Skyrme SIDM Pocket Reference (no mediator)

Skyrme $\to$ HLS hygiene $\to$ Hedgehog profile $\to$ $\sigma_T/m(v)$ (two-phase baseline)

Working notes for discussions with physicists

August 20, 2025

Premise (one line). Dark matter is a topological soliton of an SU(2) Skyrme EFT. A single, locked hedgehog profile fixes dimensionless shape constants c_m, c_R, c_σ, which in turn lock the mass–size relation, the low–v transfer cross–section normalization, and a finite–size suppression scale. Calibrated at dwarfs by $\big(m, (\sigma_T/m)_0\big)$, the micro theory predicts a *single*, monotonic $\sigma_T/m(v)$ from dwarfs to clusters with no per–halo retuning. No additional degrees of freedom are used on this sheet.

Core equations (what is actually needed)

Skyrme setup (locked normalization). Field $U(x) \in \mathrm{SU}(2)$, left current $L_\mu \equiv U^\dagger \partial_\mu U$, trace $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$:

$$\mathcal{L} = \frac{F^2}{16}\,\mathrm{Tr}(L_\mu L^\mu) + \frac{1}{32e^2}\,\mathrm{Tr}([L_\mu, L_\nu]^2), \qquad K_s \equiv \frac{F}{2}, \quad X \equiv eK_s.$$

Hedgehog $U(\mathbf{r}) = \cos f(r) + i(\hat{\mathbf{r}}\boldsymbol{\tau})\sin f(r)$, dimensionless $x \equiv Xr$. Static energy $E = (K_s/e)\,4\pi\int_0^\infty \mathrm{d}x\,\varepsilon(x)$ with

$$\varepsilon(x) = x^2 f_x^2 + 2\sin^2 f + 2\sin^2 f\, f_x^2 + \frac{\sin^4 f}{x^2} \quad (\text{“}x^2 \text{ inside } \varepsilon\text{”}).$$

Shape constants from the solved profile:

$$c_m \equiv 4\pi\int_0^\infty \varepsilon\,\mathrm{d}x, \qquad c_R \equiv \left(\frac{\int_0^\infty x^2\varepsilon\,\mathrm{d}x}{\int_0^\infty \varepsilon\,\mathrm{d}x}\right)^{1/2}, \qquad c_\sigma \equiv \frac{\pi c_R^2}{c_m}.$$

Micro observables and calibration:

$$m = \frac{c_m K_s}{e}, \quad R_* = \frac{c_R}{X}, \quad \left(\frac{\sigma_T}{m}\right)_{\mathrm{nat}} = \frac{c_\sigma\, e}{K_s^3}, \qquad K_s^2 = \frac{c_m c_\sigma}{m(\sigma_T/m)_{\mathrm{nat}}}, \quad e = \frac{c_m K_s}{m}.$$

Predictive scattering (two-phase baseline). Effective range (s–wave) with $k = \mu v$ (v in units of c):

$$k\cot\delta_0(k) = -\frac{1}{a} + \frac{r_e}{2}\,k^2, \qquad r_e = \xi R_*, \quad \xi = \mathcal{O}(1) \ \ (\text{baseline } \xi = 2/3).$$

Per–mass transfer cross–section:

$$\frac{\sigma_T}{m}(v) = \left(\frac{\sigma_T}{m}\right)_0 \frac{1}{\big(1 - \frac{1}{2}a r_e k^2\big)^2 + (ak)^2} \times C_T(k),$$

with transfer factor from the hedgehog profile,

$$C_T(k) = \frac{1}{4\pi}\int \mathrm{d}\Omega\,(1-\cos\theta)\,\big|F_{\mathrm{prof}}(q)\big|^2, \quad q = 2k\sin\frac{\theta}{2}, \quad F_{\mathrm{prof}}(q) = \frac{\int_0^\infty \varepsilon(x)\, j_0\big((q/X)x\big)\,\mathrm{d}x}{\int_0^\infty \varepsilon(x)\,\mathrm{d}x}.$$

HLS renormalization hygiene (one-loop sketch). Matched hidden–local–symmetry scheme (background-field gauge): along $a_{\mathrm{HLS}} = 1$, divergent parts renormalize $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$ equally, hence

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln(eK_s) = 0 \quad \Rightarrow \quad X = eK_s \;\text{ marginal at one loop.}$$

This removes a leading RG drift of the calibrated mass/size (finite pieces may shift r_e mildly; baseline remains predictive).

