_0) \sum_{n=1} n^2 p_n - \bar{n}^2 \right) \equiv \frac{1}{(1 - p_0)^2} \tilde{\sigma}_\gamma^2. \quad (24)$$

With this notation the pertinent k_γ parameter is given by

$$k_\gamma = \frac{(\bar{n} - (1 - p_0))^2}{\tilde{\sigma}_\gamma^2 - (1 - p_0)(\bar{n} - (1 - p_0))}. \quad (25)$$

Hence, we assume that for $n \geq 1$

$$p_{\gamma n}(y) \stackrel{?}{=} p_{\gamma n}^{\text{NBD}}(y) \equiv P_{n-1}^{\text{NBD}}(k_\gamma, \bar{n}_\gamma - 1). \quad (26)$$

Therefore the probability distribution is assumed to be

$$p_n = \{p_0, (1 - p_0) p_{\gamma n}^{\text{NBD}}\}. \quad (27)$$

Recall that \bar{n} is the multiplicity of the γ -cascade, whereas \bar{n}_γ is the multiplicity of the properly normalized remainder, after discarding p_0 . Likewise, k_γ is the k parameter of the NBD approximating the remainder. Therefore, distribution (27) depends on three parameters p_0 , \bar{n} and k_γ .

In Fig. 10 we plot the probability distributions $p_n(y)$ for the γ -cascade with $\alpha = 0.5$ and $s = 0.5$, and the corresponding NBD for selected values of $y = 3$ and 15. We can see that probability p_0 grows with y and the growth is faster for larger r . The remainder is very well approximated by the NBD (27). In the left panel of Fig. 11 we plot rapidity dependence of k_γ for different values of s . Unlike \bar{n} and σ^2 , which decrease with y , see Fig. 9, parameter k_γ saturates for large y at a value smaller than for the β -cascade, see the right panel of Fig. 11. Therefore, an interesting picture arises: in the case of the γ -cascade $p_0 \rightarrow 1$ asymptotically, and the remaining probabilities tend to 0, preserving, however, the shape of the NBD with constant k_γ . The driving force behind nullification of $p_{n>0}$ is vanishing \bar{n}_γ at constant k_γ .

