

# 1 Generating Random Variables

## 1.1 Quantiles

Suppose that we are given  $X$  and a value  $p \in (0, 1)$  and we are interested in computing the value of  $t$  such that

$$F_X(t) = \mathbb{P}(X \leq t) = p.$$

If  $F_X$  is invertible, then  $t = F_X^{-1}(p)$ . However, not all CDFs are invertible so how does one define such a  $t$  in general. This generalized notation of an inverse is called a quantile function.

**Definition 1** (Quantile). Let  $p \in [0, 1]$ . The  $p$ -quantile (or  $100 \times p$ th percentile) of the distribution of  $X$  with CDF  $F_X$  is the smallest number  $c_p$  that satisfies  $F_X(c_p) \geq p$ . In other words,

$$c_p = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}.$$

**Definition 2** (Median). The *median* of a distribution is its 0.5 quantile.

The quantile function takes a probability  $p$  and returns its  $p$ -quantile.

**Definition 3** (Quantile Function). The *quantile function*  $F_X^{-1}[0, 1] \mapsto \mathbb{R}$  is the function given by

$$F_X^{-1}(p) := c_p = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}.$$

The quantile function is also called *generalized inverse function*, because it is a well defined function even if  $F_X$  is not strictly increasing like in the case of discrete random variables. This is why we use the same notation  $F_X^{-1}$ , even though it is not the inverse in the traditional sense. Recall that if  $F_X$  is an invertible function, then

$$F_X^{-1}(F_X(x)) = x \text{ for all } x \in \mathbb{R} \quad \text{and} \quad F_X(F_X^{-1}(p)) = p \text{ for all } p \in [0, 1].$$

In fact, the quantile behaves exactly like an inverse function, but the equalities are often replaced by inequalities.

**Proposition 1** (*Properties of the Generalized Inverse*)

The quantile function satisfies  $F_X^{-1}$  for  $F_X$  satisfies

1. For all  $x \in \mathbb{R}$ ,  $F_X^{-1}(F_X(x)) \leq x$
2. For all  $p \in [0, 1]$ ,  $F_X(F_X^{-1}(p)) \geq p$
3.  $F_X^{-1}(p) \leq x \Leftrightarrow p \leq F_X(x)$
4.  $F_X^{-1}(p)$  is non-decreasing and left-continuous (except for the endpoints  $p = 0$  or  $p = 1$ )

**Remark 1.** To remember which way the inequalities go, recall that  $F_X$  “jumps up” at discontinuities so  $F_X(F_X^{-1}(p)) \geq p$  and the quantile function “jumps down” at discontinuities to  $F_X^{-1}(F_X(x)) \leq x$ .

We can compute the quantile in the following way

- If the distribution function  $F_X$  is continuous and strictly increasing, it has an inverse  $F_X^{-1}$  so

$$c_p = F_X^{-1}(p).$$

- If  $F_X$  has jumps or flat regions, then  $F_X(x) = p$  may not have any solution or it might have infinitely many. In this case, the function  $F_X^{-1}(p)$  is the left continuous step function that interpolates between the points  $(p, x)$  where  $x$  is the location of the jumps of  $F_X$ .

## 1.2 Inverse Transform Sampling

It is “easy” to sample from the continuous uniform distribution  $\text{Unif}(0,1)$  on a computer. These uniform random variables can be used to generate samples from any distribution.