Variables & pronunciations (with units)

Symbol	**Say it as**	**Meaning**	**Units**
U	"U"	SU(2) field	—
L_μ	"ell mu"	left current $U^\dagger \partial_\mu U$	—
F	"F"	chiral scale in $O(p^2)$ term	GeV
e	"e"	Skyrme coupling (dimensionless)	—
K_s	"K sub s"	$F/2$ (convenient normalization)	GeV
X	"X"	eK_s (micro scale)	GeV
$\varepsilon(x)$	"epsilon of x"	dimensionless profile energy density (with x^2 inside)	—
c_m, c_R, c_σ	"c m, c R, c sigma"	shape integrals from ε	—
m	"mass"	soliton mass $c_m K_s/e$	GeV
R_*	"R star"	rms radius c_R/X	GeV^{-1}, cm
μ	"mu"	reduced mass $m/2$	GeV
v	"vee"	relative speed (use v/c in formulas)	—
k	"kay"	wavenumber μv (with v/c)	GeV
q	"cue"	momentum transfer $2k\sin(\theta/2)$	GeV
a	"a"	s-wave scattering length	GeV^{-1}
r_e	"r e"	effective range, often $= \xi R_*$	GeV^{-1}
ξ	"xi" (zai)	$\mathcal{O}(1)$ proportionality for r_e	—
δ_ℓ	"delta ell"	phase shift in partial wave ℓ	—
σ_T/m	"transfer sigma over m"	transfer cross-section per mass	cm^2/g
$F_{\mathrm{prof}}(q)$	"F profile"	profile form factor (dimensionless)	—
$C_T(k)$	"C T"	transfer factor from profile	—
g_H, a_{HLS}	"g H, a H L S"	HLS gauge coupling, VM-line parameter	—

Calibrated constants & numeric plug-ins

Locked profile constants (dimensionless): $c_m = 147.565$, $c_R = 1.24220$, $c_\sigma = 0.032851$.
Dwarf-scale inputs: $m = 6.283\,\mathrm{GeV}$, $(\sigma_T/m)_0 = 0.10\,\mathrm{cm}^2\,\mathrm{g}^{-1}$.
Unit conversion: 1 cm^2/g = 4,578.21324 GeV^{-3}. Hence $(\sigma_T/m)_{\mathrm{nat}} = 457.821324\ \mathrm{GeV}^{-3}$.
Couplings (algebraic calibration): $K_s = 0.041\,049\,58\,\mathrm{GeV}$, $e = 0.96410663 \Rightarrow X = eK_s = 0.039\,576\,176\,\mathrm{GeV}$.

Size & velocities: $R_* = c_R/X = 31.387\,57\,\mathrm{GeV}^{-1} = 6.1936 \times 10^{-13}\,\mathrm{cm}$. Reduced mass $\mu = m/2 = 3.1415\,\mathrm{GeV}$. The profile scale

$$v_{R_*} \equiv (\mu R_*)^{-1} = 0.01014\,c \;\approx\; 3.04 \times 10^3\,\mathrm{km\,s^{-1}},$$

and $C_T \simeq 0.48$–0.50 near $v \approx 3{,}000\ \mathrm{km\,s^{-1}}$ in this calibration.
Low-v scattering length: $a = \sqrt{m(\sigma_T/m)_{\rm nat}/(4\pi)} = 15.1296\,\mathrm{GeV}^{-1}$.
Effective range choice: $r_e = \xi R_*$ with baseline $\xi = \frac{2}{3}$; for presentations, a one-phase ER band $\xi \in [0.5, 1.0]$ is shown as model spread (no extra fit).

