$\Sigma = 94$

MAT2100

Assignment 2

Justin Ross

0050040865

Question 1

a)

For first half of the equation:

$\Omega\ = -2 \leq x \leq -\sqrt{3}, 0 \leq y \leq \sqrt{4 - x^2}$

x limits run from -2 to $-\sqrt{3}$

y limits run from 0 to $\sqrt{4 - x^2}$

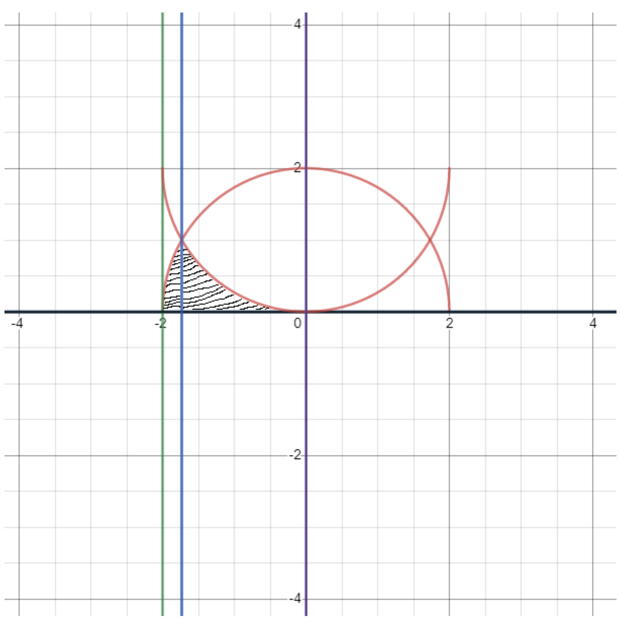
For the second half of the equation:

$\Omega\ = -\sqrt{3} \leq x \leq\ 0, 0 \leq y \leq 2 - \sqrt{4 - x^2}$

x limits run from $-\sqrt{3}$ to 0

y limits run from 0 to $2 - \sqrt{4 - x^2}$

Ans:



*Note: area with black, hand drawn lines is the region of integration.

b)

$$I = \int_{-2}^{-\sqrt{3}} \left(\int_{0}^{\sqrt{4-x^2}} f(x,y)dy \right) dx + \int_{-\sqrt{3}}^{0} \left(\int_{0}^{2-\sqrt{4-x^2}} f(x,y)dy \right) dx$$

For first half of the equation:

The original omega was:

$$\Omega \;= -2 \le x \le -\sqrt{3}, 0 \le y \le \sqrt{4-x^2}$$

We need to transform it as follows:

$$\Omega \;= -2 \le y \le -\sqrt{3}, 0 \le x \le \sqrt{4-y^2}$$

$$\int_{0}^{1} \left(\int_{-\sqrt{4-y^2}}^{-\sqrt{3}} f(x,y)dx \right) dy$$

For second half of the equation:

The original omega was:

$$\Omega \;= -\sqrt{3} \le x \le \; 0, 0 \le y \le 2 - \sqrt{4-x^2}$$

$$\int_{0}^{1} \left(\int_{-\sqrt{3}}^{-\sqrt{-y^2+4y}} f(x,y)dx \right) dy$$

Ans:

$$\boldsymbol{I = \int_{0}^{1} \left(\int_{-\sqrt{4-y^2}}^{-\sqrt{3}} f(x,y)dx \right) dy + \int_{0}^{1} \left(\int_{-\sqrt{3}}^{-\sqrt{-y^2+4y}} f(x,y)dx \right) dy}$$

10

Question 2

a and b are the sides of a rectangular plate.

$p = kr^4$

Where:

p is the density of the plate alloy

r is the distance from one corner to the fourth power.

K is just some coefficient, basically density coefficient of some kind.

