SOLUTIONS - PROBLEM SET 4

① Budget constraint: $P_x X + P_y Y = I$ But since $\frac{Y}{X} = 3$, or $Y = 3X$ we can sub in budget constraint to get $P_x X + P_y 3P_x = I$

$\Rightarrow X(P_x + 3P_y) = I \Rightarrow X^* = I/(P_x + 3P_y)$ and since $Y = 3X$,

$Y^* = 3I/(P_x+3P_y)$

② a) $u(x,y) = xy^{\frac{1}{2}}$ $\quad \mathcal{L} = xy^{\frac{1}{2}} + \lambda(I - P_x X - P_y Y)$

$$\frac{\partial \mathcal{L}}{\partial X} = Y^{\frac{1}{2}} - \lambda P_x = 0 \qquad \frac{\partial \mathcal{L}}{\partial Y} = \frac{1}{2} XY^{-1/2} - \lambda P_y = 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = I - P_x X - P_y Y = 0$$

$$\lambda = \frac{Y^{1/2}}{P_x} \qquad \lambda = \frac{\frac{1}{2}XY^{-1/2}}{P_y} \qquad \text{So } \frac{Y^{1/2}}{P_x} = \frac{\frac{1}{2}XY^{-1/2}}{P_y}$$

$\Rightarrow \dfrac{Y^{1/2}}{\frac{1}{2}XY^{-1/2}} = \dfrac{P_x}{P_y} \Rightarrow \dfrac{2Y}{X} = \dfrac{P_x}{P_y} \Rightarrow Y = \dfrac{XP_x}{2P_y}$. Sub in budget constraint:

$$I - P_x X - P_y\left(\frac{XP_x}{2P_y}\right) = 0 \Rightarrow I - P_x X - \tfrac{1}{2}P_x X = 0 \Rightarrow I = \tfrac{3}{2} X P_x$$

$\Rightarrow X^* = \frac{2}{3}(I/P_x)$ and since $Y = \dfrac{XP_x}{2P_y}$, $Y^* = \dfrac{(2/3\, I/P_x)P_x}{2P_y} = \frac{1}{3}(I/P_y)$

b) $u(x,y) = 3x^{\frac{1}{3}} + 3y^{\frac{1}{3}}$ $\quad \mathcal{L} = 3x^{\frac{1}{3}} + 3y^{\frac{1}{3}} + \lambda(I - P_x X - P_y Y)$

$$\frac{\partial \mathcal{L}}{\partial X} = X^{-2/3} - \lambda P_x = 0 \qquad \frac{\partial \mathcal{L}}{\partial Y} = Y^{-2/3} - \lambda P_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - P_x X - P_y Y = 0$$

$$\lambda = \frac{X^{-2/3}}{P_x} \qquad \lambda = \frac{Y^{-2/3}}{P_y} \qquad \frac{X^{-2/3}}{P_x} = \frac{Y^{-2/3}}{P_y} \Rightarrow \frac{X^{-2/3}}{Y^{-2/3}} = P_x/P_y$$

$$\Rightarrow (Y/X)^{2/3} = P_x/P_y \Rightarrow Y/X = (P_x/P_y)^{3/2} \Rightarrow Y = \left(\frac{P_x}{P_y}\right)^{3/2} X$$

$$I - P_x X - P_y\left(\frac{P_x}{P_y}\right)^{3/2} X = 0 \Rightarrow I - P_x X - P_y^{-1/2} P_x^{3/2} X = 0$$

$$\Rightarrow I = X\left[P_x + P_y^{-1/2}P_x^{3/2}\right] \Rightarrow \boxed{X^* = I\left[\frac{1}{P_x(P_x^{1/2}P_y^{-1/2}+1)}\right]}$$

$$Y = \left(\frac{P_x}{P_y}\right)^{3/2} X \Rightarrow Y = (P_x/P_y)^{3/2} I\left[\frac{1}{P_x(P_x^{1/2}P_y^{-1/2}+1)}\right]$$

$$\Rightarrow Y = I\left[\frac{1}{P_x^{-3/2}P_y^{3/2}\left[P_x^{3/2}P_y^{-1/2} + P_x\right]}\right]$$

$$\Rightarrow Y = I \frac{1}{(P_y + P_x^{-1/2}P_y^{3/2})} \Rightarrow \boxed{Y^* = I\left[\frac{1}{P_y(P_x^{-1/2}P_y^{1/2}+1)}\right]}$$

(3) a) For 2a) $X^* = 2I/3P_x \Rightarrow P_x X^* = \frac{2}{3} I \Rightarrow$
Consumer spends a constant fraction (2/3) of income on X. And since $Y^* = I/3P_y \Rightarrow P_y Y^* = (\frac{1}{3})I$,
Consumer spends 1/3 of income on Y (constant fraction).

For 2b) $X^* = I\left[\frac{1}{P_x(P_x^{1/2}P_y^{-1/2}+1)}\right] \Rightarrow P_x X = I\Bigg[\underbrace{\frac{1}{(P_x^{1/2}P_y^{-1/2}+1)}}_{\text{fraction of income spent on X}}\Bigg]$ not constant

$Y^* = I\left[\frac{1}{P_y(P_x^{-1/2}P_y^{1/2}+1)}\right] \Rightarrow P_y Y^* = I\Bigg[\underbrace{\frac{1}{(P_x^{-1/2}P_y^{1/2}+1)}}_{\text{Fraction of income spent on Y}}\Bigg]$

b) For 2a): $X = 2I/3P_x$; $X' = 2tI/3tP_x = 2I/3P_x = X$
$\Rightarrow$ homogeneous of degree 0

