SOLUTIONS TO PROBLEM SET 1

① $F'(x) = 4x^3 - 12x^2 + 8x$ Find critical values x^* where $F'(x^*) = 0$. $4x^3 - 12x^2 + 8x = 0 \Rightarrow$ Factor:
$4x(x^2 - 3x + 2) = 0$ or $4x(x-2)(x-1) = 0$
The 3 critical values are therefore $x^* = 0, 2, 1$
$F''(x) = 12x^2 - 24x + 8$ $F''(0) = 8 > 0 \Rightarrow$ local minimum
$F''(2) = 12(2^2) - 24(2) + 8 = 8 > 0 \Rightarrow$ local minimum
$F''(1) = -4 < 0 \Rightarrow$ local maximum

② a) $f_1 = 2x_2^3$ $f_2 = 6x_1x_2^2$ $f_{11} = 0$ $f_{22} = 12x_1x_2$
$f_{12} = 6x_2^2$ $f_{21} = 6x_2^2$ $f_{12} = f_{21} \Rightarrow$ Young's th$^{\underline{m}}$ holds.

b) $f_1 = 14 + 2x_2 - 2x_1$ $f_2 = 2x_1 - 4x_2$ $f_{11} = -2$ $f_{22} = -4$
$f_{12} = 2$ $f_{21} = 2$ $f_{12} = f_{21} \Rightarrow$ Young's th$^{\underline{m}}$ holds

c) $f_1 = -4 + 2x_1 - x_2$ $f_2 = -6 - x_1 + 4x_2$ $f_{11} = 2$ $f_{22} = 4$
$f_{12} = -1$ $f_{21} = -1$ $f_{12} = f_{21} \Rightarrow$ Young's th$^{\underline{m}}$ holds

d) $f_1 = \ln x_2$ $f_2 = x_1/x_2$ $f_{11} = 0$ $f_{22} = -x_1/x_2^2$
$f_{12} = 1/x_2$ $f_{21} = 1/x_2$ $f_{12} = f_{21} \Rightarrow$ Young's th$^{\underline{m}}$ holds

e) $f_1 = x_1 e^{(x_1+x_2^2)} + e^{(x_1+x_2^2)} = (1+x_1)(e^{(x_1+x_2^2)})$
$f_2 = 2x_2x_1e^{(x_1+x_2^2)}$ $f_{11} = (1+x_1)(e^{(x_1+x_2^2)}) + e^{(x_1+x_2^2)}$
$= (2+x_1)(e^{(x_1+x_2^2)})$
$f_{22} = 4x_1x_2^2e^{(x_1+x_2^2)} + e^{(x_1+x_2^2)} \cdot 2x_1$

$f_{12} = x_1 e^{(x_1+x_2^2)} \cdot 2x_2 + 2x_2 e^{(x_1+x_2^2)} = (1+x_1)2x_2 e^{(x_1+x_2^2)}$

$f_{21} = 2x_2(x_1 e^{(x_1+x_2^2)} + e^{(x_1+x_2^2)}) = (1+x_1)2x_2 e^{(x_1+x_2^2)}$

$f_{12} = f_{21} \Rightarrow$ Yang's $th^{\underline{m}}$ holds

③ $f_1 = 14 + 2x_2 - 2x_1$ $\quad f_2 = 2x_1 - 4x_2$ $\quad f_1 = f_2 = 0$ at

Critical x_i^* : $f_2 = 0 \Rightarrow 2x_1 - 4x_2 = 0 \Rightarrow 2x_1 = 4x_2$ sub

Sub in for $2x_1$ in f_1 set equal to zero : $14 + 2x_2 - (4x_2) = 0$

$\Rightarrow 14 - 2x_2 = 0 \Rightarrow \boxed{x_2^* = 7}$ and since $2x_1 = 4x_2 \Rightarrow x_1 = 2x_2$

$\boxed{x_1^* = 14}$ From 2b), we have $f_{11} = -2 < 0$ and $f_{22} = -4 < 0$

Now check $f_{11}f_{22} - f_{12}^2$: $(-2)(-4) - 2^2 = 8 - 4 > 0$

Since $f_{11} < 0$ $f_{22} < 0$ and $f_{11}f_{22} - f_{12}^2 > 0$ at $x_1^*, x_2^* \Rightarrow$ local maximum (in fact since these conditions hold everywhere, we have a global maximum)