Baseline $\sigma_T/m(v)$ checkpoints (two-phase ERE $\oplus$ profile; baseline $\xi = \frac{2}{3}$):

v [$\mathrm{km\,s^{-1}}$]	≈ 10	1000	2000	3000
σ_T/m [$\mathrm{cm^2\,g^{-1}}$]	0.100	0.092	0.075	0.053
$C_T(k)$ [dimensionless]	1.000	0.915	0.730	0.500

(Anchor is recovered at low v; the decline is driven primarily by $C_T(k)$ once $kR_* \sim 1$.)

Logic in one slide (what to say)

1. Solve one hedgehog once $\Rightarrow \{c_m, c_R, c_\sigma\}$ fixed.
2. Dwarf inputs $\big(m, (\sigma_T/m)_0\big) \Rightarrow (K_s, e)$ via algebra; then R_*, v_{R_*}, a, r_e follow.
3. Predict $\sigma_T/m(v)$ from ER formula $\times$ profile transfer $C_T(k)$; no per–halo knobs.
4. HLS one-loop hygiene: along $a_{\rm HLS} = 1$, $X = eK_s$ is marginal $\Rightarrow$ no leading RG runaway of calibrated scales.
5. Hygiene checks used here: unitarity (partial-wave), analytic forward limit, finite-range C_T (causality), and behavior consistent with Froissart–Martin bounds.

Q&A

Q1: What does K_s do in the Skyrme Lagrangian? Why does it appear with dimensions of energy (GeV)?

K_s is the energy-scale parameter set by $F/2$ (with F the sigma-model decay constant). It sets the strength of the $O(p^2)$ kinetic term and therefore carries energy units (GeV) because it multiplies derivatives. Physically, it sets the overall mass/size scale: $m \propto K_s/e$ and $R_* \propto 1/(eK_s)$.

Q2: What does e do in the Skyrme Lagrangian, and how does it relate to K_s?

e is the dimensionless Skyrme coupling multiplying the $O(p^4)$ (stiffness) term. Together with K_s it defines the single micro scale $X \equiv eK_s$, which sets the dimensionless radius $x = Xr$. In observables, $m \propto K_s/e$ and $R_* \propto 1/(eK_s)$: K_s sets the energy dimension; e tunes the mass/size tradeoff.

Q3: Why use an $SU(2)$ Skyrme model for dark matter instead of a simpler single real scalar field theory?

It is the minimal non-Abelian EFT that naturally supports finite-energy, topologically stable solitons (stability from $\pi_3(SU(2)) = \mathbb{Z}$). The Skyrme term evades Derrick's theorem, giving a finite radius. One universal hedgehog profile then yields locked dimensionless shape constants (c_m, c_R, c_σ) that fix mass–size and the low-v normalization of σ_T/m; generic single scalars lack this built-in topological stability and finite-size structure.

Q4: Why keep both the $SU(2)$ and $O(3)$ representations, and where do K_s and e enter conceptually in each?

$SU(2)$ is natural for topology/symmetry (currents, commutators); $O(3)$ (unit vector $|H| = 1$) is numerically convenient and maps cleanly to hedgehog variables. Conceptually, K_s multiplies the $O(p^2)$ sigma term (energy scale), while e multiplies the $O(p^4)$ Skyrme term (stiffness). Using both views lets you argue stability/topology cleanly and compute profile integrals efficiently.

Q5: Define in words the hedgehog ansatz, its boundary conditions, and what $a_{\mathbf{shoot}}$ represents.

Hedgehog: Spherical configuration with isospin aligned to the radial direction; the field is encoded by a single function $f(r)$ (or $f(x)$). *Boundary conditions:* $f(0) = \pi$ (regular), $f(\infty) = 0$ (vacuum), ensuring finite energy and the correct topological sector. a_{shoot}*:* The dimensionless initial slope at the origin in $f(x) = \pi - ax + \cdots$. It is fixed *once* by requiring the far-field boundary condition; it is a solver datum, not a tunable physical parameter (and unrelated to the physical scattering length a).

Q6: What is the dimensionless energy density $\varepsilon(x)$, what does the "x^2 inside ε" convention mean, and which integrated quantities are invariant?

$\varepsilon(x)$ is the dimensionless radial energy density built from $f(x)$ after $r \to x/(eK_s)$, so $E = (K_s/e)\, 4\pi \int \varepsilon\, dx$. Putting x^2 "inside" ε is a bookkeeping choice (equivalent to keeping it in the measure); the integrated constants c_m, c_R, c_σ and derived observables m, R_*, $(\sigma_T/m)_{\text{nat}}$, and $X = eK_s$ are invariant under the switch.

Q7: Define the three shape constants c_m, c_R, and c_σ and what each one locks.

c_m fixes the mass normalization ($m \propto c_m K_s/e$). c_R fixes the geometric size ($R_* \propto c_R/(eK_s)$) and the scale where finite-size effects turn on. $c_\sigma \equiv \pi c_R^2/c_m$ fixes the low-v normalization of $(\sigma_T/m)_{\text{nat}}$.

All three are dimensionless integrals of the unique hedgehog profile and are computed once.

Q8: Define X and R_* in words, and explain how $vR_* \equiv 1/(\mu R_*)$ sets the turnover in $\sigma_T/m(v)$.

$X \equiv eK_s$ is the single micro scale that sets $x = Xr$. The size is $R_* = c_R/X$ (units "GeV inverse"). Finite-size suppression in the *transfer* cross section turns on when $kR_* \sim 1$ with $k = \mu v$, i.e. $v \sim vR_* \equiv 1/(\mu R_*)$. For $v \ll vR_*$, $C_T(k) \to 1$ (plateau); near vR_*, $C_T(k)$ drops and $\sigma_T/m(v)$ declines.

Q9: What are the meanings of the scattering length a and effective range r_e, how do they enter $\sigma_T/m(v)$, and why set $r_e = \xi R_*$ with $\xi = O(1)$ (e.g. $\xi = 2/3$)?

a is the leading low-energy s-wave parameter (units GeV^{-1}) and fixes the zero-velocity normalization; in our pipeline it is set by the dwarf anchor. r_e is the next finite-range length scale, entering via $k \cot \delta_0 = -1/a + (r_e/2)k^2 + \cdots$, and controls curvature away from the plateau. We use $r_e = \xi R_*$ with $\xi = O(1)$ because finite-range interactions have r_e of order the size; $\xi = 2/3$ is a conservative baseline pending the full phase-shift computation.

Q10: Why use the momentum-transfer cross section σ_T instead of the total σ, and how does finite size enter via $C_T(k)$?

σ_T weights by $(1 - \cos\theta)$ and thus measures momentum exchange and heat transport; forward scatterings that inflate σ but move little momentum are down-weighted. Finite size enters through a profile-derived form factor inside the transfer integral, giving $C_T(k) \in (0, 1]$: $C_T \to 1$ for $kR_* \ll 1$ and drops once $kR_* \sim 1$, causing the turnover and decline with v.

Q11: After anchoring to $\{m, (\sigma_T/m)_0\}$, what parts of $\sigma_T/m(v)$ are locked, what choices remain, and what would falsify the claim?

Locked by the profile: (c_m, c_R, c_σ) (data-independent). The anchors then fix (K_s, e), hence X and R_*; the $\sigma_T/m(v)$ *shape* is determined by the ER factor and $C_T(k)$. Residual global choice: $r_e = \xi R_*$ with a single $\xi = O(1)$ (held globally). Optional knob (off by default): one internal mode scale $m_\phi \sim R_*^{-1}$ if data demand extra high-v suppression. Falsification: inconsistent normalizations at the same v across systems or a velocity dependence outside what any $O(1)$ ξ (and at most one m_ϕ) can produce.

Q12: Non-technical intuition for the decline of σ_T/m with v and the nucleon analogy.

At low v the de Broglie wavelength is $\gg R_*$ so the soliton looks pointlike (plateau). As v rises to $kR_* \sim 1$, internal structure is resolved, the elastic amplitude acquires a form factor, and large-angle contributions to the *transfer* integral are suppressed, so σ_T/m declines. This mirrors nucleon form factors falling with Q^2 due to finite charge/magnetization radii.

Q13: How do the dwarf-scale inputs $\{m, (\sigma_T/m)_0\}$ fix (K_s, e) and therefore R_*? Why convert $(\sigma_T/m)_0$ to natural units?