### Theorem 1 (*Inverse Transform Sampling*)

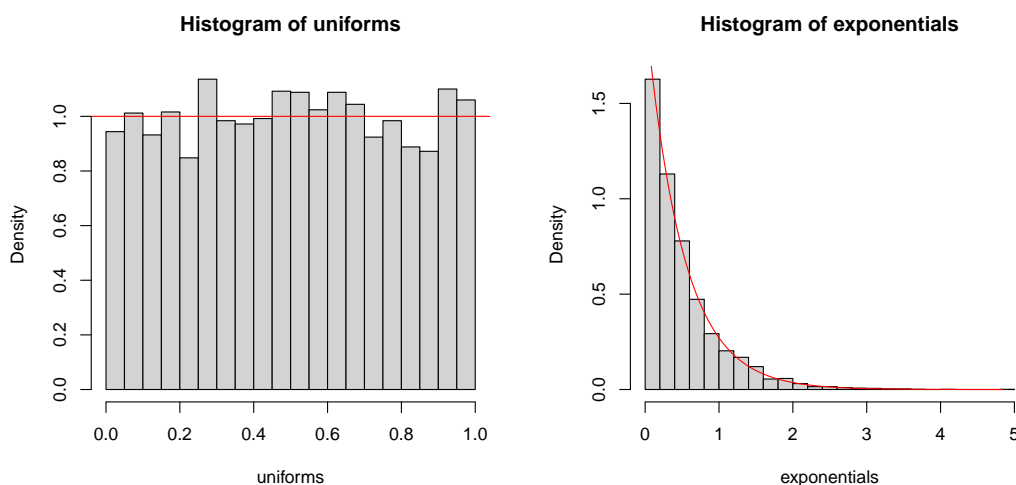
Let  $F_X$  be any cumulative distribution function of some random variable  $X$  and  $U \sim \text{U}(0,1)$ . Then the random variable  $Y = F_X^{-1}(U)$  has the same distribution as  $X$ , i.e.  $Y$  has the CDF  $F_X$ .

**Remark 2.** This is a generalization of a simple concept. For instance, if we want to generate a flip of a coin (a  $\text{Ber}(0.5)$  random variable), then we can sample a number uniformly  $u$  from  $[0,1]$  and define  $x = X(u) = 0$  if  $u_1 \in [0, 0.5]$  and  $x = X(u) = 1$  if  $u_1 \in [0.5, 1]$ . One can check that this coincides with the inverse transform sampling method (see Problem 1.4.)

## 1.3 Sampling Algorithm

1. No matter what CDF  $F_X$  (discrete or continuous), we can sample observations as follows:
  - (a) Sample  $u \sim U(0,1)$  (eg via `runif()`)
  - (b) Return  $x = F_X^{-1}(u)$ .
2. Repeating this  $n$  times independently gives  $n$  realizations of  $X$ .

**Example 1.** We sample `uniforms <- runif(5000)` and then `exponentials <- -log(1-uniforms)/2`.



## 1.4 Example Problems

**Problem 1.1.** Consider the random variable  $X$  with

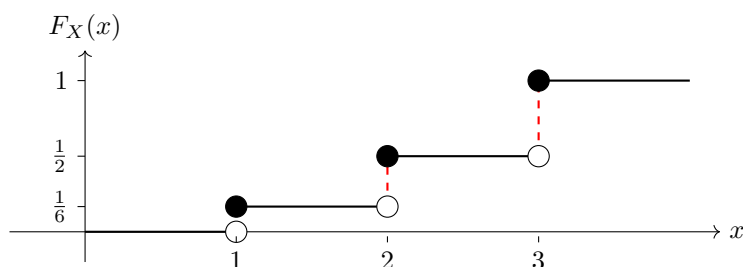
$$\mathbb{P}(X = 1) = 1/6, \quad \mathbb{P}(X = 2) = 2/6 \quad \mathbb{P}(X = 3) = 3/6.$$

Sketch the CDF of  $X$  and compute  $F_X^{-1}(p)$  for  $p \in (0,1)$ .

**Solution 1.1.**

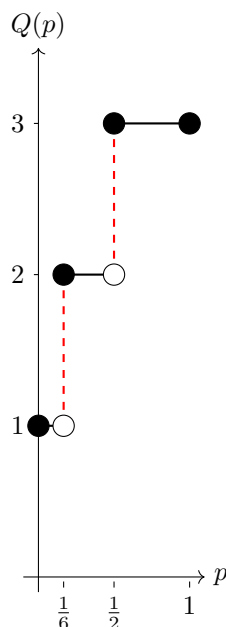
1. To draw the CDF, we notice that the discontinuities of the CDF occur at  $(1, \frac{1}{6})$ ,  $(2, \frac{1}{2})$ ,  $(3, 1)$ . Extending this to make the function right continuous implies the CDF is

$$F_X(x) = \begin{cases} 0, & x < 1, \\ \frac{1}{6}, & 1 \leq x < 2 \\ \frac{1}{2}, & 2 \leq x < 3, \\ 1 & 3 \leq x \end{cases}$$



