SOLUTIONS – Problem Set 8

① a) $\frac{10(tK)(tL)}{tK + tL} = t\left[\frac{10KL}{K+L}\right] = tq \Rightarrow$ CRS

b) $q = \frac{10\bar{K}L}{\bar{K}+L}$ (short-run production fn)

i) $AP_L = q/L = \frac{10\bar{K}}{\bar{K}+L}$

ii) $MP_L = \partial q/\partial L = \frac{(\bar{K}+L)10\bar{K} - 10\bar{K}L}{(\bar{K}+L)^2} = \frac{10\bar{K}^2}{(\bar{K}+L)^2}$

c) Yes, MP_L is diminishing – $\frac{\partial MP_L}{\partial L} = \frac{-10\bar{K}^2 \cdot 2(\bar{K}+L)}{(\bar{K}+L)^4} < 0$

$\frac{\partial MP_L}{\partial \bar{K}} = \frac{(\bar{K}+L)^2 \cdot 20\bar{K} - 10\bar{K}^2 \cdot 2(\bar{K}+L)}{(\bar{K}+L)^4} = \frac{20\bar{K}(\bar{K}+L)^2 - 20\bar{K}^2(\bar{K}+L)}{(\bar{K}+L)^4}$

$= \frac{20\bar{K}}{(\bar{K}+L)^2} - \frac{20\bar{K}^2}{(\bar{K}+L)^3}$. $\frac{20\bar{K}}{(\bar{K}+L)^2} - \frac{20\bar{K}^2}{(\bar{K}+L)^3} \gtrless 0$? $\Rightarrow 1 - \frac{\bar{K}}{\bar{K}+L} \gtrless 0$

Since $\bar{K}/\bar{K}+L < 1$, $1 - \frac{\bar{K}}{\bar{K}+L} > 0 \Rightarrow \frac{\partial MP_L}{\partial \bar{K}} > 0$ (MP_L is increasing as more K is employed)

d) $RTS_{L,K} = \frac{\partial q/\partial L}{\partial q/\partial K} = \frac{10K^2}{(K+L)^2} \Big/ \frac{10L^2}{(K+L)^2} = \frac{K^2}{L^2}$

The $RTS_{L,K}$ is diminishing as more L is employed.
Since $RTS_{L,K} = g(K/L) \Rightarrow$ the production function is homothetic.

2 a) $\mathcal{L} = wL + vK + \lambda\left(10 - \frac{10KL}{K+L}\right)$

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda \underbrace{\left[\frac{(K+L)10K - 10KL}{(K+L)^2}\right]}_{\text{"A"}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial K} = v - \lambda \underbrace{\left[\frac{(K+L)10L - 10KL}{(K+L)^2}\right]}_{\text{"B"}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 10 - \frac{10KL}{K+L} = 0$$

$\lambda = w/A \quad \lambda = v/B \Rightarrow w/A = v/B \Rightarrow A/B = w/v$

$$A/B = \frac{(K+L)10K - 10KL}{\cancel{(K+L)^2}} \cdot \frac{\cancel{(K+L)^2}}{(K+L)10L - 10KL} = \frac{10K^2 + 10KL - 10KL}{10L^2 + 10KL - 10KL}$$

$= K^2/L^2$. And since at optimum, $A/B = w/v$, or $K^2/L^2 = w/v$ so we have $K^2/L^2 = 1/4 \Rightarrow K^2 = 1/4\, L^2 \Rightarrow K = \frac{1}{2}L$

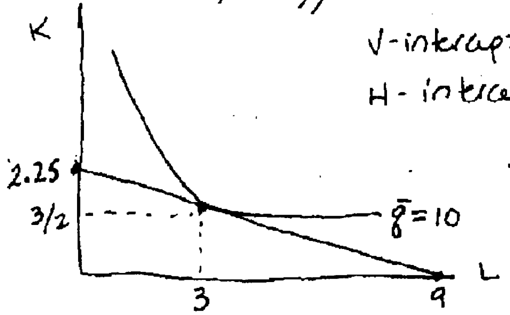
Sub into constraint: $\dfrac{10\left(\frac{1}{2}L\right)L}{\frac{1}{2}L + L} = 10 \Rightarrow \dfrac{5L^2}{\frac{3}{2}L} = 10$

$\Rightarrow \frac{10}{3}L = 10 \Rightarrow L^* = 3$ and $K^* = 1/2\, L^* = 3/2$

The minimum cost of producing $q = 10$ is $wL^* + vK^* =$

$(1)3 + (4)(3/2) = 9$ //

b)



V-intercept $= \frac{TC}{v} = \frac{9}{4} = 2.25$

H-intercept $= \frac{TC}{w} = \frac{9}{1} = 9$

Slope of isocost $= \frac{-w}{v} = -\frac{1}{4}$

c) From part a), we have that the RTS = K^2/L^2 and at the optimum RTS = w/v, so we have $K^2/L^2 = w/v \Rightarrow$ $K = (w/v)^{1/2} L$. Subbing this into the general constraint (with "10" replaced by arbitrary "q"), we have

$$\frac{10\,(w/v)^{1/2}L^2}{(w/v)^{\frac{1}{2}}L + L} = q \Rightarrow 10\,(w/v)^{1/2}L^2 = q\,(w/v)^{1/2}L + qL$$

$$\Rightarrow 10\,(w/v)^{1/2}L = q\,(w/v)^{1/2} + q \Rightarrow \underline{L = q/10 + q/10\,(w/v)^{-1/2}}$$

