Week 3: The Wave Equation on $\mathbb{R}$

Introduction

Consider the wave equation on the whole line

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi(x) & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x) & x \in \mathbb{R}. \end{cases}$$

The particular solution to this PDE is given by

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$

This solution to the initial value problem is called *d'Alembert's formula*.

Problems

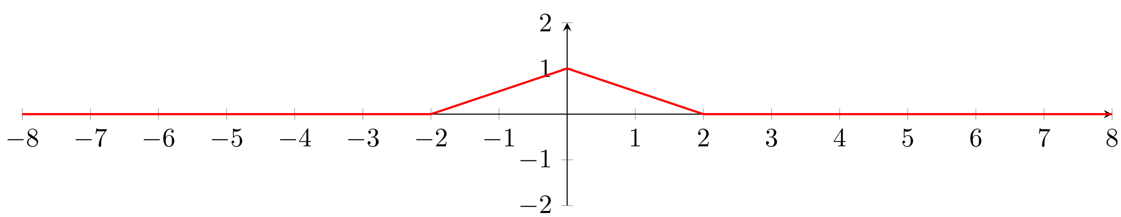
Problem 1. *(Plucked String)* Solve the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi(x) & x \in \mathbb{R}, \\ u_t(x,0) = 0 & x \in \mathbb{R}. \end{cases}$$

where for $a > 0$ and $b > 0$, the initial position $\phi(x)$ is given by

$$\phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| \le a \\ 0 & \text{for } |x| > a. \end{cases}$$

Solution 1. The graph of $\phi(x)$ for $b = 1$ and $a = 2$ is given below



In general, the b controls the amplitude of the triangle, and a denotes the width of the triangle.

From d'Alembert's formula, we know

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2}.$$

Notice that the first term is a wave with height $b/2$, centered at $-ct$ with width a,

$$\phi(x+ct) = \begin{cases} b - \frac{b|x+ct|}{a} & \text{for } |x+ct| < a \\ 0 & \text{for } |x+ct| > a \end{cases} = \begin{cases} b + \frac{b(x+ct)}{a} & \text{for } -a-ct \le x < -ct \\ b - \frac{b(x+ct)}{a} & \text{for } -ct \le x \le a - ct \\ 0 & \text{for } x > a - ct \text{ or } x < a - ct. \end{cases}$$

and the second term is a wave with height $b/2$, centered at ct with width a,

$$\phi(x-ct) = \begin{cases} b - \frac{b|x-ct|}{a} & \text{for } |x-ct| < a \\ 0 & \text{for } |x-ct| > a \end{cases} = \begin{cases} b + \frac{b(x-ct)}{a} & \text{for } -a+ct \le x < ct \\ b - \frac{b(x-ct)}{a} & \text{for } ct \le x \le a+ct \\ 0 & \text{for } x > a-ct \text{ or } x < a-ct. \end{cases}$$

Our solution is of a different form depending on large and small times

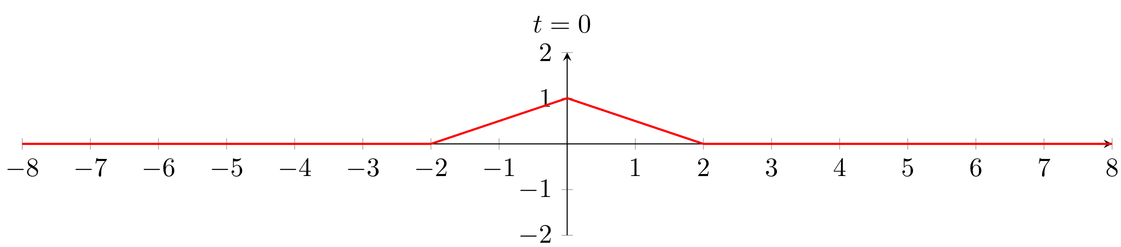
1. $t < \frac{a}{c}$: In this case we have $a > ct$, so we have $-a-ct < -a+ct < a-ct < a+ct$. In this region, the two separate waves overlap in the region $-a+ct < x < a-ct$. Therefore, we have

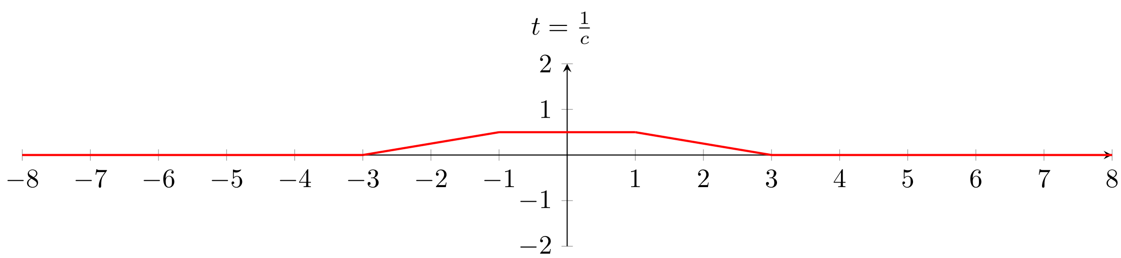
$$u(x,t) = \begin{cases} 0 & x \le -a-ct \\ \frac{1}{2}\Big(b + \dfrac{b(x+ct)}{a}\Big) & -a-ct < x \le -a+ct \\ \frac{1}{2}\Big(b - \dfrac{b(x+ct)}{a} + b + \dfrac{b(x-ct)}{a}\Big) = b \cdot \Big(1 - \dfrac{ct}{a}\Big) & -a+ct < x \le a-ct \\ \frac{1}{2}\Big(b - \dfrac{b(x-ct)}{a}\Big) & a-ct < x \le a-ct \\ 0 & x > a+ct \end{cases}$$