Solve the hedgehog once to get (c_m, c_R, c_σ). Convert $(\sigma_T/m)_0$ to GeV^{-3} (natural units), then solve the two algebraic relations to get (K_s, e). From these, $X = eK_s$ and $R_* = c_R/X$ follow. The unit conversion is essential for dimensional consistency; otherwise the algebra mixes SI/CGS with natural units.

Q14: What is the optional internal mode with $m_\phi \sim R_*^{-1}$, why is it off in baseline, and when would you turn it on?

It encodes a single monopole ("breathing") excitation; its natural scale tracks R_*^{-1}. Baseline leaves it off to keep falsifiability tight because ER+profile already predicts a finite-size turnover. Turn it on only if multiple systems at similar v exhibit a decline steeper than ER+profile can deliver; then one global m_ϕ adds high-q suppression without per-halo tuning.

Q15: Order-of-magnitude sanity check for R_* and vR_*. Why do these scales make sense?

$R_* \approx 6.2 \times 10^{-13}$ cm ≈ 6 fm (a few nuclear radii). $vR_* \approx 3.0 \times 10^3$ $\mathrm{km\,s^{-1}}$ ("kilometers per second"). Dwarfs ($\ll 10^3$ $\mathrm{km\,s^{-1}}$) lie on the plateau; clusters ($\sim 10^3$–$10^{3.5}$ $\mathrm{km\,s^{-1}}$) probe $kR_* \sim 1$ where the profile form factor suppresses large-q contributions in the momentum-transfer integral.

Q16: How is unitarity respected, and why don't the profile factors violate it?

The s-wave amplitude uses the standard effective-range form, which is unitary by construction (partial-wave unitarity). σ_T is just a weighted angular integral of the same elastic amplitude, and the profile factors are bounded (≤ 1) multiplicative suppressions inside the transfer integral. They cannot push probabilities above the unitarity bound; they only attenuate high-q contributions once the object is resolved.

Q17: What prevents collapse of the soliton under scaling?

The four-derivative Skyrme term supplies gradient energy that grows under contraction, evading Derrick's theorem and stabilizing a finite radius. Without it, finite-energy hedgehogs are unstable.

Q18: What is the topological charge here and does it equal particle number?

It is a dark-sector winding number in $SU(2)$ (analogous to baryon number for nucleon Skyrmions). It labels dark soliton number; it is not SM baryon number and does not couple to SM currents.

Q19: Could you use a global $O(3)$ sigma model or $U(1)$ instead of $SU(2)$?

$SU(2)$ is the minimal non-Abelian group with $\pi_3(SU(2)) = \mathbb{Z}$ supporting hedgehog solitons. The $O(3)$ unit-vector form is the equivalent real representation used for numerics; a single $U(1)$ does not supply the needed topology.

Q20: Why is r_e taken as ξR_* and why choose $\xi \approx 2/3$? *(See Q9 for full context)*

Finite-range interactions generically have r_e of order the size, so $r_e = \xi R_*$ is the minimal closure. $\xi = 2/3$ is a conservative baseline pending a microscopic phase-shift computation of r_e.

Q21: When do higher partial waves matter, and would they change your conclusion?

When $kR_* \gtrsim 1$ (cluster velocities). They modify angular distributions, but the finite-size form-factor suppression and the turnover remain; this is a refinement, not a reversal.

Q22: How does this differ from Yukawa or atomic SIDM models?

Those introduce mediator/atomic scales and often use per-halo fits. Here, a single solved profile fixes (c_m, c_R, c_σ); after one dwarf anchor, the entire $\sigma_T/m(v)$ curve is predicted without per-halo retuning.Yukawa SIDM is like imposing an adjustable potential, while our Skyrme soliton is like a composite nucleus — the scattering cross section reflects its intrinsic size and form factor. That makes the result predictive and rooted in microphysics rather than parametric fitting.

Q23: How do you convert $\mathrm{cm}^2\,\mathrm{g}^{-1}$ to GeV^{-3} quickly?

1 cm $= 5.0677 \times 10^{13}$ GeV^{-1}, 1 g $= 5.6096 \times 10^{23}$ GeV. Thus 0.10 $\mathrm{cm}^2\,\mathrm{g}^{-1} \approx 0.10\,(5.0677 \times 10^{13})^2/(5.6096 \times 10^{23}) \approx 4.58 \times 10^2$ GeV^{-3}.

Q24: When do you turn on the optional internal mode m_ϕ? *(See Q14 for full context)*

Only if multiple systems at similar v show a decline steeper than ER+profile; then one global scale $m_\phi \sim R_*^{-1}$ adds high-q suppression without per-halo tuning. Baseline keeps m_ϕ off to maximize falsifiability.

Q25: Does the cosmology "macro bridge" alter the ΛCDM background?

Not in baseline: it is off, so the background reduces to ΛCDM and the micro predictions stand alone. If turned on later, it is an explicit, testable source term to be calibrated empirically.

Q26: The Skyrme model's nucleon analogy—what carries over and what does not?

Carries over: Topological stability, finite radius, two-parameter control (K_s, e), and finite-size form-factor suppression at higher momentum transfer. *Differs:* Dark-sector interpretation (no strong SM couplings), calibration to astrophysical anchors $\{m, (\sigma_T/m)_0\}$ rather than hadron data, observables are halo transport $\sigma_T/m(v)$, and excitations are limited to an optional breathing mode kept off in baseline.

Q27: What are the only inputs and how many free parameters remain after anchoring?

Inputs: particle mass m and low-v anchor $(\sigma_T/m)_0$ (converted to natural units). After solving the profile once to get (c_m, c_R, c_σ), these two inputs fix (K_s, e), hence X and R_*. Baseline leaves a single global ξ in $r_e = \xi R_*$ (held $O(1)$, e.g. $\xi = 2/3$). An optional global m_ϕ may be introduced only if data require extra high-v suppression. No per-halo parameters.

Q28: At cluster velocities (v 2000–3000 km/s), your baseline curve predicts T/m 0.05–0.07 cm²/g . Why is this phenomenologically significant compared to both (i) collisionless CDM and (ii) hand-fit SIDM models?

At cluster velocities the baseline predicts T/m 0.07 at 2000 km/s and 0.05 at 3000 km/s. That's phenomenologically important in two ways. First, versus collisionless CDM, it is small but nonzero, so clusters are largely collisionless while dwarfs can still be collisional; the decline is fixed by the finite soliton size through the profile transfer factor CT(k) and the turnover scale vR* = 1/(µR*), which is set by the calibrated R*. Second, versus hand-fit SIDM forms, these cluster-scale values aren't retuned; once we anchor at dwarfs, (Ks, e) fix R*, and CT(k) gives a universal monotonic suppression that lands in the 0.05–0.07 cm²/g regime at cluster speeds.

Consistency checks (interview excerpt)

One–line construction (for context). Use a Lorentz–invariant $2 \to 2$ amplitude

$$\mathcal{M}(s,t) = 8\pi\sqrt{s}\, f_{\rm ERE}(k)\, H(s,t),$$

where $f_{\rm ERE}(k) = \big[-1/a + (r_e/2)k^2 - ik\big]^{-1}$ is the standard s–wave effective–range (ER) amplitude and $H(s,t)$ carries the composite finite–size physics via your profile form factor (and optional internal mode). Normalize H so that $\frac{1}{4\pi}\int d\Omega\, H^2 = 1$ at each k; then $\sigma_{\rm tot} = 4\pi|f_{\rm ERE}|^2$ and

$$\frac{\sigma_T}{m}(v) = \Big[\text{ ER factor }\Big] \times \underbrace{\Big[\, C_T(k)\, C_0(k)\,\Big]}_{\text{profile transfer with unitary normalization}},$$

with $C_0(k) = \frac{1}{4\pi}\int d\Omega\, H^2$ and $C_T(k) = \frac{1}{4\pi}\int d\Omega\,(1-\cos\theta)\, H^2$. At $k \to 0$, $C_0 \to 1$ so your dwarf anchor is unchanged.

1) Optical theorem (forward unitarity). With $H(s,0) = 1$, the forward limit depends only on $f_{\rm ERE}$: $\mathrm{Im}\,\mathcal{M}(s,0) = 2k\sqrt{s}\,\sigma_{\rm tot}(s)$ holds exactly. No hidden negative terms appear since $\mathrm{Im}\, f_{\rm ERE} = k\,|f_{\rm ERE}|^2 \geq 0$. The $C_0(k)$ normalization makes the transfer piece consistent with the exact forward normalization while leaving the $k \to 0$ anchor unchanged.

2) Analyticity / positivity (forward–limit dispersion). $\mathcal{M}(s,0)$ is analytic in s (right–hand cut at $s \geq 4m^2$). A twice–subtracted dispersion relation gives $\partial_s^2 \mathrm{Re}\,\mathcal{M}(s,0) \geq 0$ below threshold, because $\mathrm{Im}\,\mathcal{M}(s,0) \propto k\sqrt{s}\,\sigma_{\rm tot} \geq 0$. In the strict forward limit only the ER parameters (a, r_e) enter; composite structure (profile, internal mode) affects angular moments away from $t = 0$. At small k, $C_T(k)C_0(k) = 1 - \alpha\, k^2 + \mathcal{O}(k^4)$ with $\alpha \geq 0$ set by (c_R, X, m_ϕ), i.e. it *reduces* σ_T as k grows.

3) Froissart–Martin high–energy bound. At $t = 0$ and $s \to \infty$, $f_{\rm ERE}(k) \to i/k$ so $\mathcal{M}(s,0) \to$ const and $\sigma_{\rm tot}(s) \propto 1/s$, far below the $\log^2 s$ bound. For $t \neq 0$, $H(s,t)$ (Fourier transform of a finite–size profile, optionally times $1/(1 + q^2/m_\phi^2)$) further suppresses large $|t|$, improving UV behavior and keeping cross-sections well within Froissart–Martin expectations.

4) Microcausality / no superluminal propagation. The micro–EFT is local and Lorentz invariant. Linearizing about the finite–energy hedgehog background yields a hyperbolic quadratic form with characteristics inside the light cone (no higher–time derivatives, no Ostrogradski ghosts). Normal modes, including the breathing mode, have subluminal front velocities. On–shell amplitudes inherit causality/analyticity via LSZ, consistent with the dispersion relation used above.

Where parameters enter (sound bite). ER sector (a, r_e) controls the forward amplitude and fixes the anchored low–v normalization/curvature; composite structure (c_R, X) and the optional $m_\phi \sim R_*^{-1}$ enter only through $H(s,t)$ and thus the angular (transfer) integral $C_T \times C_0$. In the strict forward limit, a composite with fixed total "charge" scatters as if pointlike.

What would signal a failure (concise). (i) Enforcing the exact forward optical theorem while *omitting* $C_0(k)$ leads to an $\mathcal{O}(k^2)$ mismatch between $\mathrm{Im}\,\mathcal{M}(s,0)$ and $\int d\Omega\,|\mathcal{M}|^2$. (ii) Wrong–sign coefficients in the forward–limit dispersion (e.g., $\partial_s^2 \mathrm{Re}\,\mathcal{M} < 0$) would indicate a pathological ER choice, not a profile problem. Neither arises with the normalized construction above.

Appendix: numeric and unit quickies

- Natural units: $\hbar = c = 1$. $1\ \mathrm{GeV}^{-1} = 1.97327 \times 10^{-14}$ cm, so $R_* = 31.387\,57\,\mathrm{GeV}^{-1} = 6.1936 \times 10^{-13}\,\mathrm{cm}$.
- $1\ \mathrm{cm}^2/\mathrm{g} = 4{,}578.21324\ \mathrm{GeV}^{-3}$. For $(\sigma_T/m)_0 = 0.10\ \mathrm{cm}^2\,\mathrm{g}^{-1}$, $(\sigma_T/m)_{\mathrm{nat}} = 457.821324\ \mathrm{GeV}^{-3}$.
- Reduced mass $\mu = m/2 = 3.1415\,\mathrm{GeV}$; velocity scale $v_{R_*} = (\mu R_*)^{-1} = 0.01014\,c \approx 3.04 \times 10^3\,\mathrm{km\,s^{-1}}$.