Numerically solving Tolman-Oppenheimer-Volkoff Equations to study internal structure of Neutron stars

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Abstract

Neutron stars are considered to be the most dense object in the universe. In this project, Tolman-Oppenheimer-Volkhof (TOV) equations are solved numerically using a python based solver for a given EOS, central densities and total masses. The pressure, density, mass profile is then calculated for a mass range of 1.5 - 2.5 $M_{\odot}$. Then, the maximum mass attained by a neutron star is calculated using range of central densities as input and analysed how the mass is effected by central density.

1 Introduction

Determining the correct EOS for understanding the NS structure is an age old problem The structure of neutron stars and white dwarf should be studied within the framework of general relativity. In the limit of hydrostatic equilibrium, the field equations reduce into the so-called TOV equation, which, because of non-linearities, must be solved numerically.TOV equations describe the structure of spherically symmetric body of isotropic material which is hydrostatic equilibrium as modelled by general relativity. Neutron stars are formed from the core collapse of massive star. They are composed of neutrons and they typically have a mass of about $1.18 M_{\odot}$ - $2.8 M_{\odot}$. This study use numerical methods such as RK4 method to study the interior structure of neutron stars and establish the parameters that define the neutron stars.

2 Theory

TOV equations are derived from field equations of general relativity given by

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Where the $G^{\mu\nu}$ is the Einstein curvature tensor describing the curvature of space and $T^{\mu\nu}$ is the stress energy tensor for describing the pressure and density distribution of matter. In deriving TOV equations, we assume matter is composed of perfect fluid and object is spherically symmetric. Metric for spherically symmetric space is given by

$$ds^2 = e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 d\theta^2 - r^2 d\Omega^2$$

where

$$d\Omega^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

From the metric, we can calculate the curvature tensor $G^{\mu\nu}$. The curvature tensor is given by

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

where R is the Ricci scalar and $R^{\mu\nu}$ is the Ricci tensor. The stress energy tensor is given by

$$T^{\mu\nu} = -p g^{\mu\nu} + (p + \epsilon) u^{\mu} u^{\nu}$$

where ϵ is the energy density and is the pressure. The elements of metric is given as

$$g^{00} = e^{2\nu}, g^{11} = -e^{2\lambda}, g^{22} = -r^2, g^{33} = -r^2 sin^2\theta$$

The components of ricci tensor are then calculated according to the expression,

$$R^{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}$$

where

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}[g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}]$$

From this we can calculate the components of Ricci tensor,

$$R^{00} = (-\nu'' + \nu'' - (\nu')^2 - \frac{2\nu'}{r})e^{2(\nu-\lambda)}$$

$$R^{11} = \nu'' - \nu'\lambda' + (\nu')^2 - \frac{2\lambda'}{r}$$

$$R^{22} = (1 + r\nu' - r\lambda')e^{-2\lambda} - 1$$

$$R^{33} = R^{22} sin^2\theta$$

We calculate Ricci scalar inside star

$$R = g^{\mu\nu}R_{\mu\nu} = e^{-2\nu}R_{00} - e^{-2\lambda}R_{11} - \frac{2}{r^2}R_{22}$$

using mix tensor formulation,

$$G^0_0 = R^0_0 - \frac{1}{2}g^0_0 R = e^{-2\lambda}(\frac{1}{r^2} - \frac{2\lambda'}{r}) - \frac{1}{r^2} = -8\pi G\rho$$

$$G^1_1 = R^1_1 - \frac{1}{2}g^1_1 R = e^{-2\lambda}(\frac{1}{r^2} + \frac{2\nu'}{r}) - \frac{1}{r^2} = 8\pi G\rho$$

$$G^2_2 = R^2_2 - \frac{1}{2}g^2_2 R = e^{-2\lambda}(\nu'' + (\nu')^2 - \lambda'\nu' + \frac{\nu' - \lambda'}{r}) = 8\pi G\rho$$

$$G^3_3 = G^2_2$$

with some algebraic manipulations, we get the TOV equation given by,

$$\frac{dp}{dr} = -\frac{G}{r^2}\frac{(\rho + \frac{p}{c^2})(m(r) + 4\pi r^3\frac{p}{c^2})}{1 - \frac{r_s}{r}}$$

where

$$r_s = \frac{2Gm(r)}{c^2}$$

The mass distribution of spherical object is given as

$$\frac{dm}{dr} = 4\pi r^2\rho$$

To solve the TOV equation for pressure, we associate pressure with density using polytropic equation of state

$$p = K\rho^{\gamma}$$

where K is the adiabatic coeffiecient and γ is the adiabatic index.

