1 Set Notation

Let A, B be sets.

1. *Element*: $x \in A$ means that x is in the set A.
2. *Empty set*: The empty set $\emptyset$ denotes the set with no elements in it.
3. *Complement*: $A^c = \{x \mid x \in \Omega, \ \ x \notin A\}$
4. *Cardinality:* $|A|$ is the number of elements in the set A
5. *Union*: $A \cup B = \{x \mid x \in A \ \ or \ \ x \in B\}$
6. *Intersection*: $A \cap B = \{x \mid x \in A \ \ and \ \ x \in B\}$
7. *Set Difference*: $A \setminus B = A \cap B^c = \{x \mid x \in A \text{ and } x \notin B\}$
8. *Disjoint*: Two events A and B are said to be disjoint if $A \cap B = \emptyset$.

Remark 1. Set theory is the language we use to describe events. Let $\omega \in \Omega$ be the outcome of an event, then

1. *Element*: $\omega \in A$ means that A has occurred
2. *Complement*: $\omega \in A^c$ means that A has not occurred
3. *Union*: $\omega \in A \cup B$ means that A or B has occurred
4. *Intersection*: $\omega \in A \cap B$ means that A and B has occurred
5. *Disjoint*: A and B are disjoint if it is impossible for both A and B to occur.

1.1 Set Operations

1. **Commutativity:** $A \cup B = B \cup A, \quad A \cap B = B \cap A$
2. **Associativity:** $(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$
3. **Distributivity:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. **DeMorgan's Laws:** $(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$

2 Counting Techniques

2.1 Addition Rule

Intuitive Definition: Suppose we want to either do Task 1 **or** Task 2. If there are p ways to do Task 1 and q ways to do Task 2, then there is a total of $p+q$ ways of doing the Task.

Mathematical Definition: If A and B are disjoint then the size of all distinct elements from A **or** B is

$$|A \cup B| = |A| + |B|.$$

From this fundamental counting techniques we can recover many fundamental counting techniques for the concatenation of events.

1. **Complement:**
$$|A^c| = |\Omega| - |A|$$
2. **Inclusion Exclusion:** If A and B are not necessarily disjoint then
$$|A \cup B| = |A| + |B| - |A \cap B|$$
or more generally,
$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n-1} \sum_{i<\cdots<n} |\cap_{i=1}^n A_i|.$$

2.2 Multiplication Rule

Intuitive Definition: Suppose we want to do Task 1 **and** Task 2. If there are p ways to do Task 1 (no matter how we do Task 2) and q ways to do Task 2 (no matter how we do Task 1) then there are $p \cdot q$ ways to do Task 1 and Task 2.

Mathematical Definition: The size of all ordered pairs (a, b) of elements $a \in A$ **and** $b \in B$ is

$$|A \times B| = |A| \cdot |B|.$$

From this fundamental counting techniques we can recover many fundamental counting techniques for sequences (either ordered or unordered).

1. **Partial Permutations:** Given n distinct objects, a *partial permutation* of size k is an *ordered* subset of k objects. The number of partial permutations of size k taken from n objects is given by the *falling factorial*
$$n^{(k)} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$
If $k = n$, then this formula gives the number of *permutations* or arrangements of a set.
2. **Combinations:** Given n distinct objects, a *combination* of size k is an *unordered* subset of k of the individuals. The number of combinations of size k taken from n objects is given by the *binomial coefficient*
$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{(n-k)!k!}.$$
3. **Repeated Arrangements:** Consider n objects which consist of k types. Suppose that there are n_1 objects which are of type 1, n_2 which are of type 2, and in general n_i objects of type i. The number of distinguishable arrangements of the n objects is given by the *multinomial coefficient*
$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \ldots n_k!}$$

2.2.1 General Tips to Count Complicated Objects

- Find a bijection of the sample space Ω with an easier object to count.
- To avoid over counting, if the words "more than", "at most", "fewer than" are used in the problem, it is safer to break the problem into cases.
- Sometimes counting the complement of an event is easier, especially if we want to count many different cases. You can use the fact that $|A| = |\Omega| - |A^c|$.
- When computing uniform probabilities, make sure to pick the sample space with equally likely outcomes if there are more than 1 natural choice for the sample space.