The Mass Integrand:

$$M = \int\int p(x, y)dxdy$$

The density function:

$$p(x, y) = kr^4$$

$$M = \int\int kr^4 dxdy$$

$$M = \int\int kr^4 dxdy$$

We can use the Pythagorean Theorem to draw two conclusions:

$$r^2 = a^2 + b^2$$

$$r^2 = x^2 + y^2$$

If we replace r with x^2 and y^2 this we get:

$$M = \int\int k(x^2 + y^2)(x^2 + y^2)dxdy$$

$$M = \int_0^b\int_0^a k(x^2 + y^2)(x^2 + y^2)dxdy$$

$$M = \int_0^b\int_0^a k(x^2 + y^2)^2 dxdy$$

Now we integrate:

$$M = \int_0^b \int_0^a k(x^2 + y^2)^2 dxdy$$

$$\int_0^b k(ay^4 + 2y^2 \frac{a^3}{3} + \frac{a^5}{5})^2$$

Ans:

$$\boldsymbol{M = k\left(a\frac{b^5}{5} + \frac{2a^3b^3}{9} + b\frac{a^5}{5}\right)}$$

10

Question 3

$$Work\ done = \int F \cdot dr$$

$$F(x, y, z) = -xyi + 4j + xk$$

$$F(t) = -(2\cos t) \cdot (2sin\ t)i + 4j + (2\cos t)k$$

$$r(t) = 2costi + 2sintj + 4k$$

$$dr(t) = r'(t) = -2\ \text{sint dt i} + 2\ \text{cost dt j} + 0k$$

$$W = \int F \cdot dr = \int_{\pi/4}^{7\pi/4} [-(2\cos t) \cdot (2sin\ t)i + 4j + (2\cos t)k] \cdot [-2\ \text{sint dt i} + 2\ \text{cost dt j}]$$

Multiply term by term:

$$\int_{\pi/4}^{7\pi/4} [-(2\cos t) \cdot (2sin\ t)i + 4j + (2\cos t)k] \cdot [-2\ \text{sint dt i} + 2\ \text{cost dt j} + 0\ dt\ k]$$

$$\int_{\pi/4}^{7\pi/4} [-(2\cos t) \cdot (2sin\ t) \cdot (-2\ \text{sint dt})i + 4 \cdot 2\ \text{cost dt j} + (2\cos t) \cdot 0k]$$

$$\int_{\pi/4}^{7\pi/4} [-(2\cos t) \cdot (2sin\ t) \cdot (-2\ \text{sint dt}) + 4 \cdot 2\ \text{cost dt} + (2\cos t) \cdot 0]$$

$$\int_{\pi/4}^{7\pi/4} [-(2\cos t) \cdot (2sin\ t) \cdot (-2\sin t) + 4 \cdot 2\ \text{cost} + (2\cos t) \cdot 0]\, dt$$

$$\int_{\pi/4}^{7\pi/4} [8sin^2(t)\cos(t) + 8\cos(t)]\, dt$$

$$[8sin^2(t)\cos(t) + 8\cos(t)]_{\pi/4}^{7\pi/4}$$

$$-\frac{4\sqrt{2}}{3} - 8\sqrt{2}$$

Ans:

$$\boldsymbol{W = -\frac{4\sqrt{2}}{3} - 8\sqrt{2}} = -\frac{(4+24)\sqrt{2}}{3} = -\frac{28\sqrt{2}}{3}$$

10

Question 4

$F = 2xe^{x^2} \sin y\, i + e^{x^2} \cos y\, j$

Trajectory: (0, 0) to (1, 3π/2) which is x=1, y=3π/2.

$F(x, y) = \nabla \Phi(x, y)$

$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j$

$W = \int \nabla \Phi \cdot dr = \Phi(r_b) - \Phi(r_a)$

$\frac{\partial \Phi}{\partial x} = F_1 = 2xe^{x^2} \sin y$

$\Phi(x, y) = e^{x^2} \sin(y) + g(y)$

$\frac{\partial \Phi}{\partial y} = F_2 = e^{x^2} \cos y$

$\Phi(x, y) = e^{x^2} \sin(y) + h(x)$

$\Phi(x, y) = e^{x^2} \sin(y) + \cancel{e^{x^2} \sin(y)} + C$

$W = \Phi\left(1, \frac{3\pi}{2}\right) - \Phi(0,0) = \left(e^{x^2} \sin(y) + \cancel{e^{x^2} \sin(y)} + C\right)\Big|_{(0,0)}^{\left(1, \frac{3\pi}{2}\right)}$

$W = \Phi\left(1, \frac{3\pi}{2}\right) - \Phi(0,0) = \left(e^{1^2} \sin\left(\frac{3\pi}{2}\right) + e^{1^2} \sin\left(\frac{3\pi}{2}\right) + C\right)$

$W = -\cancel{2}e$

Ans:

As the work is -2e and not 0, the force is not conservative.

