Econ 701

**Problem Set 1**

**1.**

* Find the critical values of . (note: there are three of them).
* For each one, use the second-order condition to indicate whether the critical value yields a local maximum or minimum.

Find critical values where .

Factor:

or

The 3 Critical Values are therefore

**2.** For each multivariate function below, calculate .

* Verify in each case that Young’s theorem holds.

|  |  |
| --- | --- |
| a. | b. |
| c. | d. |

e.

**3.** Consider the function in question 2b above.

* Find the critical values of and that maximize and minimize the function.
* Use second-order conditions to determine if these values yield a local maximum or minimum.

at critical :

Sub in for in set equal to zero:

And since

From 2b we have and

Now check

Since , , and at local maximum

(in fact, since these conditions hold everywhere, we have a global maximum)

**4.** Repeat Question 3 but use the function in question 2c above.

at critical :

|  |  |  |
| --- | --- | --- |
|  | → Multiple both sides of second equation by 2 → |  |

sub into one of the equations:

Since , and at , , we have a local minimum.

And since these conditions hold everywhere, it is a global minimum.

**5.** Consider the function

1. Derive the total differential of the function.

so the total differential is:

1. Use the total differential to estimate the change in if initially, , and increases by 0.5 and simultaneously increased by 0.2. Compute the *actual* change in .

For actual change in :

so actual

**6.** Consider the implicit function: .

* Use the implicit function theorem to calculate and evaluate this derivative at .

Given , using the implicit function theorem we know that

Here, and

So, and at we have

**7.** Consider the function , where is a strictly positive parameter.

1. Use the first order condition for a maximum to derive the critical value of , which will be a function of .
   * In parts **b** and **c** below, you will calculate the effect by substitution and taking the derivative.
   * In part **c**, you will apply the envelope theorem.
2. Substitute from part **a** into to obtain . Calculate .
3. Now use the envelope theorem to compute .
4. Suppose initially. Use a differential approximation to estimate the change in the maximum value of if were to increase by 0.2.
5. Continuing with part **d**, calculate the actual change in the maximized value of .

a.

b. ;

c.

d. If , then

So, if were to increase by then we have , approximated by or

e. From part **b**, we have

so when ,

and when ,

so actual change is 1.04

**Problem Set 2**

**1.** Determine whether or not the following functions are quasi-concave. Explain.

**a**.

We can say is quasi-concave, since , and any monotonically increasing or decreasing function is both quasi-concave and quasi-convex.

More formally, if where , then the function is quasi-concave. Here, we need to show .

Consider any 2 points, , , where . Since , then .

Therefore, we need to show , or taking logs of both sides:

, which holds by assumption.

**b**. , where (hint: is this function concave?)

Since , , and function is concrete and therefore also quasi-concave.

**2**. Use the Method of Lagrange to solve for and in the follow problem:

Maximize subject to

What is , the maximized value (subject to the constraint) of ?

Since and since , so or

Sub into F.O.C. for : and

Maximized value of

**3**. Solve for and in the dual to the problem in question 2. Here, the dual problem is:

Minimize subject to

Where you will substitute for the maximized value (subject to the constraint) of from question 2. Are and here the same as in question 2?

Minimize subject to

sub in F.O.C.

or

(same as in **2** above)

**4**. If thousand dollars is spent on labor and thousand dollars is spent on equipment, a certain factory produces units of output.

**a**. How should $80,000 be allocated between labor and equipment to yield the largest possible output?

The problem can be written: subject to

plug into :

**b**. Use the critical value of the Lagrange multiplier to estimate the change in maximum output if this allocation decreases by $1,000.

From above, we have , so

Therefore, since measures , we estimate the maximized output will decrease by .

**c**. Compute the exact change in part b.

Now in part *a*,

Everything in the same in part *a* (i.e. ) until we sub into , since we now have maximum output before was .

Maximum output now is , so the actual change of is equal to the estimated change of 25 in this case.

**5**. Write down the dual problem to that in question 4 part **a** and solve for and .

Minimize subject to

Sub into constraint:

**6**. Show that the following functions are homogenous (specify of what degree) and verify that Euler’s theorem holds:

a.

b.

