Optimal Splits and Conjugates

- $w = s_1 \dots s_m$ is a string in the free generators
- $$C(w) = \begin{cases} \rho(s_2 \dots s_{m-1}) & s_1 = s_m^{-1} \\ \infty & \text{otherwise} \end{cases}$$
- $$\rho(w) = \begin{cases} m & m \leq 1 \\ \min(C(w), \min_{1<i<m}(\rho(s_1, \dots, s_{i-1}) + \rho(s_i, \dots s_m))) & \text{otherwise.} \end{cases}$$

Proof of Correctness

- $\rho(w) \geq rk_F(w)$ (from splitting and conjugacy)
- $\rho(w') = rk_F(w)$ when w' is a minimal band presentation for w.
- $\rho(w') \leq \rho(w)$ when w' is w after free reduction.

$\Rightarrow \rho(w) = rk(w)$ for reduced words.