MXB103 Project: Bungee!

Contents

1 Background

As part of Brisbane's "New World City" transformation, the Brisbane City Council is proposing to allow bungee jumping off the Story Bridge. A commercial bungee jump company has expressed interest, and provided Council with facts and figures concerning the proposal. Your group has been hired as consultants to verify certain aspects of this information, and to answer several key questions that have been raised about the proposal.

To answer these questions you will develop a mathematical model of the bungee jumping process, solve it numerically using MATLAB, and prepare a report to present to Council. As the proposal is still only in preliminary stages, the scope of this report is limited to investigating a few key aspects of the proposal (as will be outlined below).

2 Model

The proposal calls for a platform to be installed at the very top of the bridge, from which the bungee jumps will take place. Suppose this platform is at height H from the water level. Now, let y represent the distance the jumper has *fallen*. Hence, $y = 0$ corresponds to the platform, and y increases as the jumper falls towards the river. Later, we will also be interested in the height of the deck of the bridge, which will be distance D from the water level.

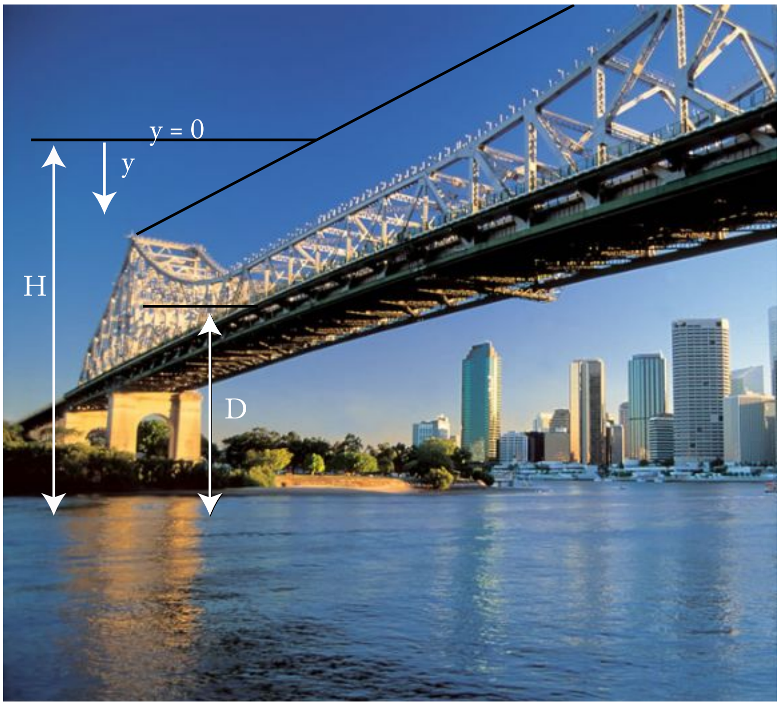

Figure 1: Illustration of platform height H, deck height D and jumper's position y, where the positive direction of y is downwards.

The mathematical model of bungee jumping can be derived by considering the forces acting on the jumper at all times. There are three forces we must consider:

1. Gravity;
2. Drag (also called air resistance): the frictional force due to the air;
3. Tension: the pull of the bungee rope when it is taut.

As Figure 2 (next page) illustrates, there are four different parts of the jump: falling with a slack bungee rope, falling with a taut bungee rope, upward bounce with a taut bungee rope, and upward bounce with a slack bungee rope (after which the cycle repeats). Depending on which part of the jump we consider, some of the above three forces may change direction, or be absent completely.

The simplest force to model is gravity: it always acts downwards, and its value is given by mg, where m is the mass of the jumper, and g is the gravitational acceleration.

The next force we consider is drag. This force always acts in the opposite direction to motion, so that it is always slowing the jumper down. Its value is given by $-c|v|v$, where c is the drag coefficient, and v is the velocity of the jumper. Notice that the strength of the drag force is proportional to the square of the velocity. The absolute value sign on the first factor of v ensures it always acts in the opposite direction to motion.

