Computing the Rank of Braids

Overview: the rank and lower bounds

- What are braid groups?
- What is the rank of a braid?
- Ways to find bounds on the rank.

Braid groups

- **Braid group on n-strands B_n:**
  - positive generators: $\{\sigma_1, \ldots \sigma_{n-1}\}$
  - $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
  - $\sigma_j \sigma_i = \sigma_i \sigma_j, |i - j| \geq 2$

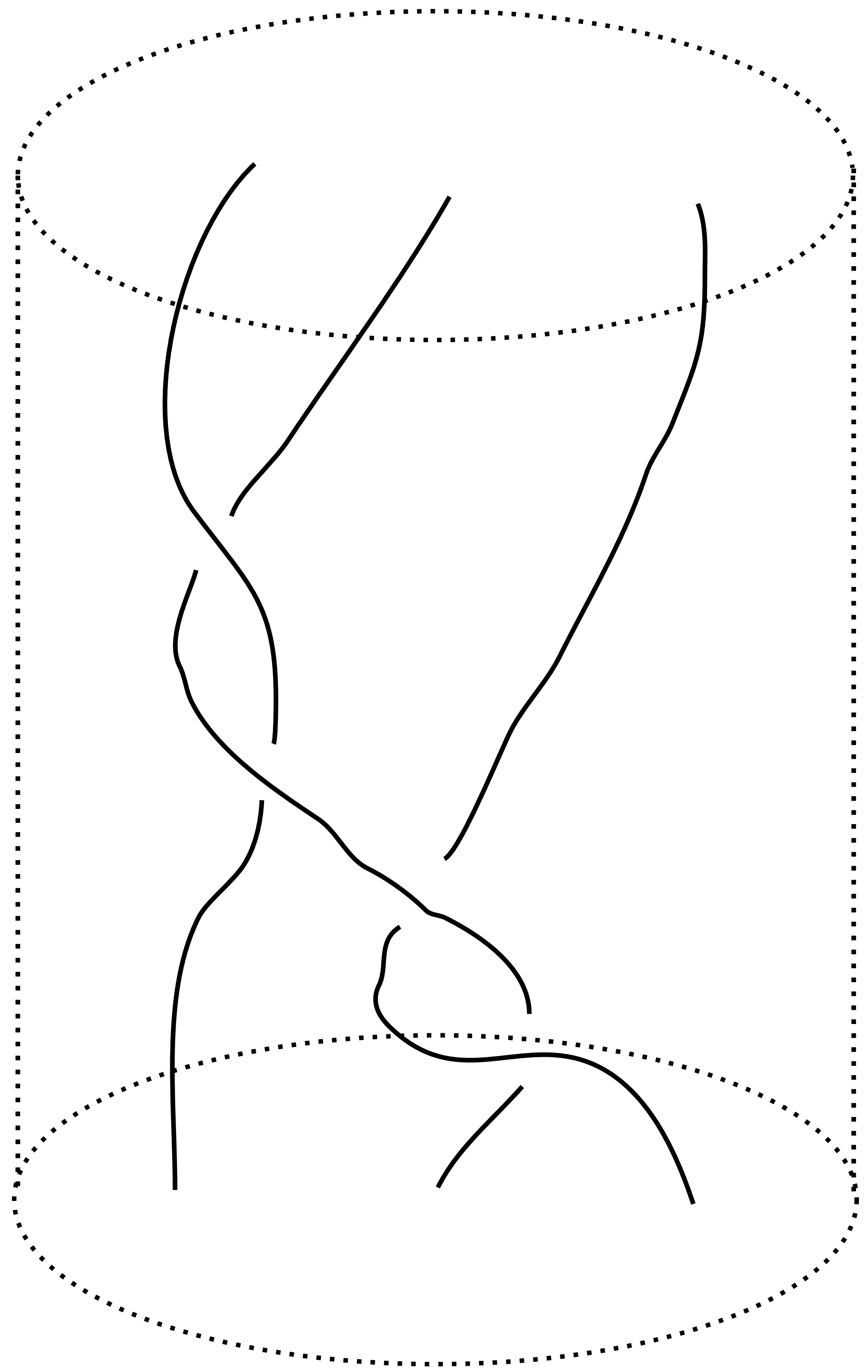
$$12\bar{3}2\bar{1}4 \in B_5$$

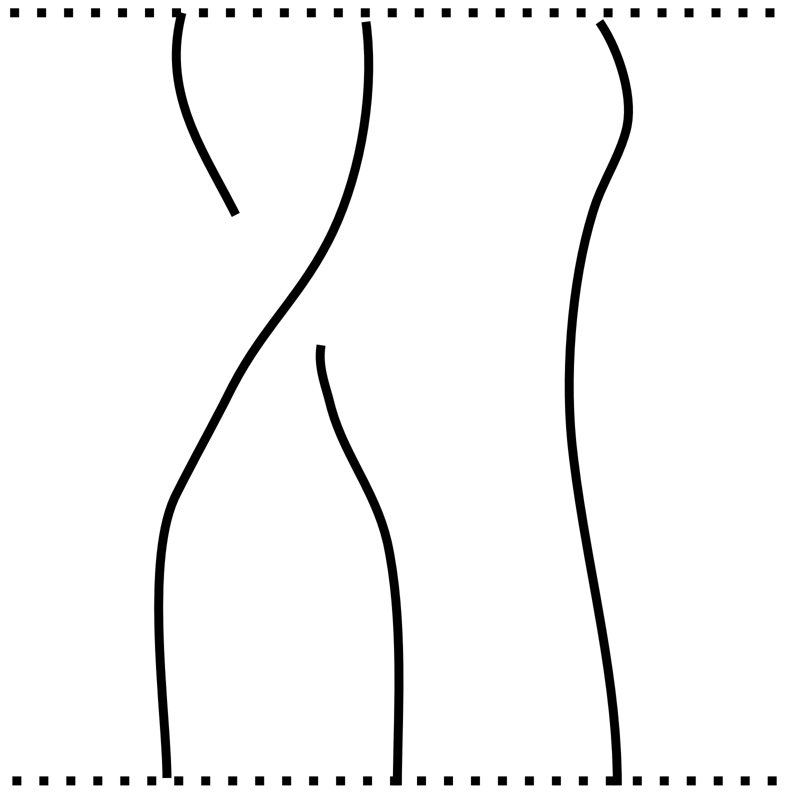
Topological braids

- 1D Submanifold M of $D^2 \times I$ with boundary, having n components, so that projection $\pi_I\colon M \to I$ is a covering map.

- Braid group describes isotopy classes.

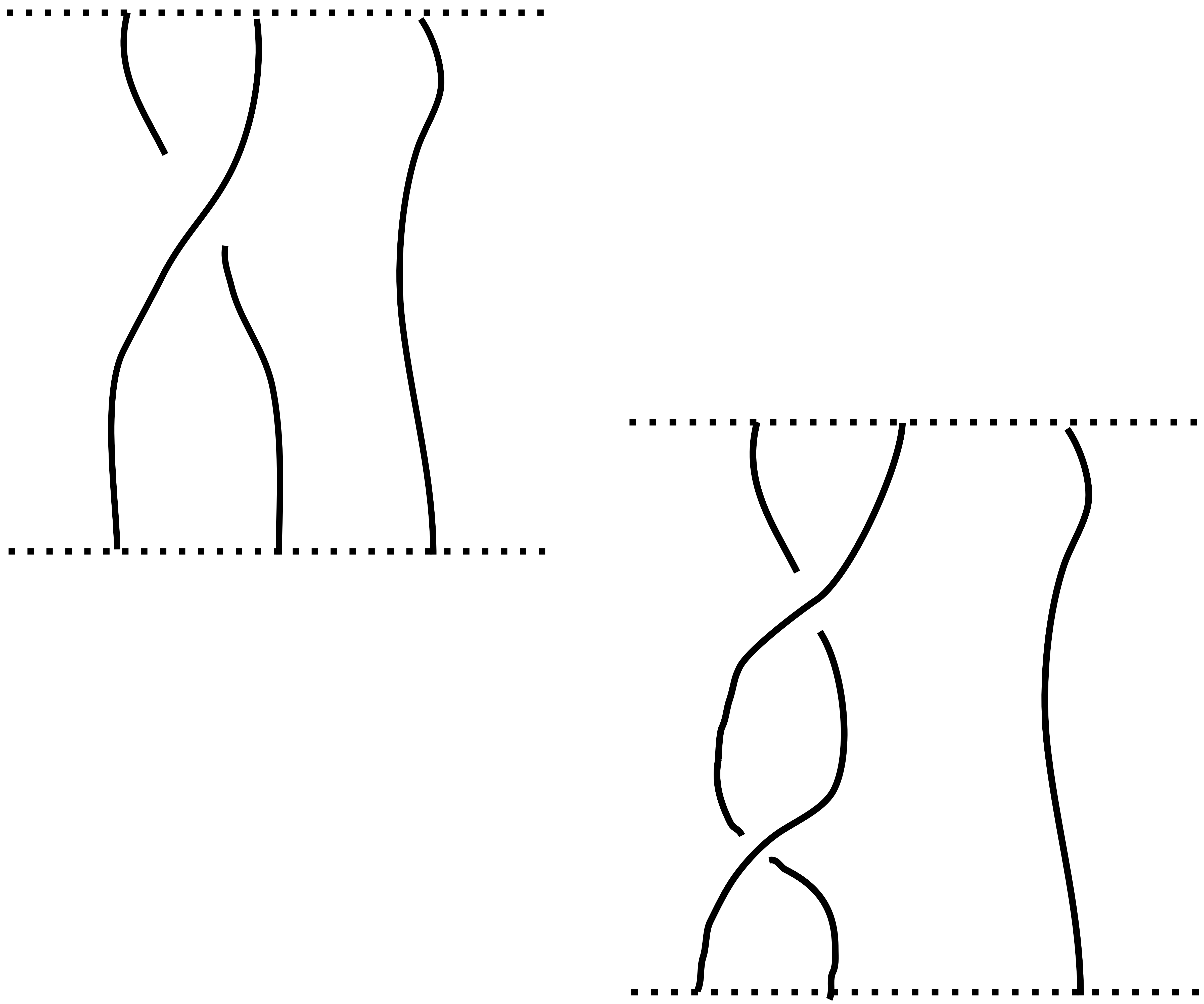
- σ_i is crossing of ith strand underneath (i+1)th strand.





Fundamental quotients

- Exponential sum $\gamma \colon B_n \to \mathbb{Z}$.
  - $\gamma(\sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_m}^{\epsilon_m}) = \sum_{i=1}^{m} \epsilon_i$
- Permutation: $\phi \colon B_n \to S_n$
  - $\phi(\sigma_i) = (i \ i+1)$



The rank

- A **positive band** is a conjugate of a positive standard generator.
- A **negative band** is the inverse of a positive band.
- A **band presentation** for $x \in B_n$ is a tuple of bands $(\alpha_1 \sigma_{j_1}^{\epsilon_1} \alpha_1^{-1}, \ldots, \alpha_k \sigma_{j_k}^{\epsilon_k} \alpha_k^{-1})$ of any length such that $x = \prod_{i=1}^{k} \alpha_i \sigma_{j_i}^{\epsilon_i} \alpha_i^{-1}$.
- The rank $rk(x)$ is the minimal number of bands required to write a presentation.

Quotients give lower bounds

- For a surjective homomorphism $\phi : B_n \to H$:
  - A **band** in H is a conjugate of $\phi(\sigma_i)$
  - $rk_H(x)$ is then defined similarly and $rk_H(\phi(x)) \le rk(x)$.
  - Because $\phi(x) = \prod_{i=1}^{k} \phi(\alpha_i)\phi(\sigma_{j_i})^{\epsilon_i}\phi(\alpha_i)^{-1}$ gives a band presentation in H.

Example

- $x = 123\overline{1}$
- Length $\Rightarrow rk(x) \leq 4$
- Exponential Sum: $\omega(x) = 2 \Rightarrow rk(x) \geq 2$
- Band presentation: $(123\overline{1}) = (12\overline{1})(13\overline{1}) \Rightarrow rk(x) = 2$

Overview: computing ribbon genus

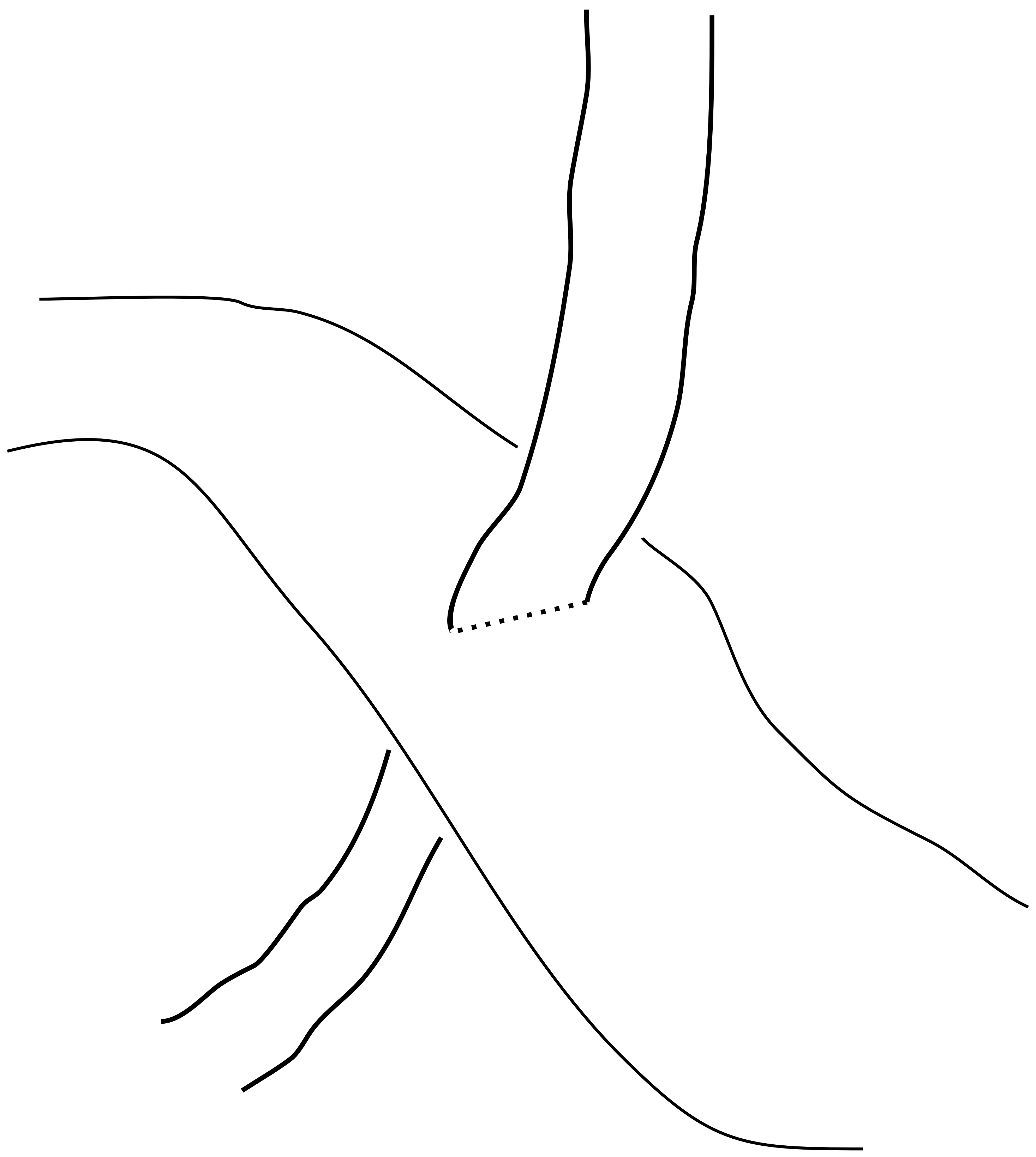
- What is the slice genus and ribbon genus of a knot?
- How is it related to the rank?