2. To compute the quantile function, we notice that the discontinuities of the CDF occur at  $(1, \frac{1}{6})$ ,  $(2, \frac{1}{2})$ ,  $(3, 1)$ . Therefore, the discontinuities for the quantile function occur at  $(\frac{1}{6}, 1)$ ,  $(\frac{1}{2}, 2)$ ,  $(1, 3)$ . Extending this to make the function left continuous implies

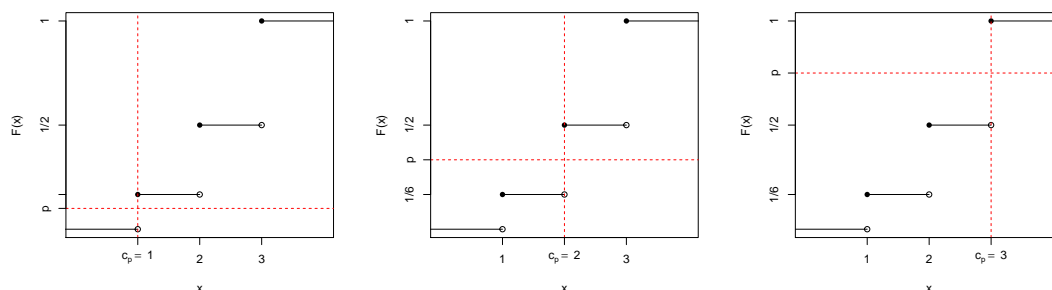
$$F_X^{-1}(p) = c_p = \inf\{x \in \mathbb{R} : F_X(x) \geq p\} = \begin{cases} 1, & 0 < p \leq \frac{1}{6} \\ 2, & \frac{1}{6} < p \leq \frac{1}{2} \\ 3, & \frac{1}{2} < p \leq 1. \end{cases}$$



**Remark 3.** Visually this corresponds to a reflection the line  $y = x$ . The dashed red line of the CDF becomes the solid line of the quantile and vice versa.

**Remark 4.** The end points of the intervals in the quantile function are the same as the  $p$  values of the CDF at the jumps. Furthermore, the  $<$  inequality is always on the left of the  $x$  and the  $\leq$  inequality is always to the right of the  $x$ . This implies the quantile function is left continuous.

**Remark 5.** To find individual points of the quantile at  $p$ , we find the smallest point where the graph  $F_X(x)$  lies on or above the horizontal line  $p$ . This is demonstrated for  $p \in (0, 1/6]$  (left),  $p \in (1/6, 1/2]$  (middle) and  $p \in (1/2, 1]$  (right).



**Problem 1.2.** Let  $U \sim \text{Unif}(0, 1)$ . We want to sample from the  $\text{Exp}(2^{-1})$  distribution with density

$$f_X(x) = 2e^{-2x}, \quad x > 0$$

and 0 otherwise. Write  $Y$  as a function of  $U$  such that  $Y$  is equal in distribution to  $X$ .

**Solution 1.2.** The CDF on the support of  $X$

$$F_X(x) = \int_0^x 2e^{-2t} dt = 1 - e^{-2x},$$

which is strictly increasing for on its support  $x \geq 0$ . Solving for  $F_X(y) = x$  to recover the inverse gives  $y = F_X^{-1}(x) = -\frac{1}{2} \log(1 - x)$ , so

$$F_X^{-1}(x) = -\frac{1}{2} \log(1 - x)$$

for  $x \in (0, 1)$ . Therefore, by Theorem 1

$$Y = -\frac{1}{2} \log(1 - U)$$

has the same distribution  $Y \sim \text{Exp}(2^{-1})$ .

**Problem 1.3.** Suppose that we wish to generate a random observation,  $x$ , from a distribution with PDF given by

$$f_X(x) = \frac{1}{8\sqrt{x}}, \quad 0 < x < 16$$

and 0 otherwise. We generate an observation,  $u$ , from a continuous  $\text{Unif}(0, 1)$  distribution (using software) and get 0.1348. Determine the value  $x = x(u)$ , that this value  $u$  will produce.