2.3 Sampling from Urns

We define some terminology related to drawing objects uniformly at random balls from an urn. Mathematically, this is a way to generate uniformly at random subsets or tuples of fixed size from a set of objects. There are two standard ways of generating samples from this urn.

Example 1 (Sampling with versus without replacement)**.** Suppose you have an urn with n distinct balls, and you select $k \leq n$ balls in order. You can do that either

1. "**with** replacement": Every time an object is selected, it is put back into the pool of possible objects. This is sampling uniformly from $\{1, \dots, n\}^k = [n]^k$.
2. "**without** replacement": Once an object is selected, it stays out of the pool of possible objects. This is sampling a subset or tuple of size k uniformly from the set $\{1, \dots, n\}$.

Furthermore, there are two main ways of encoding the outcomes of the sample

Example 2 (Sets versus Tuples)**.** Suppose you have an urn with n distinct numbers, say $[n] = \{1, 2, \dots, n\}$, and you select k numbers of them without replacement

- If you **do consider the order**, the sample space Ω consists of tuples, and there are

  $$n^{(k)} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

  elements in the sample space. For example, if $n = 10$ and $k = 3$, Ω is of the form

  $$\{(1,2,3), (2,1,3), (1,3,2), (1,2,4), \dots\}$$

  and note that $(1,2,3)$ and $(2,1,3)$ are counted as two different outcomes.

  Remark 2. Often, assigning distinct labels to the objects has the same effect as ordering.

- If you **do not care about the order**, the sample space Ω consists of sets, and there are

  $$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{(n-k)!k!}.$$

  terms in the sample space. For example, if $n = 10$ and $k = 3$, then Ω is of the form

  $$\{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \dots\}$$

  and note that we are counting $\{1,2,3\}$ and $\{2,1,3\}$ as the same outcome.

2.4 Properties of Factorials and Binomial Coefficients

1. $$n^{(k)} = n(n-1)^{(k-1)}$$
   for $k \geq 1$

2. $$\binom{n}{k} = \frac{n^{(k)}}{k!}$$

3. $$\binom{n}{k} = \binom{n}{n-k}$$
   for $k \geq 0$

4. $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

5. Binomial theorem:
   $$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

6. Sterling's formula:
   $$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

7. $\binom{n}{k}$ is equal to the kth entry (starting from 0) in the nth row of **Pascal's triangle**

n \ k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

2.5 Example Problems

2.5.1 Addition Rule

Problem 2.1. Suppose I have 3 pairs of pants and 4 pairs of shorts. If I need to pick either a pair of pants or a pair of shorts, how many possible picks are there?

Solution 2.1. Denote by P_i for $i = 1, 2, 3$ my pants and by S_j for $j = 1, 2, 3, 4$ my shorts. Then my picks are in the set
$$\{P_1, P_2, P_3, S_1, S_2, S_3, S_4\}$$
with 7 elements.

Problem 2.2. Suppose two six sided die are rolled, how many outcomes would result in the sum of the die rolls being larger than 8?

Solution 2.2. Denote by A_j the event "the sum is j" for $j = 2, \dots, 12$ (e.g., $A_{12} = \{(6,6)\}$). Denote by B the event "the sum is larger than 8". The A_j are disjoint and

$$B = \bigcup_{j=9}^{12} A_j = A_9 + A_{10} + A_{11} + A_{12}$$

so that

$$|B| = \sum_{j=9}^{12} |A_j| = |A_9| + |A_{10}| + |A_{11}| + |A_{12}| = 4 + 3 + 2 + 1 = 10.$$

2.5.2 Multiplication Rule

Problem 2.3. Suppose I have 4 different pairs of pants and 6 different shirts, and I need to pick one pair of pants and one shirt. How many options do I have?