The final force we must consider is tension. When the bungee rope is taut, it exerts a force on the jumper proportional to how much it has been stretched (this is called *Hooke's law*). This force always acts upwards, pulling the jumper back up. If we let the length of the *unstretched* rope be L, then the tension when the rope is taut is given by $k(y - L)$, where k is the "spring constant", which measures the elasticity of the bungee rope. When the rope is slack (like at

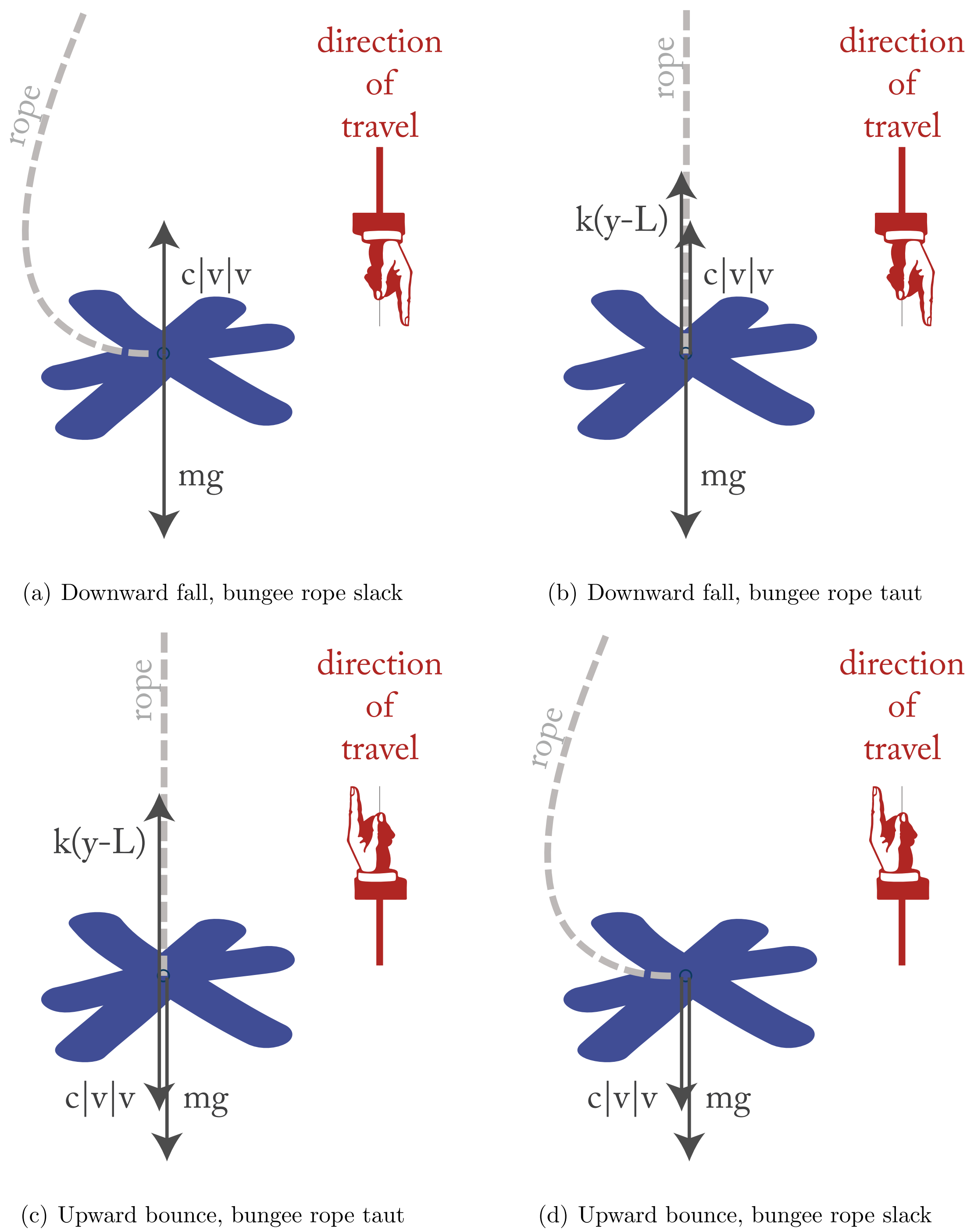

Figure 2: The four stages of a bungee jump and the forces acting on the jumper.