**Solution 1.3.** We first compute the CDF on the support of  $X$

$$F_X(x) = \int_0^x \frac{1}{8\sqrt{t}} dt = \frac{1}{4}\sqrt{x}, \quad 0 < x < 16.$$

which is strictly increasing on its support  $0 < x < 16$ . Solving for  $F_X(y) = x$  to recover the inverse gives  $y = F_X^{-1}(x) = (4x)^2$ , so

$$F_X^{-1}(x) = (4x)^2$$

for  $x \in (0, 1)$ . By the sampling algorithm, if  $u = 0.1348$  the corresponding observation of  $x$  is

$$x = F_X^{-1}(u) = (4 \cdot 0.1348)^2 = 0.2907.$$

**Problem 1.4.** Explain how you would sample a biased flip of a coin with probability of heads  $p$  using a uniform random variable.

**Solution 1.4.** If  $X$  is the outcome of a biased flip of a coin with probability of heads  $p$ , then  $X \sim \text{Bern}(p)$ . This means that  $f_X(1) = p$  and  $f_X(0) = 1 - p$ . The CDF and quantile function is therefore,

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases} \quad F_X^{-1}(x) = \begin{cases} 0 & 0 < x \leq 1 - p \\ 1 & 1 - p < x \leq 1 \end{cases}.$$

If  $U \sim \text{Unif}(0, 1)$ , we know that  $X$  has the same distribution as  $F_X^{-1}(U)$ . Therefore, to generate a biased coin flip, we sample  $u \sim \text{Unif}(0, 1)$  and define  $x(u) = 0$  if  $u < 1 - p$  and  $x(u) = 1$  if  $u > 1 - p$ .

**Problem 1.5.**

1. 75th percentile of the standard normal distribution
2. 58th percentile of the  $N(5, 9)$  distribution
3. Let  $Z \sim N(0, 1)$ . Find  $c$  such that

$$\mathbb{P}(-c \leq Z \leq c) = 0.95$$

**Solution 1.5.**

1. We find

$$\Phi^{-1}(0.75) = 0.6745$$

2. We need to find the 0.58 quantile of  $X$  where  $\mu = 5$  and  $\sigma = \sqrt{9} = 3$ ,

$$F_X^{-1}(0.58) = 5 + 3F_Z^{-1}(0.58) = 5 + 3\Phi^{-1}(0.58) = 5 + 3 \cdot 0.2019 = 5.6057$$

3. We solve for  $c$  using the quantile function,

$$\begin{aligned} 0.95 &= \mathbb{P}(-c \leq Z \leq c) = \Phi(c) - \Phi(-c) \\ &\Leftrightarrow 0.95 = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1 \\ &\Leftrightarrow 0.975 = \Phi(c) \\ &\Leftrightarrow c = \Phi^{-1}(0.975) = 1.96 \end{aligned}$$

## 2 Proofs of Key Results

**Problem 2.1.** Prove Theorem 1 in the simpler case when  $F_X$  is invertible.

**Solution 2.1.** Let  $F_Y$  denote the CDF of the random variable  $Y = F_X^{-1}(U)$ . Then,

$$F_Y(x) = \mathbb{P}(F_X^{-1}(U) \leq x) = \mathbb{P}(F_X(F_X^{-1}(U)) \leq F_X(x)) = \mathbb{P}(U \leq F_X(x)).$$

Furthermore, if  $U \sim U(0, 1)$  then

$$F_Y(x) = \mathbb{P}(U \leq F_X(x)) = \int_0^{F_X(x)} t \, dt = F_X(x).$$

The random variable  $Y = F_X^{-1}(U)$  has the CDF  $F_X$ , as desired.