Solution 2.3. All pants and shirt combinations can be of the form

$$\{P_1, \dots, P_4\} \times \{S_1, \dots, S_6\}$$

By the multiplication rule, there are

$$4 \cdot 6 = 24$$

combinations of such sequences.

Problem 2.4. Let $\Omega = \{a_1, \dots, a_n\}$ be a finite sample space. Count the number of subsets of Ω.

Solution 2.4. The number of subsets is a difficult problem to count directly, so we will find a bijection between an object that is much easier to count. If we knew if any particular element was in a A or not, then we can recover what the subset is. To this end, given a subset A, we can write $a_i = 1$ if $a_i \in A$ and $a_i = 0$ if $a_i \notin A$, which encodes every subset of Ω by a sequence of the form $\{0,1\}^n$. By the multiplication rule, there are

$$|\{0,1\}^n| = 2^n$$

such encodings. Since there is a bijection between the number of subsets and the number of ordered sequences we can conclude that there are 2^n subsets.

Remark 3. This fact can be used to prove that the sum of the rows of Pascal's Triangle is

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

One simply recognizes the left hand side as the sum of subsets of $[n]$ of size $k = 0, k = 1, \dots k = n$. This combinatorial proof technique is called *double counting.*

Problem 2.5. Show that the number of ways to draw k distinct balls from an urn of size n without replacement is

1. (if order matters)

$$n^{(k)} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

2. (if order does not matter)
$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{(n-k)!k!}.$$

using the multiplication rule.

Solution 2.5.

1. Our draws from the urn are encoded by strings of length k of the form
$$\_\ \_\ \_\ \cdots\ \_$$
where there are no repeated objects (since we are sampling without replacement). There are n ways to fill in the first blank, $n-1$ ways to fill in the second blank (since we can't repeat the object in the first blank), $n-2$ ways to fill in the third blank (since we can't repeat the object in the first or second blank), etc. Therefore, by induction and the multiplication rule, we see that there are
$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$
ordered draws.

   Remark 4. Notice that the sample space from this experiment is also an ordered subset of size k from n distinct objects, which gives us the derivation for the formula for a permutation.

2. By above, there are
$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$
ordered subsets of the form
$$\_\ \_\ \_\ \cdots\ \_$$
However, we are over counting by a factor of the number of orderings of strings of length k. By part 1, there are $k!$ different arrangements of a string of k distinct objects. Normalizing the total number of ordered subsets by this amount gives
$$\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{(k)}}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$
unordered draws.

   Remark 5. Notice that the sample space from this experiment is also an unordered subset of size k from n distinct objects, which gives us the derivation for the formula for a combination.

Problem 2.6. Find the number of non-negative integer solutions that sum to n. That is, find the number of solutions to
$$x_1 + \cdots + x_k = n \qquad x_i \geq 0,\ x_i \in \mathbb{Z}.$$

Solution 2.6. We can find a bijection between the number of integer solutions with a simpler object called stars and bars. A solution to the equation $x_1 + \cdots + x_k = n$, can be expressed as a sequence of the form
$$(\underbrace{*\cdots*}_{x_1}|\underbrace{*\cdots*}_{x_2}|\ldots|\underbrace{*\cdots*}_{x_k})$$

where the $|$ are separations between the number of terms. Notice that such solutions always have n " $*$ " terms and $k-1$ " $|$ " terms. There are

$$\frac{(n+k-1)!}{n!(k-1)!} = \binom{n+k-1}{k-1} \tag{1}$$

distinguishable arrangements of such sequences using the multinomials.