For 2b): $X = I\left[\frac{1}{P_x(P_x^{1/2}P_y^{-1/2}+1)}\right]$

$$X' = tI\left[\frac{1}{tP_x((tP_x)^{1/2}(tP_y)^{-1/2}+1)}\right]$$

$$= \frac{t}{t} I\left[\frac{1}{t^{1/2}t^{-1/2}(P_x^{1/2}P_y^{-1/2}+1)}\right] = X$$

$\Rightarrow$ homogeneous of degree 0

(4) a) $V(P_x, P_y, I) = \left(\frac{2}{3}\frac{I}{P_x}\right)\left(\frac{1}{3}\frac{I}{P_y}\right)^{1/2}$

$$= \frac{2}{3}\left(\frac{1}{3}\right)^{1/2} \frac{I^{3/2}}{P_x P_y^{1/2}}$$

b) $\mathcal{L} = P_x X + P_y Y + \lambda(u - XY^{1/2})$

$\frac{\partial \mathcal{L}}{\partial X} = P_x - \lambda Y^{1/2} = 0$ $\quad \frac{\partial \mathcal{L}}{\partial Y} = P_Y - \lambda \frac{1}{2} X Y^{-1/2} = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = u - XY^{1/2}$

$\lambda = P_x / Y^{1/2}$ $\quad \lambda = P_Y / \frac{1}{2}XY^{-1/2}$

$\frac{P_x}{Y^{1/2}} = \frac{P_Y}{\frac{1}{2}XY^{-1/2}} \Rightarrow \frac{P_x}{P_Y} = \frac{2Y}{X} \Rightarrow Y = \frac{X}{2}(P_x/P_y)$

$u - X\left(\frac{X}{2} P_x/P_y\right)^{1/2} = 0 \quad u = X\left(\frac{X}{2} P_x/P_y\right)^{1/2}$

$u = x^{3/2}\left(\frac{1}{2}\,P_x/P_y\right)^{1/2} \Rightarrow u\left(\frac{1}{2}P_x/P_y\right)^{-1/2} = x^{3/2}$

$\Rightarrow x = u^{2/3}\left(\frac{1}{2}P_x/P_y\right)^{-1/3} = u^{2/3}\left(\frac{2P_y}{P_x}\right)^{1/3} = \left(\frac{2u^2P_y}{P_x}\right)^{1/3}$

and since $y = \left(\frac{1}{2}\right)x\,(P_x/P_y)$, $y = \frac{1}{2}\left(\frac{2u^2P_y}{P_x}\right)^{1/3}\left(\frac{P_x}{P_y}\right)$

or $y = \frac{1}{2}(2u^2)^{1/3}(P_x/P_y)^{2/3}$

So, $E(P_x, P_y, u) = P_x\left(\frac{2u^2P_y}{P_x}\right)^{\frac{1}{3}} + P_y\left[\frac{1}{2}(2u^2)^{1/3}(P_x/P_y)^{2/3}\right]$

$= (2u^2)^{1/3}P_x^{2/3}P_y^{1/3} + \frac{1}{2}(2u^2)^{1/3}P_x^{2/3}P_y^{1/3}$

$= \frac{3}{2}(2u^2)^{1/3}P_x^{2/3}P_y^{1/3} = 3\,(u/2)^{2/3}P_x^{2/3}P_y^{1/3}$

c) $u = (2/3)(1/3)^{1/2}\frac{E^{3/2}}{P_x P_y^{1/2}} \Rightarrow u^{2/3} = \left(\frac{2}{3}\right)^{2/3}\left(\frac{1}{3}\right)^{1/3}E\,P_x^{-2/3}P_y^{-1/3}$

$\Rightarrow E = u^{2/3}(3/2)^{2/3}(3)^{1/3}P_x^{2/3}P_y^{1/3} = (u/2)^{2/3}\,3\,P_x^{2/3}P_y^{1/3}$

d) $x = \frac{2}{3}\left(\frac{30}{1/2}\right) = 40 \quad y = \frac{1}{3}\left(\frac{30}{1}\right) = 10$

e) $x = \frac{2}{3}\left(\frac{30}{1}\right) = 20 \quad y = \frac{1}{3}\left(\frac{30}{1}\right) = 10$

f) $1.00(40) + 1.00(10) = 50$ Given $I = 30$, we would have to give him 20.

g) $u = 40(10)^{1/2} = 126.491$. Alternatively you can use indirect utility function: $\frac{2}{3}\left(\frac{1}{3}\right)^{1/2}\left[\frac{30^{3/2}}{1/2}\right] = 126.491$

h) Use expenditure function: $3\left(\frac{(126.491)}{2}\right)^{2/3}(1)^{2/3}(1)^{1/3} = 47.62$

So we would have to increase his income of 30 by 17.62

i) The lump sum in part g which keeps the original consumption bundle affordable is greater than the lump sum in part i, which keeps the original utility level the same.

5.

If $U(x,y)=\min(x,y)$, utility maximization requires $x=y$. Substitution into the budget constraint yields $x=I/(p_x+p_y)=y$. Hence,

$$V(p_x,p_y.I)=\frac{I}{p_x+p_y},$$
$$E(p_x,p_y,V)=(p_x+p_y)V.$$

If $U(x,y)=x+y$, utility maximization requires the purchase of whichever of these two perfect substitutes has the lower price. So, if $p_x>p_y$, $x=0, y=I/p_y$. If $p_x<p_y$, $x=I/p_x, y=0$. Given these results,

$$V(p_x,p_y.I)=\frac{I}{\min(p_x,p_y)},$$
$$E(p_x,p_y,V)=\min(p_x,p_y)V.$$