Solutions Problem Set 3

① a) equivalent; $h(f)$, or here, $h(u) = 7u + 2$

b) equivalent; $h(u) = \ln u + 1$

c) not equivalent; it is not a monotonic transformation
So, for example, consider bundles (X,Y): Bundle $A = (2,3)$ and Bundle $B = (3,2)$. If utility is $u(x,y) = xy$, the consumer is indifferent between the bundles. However, if utility is $u(x,y) = x^2y$, the consumer strictly prefers B to A.

② By the implicit function theorem, the $MRS_{XY} = \dfrac{\partial u/\partial X}{\partial u/\partial Y} = \left.\dfrac{-dY}{dX}\right|_{\bar{u}}$

i) $u(X,Y) = XY$, $\frac{\partial u}{\partial X} = Y$ $\frac{\partial u}{\partial Y} = X$, so $MRS_{XY} = Y/X$ at $(2,1)$, $MRS = 1/2$
$MU_X = \frac{\partial u}{\partial X} = Y$, and at $(2,1)$, $MU_X = 1$

ii) $u(X,Y) = 7XY + 2$ $\frac{\partial u}{\partial X} = 7Y$ $\frac{\partial u}{\partial Y} = 7X$, so $MRS_{XY} = \frac{7Y}{7X} = \frac{Y}{X}$, and
so MRS at $(2,1) = 1/2$
$MU_X = 7Y$, and at $(2,1)$, $MU_X = 7$

iii) $u(X,Y) = \ln X + \ln Y + 1$ $\frac{\partial u}{\partial X} = \frac{1}{X}$ $\frac{\partial u}{\partial Y} = \frac{1}{Y}$, so $MRS_{XY} = \frac{1/X}{1/Y} = \frac{Y}{X}$
so MRS at $(2,1) = 1/2$
$MU_X = 1/X$ and at $(2,1)$, $MU_X = 1/2$

The MRS is invariant to order preserving transformations of the utility function, and therefore its value is based on ordinal, not cardinal properties. In contrast, the marginal utilities are not invariant to such transformations of utility, and they are therefore a cardinal property.

③ a) $\frac{\partial u}{\partial X} = \alpha X^{\delta-1}$ $\frac{\partial u}{\partial Y} = \beta Y^{\delta-1}$ $MRS_{XY} = \frac{\partial u}{\partial X} \Big/ \frac{\partial u}{\partial Y} = \frac{\alpha X^{\alpha-1}}{\beta Y^{\delta-1}}$
$= \frac{\alpha}{\beta}\left(\frac{X}{Y}\right)^{\alpha-1}$. Since the derivative of the level curves (i.e. the MRS, the derivative of the indifference curves) is a function of the ratio Y/X (or X/Y), the function $g(X/Y)$, a function of the ratio is homothetic.

b) $MRS_{X,Y}$ when $\delta = 1$ is α/β. $\sigma = \frac{1}{1-\delta}$ and at $\delta = 1$, $\sigma \Rightarrow \infty$. This describes perfect substitutes.

c) This individual is indifferent between 1 unit of X and α/β units of Y; 1X is a perfect substitute for α/β units of Y. They will therefore spend their entire income on whichever is cheaper: 1 X or α/β Y. 1X costs P_X and α/β Y costs $\alpha/\beta P_Y$. We are given $P_X < \alpha/\beta P_Y$, so the consumer will spend 100% of income on good X only.

d) $u(X,Y) = 3X^{1/3} + 3Y^{1/3}$ $\quad \sigma = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$ $\quad \sigma$ measures $\frac{\%\Delta(Y/X)}{\%\Delta MRS_{XY}}$

Since at the consumer optimum, the consumer equates their $MRS_{XY} = P_X/P_Y$, the 5% increase in the price ratio $\Rightarrow$ the MRS will have increased 5% as well. So the percentage Δ in Y/X is approximately: $\%\Delta(Y/X) \approx \%\Delta(MRS_{XY})\,\sigma$ or, $\%\Delta(Y/X) \approx 5(3/2) = 7.5$

e) $\mathcal{L} = 3X^{\frac{1}{3}} + 3Y^{\frac{1}{3}} + \lambda(60 - 4X - 1Y)$

$\frac{\partial \mathcal{L}}{\partial X} = X^{-2/3} - \lambda 4 = 0$ $\qquad \frac{\partial \mathcal{L}}{\partial Y} = Y^{-2/3} - \lambda = 0$ $\qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 60 - 4X - 1Y = 0$

$\downarrow \qquad\qquad\qquad\qquad \downarrow$

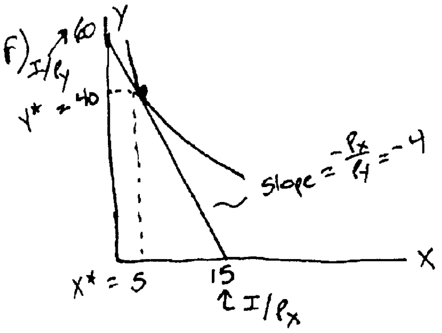
$\lambda = \frac{X^{-2/3}}{4} \qquad\qquad \lambda = Y^{-2/3}$