Problem 2.7. Find the number of strictly positive integer solutions to that sum to n. That is, find the number of solutions to

$$x_1 + \cdots + x_k = n \qquad x_i > 0,\ x_i \in \mathbb{Z}.$$

Solution 2.7. We can find a bijection between the number of integer solutions with a simpler object called stars and bars. A solution to the equation $x_1 + \cdots + x_k = n$, can be expressed as a sequence of the form

$$(\underbrace{* \cdots *}_{x_1} | \underbrace{* \cdots *}_{x_2} | \ldots | \underbrace{* \cdots *}_{x_k})$$

where the $|$ are separations between the number of terms. Notice that such solutions always have n " $*$ " terms and $k-1$ " $|$ " terms. Furthermore, there cannot be a $|$ as the first or last term in the sequences and no two $|$ can appear consecutively since $x_i > 0$. Another way of saying this is that the " $|$ " terms can be placed in between any $*$ terms. There are $n-1$ viable spots to place the " $|$ " and there are $k-1$ " $|$ " terms in total, so there are

$$\binom{n-1}{k-1} \tag{2}$$

such sequences.

Alternative Solution: We can subtract k from both sides to see that the number of solutions to

$$x_1 + \cdots + x_k = n \qquad x_i > 0,\ x_i \in \mathbb{Z}$$

is equivalent to the number of solutions to

$$(x_1 - 1) + \cdots + (x_k - 1) = n - k \qquad x_i > 0,\ x_i \in \mathbb{Z}$$

Using the change of variables $y_i = x_i - 1$ and noticing that $x_i > 0 \implies y_i \geq 0$ means it suffices to solve

$$y_1 + \cdots + y_k = n - k \qquad y_i \geq 0,\ y_i \in \mathbb{Z}.$$

Therefore, using the solution to the previous problem (1) we have

$$\frac{((n-k)+k-1)!}{n!(k-1)!} = \binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$$

total solutions.

Problem 2.8. At an icecream parlor, there are flavors chocolate, vanilla, strawberry, and mango. How many ways can I order 3 scoops?

Solution 2.8. The 3 scoops of icecream can be encoded by stars and bars. We have bins

$$(\underbrace{\quad}_{\text{chocolate}} | \underbrace{\quad}_{\text{vanilla}} | \underbrace{\quad}_{\text{strawberry}} | \underbrace{\quad}_{\text{mango}})$$

and our selection can be encoded by picking 3 scoops or balls into each bin. For example, two scoops of chocolate and 1 scoop of mango looks like

$$\{2,0,0,1\} = (\underbrace{**}_{\text{chocolate}} | \underbrace{\quad}_{\text{vanilla}} | \underbrace{\quad}_{\text{strawberry}} | \underbrace{*}_{\text{mango}}) = (* \; * \mid \mid \mid *)$$

Therefore, every choice of scoops can be encoded by a permutation of sets of the form

$$\underbrace{(* \; * \mid\mid\mid *)}_{\{2,0,0,1\}}, \underbrace{(* \mid * \mid * \mid)}_{\{1,1,1,0\}}, \underbrace{(\mid * \; * \; * \mid\mid)}_{\{0,3,0,0\}}, \text{ etc}$$

where there are 3 $*$ terms and 3 $|$ terms. There are

$$\frac{6!}{3!3!} = 20$$

arrangements of such sequences, and thus 20 different choices.

Alternative Solution: Let $x_1, x_2, \ldots, x_4$ denote the number of scoops of icecream. This problem is equivalent to finding the number of integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 3 \qquad x_i \geq 0.$$

By equation 1 (with $n = 3$ and $k = 4$) the number of solutions are given by

$$\frac{6!}{3!3!} = 20.$$

Problem 2.9. How many ways can you put 10 coloured balls into a cup such that the cup contains exactly 4 different colors.

