SOLUTIONS PROBLEM SET 2

① a) We can say e^x is quasi-concave, since $f'(x) > 0$, and any monotonically increasing or decreasing function is both quasi-concave and quasi-convex. More formally, if $f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\}$ where $0 \leq \lambda \leq 1$, then the function is quasi-concave. Here, we need to show $e^{\lambda x_1 + (1-\lambda)x_2} \geq \min\{e^{x_1}, e^{x_2}\}$. Consider any 2 points, x_1, x_2 where $x_1 \leq x_2$. Since $f'(x) > 0 \ \forall x$, then $e^{x_1} < e^{x_2}$. Therefore, we need to show $e^{\lambda x_1 + (1-\lambda)x_2} \geq e^{x_1}$, or, taking logs of both sides: $\lambda x_1 + (1-\lambda)x_2 \geq x_1 \Rightarrow x_2 \geq x_1$, which holds, by assumption.

b) Here, we can show the function is concave, since concavity ⇒ quasi-concavity.

$f_1 = x_1^{-1/2}$ $\quad f_2 = 2x_2^{-1/2}$ $\quad f_{11} = -\frac{1}{2}x_1^{-3/2}$ $\quad f_{22} = -x_2^{-3/2}$ $\quad f_{12} = 0$ $\qquad x_1, x_2 > 0$

Since $f_{11} < 0$, $f_{22} < 0$ and $f_{11}f_{22} - f_{12}^2 = \frac{1}{2}x_1^{-3/2}x_2^{-3/2} > 0$ ⇒ function is concave and therefore, also quasi-concave.

② $\mathcal{L} = x_1x_2 + \lambda(16 - x_1 - 4x_2)$

$\frac{\partial \mathcal{L}}{\partial x_1} = x_2 - \lambda = 0 \qquad \frac{\partial \mathcal{L}}{\partial x_2} = x_1 - 4\lambda = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 16 - x_1 - 4x_2 = 0$

Since $x_2 - \lambda = 0 \Rightarrow \lambda = x_2$ and since $x_1 - 4\lambda = 0$

$\Rightarrow \lambda = x_1/4$ so, $x_2 = x_1/4$, or $x_1 = 4x_2$

Sub into F.O.C. for $\partial\mathcal{L}/\partial\lambda$: $16 - 4x_2 - 4x_2 = 0$

$\Rightarrow 16 - 8x_2 = 0 \Rightarrow \boxed{x_2^* = 2}$ and $\boxed{x_1^* = 4(2) = 8}$

maximized value of f, $\boxed{f^* \text{ is } 2 \cdot 8 = 16}$

③ minimize $x_1 + 4x_2$ subject to $x_1x_2 = 16$

$\mathcal{L} = x_1 + 4x_2 + \lambda(16 - x_1x_2)$

$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda x_2 = 0 \qquad \frac{\partial \mathcal{L}}{\partial x_2} = 4 - \lambda x_1 = 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 16 - x_1x_2 = 0$

$1 - \lambda x_2 = 0 \Rightarrow \lambda = 1/x_2 \quad 4 - \lambda x_1 = 0 \Rightarrow \lambda = 4/x_1$

$1/x_2 = 4/x_1 \Rightarrow x_1 = 4x_2$. Sub in F.O.C. $\frac{\partial \mathcal{L}}{\partial \lambda}$: $16 - 4x_2x_2 = 0$

or $16 - 4x_2^2 = 0 \Rightarrow 4 = x_2^2 \Rightarrow \boxed{x_2^* = 2}$ $\boxed{x_1^* = 4(2) = 8}$ (Same as in ② above)

④ The problem can be written: max $Q(x,y) = 50x^{\frac{1}{2}}y^{\frac{1}{2}}$

subject to $x + y = 80$

a) $\mathcal{L} = 50x^{1/2}y^{1/2} + \lambda(80 - x - y)$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2}(50)x^{-1/2}y^{1/2} - \lambda = 0 \quad \frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2}(50)x^{1/2}y^{-1/2} - \lambda = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 80 - x - y = 0 \quad \lambda = 25x^{-1/2}y^{1/2}$ and $\lambda = 25x^{1/2}y^{-1/2}$

$\Rightarrow 25x^{-1/2}y^{1/2} = 25x^{1/2}y^{-1/2} \Rightarrow x = y$ Plug into $\frac{\partial \mathcal{L}}{\partial \lambda}$:

$80 - x - x = 0 \Rightarrow 80 - 2x = 0 \Rightarrow \boxed{x^* = 40 \text{ and } y^* = 40}$

b) From above, we have $\lambda = 25x^{-1/2}y^{1/2}$, so $\lambda^* = 25\left(\frac{40}{40}\right)^{1/2} = 25$
Therefore, since λ^* measures dQ^*/d(total dollars available), we estimate the maximized output will **decrease** by $25 \times 1 = 25$

c) Now in part a, $\mathcal{L} = 50x^{1/2}y^{1/2} + \lambda(79 - x - y)$
Everything is the same in part a (i.e. $x^* = y^*$) until we sub $x = y$ into $\partial\mathcal{L}/\partial\lambda$, since now we have
$79 - 2x = 0 \Rightarrow \boxed{x^* = y^* = 39.5}$ Maximum output before was $50(40)^{1/2}(40)^{1/2} = 50 \cdot 40 = 2000$. Maximum output now is $50(39.5)^{1/2}(39.5)^{1/2} = 50 \cdot (39.5) = 1975$, so the actual change of $2000 - 1975$ is equal to the estimated change of 25 in this case.

5 minimize $x + y$ subject to $50x^{1/2}y^{1/2} = 2000$

$\mathcal{L} = x + y + \lambda(2000 - 50x^{1/2}y^{1/2})$

$\frac{\partial \mathcal{L}}{\partial x} = 1 - \frac{1}{2}(50)x^{-1/2}y^{1/2}\lambda = 0 \quad \frac{\partial \mathcal{L}}{\partial y} = 1 - \frac{1}{2}(50)x^{1/2}y^{-1/2}\lambda = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 2000 - 50x^{1/2}y^{1/2} = 0 \quad \lambda = \frac{x^{1/2}}{25y^{1/2}} \quad \lambda = y^{1/2}/25x^{1/2}$

$x^{1/2}/25y^{1/2} = y^{1/2}/25x^{1/2} \Rightarrow x = y$. Sub into constraint:

$2000 - 50x^{1/2}x^{1/2} = 0 \Rightarrow 50x = 2000 \Rightarrow \boxed{x^* = 40 = y^*}$

(6) a) $f(tx_1, tx_2) = (tx_1)(tx_2)^2 = t^3 x_1 x_2^2 = t^3 f(x_1, x_2)$

$\Rightarrow$ homogeneous of degree 3

$\frac{\partial f}{\partial x_1} = x_2^2 \quad \frac{\partial f}{\partial x_2} = 2x_1 x_2$

$\frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 = x_2^2 \cdot x_1 + 2x_1 x_2 \cdot x_2$

$= 3x_1 x_2^2 = 3f(x_1, x_2) \Rightarrow$ Euler's th^m holds

b) $f(tx_1, tx_2) = (tx_1)(tx_2) + (tx_2)^2 = t^2 x_1 x_2 + t^2 x_2^2 = t^2 f(x_1, x_2)$

$\Rightarrow$ homogeneous of degree 2

$\frac{\partial f}{\partial x_1} = x_2 \quad \frac{\partial f}{\partial x_2} = x_1 + 2x_2$

$\frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 = x_1 x_2 + x_1 x_2 + 2x_2^2 = 2(x_1 x_2 + x_2^2) \Rightarrow$ Euler's th^m holds

(7) a) Level curve defined by x_1, x_2 such that

$\ln(x_1) + \ln(x_2) - c = 0 \qquad c = \text{constant}$

By implicit fctn. th^m, $\frac{dx_2}{dx_1} = -f_{x_1}/f_{x_2}$

$f_{x_1} = 1/x_1 \quad f_{x_2} = 1/x_2$, so $dx_2/dx_1 = -(1/x_1)/(1/x_2) = -\frac{x_2}{x_1}$

b) $f_{x_1} = 2x_1 x_2^2 - x_2 \qquad f_{x_2} = 2x_1^2 x_2 - x_1$

Using implicit fctn. th^m again,

$$\frac{dx_2}{dx_1} = -\left[\frac{2x_1 x_2^2 - x_2}{2x_1^2 x_2 - x_1}\right] = -\left[\frac{x_2(2x_1 x_2 - 1)}{x_1(2x_1 x_2 - 1)}\right] = -\frac{x_2}{x_1}$$

At any point along a ray from the origin, the proportion x_2/x_1 is constant. Since the derivatives in a) and b) above are functions of (x_2/x_1) - in fact in both cases here, they are equal to $-x_2/x_1$, the derivatives of the level curves at points along a given ray from the origin are the same, and therefore the functions are homothetic. $g(x_2/x_1) = -x_2/x_1$