$\Rightarrow \frac{X^{-2/3}}{4} = Y^{-2/3} \Rightarrow \left(\frac{Y}{X}\right)^{2/3} = 4 \Rightarrow \frac{Y}{X} = 8 \Rightarrow Y = 8X$

Sub in constraint (3rd FOC): $60 - 4X - 8X = 0 \Rightarrow \boxed{X^* = 5}$

and $\boxed{Y^* = 8 \cdot 5 = 40}$

f)



g) From above, we have $\lambda^* = \frac{X^{*\,-2/3}}{4}$

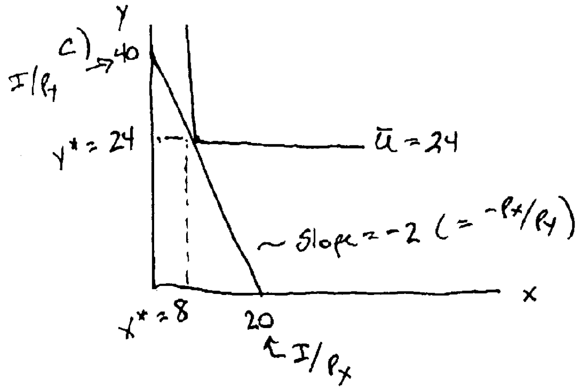
$= \frac{5^{-2/3}}{4} = .0855$ = marginal utility of income at the optimum

(4) a) $Y/X = 3$, or $Y = 3X$ For every X, they consume 3Y.

b) Budget constraint: $10X + 5Y = 200$. But since they maintain $Y = 3X$, we have $10X + 5(3X) = 200 \Rightarrow \boxed{X^* = 8}$

and $\boxed{Y^* = 3(8) = 24}$

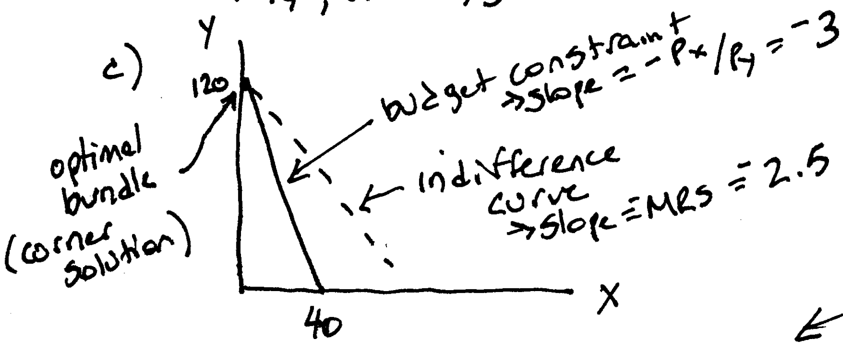
c)



⑤ a) $u(X,Y) = 5X + 2Y$ or any $u(X,Y) = \alpha X + \beta Y$ where $\alpha/\beta = 2.5$ will do.

b) Five units of Y cost $5 \times \$3 = \$15$, which is cheaper than the perfect substitute of two units of X (which cost $2 \times \$9 = \$18$). Therefore the consumer will spend his entire income on the cheaper substitute (good Y). He can afford to buy a maximum amount of good Y of I/P_Y, or $360/3 = 120$ units. He buys 0 units of X.

c)



note: we want $MRS_{XY} \propto \frac{\Delta Y}{\Delta X}$ $(=\frac{1}{3})$ not $MRS_{YX} = \frac{\Delta X}{\Delta Y}$ $(=3)$

⑥ a) MRS_{XY} at $(6,5) = 1/3 = MRS_{XY}$ at $(12,3)$
We are also given that these bundles lie on the same IC
So, X and Y are perfect substitutes where 1/3 unit of Y is a perfect substitute for 1 unit of X : $u(X,Y) = X + 3Y$ for example. ICs are linear with slope $= -1/3$.

b) For Cobb-Douglas $u(x,y) = X^{\alpha} Y^{\beta}$, $MRS_{XY} = u_X/u_Y = \frac{\alpha}{\beta}\frac{Y}{X}$
If at bundle $(8,1)$, $MRS_{XY} = 1/4$, we have: $\alpha/\beta\,(1/8) = 1/4$
So $\alpha/\beta = 2$. So $u(x,y)$ can be X^2Y, for example.

c) Didn't need info. about the other bundle.

⑦ Given $u(x,y)$, $MRS_{XY} = u_X/u_Y$ (ratio of marginal utilities)

$$\frac{\partial MRS_{XY}}{\partial X} = \frac{\overset{+}{u_Y}\,\overset{-}{u_{XX}} - u_X \overset{=0}{u_{YX}}}{\underset{+}{(u_Y)^2}}$$

Since we are given $u_{YX} = 0$, we see that this expression is negative.

Strictly speaking as we move down along IC, Y is changing (decreasing) and by similar analysis to the above $\frac{\partial MRS_{XY}}{\partial Y} > 0$
So as we move along IC, X is increasing and Y is decreasing, both of which cause MRS_{XY} to decrease.

More formally when $U_{XY} = 0$, $U_{XX} < 0$ $U_{YY} < 0$, the utility function can be shown to be strictly quasi-concave ⇒ convexity of ICs (diminishing MRS) - see footnote 7 on pg. 100 of text.