Solution 2.9. The number balls of a certain color can be encoded by stars and bars. We have bins

$$(\underbrace{\quad}_{\text{colour \#1}} | \underbrace{\quad}_{\text{colour \#2}} | \underbrace{\quad}_{\text{colour \#3}} | \underbrace{\quad}_{\text{colour \#4}})$$

and we have to assign 10 balls to each color (such that there is at least 1 ball in each bin). For example, the assignment 6 balls of the first color, 2 of the second, 1 of the third and 1 of the fourth can be encoded by

$$\{6,2,1,1\} = (* \; * \; * \; * \; * \; * \mid * \; * \mid * \mid *).$$

Every assignment of balls can be encoded by such sequences with 10 $*$'s and 3 $|$'s such that $|$'s do not appear as the first or least term, or consecutively (since we must have exactly 4 colors). Another way of saying this is that the " $|$ " terms can be placed in between any $*$ terms. There are 9 viable spots to place the 3 " $|$ "'s so there are

$$\binom{9}{3} = 84$$

distinct cups we can make.

Alternative Solution: Let $x_1, x_2, \ldots, x_4$ denote the number of balls of each color. This problem is equivalent to finding the number of integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 10 \qquad x_i > 0.$$

By equation 2 (with $n = 10$ and $k = 4$) the number of solutions are given by

$$\binom{9}{3} = 84.$$

2.5.3 Uniform Probabilities

Problem 2.10. Suppose that three of the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are selected at random without replacement, and then put together the order they are drawn to form a three digit number. Consider the events $A =$"number is larger than 500" and $B =$"number is even". Compute $\mathbb{P}(A)$, $\mathbb{P}(B)$ and $\mathbb{P}(A \cap B)$.

Solution 2.10. The sample space Ω consists of $9 \times 8 \times 7 = 504$ equally likely outcomes.

Probability of A: We count the number of outcomes in Ω that numbers larger than 500,

- First digit must be 5,6,7,8 or 9 (5 options)
- Second digit can be any of the remaining 8 (8 options)
- Third digit can be any remaining 7 (7 options)

Therefore $|A| = 5 \times 8 \times 7 = 280$, so $\mathbb{P}(A) = \frac{|A|}{|S|} = \frac{280}{504} = \frac{5}{9} = 0.556$.

Probability of B: We count the number of outcomes in Ω that are even,

- Last digit must be 2,4,6 or 8 (4 options)
- First digit can be any of the remaining 8 (8 options)
- Second digit can be any remaining 7 (7 options)

Therefore, $|B| = 4 \times 8 \times 7 = 224$ so $\mathbb{P}(B) = \frac{|B|}{|S|} = \frac{224}{504} = \frac{4}{9} = 0.444$

Probability of $A \cap B$: How many outcomes in Ω are even numbers larger than 500? We use the addition rule and consider two ways:

- Way 1:
  - First digit is odd , i.e. 5,7,9 (3 options)
  - Then, last digit can be $2, 4, 6$ or 8 (4 options)
  - and the middle digit can be any of the 7 remaining numbers (7 options).

  So there are $3 \times 4 \times 7 = 84$ options in Case 1.
- Way 2:
  - First digit is even, i.e. 6 or 8 (2 options).
  - Then, last digit can be any of the remaining 3 even numbers (3 options)
  - while the second digit can be any of the 7 remaining numbers (7 options).

  So there are $2 \times 3 \times 7 = 42$ options in Case 2.

The event $A \cap B =$" even number larger than 500" can then be achieved in Way 1 *or* in Way 2. By the addition rule, $|A \cap B| = 84 + 42 = 126$. Thus, $\mathbb{P}(A \cap B) = \frac{126}{504} = 0.25$

Problem 2.11. Melissa participates in a lottery in which she selects 7 numbers between 1 and 50, and then a computer randomly picks 7 numbers between 1 and 50 (without replacement). She wins if her selected numbers match 5 or more of the randomly selected numbers, in any order. What is the probability that Melissa wins?

