Supplemental Notes on von Neumann–Morgenstern Utility and Expected Utility

This is a numerical example to help illustrate the construction of the von Neumann – Morgenstern utility index that we discussed in class.

Suppose an individual faces a lottery with 5 possible outcomes: 400, 500, 800, 1000, 2000 (in dollars). Assign the least preferred outcome, 400, a utility of 0; i.e., $U(400) = 0$ and assign the most preferred outcome, 2000, a utility of 1; i.e., $U(2000) = 1$. The objective now is to create a utility index that will give us the individual's utility of the intermediate outcomes, 500, 800, and 1000.

Consider determining the inidvdual's utility of the outcome 800; i.e., $U(800)$. Suppose we ask the individual whether given a choice of 800 for certain or a lottery involving the least and most desirable outcomes (400 and 2000, respectively), where the probability of the most desirable outcome, 2000, is equal to one, which would the individual choose? Certainly they would choose the lottery in which they receive 2000 with probability one over 800 with certainty (i.e., also with probability one). Suppose we then lower the probability of the most desirable outcome to 0.75. Now we give the individual the choice of 800 with certainty and a lottery in which they receive 400 with probabilty 0.25 and 2000 with probability 0.75. Suppose the individual still prefers the lottery. Now suppose we keep lowering the probability of getting the most desirable outcome and continue giving the individual the choice of 800 for certain or the lottery with these lowered probabilities of getting 2000. Suppose the individual keeps choosing the lottery until we have lowered the probability of receiving 2000 to 0.35. Suppose at this point the individual is *indifferent* between the 800 for certain and a lottery with probability 0.35 of receiving 2000 and probability 0.65 of receiving 400. We would then say that 800 is the **certainty equivalent (CE)** of this lottery. In other words, the individual is indifferent between having 800 with certainty and a lottery that pays 2000 with probability 0.35 and pays 400 with probability 0.65. So, in general, the certainty equivalent of a lottery is an amount that would make an individual indifferent between having that amount with certainty and engaging in the lottery.

So the next question is what utility value for our index do we assign 800? That is what value do we give $U(800)$? Well, if the individual is indifferent between having 800 for certain and the lottery described above, it must be the case that the **expected *utilities*** are the same. And since the expected utility of some value with certainty is simply equal to the utility of that value, it must be the case that the utility of 800 (the expected utility of 800) is the same as the expected utility of the lottery. That is,

$$U(800) = 0.35 \times U(2000) + 0.65 \times U(400)$$

where the right–hand side of the equation is the **expected *utility*** of the lottery. Therefore, the **U(800) is the expected utility of the lottery for which 800 is the certainty equivalent.** And since we have normalized $U(400)$ to 0 and $U(2000)$ to 1, we have:

$$U(800) = 0.35.$$

So with this normalization, the utility index of some outcome X_i is simply the probability of the most desirable outcome that would make the individual indifferent between having X_i with certainty and a lottery involving the least and most desirable outcomes.

Notice that in this example, the **expected *value*** of the lottery that provides the certainty equivalent is $0.35 \times 2000 + 0.65 \times 400 = 960$. This means that with repeated play, the individual would have, on average, 960 per time played. And of course, the **expected *value*** of 800 with certainty is 800. Repeated play of being given 800 with certainty each time gives the individual 800 per time played. Yet, the consumer is just **indifferent** between having 800 with certainty and the above lottery, despite the fact that the lottery has a higher expected value; that is, will generate a higher monetary value on average. This all goes to show that the individual is **risk averse**. In fact, for any probability of receiving 2000 lower than 0.35, the individual will **strictly prefer** the 800 with certainty. Suppose the probability of receiving 2000 is 0.30. In this case, the **expected *value*** of the lottery would be 880, still higher than 800, yet the individual would *strictly prefer* having the 800 with certainty. Why? Because in choices involving risk and uncertainty, **individuals base their decisions on expected *utilities*, not expected *values*.**

(Can you determine what the $U(800)$ would be in the above scenario if the individual were risk neutral?)

In class, we had for the more general case:

$$U(X_i) = \Pi_i \times U(X_n) + (1 - \Pi_i) \times U(X_1)$$

where the right hand side is the **expected *utility*** of the lottery involving the least and most preferred outcomes (X_1 and X_n, respectively) and Π_i is the probability of the most preferred outcome such that X_i is the certainty equivalent.

So with our normalization that $U(X_n) = 1$ and $U(X_1) = 0$, we have simply that $U(X_i) = \Pi_i$. Given this normalization, all outcomes X_i will be assigned a utility value between 0 and 1, which will be, in a lottery involving the best and worst outcomes, the probability of the best outcome for which X_i is the certainty equivalent.

Finally, as also shown in class, unlike the standard utility function we have employed previously, preferences underlying the VNM utility function are not invariant to any general monotonic transformation of the VNM utility function; they are, however, preserved under a positive linear transformation: $V(X_i) = a + bU(X_i)$, $b > 0$.