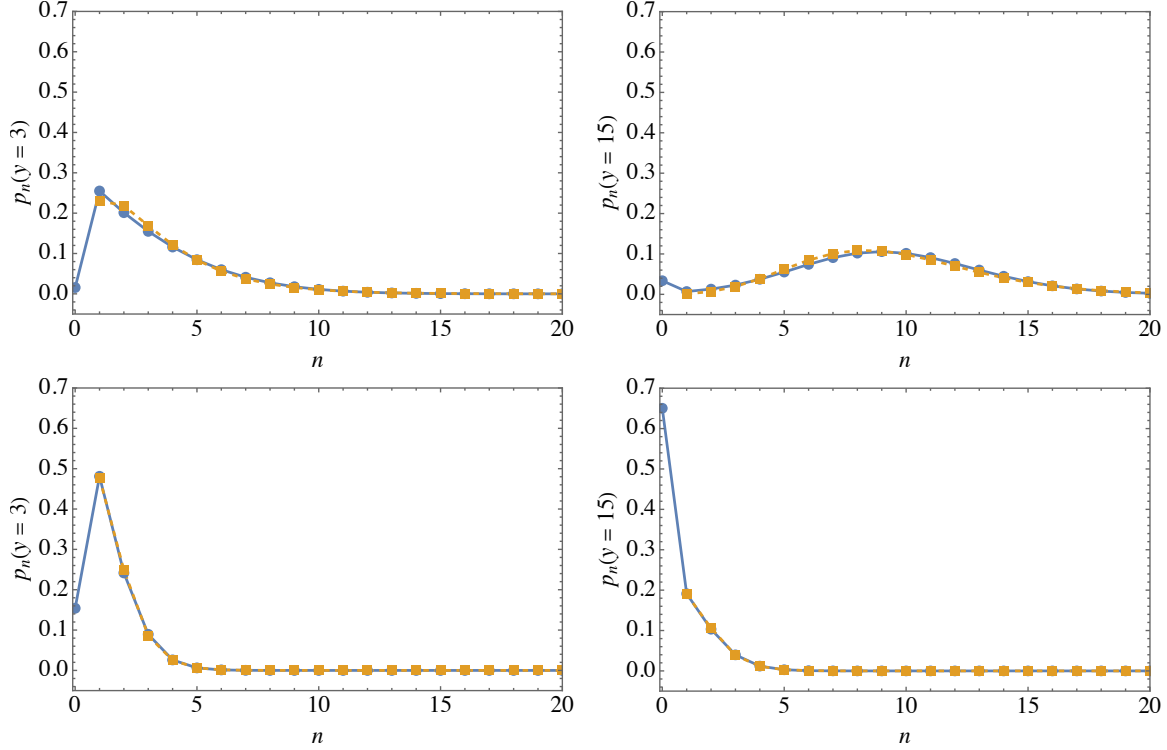


Figure 10: Probability distributions $p_n(y)$ for the γ -cascade (blue circles) with $\alpha = 0.5$ and $s = 0.5$ and the corresponding NBD (orange squares) for selected values of $y = 3$ and 15 . In the upper row we have plotted cascade for $r = 0.05$, whereas the lower row corresponds to $r = 0.5$. Lines are drawn to guide the eyes.

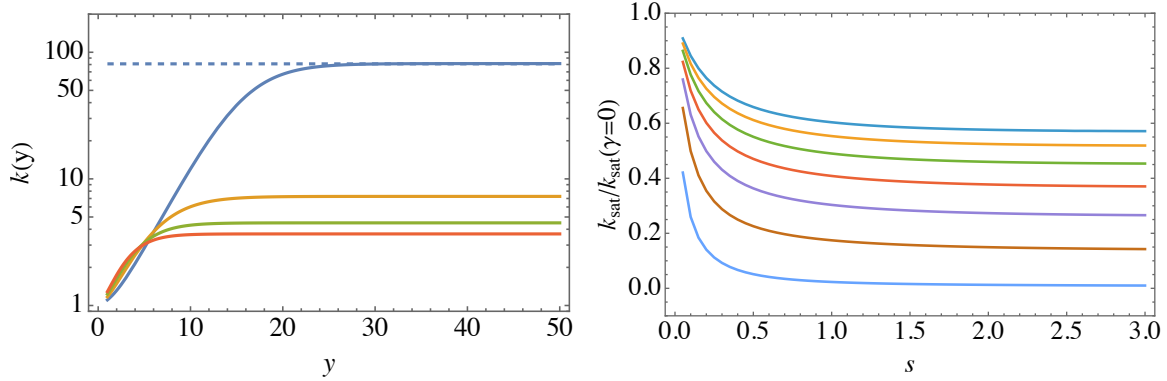


Figure 11: NBD parameter k for the γ -cascade. In left panel k is plotted as a function of y for $\alpha = 0.5$, $r = 0.1$ and different values of $s = 0, 0.5, 1, 1.5$ (from top to bottom). In the right panel ratios of saturation values of $k_{\gamma \text{ sat}}/k_{\text{sat}}$ are plotted as functions of s for different values of $r = 0.7, 0.6, 0.5, 0.4, 0.2, 0.05$ (from top to bottom).

4 Entropy and other quantum measures

In this section we discuss the behavior of entropy and other quantum measures for the cascades studied in the previous sections. The von Neuman entropy that follows from

integrating out all degrees of freedom except rapidity [11, 14] reads

$$S(y) = - \sum_n p_n(y) \ln(p_n(y)), \quad (28)$$

where $p_n(y)$ are interpreted as probabilities for a given number of partons at rapidity y . It measures how much information has been lost due to integrating out quarks and averaging over transverse and color degrees of freedom, as evolution in rapidity progresses. In the complete 3+1 D dipole model the entropy depends also on the hard scale of the measurement process, while in the 1 D model it might be introduced by promoting the partonic multiplicity to parton density function and letting it to depend on the hard scale.

In the case of the α -cascade, where the probability distribution given by (4), the entropy can be calculated analytically [11]

$$S = \bar{n} \ln \bar{n} - (\bar{n} - 1) \ln(\bar{n} - 1), \quad (29)$$

where $\bar{n}(y)$ is given by (3). For large y entropy $S \rightarrow \ln \bar{n} = \alpha y$ and grows linearly with rapidity. This is a sign of maximal entanglement [16, 19].