Solution 2.11. Without loss of generality, suppose that Melissa picks the numbers $\{1, \dots, 7\}$. The computer gives us a unordered subset of 7 numbers sampled uniformly. In particular, Ω is the subsets of $[50]$ of length 7 and the outcomes are *equally likely.* We break the problem into cases

1. Exactly 5 matching numbers: There are $\binom{7}{5}$ ways to pick 5 matching numbers and $\binom{43}{2}$ ways to pick the remaining numbers, so there are
$$\binom{7}{5}\binom{43}{2}$$
combinations that match exactly 5 numbers .

2. Exactly 6 matching numbers: There are $\binom{7}{6}$ ways to pick 6 matching numbers and $\binom{43}{1}$ ways to pick the remaining numbers, so there are
$$\binom{7}{6}\binom{43}{1}$$
combinations that match exactly 6 numbers .

3. Exactly 7 matching numbers: There are $\binom{7}{7}$ ways to pick 7 matching numbers and $\binom{43}{0}$ ways to pick the remaining numbers, so there are
$$\binom{7}{7}\binom{43}{0}$$
combinations that match exactly 5 numbers .

By the addition rule and the fact $|S| = \frac{50}{7}$,

$$\mathbb{P}(\text{Melissa wins}) = \frac{\binom{7}{5}\binom{43}{2} + \binom{7}{6}\binom{43}{1} + \binom{7}{7}\binom{43}{0}}{\binom{50}{7}}$$

Remark 6. It is very easy to over count. It might be tempting to argue that the number of combinations is given by
$$\binom{7}{5} \cdot \binom{45}{2}$$
since you can first pick 5 of the 7 winning numbers, then pick 2 from remaining 45 numbers. However, this method double counts certain combinations, since $\{1, 2, 3, 4, 5, 6, 7\}$ and $\{7, 2, 3, 4, 5, 6, 1\}$ can both be generated by picking 5 winning numbers followed by two numbers from the remainder, but they correspond to the same set. We are partially keeping track of order when we count objects in this way. It is important to make sure that the pools we draw from do not depend on the results of earlier draws. So for events with words “more than”, “at most”, “fewer than”, it is always better to break the problem down into cases.

Problem 2.12. Consider rearranging the letters at random in the word “HELLOKITTY” to form a single word.

1. How many ways can this be done?
2. What is the probability that all of the letters appear in alphabetic order?
3. What is the probability that the word begins and ends with “T”?

Solution 2.12.

1. There are 10 letters, out of which 2 are Ts and 2 are Ls and the remaining 6 are distinct. We have
$$\frac{10!}{2!2!}$$
rearrangements using multinomials.

2. There are $\frac{10!}{2!2!}$ equally likely outcomes, and only one of these is such that the letters are in alphabetical order. Thus,
$$\mathbb{P}(\text{alphabetical}) = \frac{1}{\frac{10!}{2!2!}}$$

3. If the first and last letter are Ts, we we want to count words of the following form
$$T\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ T$$
There are 8 letters for the 8 empty spots, 2 of which are Ls. Thus, there are $\frac{8!}{2!}$ possibilities to have the word start and end with T using multinomials, so
$$\mathbb{P}(\text{starts and ends with T}) = \frac{\frac{8!}{2!}}{\frac{10!}{2!2!}}$$

Problem 2.13. Suppose that each of the n people in a room is equally likely to have any of the 365 days of the year as their birthday, so that all possible combinations of birthdays are equally likely. What is the probability at least two people in the room of n people share a birthday?

Solution 2.13. Let A be the event that at least 2 people in the room share the same birthday. To count this directly, we would have to split this into cases when 2 people share the same birthday, 3 people share the same birthday, etc. In this problem, it is much easier to compute the complement A^c, which is the event that everyone has a different birthday.