**7**. Show that the following functions are homothetic by demonstrating that the derivatives of the level curves are constant along a ray from the origin:

**a**.

Level curve defined by , such that:

By implicit function theorem,

So

**b**.

Using implicit function theorem again,

At any point along a ray from the origin, the proportion is constant. Since the derivatives in a) and b) above are functions of – in fact in both cases here, they are equal to , the derivatives of the level curves at points along a given ray from the origin are the same, and therefore the functions are homothetic.

**Problem Set 3**

**1**. Which of the following utility functions are equivalent to ? For those that are, what is the monotonic transformation that proves the equivalence?

**a**.

equivalent: or here,

**b**.

equivalent:

**c**.

not equivalent; it is not a monotonic transformation.

So, for example, consider bundles : Bundle A = (2,3) and Bundle B = (3,2).

If utility is , the consumer is indifferent between the bundles.

However, if utility is , the consumer strictly prefers B to A.

**2**. For every function in 3a-c that you found equivalent to :

* Show that they all (including ) have the same marginal rates of substitution at bundle , but differing marginal utilities for good at that same bundle.
* Discuss how this illustrates the difference in the ordinality and cardinality properties of the and marginal utility.

By the implicit function theorem, the

*i*) , , so at ,

, and at ,

*ii*) , , so , and so

, and at ,

*iii*) , , so , so

, and at ,

The MRS is invariant to order preserving transformations at the utility function, and therefore its value is based on ordinal, not cardinal properties. In contrast, the marginal utilities are not invariant to such transformations of utility, and they are therefore a cardinal property.

**3**. Consider the utility function given by: (CES)

**a**. Compute the marginal rate of substitution to show that the utility function is homothetic (can it be expressed as a function of ?).

Since the derivative of the level curves (i.e. the MRS, the derivative of the indifference curves) is , a function of the ratio (or ), the function is homothetic.

**b**. What is the when ? What is the elasticity of substitution when and what type of preference does this describe?

when is . and at , . This describes perfect substitutes.

**c**. If and , what fraction of income will be spent on good ? Good ? Explain.

This individual is indifferent between 1 unit of and units of ; is a perfect substitute for units of . They will therefore spend their entire income on whichever is cheaper: or . costs and costs . We are given , so the consumer will spend 100% of income on good only.

**d**. For the remaining questions d-g consider a consumer with CES utility where and . Suppose in the past year there has been a 5% increase in the price ratio, . By how much did the consumer’s relative consumption, , change during the year? Explain. (hint: use the elasticity of substitution).

Since at the consumer optimum, the consumer equates their , the 5% increase in the price ratio the MRS will have increased 5% as well. So the percentage in is approximately: or

**e**. If the price of good is $4 and the price of good is $1, how much of goods and will the consumer purchase, given her income is $60? Show all work.

sub in constraint (3d FOC): and

**f**. Graph the optimum in part e) using a budget constraint and indifference curve (be sure to calculate the intercepts of the budget constraint and indicate the value of its slope).

Diagram

Description automatically generated

**g**. What is the value of the consumer’s marginal utility of income at the optimum?

From above, we have marginal utility of income at the optimum

**4**. Consider the utility function

**a**. In what fixed proportion does the consumer consume the goods?

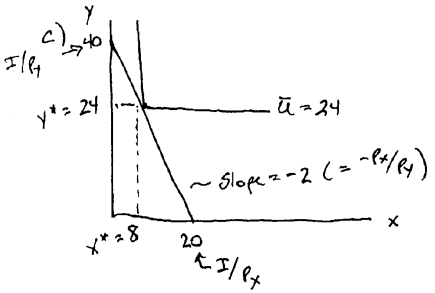
or For every x, they consume 3y

**b**. If the price of good is $10 and the price of good is $5, how much of goods and will the consumer purchase when his income is $200?

Budget constraint: .

But since they maintain , we have and

**c**. Graph the optimum in part b) using a budget constraint and indifference curve (be sure to calculate the intercepts of the budget constraint and indicate the value of its slope).



