Mathematical Theorems

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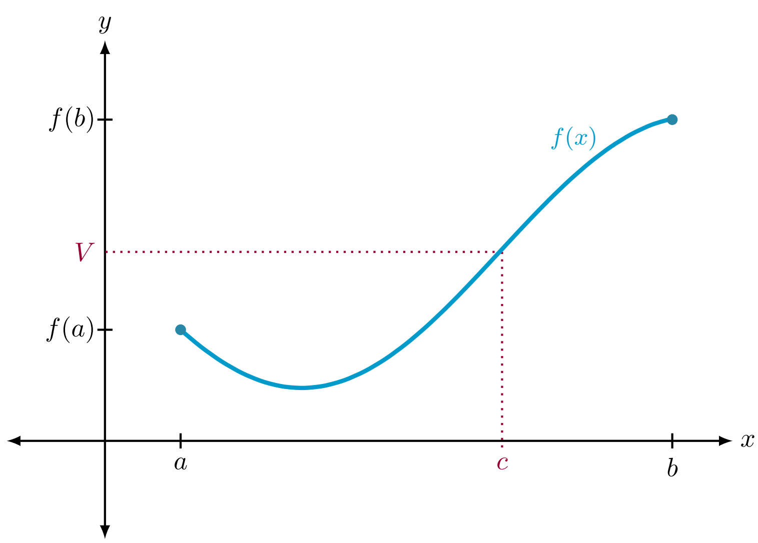
1 Functions

Theorem 1.1 (**Intermediate Value Theorem**)**.** Suppose f is a continuous function on the closed interval $[a, b]$. If V is any number between $f(a)$ and $f(b)$, then there exists at least one number c in the open interval (a, b) such that $f(c) = V$. [1]



[2]

2 Polynomials

Theorem 2.1 (**Extreme Value Theorem**)**.** If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then, f attains a maximum and minimum value. In other words, there exist $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$. [3]

Proposition 2.2 (**Local Extrema of Polynomials**)**.** If $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ is a polynomial of degree n, then the graph of P has at most $n - 1$ local extrema. [1]

Theorem 2.3 (**Remainder Theorem**)**.** If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$ [1]

Theorem 2.4 (**Factor Theorem**)**.** $P(c) = 0$ if and only if $(x - c)$ is a factor of $P(x)$. [1]

Theorem 2.5 (**Rational Zeros Theorem**)**.** If the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients (where $a_n \neq 0$ and $a_0 \neq 0$), then every rational zero of P is of the form $\frac{p}{q}$, where p and q are integers and

- p is a factor of the constant coefficient a_0
- q is a factor of the leading coefficient a_n

[1]

Theorem 2.6 (**Upper and Lower Bound Theorem**)**.** Let P be a polynomial with real coefficients.

1. If we divide $P(x)$ by $(x - b)$ (with $b \neq 0$) using synthetic division, and if the row that contains the quotient and remainder has no negative entry, then b is an upper bound for the real zeros of P.
2. If we divide $P(x)$ by $(x - a)$ (with $a \neq 0$) using synthetic division, and if the row that contains the quotient and remainder has entries that are alternately nonpositive and nonnegative, then a is a lower bound for the real zeros of P.

[1]

Theorem 2.7 (**Descartes' Ruel of Signs**)**.** Let $P(x)$ be a polynomial with real coefficients.

1. The number of positive real zeros of $P(x)$ is either equal to the number of variations in sign in $P(x)$ or is less than that by an even whole number.
2. The number of negative real zeros of $P(x)$ is either equal to the number of variations in sign in $P(-x)$ or is less than that by an even whole number.

[1]

Theorem 2.8 (**Fundamental Theorem of Algebra**)**.** Every polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad (n \geq 1, a_n \neq 0)$$

with complex coefficients has at least one complex zero. [1]

Theorem 2.9 (**Complete Factorisation Theorem**)**.** If $P(x)$ is a polynomial of degree $n \geq 1$, then there exist complex numbers $a, c_1, c_2, \ldots, c_n$ (with $a \neq 0$) such that

$$P(x) = a(x - c_1)(x - c_2)...(x - c_n)$$

[1]

Theorem 2.10 (**Zeros Theorem**)**.** Every polynomial of degree $n \geq 1$ has exactly n zeros, provided that a zero of multiplicity k is counted k times. [1]

Theorem 2.11 (**Conjugate Zeros Theorem**)**.** If the polynomial P has real coefficients and if the complex number z is a zero of P, then its complex conjugate z is also a zero of P. [1]

Theorem 2.12 (**Linear and quadratic factors theorem**)**.** Every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors with real coefficients. [1]

References

[1] James Stewart, Lothar Redlin, and Saleem Watson. *Precalculus Mathematics for Calculus*. Cengage Learning, 2015.

[2] L. K. Williams. Tikz gallery. Accessed on: 2023-12-11.

[3] Stephen Abbott. *Understanding Analysis*. Springer, New York, 2015.