The sample space of birthdays is given by $\Omega = [365]^n$, so $|\Omega| = 365^n$ by the multiplication rule. If everyone has a different birthday, then we want to count the number of tuples with n distinct elements,
$$|A^c| = 365^{(n)}.$$
Therefore,
$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{365^{(n)}}{356^n}.$$

Remark 7. Let A_n be the event that at least 2 people in the room of n share the same birthday. For $n \in \{100, 30, 23\}$ the formula we computed implies that
$$\mathbb{P}(A_{100}) = .9999997, \quad \mathbb{P}(A_{30}) = .7063 \quad \mathbb{P}(A_{23}) = .5073.$$

Problem 2.14. There are 6 stops on a subway line and 4 passengers on a subway car. Assume the passengers are each equally likely to get off at any stop. Find the probability that

1. the passengers all get off at different stops,
2. 2 passengers get off at stop two and 2 passengers get off at stop five,
3. 2 passengers get off at one stop and the other 2 passengers get off at another same stop.

Solution 2.14. Each of the 4 passengers can choose any of the 6 stops, so the sample space is

$$\Omega = [6]^4 = \{(x_1, x_2, x_3, x_4) : x_i \in \{1, \dots, 6\} \text{ for } i = 1, \dots, 4\}$$

with $|S| = 6^4$. For instance, the outcome $(1, 1, 1, 1)$ means everybody gets off at Stop 1. Furthermore, the sample space Ω has *equally likely* outcomes by assumption.

1. For the stops to be all different, we have 6 options for person 1, then 5 options for person 2, then 4 options for person 3 and 3 options for person 4, giving $6^{(4)} = 6 \cdot 5 \cdot 4 \cdot 3$ outcomes. The probability is then
$$\frac{6^{(4)}}{6^4}.$$
2. We have the stops 2, 2, 5, 5 to distribute among the 4 passengers. Since the 2s and 5s are repeated, there are $\frac{4!}{2!2!}$ outcomes, so the probability is
$$\frac{\frac{4!}{2!2!}}{6^4}.$$
3. We can pick the two stops in $\binom{6}{2}$ ways and then proceed as in b) to get the probability
$$\binom{6}{2} \cdot \frac{\frac{4!}{2!2!}}{6^4}.$$

Remark 8. The probability is uniform over the sample space $\Omega = [6]^4$ because each person is equally likely to get off at any stop. It is a common mistake to pick the sample space $\tilde{S}$ given by tuples of 6 natural numbers that sum up to 4 which encodes how many passengers got off at each stop. For instance, the outcome

$$(0, 0, 1, 0, 0, 3)$$

means that 1 passenger got off at stop 3 and 3 passengers got off at stop 6.

The sample space $\tilde{S}$ a valid sample space, but the probability is *not* uniform over this sample space $\tilde{S}$. This sample space aggregates where the passengers got off, but it contains no information about where individual passenger got off (which is what we assumed was uniform in the problem). Intuitively, it is much less likely for all the passengers to get off at stop 1 compared to passengers getting off at stops staggered stops like 1, 2, 3, 4 so the probability is clearly not uniform.

Problem 2.15. Six digits from $2, 3, 4, \dots, 8$ are chosen and arranged in a row without replacement. Find the probability that the digits 2 and 3 appear consecutively in the proper order (that is, in the order 23).

Solution 2.15. Our sample space Ω consists of ordered tuples of length 6 from the set $[8]$, so $|S| = 7^{(6)}$.

We consider the various cases. Suppose that 2 and 3 appear consecutively in the first spot

$$2\ 3\ \_\ \_\ \_\ \_$$

There are $5^{(4)}$ choices for the blanks (since the numbers 2 and 3 are taken). There are 5 cases to consider (since the 2 location can appear as the 1 to 5th digit) for 2 and 3 to be fixed such that they are consecutive,

$$2\ 3\ \_\ \_\ \_\ \_, \quad \_\ 2\ 3\ \_\ \_\ \_, \quad \dots, \quad \_\ \_\ \_\ \_\ 2\ 3$$

There are 5 cases and $5^{(4)}$ terms in each case by symmetry, so the probability is

$$\frac{5 \cdot 5^{(4)}}{7^{(6)}}.$$