Recombination processes slow down the probability evolution and – as we have shown in Sect. 3.1 – the multiplicity and the NBD parameter k saturate. We expect that also entropy saturates. In the left panel of Fig. 12 we plot entropy generated in β branching process as a function of rapidity y for small (upper solid lines) and large (lower dashed lines) value of parameter r . For small $r = 0.05$ the system is in the focal regime and the maximum of entropy occurs at some finite y corresponding to the probabilities crossing shown in the left panel of Fig. 1. This behavior has been already observed in Ref. [63]. After the bump the entropy saturates at some finite value. In the left panel Fig. 12 we also show the entropy for the cascade with $r = 0.5$ in the parallel regime (dashed lines). We see that there is no bump and the entropy monotonically reaches the saturation value. In both cases the approach to saturation is faster for larger α . Since recombination processes lead to saturation, we conclude that saturation prevents the system from reaching maximal entanglement.

It is interesting to compare our results for entropy with approximate formulas relating entropy to the average multiplicity \bar{n} . To this end we use (29) and an approximate relation between entropy and \bar{n} for the Poisson distribution [63]

$$S = \frac{1}{2} \ln(2\pi e \bar{n}) - \frac{1}{12\bar{n}} + \dots \quad (30)$$

We expect (30) to work well for the β -cascade where the probability distribution is very close to the Poisson distribution. That this is indeed the case can be seen in the right panel of Fig. 12 where (30) is shown as dashed lines. Approximation (29) is not shown, as it misses the numerical results. On the contrary, one can see that (30) very well reproduces the β -cascade for large y (upper blue lines), and to a lesser accuracy the γ -cascade. The latter is to be expected since the probability distribution (including p_0) is in this case neither NBD nor Poissonian. Nevertheless, we see that the decrease of entropy with y is driven by a decrease of \bar{n} .

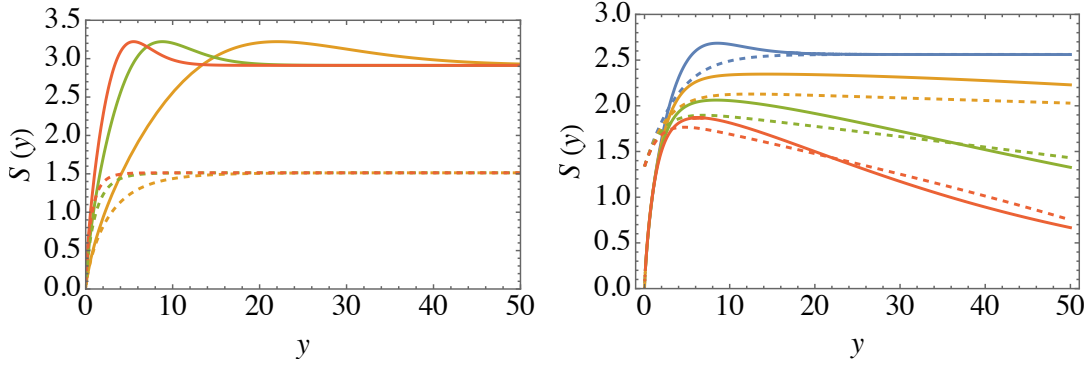


Figure 12: Entropy of the β (left) and γ (right) cascade as functions of y for $r = 0.05$ (upper left) and $r = 0.5$ (lower left) for three values of α : 0.2 (orange), 0.5 (green) and 0.8 (red). In the right panel the β cascade (upper blue line) for $\alpha = 0.5$ and $r = 0.1$ is compared with the γ cascade with $s = 0.5, 1, 1.5$ (top to bottom). Approximation (30) is shown as dashed lines.