**5.** For Ben, 5 units of good is always a perfect substitute for two units of good .

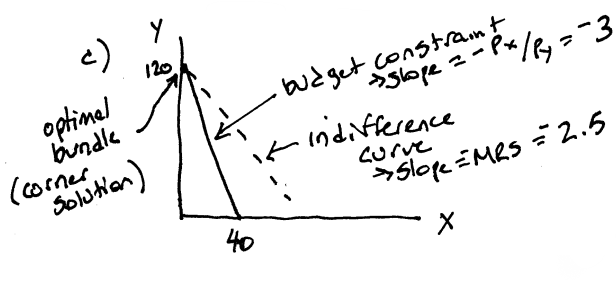
**a**. Write down a specific utility function for Ben that represents his preferences.

or any where will do.

**b**. If the price of good is $9 and the price of good is $3, how much of goods and does he purchase when his income is $360?

Five units of cpst , which is cheaper than the perfect substitute of two units of (which cost . therefore the consumer will spend his entire income on the cheaper substitute (good ). He can afford to buy a maximum amount of good of , or units. He buys 0 units of .

**c**. Graph the optimum in part b) using a budget constraint and indifference curve (be sure to calculate the intercepts of the budget constraint and indicate the value of its slope).



**6.** from question 3.7 in text

**a**. A consumer is willing to trade 3 units of for 1 unit of when she has 6 units of and 5 units of . She is also willing to trade in 6 units of for 2 units of when she has 12 units of and 3 units of . She is indifferent between bundle (6, 5) and bundle (12, 3). What is the utility function for goods and ? Hint: What is the shape of the indifference curve?

note: we want not

We are also given that these bundles lie on the same IC. So, and are perfect substitutes where unit of is a perfect substitute for 1 unit of : for example. ICS are linear with slope = .

**b**. A consumer is willing to trade 4 units of for 1 unit of when she is consuming bundle

(8, 1). She is also willing to trade in 1 unit of for 2 units of when she is consuming bundle (4, 4). She is indifferent between these two bundles. Assuming that the utility function is Cobb–Douglas of the form , where α and β are positive constants, what is the utility function for this consumer?

For Cobb-Douglas , we have: .

So . so can be , for example.

**c**. Was there a redundancy of information in part (b)? If yes, how much is the minimum amount of information required in that question to derive the utility function?

Didn’t need info about the other bundle.

**7**. (from question 3.11 in text)

Independent Utilities. Two goods have independent marginal utilities if:

This means that the quantity of one good does not affect the marginal utility of the other good. Show that if we assume diminishing marginal utility for each good, then any utility function with independent marginal utilities will have diminishing .

(Hint: express of in terms of partials of the utility function – when does it diminish as the amount of good increases?)

Given , (ratio of marginal utilities)

Since we are given , we see that this expression is negative.

Strictly speaking as we move along IC, is changing (decreasing) and by similar analysis to the above

So as we move along IC, is increasing and is decreasing, both of which cause to decrease.

More formally when , , the utility function can be shown to be strictly quesi-concave convexity of ICS (diminishing MRS) – see footnote 7 on pg. 100 of text.

**Problem Set 4**

**1**. Consider the utility function .

* Derive the demand curves for good and (consult your textbook page 126, Case)

budget constraint:

But since or , we can sub in budget constraint to get:

and since ,

**2**. For each of the following utility functions, use the method of Lagrange to derive the demand functions for good and good .

**a**. (Cobb-Douglas)

sub in budget constraint:

and since ,

**b**. (CES)

**3**. For each case above (2a & 2b)

**a**. Use the demand functions you derived to show, for each good, the proportion of income spent on the good. Are the proportions constant?

for 2a) consumer spends a constant function of income on . And since , consume spends of income on (constant function).

for 2b)

fraction of income spent on , not constant

fraction of income spent on , not constant

**b**. Show that the demand functions are homogeneous of degree zero (in both cases, you need only show this for the demand function for good ).

For 2a) ; homogenous at degree 0

For 2b)

homogenous at degree 0

**4**. Consider the utility function from question 2a) above, and for which you have already derived demand functions.

**a**. Derive , the indirect utility function.

**b**. Use the method of Lagrange to solve the consumer's dual problem to derive ,the expenditure function.

and since , or

so,

**c**. Show that the expenditure function in part b) is the inverse of the indirect utility function from part a).

**d**. Suppose initially , and . Use the demand functions from question 2a) to determine the optimal quantities of and .

**e**. Now suppose the price of rises to while everything else remains the same. Use the demand functions from question 2a) again to find the new optimal quantities of and .