Slice and ribbon surfaces

- **Knot:** smooth embedding of the circle S^1 into the 3-sphere S^3 (up to isotopy).

- **Slice surface:** for a knot $K \subseteq S^3 = \partial D^4$ is an orientable surface smoothly embedded in D^4 whose boundary is K.

- **Ribbon surface:** for a knot $K \subseteq S^3$ is an orientable surface immersed in S^3 whose boundary is K and whose singularities are all of **ribbon type**.



Slice and ribbon genus

- **Slice genus** $g_s(K)$: minimum genus of a slice surface for K.
- **Ribbon genus** $g_r(K)$: minimum genus of a ribbon surface for K.
- From a ribbon surface we can construct a slice surface of the same genus $g_s \leq g_r$ [Grigsby, 2018].

Slice surface from band presentation

- Given a band presentation we can construct a ribbon surface. The Ribbon Genus depends only the number of bands k and the number of strands n: [Rudolph, 1983]

- $g_r(K) \leq \dfrac{1 + rk(x) - n}{2}$

- Furthermore $g_s(K) = \dfrac{1 + rk(x) - n}{2} \Rightarrow g_s(K) = g_r(K)$

- **Slice-Ribbon conjecture**: Is $g_s(K) = g_r(K)$ true for all knots? (typically genus 0)

Overview: upper bound methods

- Construct an algorithm for computing rank in the free group.
- Adapt the algorithm for braid groups.
- Test its effectiveness on braids in the KnotInfo database.

Rank in the free group?

- A **band** in the free group is a conjugate of a generator. Rank rk_F is defined similarly.
- rk_F gives upper bound on rk
- Suggests how to approach computational questions.
- Identify free words with positive band presentations [Orekov, 2004].

Inductive relationships

- Splitting: $rk_F(wv) \leq rk_F(w) + rk_F(v)$ for $w, v \in F_n$,
    - Band presentations can be concatenated.
- Conjugacy: $rk_F(vwv^{-1}) = rk_F(w)$
    - Conjugate each term in a band presentation.

Example

- $aba^{-1}c \in F_3$
- $rk_F(aba^{-1}c) \leq rk_F(aba^{-1}) + rk_F(c) \leq rk_F(b) + rk_F(c) = 2$
- But how should we apply them? $a \cdots b \cdots a^{-1} \cdots b^{-1}$
- Claim: rank is computed by applying these relationships optimally.

Optimal Splits and Conjugates

- $w = s_1 \ldots s_m$ is a string in the free generators
- $C(w) = \begin{cases} \rho(s_2 \ldots s_{m-1}) & s_1 = s_m^{-1} \\ \infty & \text{otherwise} \end{cases}$
- $\rho(w) = \begin{cases} m & m \leq 1 \\ \min(C(w), \min_{1<i<m}(\rho(s_1, \ldots, s_{i-1}) + \rho(s_i, \ldots s_m))) & \text{otherwise.} \end{cases}$

Proof of Correctness

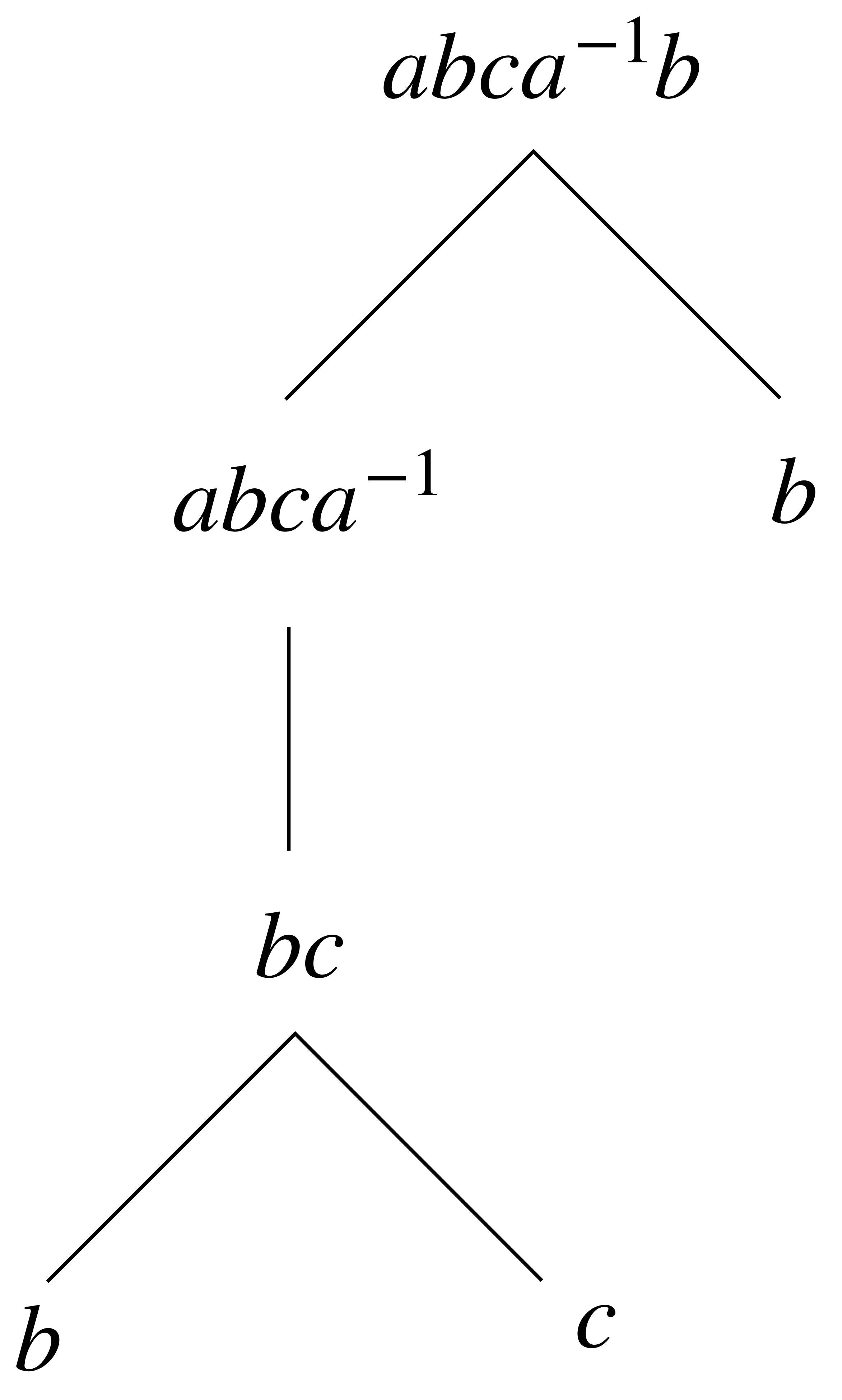
- $\rho(w) \geq rk_F(w)$ (from splitting and conjugacy)
- $\rho(w') = rk_F(w)$ when w' is a minimal band presentation for w.
- $\rho(w') \leq \rho(w)$ when w' is w after free reduction.

$\Rightarrow \rho(w) = rk(w)$ for reduced words.

Expression Trees

Expression Tree for w

- Each node is labeled with a string.
- Root is w
- If $|v| \leq 1$ then v is a leaf.
- Otherwise v has one or two children:
  - If one child v' then $v = sv's^{-1}$ (**conjugate node**)
  - If two children w_1, w_2 then $v = w_1 w_2$ (**leaf node**)



Relationship to ρ

- $L(T)$ is the number of leaves with nonempty label.
- An expression tree T is optimal if $L(T) \leq L(T')$ for all other T'.
- $\rho(w) = L(T) \Leftrightarrow T$ is optimal.

$\rho(w') \leq \rho(w)$

- $w = \cdots s_i s_{i+1} \cdots$
- Each letter s_i appears in a tree in exactly one of two ways:
    - Leaf node
    - Removed by conjugate node $s_i w s_i^{-1} \to w$
- Enumerate all possible cases. Show there is a node of interest $v = \cdots s_i s_i^{-1} \cdots$
- Replace subtree at v with a tree for the reduced word v' without adding leaf nodes.

Example Case 1

s_i and $s_{i+1}$are removed by the same conjugation node:

$v = s_i s_{i+1}$

$\rho(v) = C(s_i s_{i+1}) = 0$

After reduction:

$\rho(v') = 0$

$$\begin{tikzcd} s_i s_{i+1} \arrow[d, dash] \\ empty \end{tikzcd}$$

Example Case 2

s_i is a leaf s_{i+1} is removed by a conjugate node, and s_i is its descendent

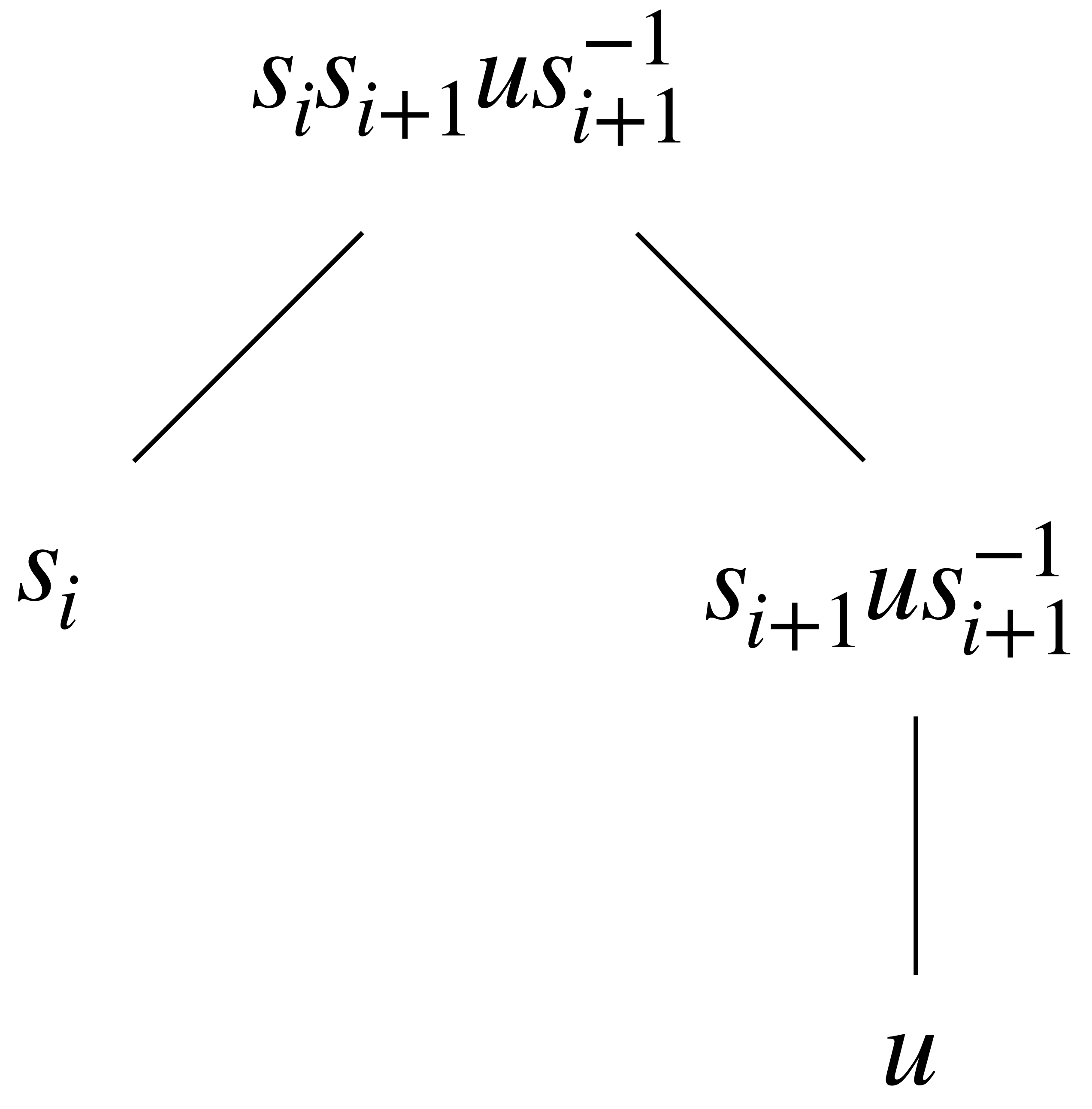
$v = s_i s_{i+1} u s_{i+1}^{-1}$ where u is a string

Assume s_i and $s_{i+1} w_1 s_{i+1}^{-1}$ are children.

$\rho(v) = \rho(s_i) + C(s_{i+1} u s_{i+1}^{-1}) = 1 + \rho(u)$

After reduction:

$\rho(v') \leq \rho(u s_{i+1}^{-1}) \leq \rho(u) + \rho(s_{i+1}^{-1}) = \rho(u) + 1$



Conclusion

- Check all 9 cases.