$\begin{cases} f = x + C_1 \\ f = x + C_2 \end{cases}$

does not mean

$f = 2x + C$,

it means

$f = x + C$

8

Question 5

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

Where y is the displacement of the mass

m is the mass

c is the friction coefficient

k is the spring constant

The parameter condition $c^2 < 4mk$ is satisfied

Initial condition:

$$y(0) = A, (\frac{dy}{dt})_{t=0} = B$$

The solution must be a formula containing m, c, k, A and B.

To attack this problem we need to break it down a bit more:

$$m\frac{d^2y}{dt^2} \text{ is equivalent to } m * a, \text{as acceleration is equivalent to the second derivative of distance}$$

$$c\frac{dy}{dt} \text{ is the frictionor damping force, to stop movement}$$

$$ky \text{ is the restoration force of the spring, trying to restore it to its original position}$$

y(0) = A means the distance and initial displacement of the string, y'(0) is the initial velocity, y''(0) is the acceleration, resistive force is B. Displacement of the mass with respect to time at time 0 is B.

Find the general solution of:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

$$y = y_h + y_p$$

Complementary Function (y_h):

The characteristic equation is:

$ax^2 + bx + c$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$\boxed{\sqrt{4mk - c^2}}$$

0

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

As $c^2 < 4mk$, the system is lightly damped and will adhere to the form:

$$y(t) = e^{\lambda t}(Bsin(\omega t) + Acos(\omega t))$$

$$y(t) = e^{-\frac{ct}{2}}\left(Bsin(\frac{\sqrt{c^2 - 4mk}}{2m}t) + Acos(\frac{\sqrt{c^2 - 4mk}}{2m}t)\right)$$

need to find A, B from the init. cond.

Ans:

$$\boldsymbol{y(t) = \sqrt{B^2 + A^2}e^{-\frac{ct}{2}}\cos\left(\frac{\sqrt{c^2 - 4mk}}{2m} - tan^{-1}\frac{A}{B}\right)}$$

6

Question 6

Find the general solution of:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 2 + 3x + sinx$$

This equation is a non-homogenous, second order ordinary differential equation so the solution will take the form:

$$y = y_h + y_p$$

Complementary Function (y_h):

This has the characteristic equation:

$$(D^2 - 2D + 4)y = 2 + 3x + sinx$$

This has the characteristic equation:

$$Its\ AE\ is\ m^2 - 2m + 4 = 0$$

Factorize:

$$m^2 - 2m + 4 = (m -)(m +)$$

The above didn't work because the roots are going to need to be complex conjugates, so we need to use the equation:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{2 \pm \sqrt{-2^2 - 4 \cdot 4}}{2}$$

$$r = \frac{2 \pm \sqrt{-12}}{2}$$

$$r = \frac{2 \pm \sqrt{12i}}{2}$$

$$r = \frac{2(1 \pm i\sqrt{3})}{2}$$

$$r = 1 \pm \sqrt{3}i$$

$$m = 1 + \sqrt{3}i, 1 - \sqrt{3}i (Complex\ and\ distinct\ roots)$$

$\therefore CF = C_1e^{1+\sqrt{3}ix} + C_2e^{1-\sqrt{3}ix}$

The roots are not real and are unequal so the particular integral needs a specific form:

$y(x) = e^{1x}\left(C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x)\right)$

Particular Integral (y_p):

$Since\ g(x) = 2 + 3x + sinx$

Because this contains both a sin function and other variables we need to split it and apply the sum rule:

$Find\ y_1 that\ satisfies\ \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 2 + 3x$

$Let\ y_1 = D + Cx$

$y_1 = \frac{3x}{4} + \frac{7}{8}$

$Find\ y_2 that\ satisfies \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = sinx$

$Let\ y_2 = \lambda cosx + \mu sinx$

$y_2 = \frac{3}{13}\sin(x) + \frac{2}{13}\cos(x)$

Ans:

The general solution is:

$\boldsymbol{y(x) = e^{x}\left(C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x)\right) + \frac{3x}{4} + \frac{7}{8} + \frac{3}{13}\sin(x) + \frac{2}{13}\cos(x)}$ ✓

10

Question 7

a) Find the length of the vector v = a -2b

$$\vec{v} = \vec{a} - 2\vec{b} = (0,3,-2,1,4) - 2 * (5,2,1,0,-1) = (0,3,-2,1,4) - (10,4,2,0,-2)$$
$$= (-10,-1,-4,1,6)$$

$$\vec{v} = (-10,-1,-4,1,6)$$

$$|v| = \sqrt{-10^2 + -1^2 + -4^2 + 1^2 + 6^2}$$

$$|v| = \sqrt{154}$$

$$|v| = 12.4097\ (rounded\ up)$$

Ans: The length(magnitutude) of the vector v is $\sqrt{\mathbf{154}}$

b) Indicate which of the vectors are parallel or orthogonal.

$$\vec{a} = (0,3,-2,1,4)$$

$$\vec{b} = (5,2,1,0,-1)$$

$$\vec{c} = (7,-3,6,21,0)$$

Check the AB vector:

$$\vec{a} \cdot \vec{b} = a_a \cdot b_a + a_b \cdot b_b$$

$$\vec{a} \cdot \vec{b} = 0 \cdot 5 + 3 \cdot 2 + -2 \cdot 1 + 1 \cdot 0 + 4 \cdot -1$$

$$\vec{a} \cdot \vec{b} = 0 \cdot 5 + 3 \cdot 2 + -2 \cdot 1 + 1 \cdot 0 + 4 \cdot -1 = 0$$

$$\vec{a} \cdot \vec{b} = 0$$

As the dot product of a and b is zero, the vectors are orthogonal.

Check the AC vector:

$$\vec{a} \cdot \vec{c} = 0 \cdot 7 + 3 \cdot -3 + -2 \cdot 6 + 1 \cdot 21 + 4 \cdot 0$$

$$\vec{a} \cdot \vec{c} = 0 \cdot 7 + 3 \cdot -3 + -2 \cdot 6 + 1 \cdot 21 + 4 \cdot 0 = 0$$

$$\vec{a} \cdot \vec{c} = 0$$

As the dot product of a and c is zero, the vectors are orthogonal.

Check the BC vector:

$\vec{b} \cdot \vec{c} = 5 \cdot 7 + 2 \cdot -3 + 1 \cdot 6 + 0 \cdot 21 + -1 \cdot 0$

$\vec{b} \cdot \vec{c} = 5 \cdot 7 + 2 \cdot -3 + 1 \cdot 6 + 0 \cdot 21 + -1 \cdot 0 = 35$

$\vec{b} \cdot \vec{c} = 35$

As the dot product of b and c is 35 and not zero, the vectors are not orthogonal.

Now we need to check if b and c are parallel:

$\vec{b} = (5,2,1,0,-1)$

$\vec{c} = (7,-3,6,21,0)$

Two vectors are parallel if one vector can be expressed as a multiple of the other:

$\vec{b} \ = \ A\vec{c}$

$(5,2,1,0,-1) = \ (A7, A-3, A6, A21, A0)$

$(5,2,1,0,-1) = \ (A7, A-3, A6, A21, A0)$

$7/5 \ = \ 1.4$

$2 * 1.4 \ = \ 2.8$

$1.4 \ \neq 2.8$

From the above we can see that the two vectors cannot be expressed as multiples of each other as the first and second items require different scalars to be applied to equal each other. Therefore, vector b and c are not parallel.

Ans: Vector a and b are orthogonal, vector a and c are orthogonal. Vector b and c are neither orthogonal nor parallel.