the beginning of the jump), there is no tension. Altogether, we may write the tension force as $-\max(0, k(y-L))$. The *max*imum function ensures that the tension only "switches on" when $y > L$ (i.e. when the rope is taut) and the minus sign in front ensures that it acts upwards.

To summarise, the forces acting on the jumper are

gravity	mg
drag	$-c\|v\|v$
tension	$-\max(0, k(y-L))$

Now, Newton's second law of motion says that the sum of these forces must equal the product of the jumper's mass and acceleration. Hence, the equation governing bungee jumping is the following ordinary differential equation (ODE):

$$m\frac{dv}{dt} = mg - c|v|v - \max(0, k(y-L)),$$

where $\frac{dv}{dt}$ is the jumper's acceleration. We can simplify this slightly by dividing through by m, to obtain

$$\frac{dv}{dt} = g - C|v|v - \max(0, K(y-L)), \tag{1}$$

where $C = c/m$, and $K = k/m$.

3 Numerical method

Equation (1) is too complicated to solve analytically, so you will need to use numerical methods to find solutions. You have developed numerical methods for solving ODEs in your lectures and practicals. Solving the bungee jumping equation will require just a slight modification of these methods. Since equation (1) actually involves *two* unknowns, v and y, we need a second equation that relates the two. This relationship is simply that the jumper's velocity v is the derivative of the jumper's position y. Hence for numerical purposes, we can think of this problem as *two* ODEs:

$$\frac{dy}{dt} = v \tag{2}$$

$$\frac{dv}{dt} = g - C|v|v - \max(0, K(y-L))\,. \tag{3}$$

We can extend any of our numerical methods for solving ODEs to two coupled ODEs:

$$\frac{dy}{dt} = f_1(t, y, v)$$
$$\frac{dv}{dt} = f_2(t, y, v).$$

For example, for Euler's method, expanding both $y(t)$ and $v(t)$ in Taylor polynomials of degree one centered at $t = t_i$ and evaluating the polynomials at $t = t_i$ (see Chapter 2 lectures) yields:

$$y_{i+1} = y_i + hy'(t_i) = y_i + hf_1(t_i, y_i, v_i)$$
$$v_{i+1} = v_i + hv'(t_i) = v_i + hf_2(t_i, y_i, v_i),$$

where h is the subinterval width and y_i and v_i are our numerical solutions for $y(t)$ and $v(t)$ at $t = t_i$ (we haven't bothered introducing a new variable w_i to represent the numerical solution, since we have enough different letters already!). Hence, for the bungee jump model where $f_1(t, y, v)$ and $f_2(t, y, v)$ are identified from the right-hand side of Equations (2)–(3), Euler's method is:

$$y_{i+1} = y_i + hv_i \tag{4}$$
$$v_{i+1} = v_i + h\big(g - C|v_i|v_i - \max(0, K(y_i - L))\big). \tag{5}$$

4 Model parameters

You have been provided with the following parameters for the model. Some of these come from facts about the bridge itself, while others have been supplied by the bungee jump company assuming an 80kg jumper.

Height of jump point	$H = 74\,\mathrm{m}$
Deck height	$D = 31\,\mathrm{m}$
Drag coefficient	$c = 0.9\,\mathrm{kg/m}$
Mass of jumper	$m = 80\,\mathrm{kg}$
Length of bungee rope	$L = 25\,\mathrm{m}$
Spring constant of bungee rope	$k = 90\,\mathrm{N/m}$
Gravitational acceleration	$g = 9.8\,\mathrm{m/s^2}$

Table 1: Model parameters.

5 Your tasks

Numerical solution

The consultancy contract with the Council requires you to implement either the Modified Euler Method *or* the Classical Fourth Order Runge-Kutta (RK4) method to solve the model to obtain the jumper's position and velocity. You only need to implement one of these methods: you may choose which.

Use your code to obtain the numerical solution for the model and plot the jumper's position against time. You have been provided with the set of model parameters in Table 1. The scope of this report focuses on this particular set of parameters only, which are for an 80kg jumper.

Analysis

1. The bungee jump company suggests that the standard jump will consist of 10 "bounces" which should take approximately 60 seconds. (Although the jumper will still be in motion at this point, the jump will be considered to be over, and the the jumper will be gently raised all the way back onto the platform above.) Do your model results agree with this timing: 10 bounces in around 60 seconds?

2. The "thrill factor" of bungee jumping is partly determined by the maximum speed experienced by the jumper. What is this maximum speed and when does it occur in relation to the overall jump? Answer this question graphically by plotting the jumper's velocity against time.

3. Another factor for thrill-seekers is the maximum acceleration experienced by the jumper. More acceleration equals bigger thrills, but too much acceleration can be dangerous. The bungee jump company boasts that the jumper will experience acceleration "up to $2g$". Use numerical differentiation to find the acceleration predicted by your model, and plot the jumper's acceleration against time. What is this maximum acceleration and when does it occur in relation to the overall jump? Is the claim of "up to $2g$" acceleration supported by the model?

4. For the writing of promotional material it is of interest to know how far the jumper actually travels in the 60 second jump. One way to answer this question is to compute the integral

$$\int_0^{60} |v| \; dt \, .$$

   Use numerical integration to compute this integral and hence determine how far the jumper travels.

5. Part of the proposal is to have a camera installed on the bridge deck, at height D from the water. As the jumper first passes this point, the camera would take a photo which could then be offered for purchase afterwards. It is hoped that the model can provide sufficiently accurate results that the camera could be set to trigger at a predetermined time, for a given set of model parameters.

   The distance the jumper falls from the platform to the deck is $H - D$. Hence you need to compute an accurate value for t such that $y(t) = H - D$. Since you only know $y(t)$ as a discrete set of points, not as a function, you will need to fit an interpolating polynomial. Construct an interpolating polynomial $p(t)$ through the nearest four values of your numerical solution that lie either side of $H - D$. That is, write MATLAB code to find values $y_i, y_{i+1}, y_{i+2}, y_{i+3}$ such that $y_i, y_{i+1} < H - D$ and $y_{i+2}, y_{i+3} > H - D$. Use a rootfinding method of your choice to find the value of t such that $p(t) = H - D$. Hence, for the model parameters provided, at what time should the camera trigger in order to capture the image of the jumper?

6. The bungee jump company has suggested a "water touch" option could be considered, whereby the jumper just touches the water at the bottom of the first bounce. For the given parameters and assuming a jumper of height 1.75m, how close does the jumper come to touching the water? Investigate how the bungee rope could be altered (its length, its spring constant, or both) to produce a true water touch experience for an 80kg jumper, while keeping as close as possible to 10 bounces in 60 seconds. Note: any combination of parameters that produces acceleration of greater than $2g$ must be rejected as too dangerous.

6 Report

You are required to submit a report that introduces the problem, outlines the mathematical model, discusses your solution methodology, presents your results/findings/analyses and concludes your work.

Your report must be typeset in a word processor or document preparation system such as Microsoft Word, Google Docs, LaTeX or similar.

Your report must not exceed 10 pages and must be typeset in 11pt using an appropriate font. All margins must be set to 2cm. Your report must be submitted in **PDF format** and must be structured as follows:

Title

Student Name 1, Student Name 2, Student Name 3, Student Name 4

1 Introduction

Introduce and motivate the study.

What is the proposal exactly? What questions have you been asked to answer? What's your approach to doing this? A few paragraphs should be enough. Finish with a sentence linking to the next section on the model.

In Section 2 of this report, the model describing ... In Section 4, etc.

2 Mathematical Model

Write a paragraph or two about the mathematical model, including where it comes from and what the symbols mean. Include all relevant equations.

Write a paragraph on the assumptions and limitations on the model, and a paragraph on the limitations on the study you have conducted.

3 Numerical Solution

How do you formulate the model to solve numerically? (hint: you write it as two differential equations).

Discuss your chosen solution methodology. State all parameter values such as the final time, number of subintervals etc. Include all relevant equations. You must provide enough detail to allow someone else to reproduce your results.

Include and discuss figure of jumper's position against time.

4 Analysis

In this section, the model predictions are analysed with respect to the key questions being asked about the proposal. Under each subsection, describe the question and then answer it. Include any support figures and equations to describe your solution approach.

4.1 Timing and bounces

4.2 Maximum speed experienced by the jumper

4.3 Maximum acceleration experienced by the jumper

4.4 Distance travelled by the jumper

4.5 Automated camera system

4.6 Water touch option

5 Conclusion

Conclude your report by summing up your findings and making any recommendations.

6 References

List any references you have cited in your report.

Some tips for writing a good report:

- **Write in sentences**. You are not solving a problem with pen and paper, where shorthand, sentence fragments, and poor grammar are acceptable. All sentences must be grammatically correct and include appropriate punctuation. This includes sentences containing numbers, symbols or equations. Read your sentences (including all the mathematics in the sentence) out loud as you write them to make sure they make sense grammatically.
- **Be professional.** Your report must be professionally presented. Equations must be typeset not handwritten. Figures should be clear, include a caption and be referenced in the text, e.g., "Figure X shows ...". Carefully check your report for typos, formatting inconsistencies etc. You are welcome to use any figures presented in this document in your report but text must first be paraphrased.
- **Keep the reader in mind.** Your group has been hired as consultants by a commercial bungee jump company. You are not writing as though the reader is the lecturer or someone else familiar with techniques taught in MXB103. Do not assume that the reader is familiar with the contents of this document. Explain all mathematical models and computational methods used. You may, however, assume the reader is familiar with the fundamental mathematical concepts of derivatives, integrals, polynomials and differential equations. Define all notation. Make it easy for the reader to understand your work.
- **Ensure the writing flows well.** An important part of report writing is to ensure the text "reads well", that is, there is a smooth flow from one sentence to the next within a paragraph without any abrupt changes in the rhythm/direction of the writing. One way to achieve good flow is to use linking or transition words/phrases, which link two sentences together or highlight the relationship between two sentences.
(see https://www.citewrite.qut.edu.au/assets/docs/qutcitewrite2018.pdf for more information).

7 Code

Your group must submit all code. This includes a single MATLAB script file (named `bungee.m`) that calls the other MATLAB functions that you write. These functions must include at a minimum, *your own functions* to perform:

- numerical solution of model ODEs;
- numerical differentiation;
- numerical integration;
- interpolation;
- rootfinding.

The script itself should run to completion without errors and should generate all results and figures in your report precisely.

Do not use MATLAB live scripts (`.mlx`).
Do not use symbolic calculations in MATLAB (e.g. commands like `syms`).

8 Marking criteria

This assignment will be marked using the criteria accompanying this document. The group's final grade on a 1-7 scale will be a **weighted** average of the grades for the criteria:

$$\text{Grade} = (4 \times \text{Tasks} + \text{Clarity} + \text{Script} + 3 \times \text{Report} + \text{Language})/10\,.$$

This grade will then be mapped to a final score out of 30.

9 Statement of contribution

Each group member must complete a statement outlining, in that person's opinion, the relative contributions made by every member of the group. A template will be provided for this purpose. Each group member should contribute *equally* to both development of code and writing the report. Failure to do so may result in lower marks being awarded to students who do not contribute sufficiently.

10 What to submit

One member of the group submits, as a single zip file:

- the MATLAB script file and all MATLAB function files written for the assignment;
- the report in PDF format.

Each member of the group submits (not in a zip file):

- a statement outlining the relative contributions made by each group member (use the template provided).