- Conclude there exists an expression tree T' for w' where $L(T') \leq L(T)$.

$\Rightarrow \rho(w') \leq \rho(w)$

ρ with dynamic programming

Algorithm:

- $w = s_1 \cdots s_k \in F_n$
- Allocate an upper triangular matrix $\{m_{i,j}\} \in M(k \times k, \mathbb{Z})$
- $m_{i,j} = \rho(s_i \cdots s_j)$
- Fill in diagonal with 1.
- Fill in entries for subwords of length 2 by computing ρ replacing recursive calls with $m_{i,j}$
- Continue up to k.

Example:

$bcab^{-1}aba^{-1}c^{-1} \in F_3$

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 3 & 4 & 3 & \mathbf{2} \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 3 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Inductive braid relationships

- Same relationships hold for braids:
  - Splitting: $rk(xy) \leq rk(x) + rk(y)$ for $x, y \in B_n$,
  - Conjugacy: $rk(\alpha x \alpha^{-1}) = rk(x)$
- Find additional subwords we can improve.

Detecting Conjugates

- Detect conjugates by comparing words.

Word Problem Algorithms:

- Lawrence-Krammer matrix representation. $O(ln^6)$
- Artin's action on free group. $O(3^l n)$
- Dehornoy handle reduction

KnotInfo Test

- KnotInfo database 2977 knots and their braid representations.
- Our algorithm identifies quasipositive knots ($rk(x) = |\gamma(x)|$) with a 75% success rate.
- Shows that $g_S = g_r$ for 498 knots.

Overview: Brute Force Search

- Algorithm for generating bands instead of “seeing them”.
- Is rk computable?
- Construct practical version of the algorithm.

Search Algorithm

- Does there exist a band presentation for x having k bands?

Naive Search Algorithm:

- Generate all bands having maximum conjugate length l
- Generate all products of k bands. If any are equal to x the result is true.
- If none are found increase l and repeat.

Comparison with Halting Problem

Does x have a band presentation of k bands?

- If a presentation exists, the search will terminate.
- If the search runs for a long time, it may not exist, or we haven't waited long enough to find one.
- Given an upper bound on the conjugate length, we could conclude it doesn't exist after considering those bands.
- Such a bound exists.
- Possibly computable?

Does a Turing machine terminate on a given input?

- If the machine halts on the input, then it will terminate.
- If the machine runs for a long time, it may never terminate or we haven't waited long enough.
- Given an upper bound on the time (for machine that terminate) we could conclude it won't halt after exceeding that duration.
- Such a bound exists. [Minsky, 1967]
- It is not computable. (grows too large)

Templated Search

- Make band presentation search practical.
- Uses braid quotients to *enumerate* fewer band presentations.

Template Definition

- $\phi\colon B_n \to H$ is a homomorphism.
- Define $\beta_\phi = \{\phi(\alpha\delta^{\pm}\alpha^{-1})\colon \alpha \in B_n, \delta \in \{\sigma_1 \ldots \sigma_{n-1}\}\}$
- A k-template for a braid x is a tuple $(b_1, b_2, \ldots b_k) \in \beta_\phi^k$ such that $b_1 b_2 \ldots b_k = \phi(x)$

Permutation templates

- A k-**template** in S_n is tuple of transpositions $((i_1 j_1), \ldots, (i_k, j_k))$ such that $\phi(x) = (i_1 j_1) \cdots (i_k j_k)$
- Example: $x = \bar{1}211$
  - 2-templates: $((13), (12)), ((23), (13)), ((12), (23))$
  - The presentation $(\bar{1}21)(1)$ satisfies the template: $(13), (12)$

Exponential sum templates

- A **k-template** in $\mathbb{Z}$ is a tuple in $(s_1, \ldots s_k) \in \{1, -1\}^k$ such that $\sum_{i=1}^{k} s_i = \gamma(x)$.
- Alternatively written as string of signs: $\{+,-\}^k$
- Example: $w = \bar{2}\bar{2}1\bar{2}1$ and $\gamma(w) = -1$
  - 3-templates: $+--, -+-, --+$
  - The presentation $(\bar{2}12)(2\bar{1}\bar{2})(\bar{1}\bar{2}1)$ satisfies the template $+--$

Signed permutation templates

- Consider the homomorphism: $\psi\colon B_n \to S_n \times \mathbb{Z}$.
- The k-templates are the cartesian product of permutation and exponential sum k-templates.

Naive search example

- $x = \bar{2}\bar{2}1\bar{2}1$ and $1 \leq rk(x) \leq 5$. Does $rk(x) = 3$?
- Construct the set of bands B having maximum conjugate length $l = 3$.
    - $|B| = 1029$
- Consider every band in every position:
    - $(b_1, b_2, b_3) \in B \times B \times B$
    - $|B|^3 = 1{,}089{,}547{,}389$ braids to compare.

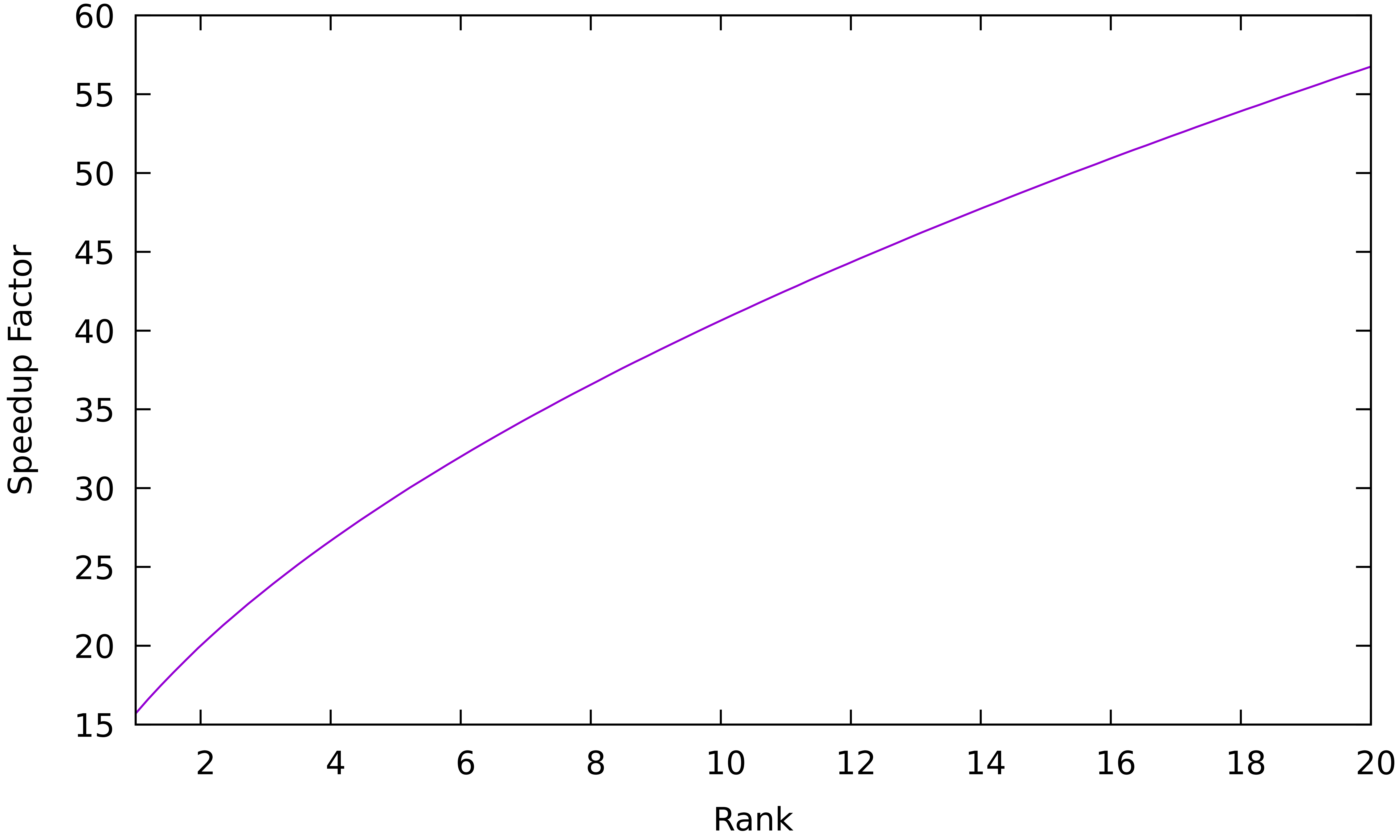
Templated search

- Generate each band β having maximum conjugate length $l = 3$. Place in set $C_{\phi(\beta)}$
- Construct each 3-template for x:
  - $-(13), -(12), +(12), \quad +(12), -(13), -(23)\ , \quad \ldots$ (27 total)
- Consider only the bands which match the template in each position:
  - $(b_1, b_2, b_3) \in C_{-(13)} \times C_{-(12)} \times C_{+(12)}$
  - $|C_{-(13)}||C_{-(12)}||C_{+(12)}| = 810$
  - Fewer than $27{,}000$ total braids to compare.

Results

- Found 5 band presentations for x including:
  - $(\bar{2}\bar{2}1\bar{2}\bar{1}22)(3\bar{1}2\bar{3}\bar{2}1\bar{3})(\bar{1}\bar{1}\bar{1}2111) = \bar{2}\bar{2}1\bar{2}1$

Speedup factor



n = 5

$$\frac{2^k \binom{n}{2}}{\binom{k}{\lfloor \frac{k+1}{2} \rfloor}}$$

Thank You!