And since $K = (w/v)^{1/2}L$, we have $K = (w/v)^{1/2}\left[q/10 + q/10\left(\frac{w}{v}\right)^{-1/2}\right]$

$$= \underline{q/10 + q/10\,(w/v)^{1/2}}$$

They are both homogeneous of degree 0 in input prices.

d) $TC = wL + vK$ Sub in conditional demands for L & K:

$$TC = w\,q/10\left[1 + (w/v)^{-1/2}\right] + v\,q/10\left[1 + (w/v)^{1/2}\right] = \frac{q}{10}\left[w + (vw)^{\frac{1}{2}}\right] + \frac{q}{10}\left[v + (vw)^{\frac{1}{2}}\right]$$

$$= q/10\left[w^{\frac{1}{2}} + v^{\frac{1}{2}}\right]^2$$

$$MC = \frac{\partial TC}{\partial q} = 1/10\left[w^{\frac{1}{2}} + v^{1/2}\right]^2$$

$$AC = \frac{TC}{q} = \text{same as } MC$$

e) $$\frac{\partial TC}{\partial w} = q/10\left[1 + (w/v)^{-1/2} + w\,(-1/2)(w/v)^{-3/2}(1/v) + 1/2\,v\left(\frac{w}{v}\right)^{-1/2}\left(\frac{1}{v}\right)\right]$$

$$= q/10\left[1 + (w/v)^{-1/2} + (w/v)^{-1/2}(-1/2) + \tfrac{1}{2}(w/v)^{-1/2}\right]$$

$$= q/10\left[1 + (w/v)^{-1/2}\right] \checkmark$$

$$\frac{\partial TC}{\partial v} = q/10\left[\frac{1}{2}w\left(\frac{v}{w}\right)^{-1/2}\left(\frac{1}{w}\right) + 1 + \frac{1}{2}v\left(\frac{w}{v}\right)^{-1/2}\left(-\frac{w}{v^2}\right) + \left(\frac{w}{v}\right)^{\frac{1}{2}}\right]$$

$$= q/10\left[\frac{1}{2}\left(\frac{w}{v}\right)^{1/2} + 1 - \frac{1}{2}(w/v)(w/v)^{-1/2} + \left(\frac{w}{v}\right)^{\frac{1}{2}}\right]$$

$$= q/10\left[\frac{1}{2}\left(\frac{w}{v}\right)^{1/2} + 1 - \frac{1}{2}\left(\frac{w}{v}\right)^{1/2} + \left(\frac{w}{v}\right)^{\frac{1}{2}}\right]$$

$$= q/10\left[1 + (w/v)^{\frac{1}{2}}\right] \checkmark$$

f) $TC = (1)\,{}^{q}/_{10}\left[1 + \left({}^{1}/_{4}\right)^{-1/2}\right] + 4\,{}^{q}/_{10}\left[1 + \left({}^{1}/_{4}\right)^{1/2}\right]$

$= {}^{q}/_{10}\left[1 + 2 + 4 + 2\right] = {}^{9q}/_{10}$

$AC = {}^{9}/_{10} = MC$

g) When $K_0 = 4$, the short-run production function is $q = \frac{40L}{4+L}$

$\Rightarrow 40L = 4q + qL \Rightarrow 40L - qL = 4q \Rightarrow L(40-q) = 4q \Rightarrow L = \frac{4q}{40-q}$

So $STC = (1)\,{}^{4q}/_{(40-q)} + (4)4 = {}^{4q}/_{(40-q)} + 16$

$SAC = {}^{4}/_{40-q} + {}^{16}/_{q}$ $\quad VC = {}^{4q}/_{40-q}$ $\quad AVC = {}^{4}/_{40-q}$ $\quad AFC = {}^{16}/_{q}$

$SMC = \frac{\partial STC}{\partial q} = \frac{(40-q)4 + 4q}{(40-q)^2} = {}^{160}/_{(40-q)^2}$

h) $STC(10) = {}^{4(10)}/_{(40-10)} + 16 = 17.33$ The minimum cost from part a) is less (9). The short-run cost is higher because capital is fixed at 4, whereas we saw in part a) that the optimal capital level for producing $q = 10$, given $w = 1, v = 4$, is $K^* = {}^{3}/_{2}$.

i) Use the conditional demand for capital: when $w = 1, v = 4$, the conditional demand for capital is $K = {}^{q}/_{10} + {}^{q}/_{20} = {}^{3q}/_{20}$, so if $K = 4$, $4 = {}^{3q}/_{20} \Rightarrow q = {}^{80}/_{3}$

Long Run TC for producing $q = {}^{80}/_{3}$ is $\frac{9(80/3)}{10} = \underline{24}$

Short Run STC for producing $q = {}^{80}/_{3}$ (given $K_0 = 4$) is

$16 + \frac{40(80/3)}{(40 - 80/3)} = \frac{16 + 320/3}{40/3} = \frac{320}{40} + 16 = \underline{24}$