3 Numerical Method

TOV equation can be numerically solved by runge-kutta method. Runge-Kutta method is classified into different orders such as RK1 (eulers method),RK2,RK3,RK4 etc, according to their complexities. In general, if a differential equation is given as

$$\frac{dy}{dt} = f(t, y)$$

and initial condition $y(t_0) = y_0$ eulers method is given as

$$y_{n+1} = y_n + f(t_n, y_n)dt$$

And similarly, RK4 method is given as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)dt$$

where

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + \frac{dt}{2}, y_n + hk_1/2)$$
$$k_3 = f(t_n + \frac{dt}{2}, y_n + hk_2/2)$$
$$k_4 = f(t_n + dt, y_n + hk_3)$$

TOV equations for pressure and mass can be rewritten for numerical treatment in the following manner to give

$$p_{n+1} = p_n + \frac{dP}{dr}(n)\Delta r$$
$$m_{n+1} = m_n + \frac{dm}{dr}(n)\Delta r$$

where Δr is the step size for numerical integration.

The initial conditions for the integration are given in the central region and the pressure and mass equation are taken as a system and is integrated till the value of p reaches zero, which implies that surface of star is reached.

4 Solving TOV equation

The TOV equations are solved for different physical situations using both euler and RK4 method. As a first step, we define our initial parameters.

Parameters	Values (c.g.s units)
adiabatic constant K	1.982x10^{-6}
adiabatic coefficient γ	2.75
central density ρ_0	5.0x10^{-14}
Zones N	2000

Table 1: Parameters for TOV solver.

After running the solver for this values, we find the final mass $1.5M_\odot$ and radius of 14.2 km for the neutron star.

4.1 Internal structure of a Neutron star

The solver can also be used to find the central density and calculate the internal structure of the neutron star that has a given mass. For instance, if we have to calculate the internal structure of a neutron star of mass $1.5M_{\odot}$, we can give the parameters as in Table 1 and with a central density value that is greater than nuclear density. The solver runs through various central density value till the given mass is reached. Hence, we can calculate the internal structure of neutron star for a given set of parameters. For $1.5M_{\odot}$, the pressure,mass,density profile is similar to body in gravitational equilibrium.

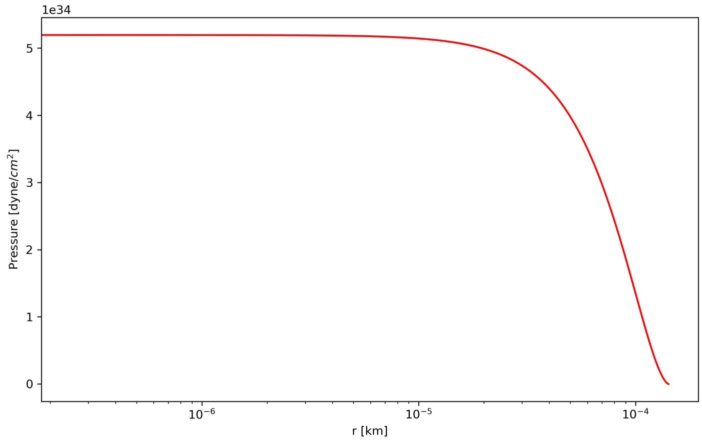

Figure 1: pressure-radius plot of TOV model.

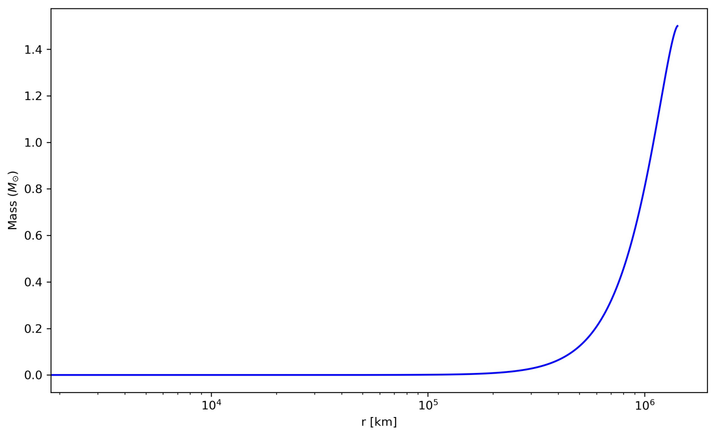

Figure 2: mass-radius plot of TOV model.