**f**. At the new prices, how much would the original consumption bundle (i.e., the bundle in part d) cost? If we wanted to give this consumer a lump sum cash subsidy to make the original bundle affordable at the new prices, how much would we have to give the consumer?

Given , we would have to give him 20.

**g**. What is the consumer's level of utility from the original consumption bundle in part d?

Alternatively you can use indirect utility function:

**h**. What is the size of the lump sum cash grant we would have to give the consumer to keep him at the original level of utility, given the new prices? (hint: use the expenditure function)

Use expenditure function:

**i**. Compare the size of the cash grants in parts f and h. (Observe that this is an illustration of the potential consequences of the well-known substitution bias inherent in certain price indicies such as the consumer price index. It is argued that cost-of-living adjustments pegged to the CPI overcompensate the consumer for price increases, as the CPI is based on a fixed consumption bundle and compensation sufficient to maintain affordability of a fixed bundle overcompensates the consumer, as it does not account for the fact that consumers make welfare-improving substitutions in consumption when relative prices change. Ideally, the appropriate compensation is to maintain the original level of utility, not the original consumption bundle).

The lump sum is part *g* which keeps the original consumption bundle affordable is greater than the lump sum in part *i*, which keeps the original utility level the same.

**5**. (From question 4.8 in text): Two of the simplest utility functions are:

1. Fixed proportions:
2. Perfect substitutes:

For each of these utility functions, compute the following

* Demand functions for and
* Indirect utility function
* Expenditure function

If , utility maximization requires . Substitution into the budget constraint yields . Hence,

If , utility maximization requires the purchase of whichever of these two perfect substitutes has the lower price.

So if , ,

If

Given these results:

**Problem Set 5**

Note: Both Problems 5.4 and 5.5 from the text are reproduced on page 2 of this problem set.

5.1 Uncompensated demand functions:

**1**. Problem 5.4, page 174 in text

Assume that utility is given by:

**a**. Use the uncompensated demand functions given in Example 5.1 to compute the indirect utility function and the expenditure function for this case.

Find the indirect utility function:

**b**. Use the expenditure function calculated in part (a) together with Shephard’s lemma to compute the compensated demand function for good .

**c**. Use the results from part (b) together with the uncompensated demand function for good to show that the Slutsky equation holds for this case.

Using uncompensated demand, we have:

substitution effect:

sub indirect utility for :

so,

(same as direct calculation above, save for rounding errors)

**2**. Problem 5.5, page 174, in text.

Suppose the utility function for goods and is given by:

**a**. Calculate the uncompensated (Marshallian) demand functions for and and describe how the demand curves for and are shifted by changes in or the price of the other good.

Solution from page 729 of the text.

Changes in : For both goods an increase in increases the quantity demanded at every price shifts both demands outward (both and are normal goods since and > 0).

Changes in do not affect changes in , but changes in do affect .

**b**. Calculate the expenditure function for and .

**c**. Use the expenditure function calculated in part (b) to compute the compensated demand functions for goods and . Describe how the compensated demand curves for and are shifted by changes in income or by changes in the price of the other good.

The compensated demand function for depends on , whereas the uncompensated function does not.

**3**. In question 1 above you started with the uncompensated demand to ultimately derive the Hicksian, or compensated demand. In this question you will start from the Hicksian demand to derive the uncompensated Marshallian demand, using the same numerical example in question 1 above.

**a**. In part b of question 1 you derived the compensated demand for good . Apply Shephard's Lemma once again to the expenditure function to derive the compensated demand for good .

**b**. Substitute the compensated demands into the objective function (expenditure minimization) to derive the expenditure function, which should be exactly the same as what you derived in part a of question 1.

Objective function:

subbing:

same as what we derived in part *a* of question 1

**c**. From the expenditure function, derive the indirect utility function, which again, should be exactly the same as what you derived in part a of question 1.

sub for and for in the expenditure function and invert:

same as in part *a* of question 1

**d**. Now apply Roy's identity to derive the uncompensated (Marhallian) demand for good . If you have done things correctly, it should be the same as what you were given for question 1.

**4**. Prove Roy's Identity. (Hint: consider the consumer's utility maximization problem in a simple two good case and apply the envelope theorem)

The consumer’s utility maximization problem generates the following Lagrangeon:

where maximal utility is given by

by the envelop theorem and

So