In addition to entropy, solutions to the cascade equations can be used to calculate various quantum information (QI) measures. In the paper [55] it has been demonstrated that the equation for the BFKL-like cascade can be directly related to the Schrödinger equation for a state evolved with a boost operator having $SL(2, R)$ symmetry. This relation established a link between QI measures and the 1 D dipole model. In particular in the mean number of dipoles \bar{n} , called in the QI context a complexity, measures how the underlying quantum state spreads in the Hilbert space [65]. Quantum complexity is here of lesser interest to us, because parton multiplicities are only indirectly related to measured hadronic multiplicities. The variance has been already introduced earlier, and it quantifies the fluctuations around the average. Here we use normalized variance defined as follows

$$\delta^2 = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle^2}. \quad (31)$$

Finally, the quantity that measures the purity, *i.e.* how the system departs from a pure state as evolution progresses, is defined as

$$\gamma = \sum_n p_n^2. \quad (32)$$

This quantity is of special interest in this paper as it is bounded: for the maximally mixed state the purity is zero and for the pure state it is equal to one.

In the following, we apply the Quantum Measures to the β - and γ -cascades, which is illustrated in Fig. 13. The QI measures were introduced to study dipole cascades in [55] and have already been applied to the β -cascade. However, it is instructive to contrast the results obtained for the β -cascade with those for the γ -cascade. We see that at the beginning of the evolution, the QI measures are similar, which indicates the dominance of the linear or splitting region. However, as nonlinear corrections and the transition to the vacuum become significant, the quantities differ substantially. In particular, we see that

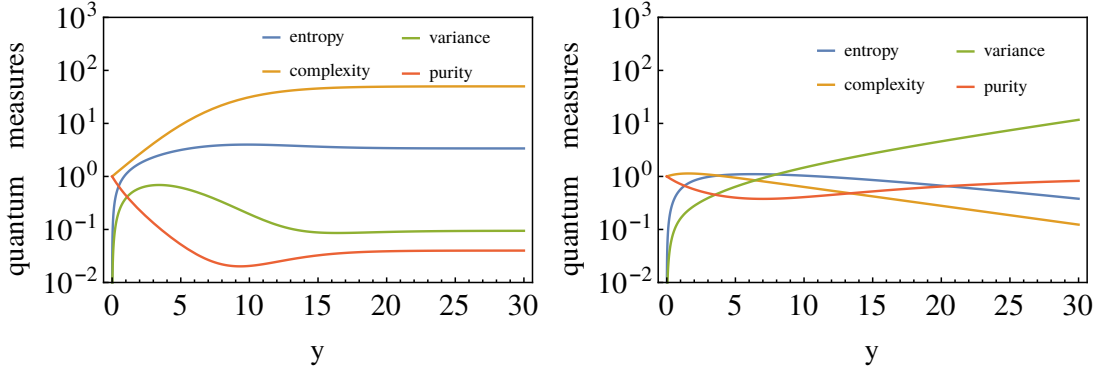


Figure 13: In the left panel: quantum measures as obtained from solutions of β -cascade Eq. (12). In the right panel: quantum measures as obtained from solutions of γ -cascade Eq. (14).

the entropy of the γ -cascade starts to drop, similarly to the complexity, whereas in the case of the β -cascade, they saturate. We also observe that as the entropy increases, the purity of the γ -cascade, after an initial drop, starts to increase, indicating that entanglement in the system decreases. On the other hand, the growing variance of the γ -cascade indicates large fluctuations, which is consistent with the growing probability of transitions to the vacuum.

5 LHCb hadronic entropy and purity vs. cascade models

In this section, we address hadronic entropy and purity, which can be derived from probability distributions measured by the LHCb collaboration [66] and compare them with partonic cascades discussed in the previous sections. See also Ref. [67] for a recent attempt to describe this data. The LHCb kinematics at $\sqrt{s} = 7$ TeV is covering the rapidity range of $2 \leq y \leq 4.5$. Therefore, the final state partons are sensitive to partonic densities with very asymmetric longitudinal momentum fractions in colliding protons, *i.e.* effectively one of the protons can be considered as a target while the other one as a projectile. The cross section for such process can be described within Hybrid Factorization version of Color Glass Condensate [68]. Here, however, we wish to apply the cascade models discussed above, to test various elements, *i.e.* recombination and transition to vacuum. The LHCb measured probability distributions in rapidity intervals $\Delta y = 0.5$. The corresponding entropy and purity can be computed in each interval (we chose for y the central value) from Eqs. (28) and (32). The results are shown in Tab. 1 and in Fig. (14).

By inspecting the data [66] one sees that the hadronic multiplicity is a non-monotonous function of rapidity. Therefore, as can be seen from Table 1, the entropy follows this behavior. Hence, it can not be described within the original 1 dimensional dipole model (2), as well as by the model accounting for merging of dipoles (12). As we have shown in

Sect. 3.1 dipole mergers stabilize the increase of entropy, but cannot make it decrease with rapidity. However, the transitions to vacuum or to states that escape the measurement, which are modeled by the γ terms (14), allow for a decrease of entropy.

y	$S(y)$	$\gamma(y)$
2.25	1.91 ± 0.11	0.19 ± 0.03
2.75	2.06 ± 0.11	0.16 ± 0.017
3.25	2.05 ± 0.06	0.16 ± 0.013
3.75	1.95 ± 0.06	0.18 ± 0.012
4.25	1.84 ± 0.06	0.2 ± 0.015

Table 1: Average entropy and purity computed from the probability distributions measured by the LHCb Collaboration [66].

In order to check whether the γ -cascade can reproduce general features of the LHCb entropy and purity data given in Table 1, we have performed two fits. In both cases we fitted entropy. In the first case we used all five LHCb points from Tab. 1 with the following result: $\alpha = 2$, $\beta = 0.074$ and $\gamma = 0.26$ with $\chi^2/\text{n.d.f} \simeq 2$. In the second fit we used four points neglecting the point at largest rapidity.² The values of fitted parameters are $\alpha = 1.2$, $\beta = 0.02$, $\gamma = 0.18$ with $\chi^2/\text{n.d.f} \simeq 2$. The values of the parameters indicate that the dominant processes are production and transitions to undetected states. The smallness of merging is consistent with the theory requirement that such processes are less probable than the splittings, and the fact that $\gamma > \beta$ indicates that the system did not saturate. The results are shown in the left panel of Fig. 14 where the orange line corresponds to the 5-point fit, whereas the green line to the fit with 4 points.

Having fixed the γ -cascade parameters we obtain *prediction* for the purity, which is shown in the right panel of Fig. 14. Interestingly, for the 5-point fit (shown as an orange line) the $\chi^2/\text{n.d.f} \simeq 1.5$ is better than in the case of entropy. On the contrary, the value of χ^2 for the 4-point fit (shown in green) is much worse: 6.3.

This is an interesting result: although the description of purity is not perfect once the entropy is described (and vice versa), apriori we did not expect the description to be so compatible with the data. Both measures — entropy and purity — originate from the same multiplicity distribution, but there was no guarantee that a model reproducing one will automatically reproduce the other. This is most likely due to the fact that numerically most important contributions to entropy and purity come from different regions of n .

6 Summary and conclusions

Multiparticle production at high-energies is a complex process that involves both perturbative and non-perturbative QCD physics. It involves parton production in a boosted projectile (or target), which is amenable to perturbative or semi-perturbative description

²During the fitting we have observed that the parameter subspace (β, γ) has flat directions that may direct fits into unphysical regions.

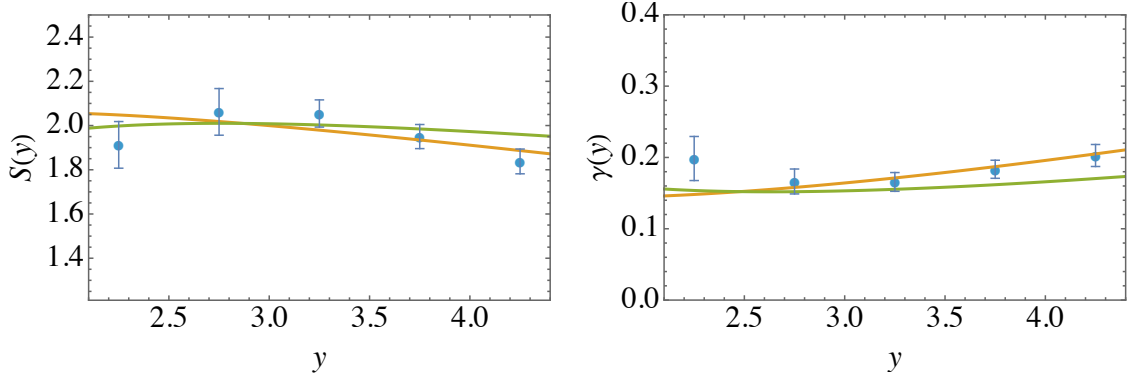


Figure 14: Entropy (left panel) and purity (right panel) obtained from the γ -cascade (solid lines) vs. data from Table 1. Parameters of the γ -cascade were obtained by minimizing the χ^2 for entropy (see text). Purity is therefore a prediction.

in terms of perturbative QCD, and non-perturbative hadronization processes, which are subject to phenomenological modeling. Multiplicity distributions of hadrons and partons are closely related, but the exact relationship is subject to large theoretical uncertainties arising, for example, from the fact that QCD predicts in most cases gluon distributions, while experimentally only charged hadrons are observed. Nevertheless, the main features of both parton and hadron distributions should be in general very similar. In contrast, statistical measures of the multiparticle final states, such as entropy, are not affected by the hadronization processes.

Implementation of the QCD cascades in the case of hadron-hadron scattering is a complicated task requiring involved analytical computations and time consuming computer simulations [13, 69–72]. However, the general features of these processes can be understood in terms of much simpler, one-dimensional “toy models” that allow us to identify the fundamental physical phenomena responsible for parton production and recombination without the complications of four-dimensional QCD cascades. In this paper we studied dipole cascade models that are motivated by the high energy limit of pQCD. These cascades are formulated in 1 dimension and describe evolution of the dipoles - gluons with rapidity. While they have been studied already for some time, our careful numerical analyses have revealed several new features that we believe are interesting in their own right.

We have studied three types of cascades described by Eqs.(2), (12) and (14) called α , β and γ -cascades, respectively. The simplest α -cascade (2) can be solved analytically and leads to the untamed production of dipoles with geometric multiplicity distribution (6). Therefore, it is typically supplemented by the recombination β -terms (12), which lead to the saturation of multiplicity. The solution of this previously known equation with splittings and recombinations can be characterized by the ratio of the couplings $r = \beta/\alpha$ corresponding to the splitting and recombination terms. We have identified two regimes of the β -cascade, for large and small r , which we call “parallel” and “focal”, respectively. In both regimes the probability distribution of the cascade is very well described by the NBD,

whose asymptotic properties such as multiplicity, dispersion and the NBD k parameter are determined solely by the value of r . However, only in the focal regime the scaling laws take the expected very simple analytical form (20).

This analysis shows that the β -cascade is not able to reproduce the energy dependence of the NBD k parameter which is decreasing with energy at the LHC [52]. It is also unable to describe the measurement of the multiplicity of charged hadrons at forward rapidity at LHCb, which is not saturating but decreasing with rapidity. Therefore, we have introduced a new model that takes into account dipole-to-vacuum transitions, so called γ -cascade. The multiplicity of the γ -cascade is indeed decreasing with rapidity. This is due to the fact, that p_0 , which is identically zero in the case of α and β -cascades, increases with y removing partons from the measured spectrum. It turns out that the probability distribution after removing p_0 can be also represented as a NBD with, however, k parameter which saturates for large y . Therefore the γ -cascade is not able to reproduce the energy dependence of the multiparticle spectra at the LHC.

Next, we analyzed the properties of the cascades using tools motivated by quantum information theory, such as entropy, complexity, variance, and purity. These measures allow us to clearly demonstrate how the cascades differ and how the local dynamics was reflected in the properties of the solutions. In particular, we demonstrated that the β -cascade leads to the saturation of entropy, while the γ -cascade leads to a non-monotonic behavior of entropy: the entropy reaches a maximum and then drops.

Finally, we addressed the description of entropy data as inferred from multiplicity of charged hadrons measured by the LHCb in the forward direction. Using the hadronic entropy as obtained from the multiplicity distributions we constrained parameters of the γ cascade and used it to describe the purity data. This result confirms earlier conjecture by [11] and evidence by comparison of theory calculations to data [16,18–20] that entropy and also purity are quantities that, although obtained for partonic degrees of freedom, can also characterize hadrons. One can also reverse the logic and state that by calculating entropy and purity of data one can have direct information about partonic degrees of freedom.

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