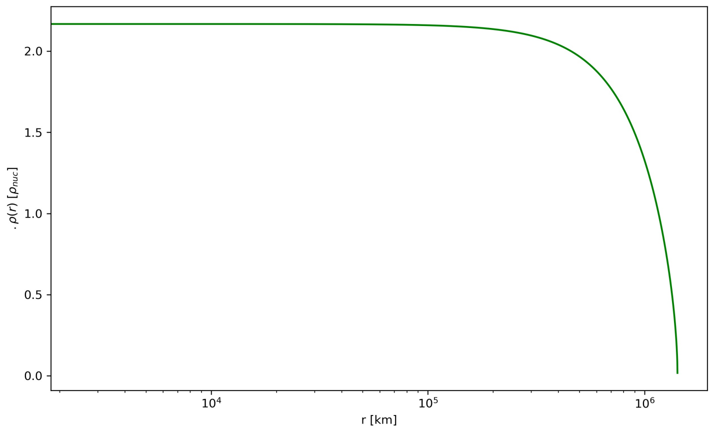

Figure 3: density-radius plot of TOV model.

4.2 Pressure,Mass,density profile for various masses

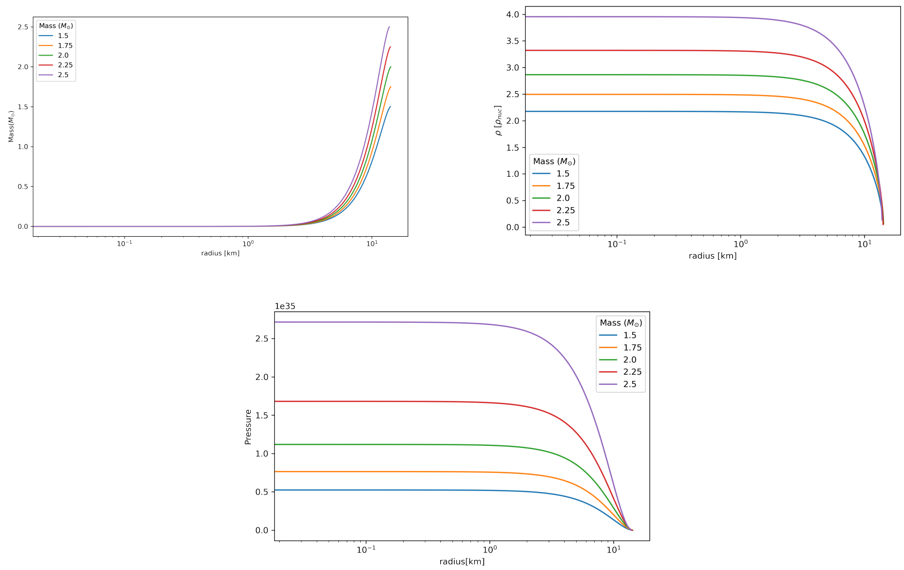

Figure 4: mass,density,pressure profile for neutron stars of mass range 1.5-2.5 $M_\odot$ range

The plots in fig.4 shows the mass,density,pressure profile of neutron stars of range 1.5 - 2.5 $M_\odot$ range. It can be observed that pressure,density stays almost constant through out the interior and drops as it reaches surface, while mass adds up as integration reaches surface. Furthermore, we can see that final pressure and density increases as the star mass increase.

5 Sensitivity of parameters of TOV equation

Parameters such as central density(ρ_0), polytropic constant (K) and adiabatic coefficient (γ) has an effect on structure of neutron star. Here, we check the effect of central density in mass and radius of neutron star.

5.1 central density effect

The central density change alters the final mass and radius of neutron star. We solve the TOV equation for values of central density from 1 to 20 times nuclear density.

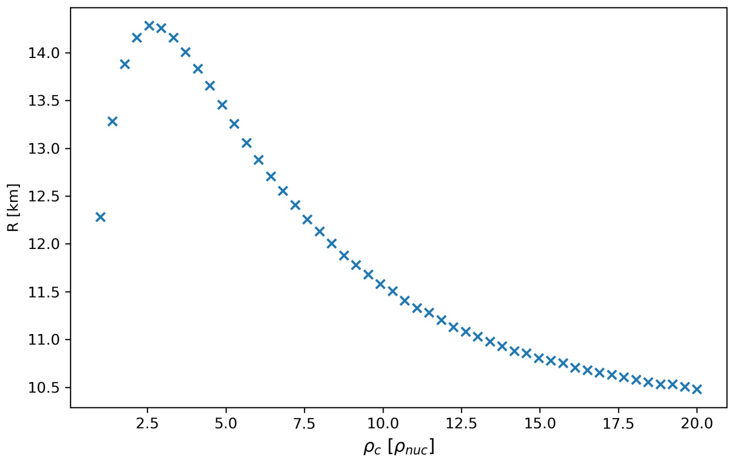

Figure 5: central density-radius plot for various central densities.