**Problem 2.2.** If  $F_X$  is a CDF, then its quantile function  $F_X^{-1}$  satisfies

$$F_X^{-1}(p) \leq x \Leftrightarrow p \leq F_X(x)$$

**Solution 2.2.** The proof relies on the fact that  $F_X^{-1}(p)$  is the infimum of all  $\{t : F_X(t) \geq p\}$ , and therefore smaller than (or equal to) any  $x \in \{t : F_X(t) \geq p\}$ .

( $\Rightarrow$ ) Suppose that  $F_X^{-1}(p) \leq x$ . This implies that  $x \in \{t : F_X(t) \geq p\}$  so  $p \leq F_X(x)$ .

( $\Leftarrow$ ) Suppose that  $p \leq F_X(x)$ . This implies that  $x \in \{t : F_X(t) \geq p\}$  so  $F_X^{-1}(p) \leq x$ .

**Problem 2.3.** Prove Theorem 1.

**Solution 2.3.** Let  $F_Y$  denote the CDF of the random variable  $Y = F_X^{-1}(U)$ . Using the properties of the quantile function (Problem 2.2) that

$$F_X^{-1}(p) \leq x \Leftrightarrow p \leq F_X(x).$$

So we can conclude that

$$\{F_X^{-1}(U) \leq x\} = \{U \leq F_X(x)\}$$

Therefore, the CDF of  $Y$  is

$$F_Y(x) = \mathbb{P}(F_X^{-1}(U) \leq x) = \mathbb{P}(U \leq F_X(x)) = F_X(x).$$

**Problem 2.4.** Prove the following properties for the quantile function

1. For all  $x \in \mathbb{R}$ ,  $F_X^{-1}(F_X(x)) \leq x$
2. For all  $p \in [0, 1]$ ,  $F_X(F_X^{-1}(p)) \geq p$
3.  $F_X^{-1}(p) \leq x \Leftrightarrow p \leq F_X(x)$
4.  $F_X^{-1}(p)$  is non-decreasing and left-continuous (except for the endpoints  $p = 0$  or  $p = 1$ )

**Solution 2.4.**

1. We have

$$F_X^{-1}(F_X(x)) = \inf_{t \in \mathbb{R}} \{F_X(t) \geq F_X(x)\} \leq x$$

since  $x \in \{t \in \mathbb{R} : F_X(t) \geq F_X(x)\}$ .

2. Since  $F_X$  is right continuous and increasing we have  $\{F_X(x) \geq p\}$  is a closed set, so it attains its infimum. Therefore,  $c_p \in \{F_X(x) \geq p\}$  so

$$F_X(F_X^{-1}(p)) = F_X(c_p) \geq p.$$

3. This was shown in Problem 2.2.

4. Suppose that  $p_1 \leq p_2$ . Then

$$F_X^{-1}(p_1) = \inf_{x \in \mathbb{R}} \{F_X(x) \geq p_1\} \leq \inf_{x \in \mathbb{R}} \{F_X(x) \geq p_2\} = F_X^{-1}(p_2)$$

since  $\{F_X(x) \geq p_1\} \subseteq \{F_X(x) \geq p_2\}$ , so  $F_X^{-1}$  is non-decreasing.

To see left continuity, notice that monotone functions can only have jump discontinuities, so it suffices to show that  $\sup_{q < p} F_X^{-1}(q) = F_X^{-1}(p)$ . For each  $q < p$  and  $\epsilon > 0$ , we have by definition of the supremum

$$\sup_{q < p} F_X^{-1}(q) + \epsilon \geq F_X^{-1}(q) \xrightarrow{(3)} F_X(\sup_{q < p} F_X^{-1}(q) + \epsilon) \geq q.$$

So taking  $\epsilon \rightarrow 0$  by right continuity of  $F_X$  implies that  $F_X(\sup_{q < p} F_X^{-1}(q)) \geq q$  for all  $q < p$  so  $F_X(\sup_{q < p} F_X^{-1}(q)) \geq p$ . Property 3 above implies that

$$\sup_{q < p} F_X^{-1}(q) \geq F_X^{-1}(p).$$

This combined with monotonicity  $\sup_{q < p} F_X^{-1}(q) \leq F_X^{-1}(p)$  implies that  $\sup_{q < p} F_X^{-1}(q) = F_X^{-1}(p)$  as required.