10

Question 8

Vectors are considered linearly independent if the only solution to the system of linear equations is the trivial solution of 0.

$$c_1 v_1 + \cdots + c_n v_n = 0$$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 2 \\ -1 \end{bmatrix} \right\}$$

$$a_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix} + a_4 \begin{bmatrix} -2 \\ -4 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -2 & 0 \\ 2 & 1 & 3 & -4 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 4 & -1 & 1 & -1 & 0 \end{bmatrix}$$

Now we need to perform some row reduction to produce a diagonal 0s:

$$\begin{bmatrix} 1 & 0 & 1 & -2 & 0 \\ 2 & 1 & 3 & -4 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 4 & -1 & 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -2 & 0 \\ 2 & 1 & 3 & -4 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 4 & -1 & 1 & -1 & 0 \end{bmatrix}$$

First column:

$$\begin{bmatrix} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 7 & 0 \end{bmatrix}$$

Second column:

$$\begin{bmatrix} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 7 & 0 \end{bmatrix}$$

Third column:

$$\begin{bmatrix} 1 & 0 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 7/2 & 0 \\ 0 & 0 & 1 & -7/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fourth Column:

$$\begin{bmatrix} 1 & 0 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 7/2 & 0 \\ 0 & 0 & 1 & -7/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The product of the diagonal values is the determinant: 1*1*1*0. Because we have a zero, the value is zero, and as the determinant of the matrix is 0, the vectors are considered linearly depend~~a~~nt.

e e

Ans: As the determinant of the matrix is 0, the vectors are linearly depend~~a~~nt.

10

Question 9

a) Find all eigenvalues of the following matrix:

$$A = \begin{bmatrix} 5 & 8/3 & -2/3 \\ 2 & 2/3 & 4/3 \\ -4 & -4/3 & -8/3 \end{bmatrix}$$

First we need to find the characteristic equation:

$$|A - \lambda I| = 0$$

$$det\begin{bmatrix} 5-\lambda & \frac{8}{3} & -\frac{2}{3} \\ 2 & \frac{2}{3}-\lambda & \frac{4}{3} \\ -4 & -\frac{4}{3} & -\frac{8}{3}-\lambda \end{bmatrix} = 0$$

Expanding the determinant and simplifying:

$$(5-\lambda)\left[\left(\frac{2}{3}-\lambda\right)\left(-\frac{8}{3}-\lambda\right)-\left(-\frac{4}{3}\right)\left(\frac{4}{3}\right)\right]-\frac{8}{3}\left[(2)\cdot\left(-\frac{8}{3}-\lambda\right)-(-4)\left(\frac{4}{3}\right)\right] + -\frac{2}{3}\left[(2)\left(-\frac{4}{3}\right)-(-4)\left(\frac{2}{3}-\lambda\right)\right] = 0$$

$$(5-\lambda)(\lambda^2+2\lambda)-\frac{8}{3}(-2\lambda)-\frac{2}{3}(-4\lambda)$$

$$(-\lambda+5)(\lambda^2+2\lambda)+\frac{16\lambda}{3}+\frac{8\lambda}{3}$$

$$(-\lambda+5)(\lambda^2+2\lambda)+8\lambda$$

$$-\lambda^3+3\lambda^2+10\lambda+8\lambda$$

$$-\lambda^3+3\lambda^2+18\lambda$$

Now we can find the eigenvalues by solving:

$$-\lambda^3+3\lambda^2+18\lambda = 0$$

Ans: The eigenvalues are:

$\boldsymbol{\lambda = 0}$

$\boldsymbol{\lambda = -3}$

$\boldsymbol{\lambda = 6}$

✓

b) Find the eigenvectors associated with the lowest eigenvalue.

The lowest eigenvalue is λ = -3.

$$\begin{bmatrix} 5-\lambda & \frac{8}{3} & -\frac{2}{3} \\ 2 & \frac{2}{3}-\lambda & \frac{4}{3} \\ -4 & -\frac{4}{3} & -\frac{8}{3}-\lambda \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5--3 & \frac{8}{3} & -\frac{2}{3} \\ 2 & \frac{2}{3}--3 & \frac{4}{3} \\ -4 & -\frac{4}{3} & -\frac{8}{3}--3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & \frac{8}{3} & -\frac{2}{3} \\ 2 & \frac{11}{3} & \frac{4}{3} \\ -4 & -\frac{4}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now wet convert this to an equation and set x to 1:

$$8 + \frac{8}{3}Y - \frac{2}{3}Z = 0$$

$$2 + \frac{11}{3}Y + \frac{4}{3}Z = 0$$

Subtract the first from the second:

$$-6 + 1Y + 2Z = 0$$

This didn’t work, so we can try setting y to 1:

$$8X + \frac{8}{3} - \frac{2}{3}Z = 0$$

$$2X + 3\frac{2}{3} + \frac{4}{3}Z = 0$$

Subtract the first from the second:

$$-6X + 1 + 2Z = 0$$

These approaches are not going to work it would seem, so row reduction is required to find the solution to the system of equations:

$$\begin{bmatrix} 8 & \frac{8}{3} & -\frac{2}{3} \\ 2 & \frac{11}{3} & \frac{4}{3} \\ -4 & -\frac{4}{3} & \frac{1}{3} \end{bmatrix}$$

R_2: R_2-1/4*R_1

$$\begin{bmatrix} 8 & \frac{8}{3} & -\frac{2}{3} \\ 0 & 3 & \frac{3}{2} \\ -4 & -\frac{4}{3} & \frac{1}{3} \end{bmatrix}$$

R_3: R_3 + 1/2 *R_1

$$\begin{bmatrix} 8 & \frac{8}{3} & -\frac{2}{3} \\ 0 & 3 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

R2: 1/3*R2

$$\begin{bmatrix} 8 & \frac{8}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

R1: R1-8/3*R2

$$\begin{bmatrix} 8 & 0 & -2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

R1:1/8*R1

$$\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

This leaves us with two equations as the third row is all zeros:

$$1x + 0y - \frac{1}{4}Z = 0$$

$$x - \frac{1}{4}Z = 0$$

$$x = \frac{1}{4}Z$$

$$0x + 1y + \frac{1}{2}Z = 0$$

$$y + \frac{1}{2}Z = 0$$

$$y = -\frac{1}{2}Z$$

$$\begin{bmatrix} \frac{1}{4}Z \\ -\frac{1}{2}Z \\ Z \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} \cdot 4 \\ -\frac{1}{2} \cdot 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

Ans: The eigenvectors for λ = -3 are:

$$\begin{pmatrix} \mathbf{1} \\ \mathbf{-2} \\ \mathbf{4} \end{pmatrix}$$ √

10

Question 10

The process of diagonalisation starts with a matrix:

$$A = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$$

We make the assumption that a matrix, the diagonal matrix (Λ), exists such that the following equation holds true:

$$S^{-1}AS = \Lambda$$

Making this assumption we can start by calculating the eigenvalues that will make up the diagonal values in the Λ matrix.

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

We then can calculate the associated eigenvectors for these values:

$$S = \begin{bmatrix} -4 & -1 \\ 5 & 1 \end{bmatrix}$$

After this we simply invert it to get the inverted matrix for our formula:

$$S^{-1} = \begin{bmatrix} 3 & 2 \\ -\frac{5}{2} & -\frac{3}{2} \end{bmatrix}$$

So the question then becomes: What is all this even achieving? The main take away is that this is a way of describing how a system linearly scales, if it scales in some non-linear way the matrix will not be diagnosable and so the assumption that the matrix is diagnosable will be shown to be false as the process progresses. When it comes to solving linear differential equations the best way to think of this is that we are trying to break down the system. Let's start with a system of linear differential equations:

$$\begin{Bmatrix} y_1'(t) = -3y_1(t) - 4y_2(t) \\ y_2'(t) = 5y_1(t) + 6y_2(t) \end{Bmatrix}$$

To solve this we convert it to matrix form:

$$y' = Ay$$

$$y = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \quad A = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$$

Straight away we can see the pattern resembling our $S^{-1}AS = \Lambda$ and how it is going to fit. So we start by diagonalizing A again, calculating the eigenvectors and inverting it:

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad S = \begin{bmatrix} -4 & -1 \\ 5 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 3 & 2 \\ -\frac{5}{2} & -\frac{3}{2} \end{bmatrix}$$

Now that we have this we can exploit a known fact concerning differential equations:

10

$$y(t) = e^{\lambda t} S$$

Adding the values in gives the general solution in matrix form:

$$y(t) = Ae^{t}\begin{bmatrix}-1\\1\end{bmatrix} + Be^{2t}\begin{bmatrix}-4\\5\end{bmatrix}$$

So, basically, we are just using mathematical tricks concerning linear scaling to solve the equation.