④ $f_1 = -4 + 2x_1 - x_2$ $\quad f_2 = -6 - x_1 + 4x_2$ $\quad f_1 = f_2 = 0$ at critical x_i^*:

$$\left.\begin{array}{l} -4 + 2x_1 - x_2 = 0 \\ -6 - x_1 + 4x_2 = 0 \end{array}\right\} \text{multiply both sides of second equation by 2} \Rightarrow \begin{array}{r} -4 + 2x_1 - x_2 = 0 \\ -12 - 2x_1 + 8x_2 = 0 \\ \hline -16 + 7x_2 = 0 \end{array}$$

ADD

$7x_2 = 16 \Rightarrow \boxed{x_2^* = 16/7}$ Sub into one of the equations:

$-4 + 2x_1 - (16/7) = 0 \Rightarrow 2x_1 = 44/7 \Rightarrow \boxed{x_1^* = 44/14 = 22/7}$

$f_{11} = 2 > 0$ $\quad f_{22} = 4 > 0$ $\quad f_{11}f_{22} - (f_{12})^2 = (2)(4) - (+1) = 7 > 0$

Since $f_{11} > 0$, $f_{22} > 0$ and $f_{11}f_{22} - (f_{12})^2 > 0$ at x_1^*, x_2^*, we have a local minimum. And actually, since these conditions hold everywhere, it is a global minimum.

⑤ a) $dy = f_1\,dx_1 + f_2\,dx_2$ $\quad f_1 = 16x_1x_2^2 - 6x_1^2x_2^3$

$f_2 = 16x_1^2x_2 - 6x_1^3x_2^2$ So the total differential is: $(16x_1x_2^2 - 6x_1^2x_2^3)\,dx_1 + (16x_1^2x_2 - 6x_1^3x_2^2)\,dx_2 = dy$

b) $\Delta y \approx (16(1)(1) - 6(1^2)(1^3))\frac{1}{2} + (16(1^2)(1) - 6(1^3)(1^2))(.2)$

$\Rightarrow \Delta y \approx 10(\frac{1}{2}) + 10(.2) = 7$

For actual change in y: $F(1,1) = 6$, $F(1.5, 1.2) \approx 14.25$ } So actual $\Delta y = 8.25$

(6) Given $g(x,y)=0$, using the implicit function theorem we know that $\frac{dy}{dx} = -g_x/g_y$. Here, $g_x = 2x - 3y$ and $g_y = -3x + 3y^2$

So, $-g_x/g_y = \frac{-(2x-3y)}{(-3x+3y^2)}$ and at $(x=4, y=3)$ we have

$\frac{dy}{dx} = \frac{-[2(4)-3(3)]}{[-3(4)+3(9)]} = 1/15$

(7) a) $\frac{\partial f}{\partial x} = -2x + 2a = 0 \Rightarrow x^* = a$

b) $y^* = -a^2 + 2a^2 + 4a$; $\frac{dy^*}{da} = -2a + 4a + 4 = 2a + 4$

c) Using the envelope theorem:

$\frac{dy^*}{da} = \frac{\partial f}{\partial a}\Big|_{x=x^*} = 2x + 4\Big|_{x=x^*=a} = 2a + 4$

d) If $a = \frac{1}{2}$, then $\frac{dy^*}{da} = 2(\frac{1}{2}) + 4 = 5$

So, if a were to increase by .2 then we have $dy^* = \frac{dy^*}{da}da$, approximated by $\Delta y \approx \frac{dy^*}{da}\Delta a$,

or $\Delta y \approx 5(.2) = 1$

e) From part b, we have $y^* = -a^2 + 2a^2 + 4a$

so when $a = \frac{1}{2}$, $y^* = -(\frac{1}{2})^2 + 2(\frac{1}{2^2}) + 4(\frac{1}{2}) = 2.25$

And when $a = 7/10$, $y^* = -(\frac{7}{10})^2 + 2(\frac{7}{10})^2 + 4(\frac{7}{10}) = 3.29$

So actual chang is 1.04