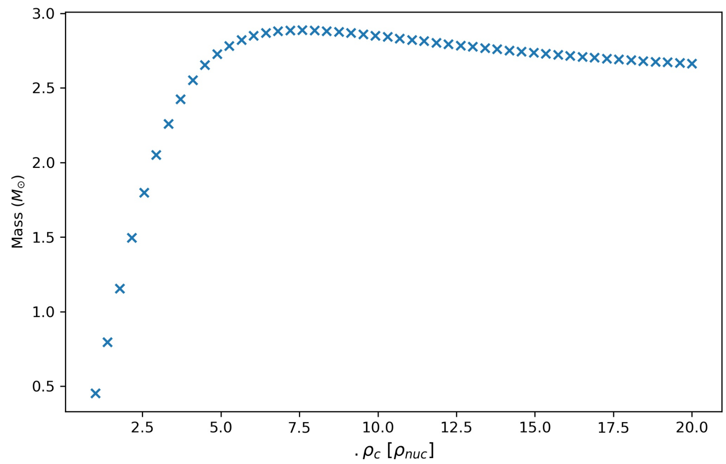

Figure 6: From the plot, it can be seen that neutron star attains the maximum mass of 2.88 $M_\odot$ at around $7.6\rho_{nuc}$

From the plot, it can be inferred that radius of neutron star reaches a maximum upto a certain value of central density and then it falls to about 10.5 km as central density increases.

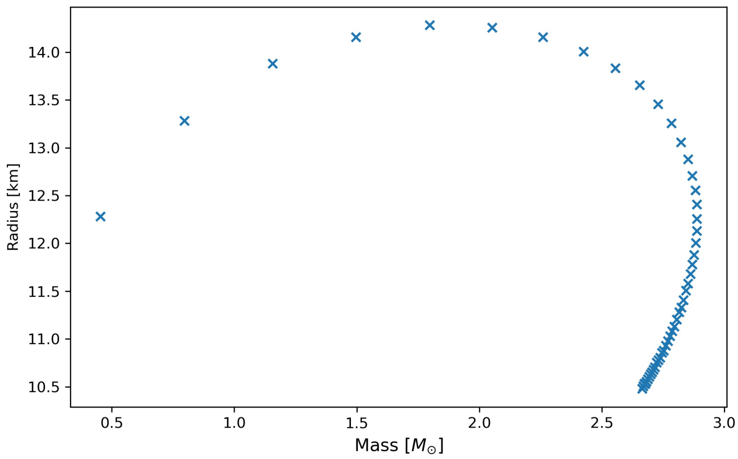

Figure 7: Radius - Mass plot for various central densities.

Similarly, the radius - mass relation of neutron stars show a limit to the maximum mass a neutron star can attain according to the K and γ value in table 1. From the plot, it is observed that neutron star reaches a maximum value of 2.88 $M_\odot$ and maximum radius of around 14 km.

6 Conclusion

In this project, structure of the neutron star is studied by solving TOV equation numerically. For this, a python based runge-kutta solver was developed. This solver can be used to study the degenerate stars for an appropriate set of parameters a neutron star of $1.5M_\odot$ and radius 14.2 km can be obtained. Then we plotted the pressure,density, mass profile of this neutron star. As expected the equation of state, the central density effects the total mass of the star and its radius significantly. Additionally, the effects of central density variations for neutron stars was also studied. Our studies show that neutron stars may reach a maximum mass of $2.88M_\odot$ whose radii can be around 14.3 km.
The TOV equations can be further modified to study the internal structure of massive neutron stars. However, it turns out that massive neutron stars with scalar fields may turn invisible and become dark energy objects[2]. The neutron stars formed from the collapse of first generation of stars may have emitted their total energies long time ago to become dark and disappear from our todays observational windows. Based on the solution of the TOV equation modified to include a universal scalar field at the background of supranuclear densities, it was shown that pulsars must be born with embryonic super-baryons (SBs). The cores of these SBs most likely are made of purely incompressible superconducting gluon-quark superfluids. Such quantum fluids have a uniform supranuclear density and governed by the critical EOSs for baryonic matter Pb = Eb and for ϕ - induced dark energy Pϕ = Eϕ. The glitch phenomena observed in NSs and pulsars can be explained if NS-cores are made of superfluids. So, we expect isolated massive NSs to metamorphose into dark objects, whose interior are made of incompressible gluon-quark superfluids and to subsequently disappear from our observation windows.

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