2. $t \ge \frac{a}{c}$: In this case we have $a \le ct$, so we have $-a-ct < a-ct \le -a+ct < a+ct$. In this region, the two separate waves do not overlap. Therefore, we have

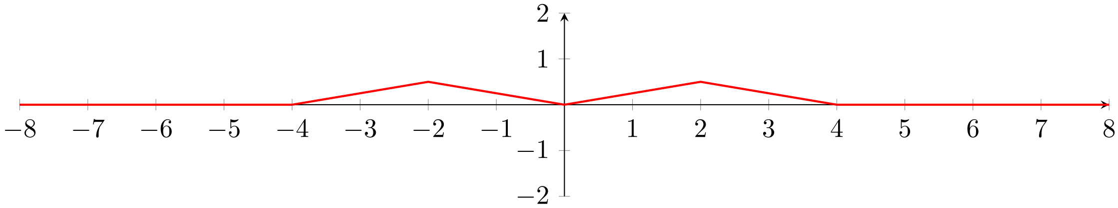
$$u(x,t) = \begin{cases} 0 & x \le -a-ct \\ \frac{1}{2}\Big(b + \dfrac{b(x+ct)}{a}\Big) & \text{for } -a-ct < x \le -ct \\ \frac{1}{2}\Big(b - \dfrac{b(x+ct)}{a}\Big) & \text{for } -ct < x \le a-ct \\ 0 & \text{for } a-ct < x \le -a+ct \\ \frac{1}{2}\Big(b + \dfrac{b(x-ct)}{a}\Big) & \text{for } -a+ct < x \le ct \\ \frac{1}{2}\Big(b - \dfrac{b(x-ct)}{a}\Big) & \text{for } ct < x \le a+ct \\ 0 & \text{for } x > a+ct. \end{cases}$$

The inequalities in the domains above don't matter, because the sum of continuous functions are continuous. The graphs of the wave for $b=1$ and $a=2$ at several times are presented below:

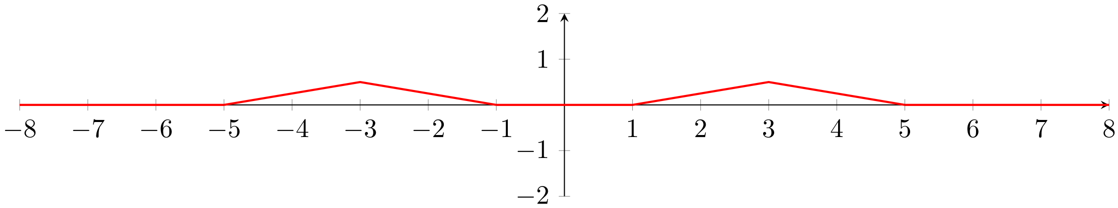




$t = \frac{2}{c}$



$t = \frac{3}{c}$



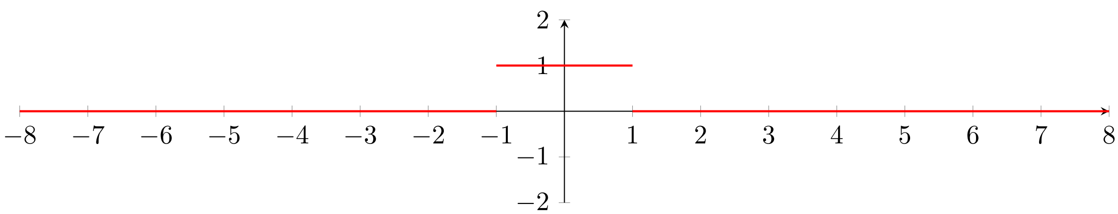
Problem 2. *(Hammer Blow)* Solve the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = 0 & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x) & x \in \mathbb{R}. \end{cases}$$

where for $a > 0$, the initial velocity $\phi(x)$ is given by

$$\psi(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| \geq a. \end{cases}$$

Solution 2. The graph of $\psi(x)$ for $a = 1$ is given below



In general, a denotes the width of the square wave.

From d'Alembert's formula, we know

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds.$$

Since ψ is constant on $[-a,a]$, the integral

$$\int_{x-ct}^{x+ct} \psi(s)\, ds$$

is simply the area of the set $[-a,a] \cap [x-ct, x+ct]$, so

$$u(x,t) = \frac{1}{2c} \big| [-a,a] \cap [x-ct, x+ct] \big|$$

where $|\cdot|$ is the length of the interval.

The graphs of the wave with $c = 1$ and $a = 1$ at several times are presented below: