Calculus of Variations

Problem 1. (Brachistochrone) We need to construct the fastest slide from $(0,0)$ to (a,h) (the y-axis points downwards). If $u(x)$ describes the shape of the slide, then

$$T[u] = \int_0^a \frac{1}{\sqrt{2gu(x)}}\sqrt{1+(u'(x))^2}\,dx.$$

We want to find the $u(x)$ that minimizes T.

1. Find the Euler–Lagrange equation for $T[u]$.
2. Find a minimizer of $T[u]$ satisfying $u(0)=0$, $u(a)=h$.

Solution 1. We want to find the minimizer of

$$T[u] = \int_0^a \frac{1}{\sqrt{2gu(x)}}\sqrt{1+(u'(x))^2}\,dx,$$

over the space of smooth functions

$$\Omega = \{u \in C^\infty([0,a]) : u(0)=0,\ u(a)=h\}.$$

The Euler–Lagrange Equations: We begin by finding some critical point conditions for the minimizer of $T[u]$. If v is a smooth function such that $v(0)=v(a)=0$, then

$$u+\epsilon v \in \Omega$$

since $u(0)+\epsilon v(0)=0$ and $u(a)+\epsilon v(a)=h$. If u is a minimizer of T, then

$$\frac{d}{d\epsilon}T[u+\epsilon v]\Big|_{\epsilon=0} = 0$$

for all v such that $v(0)=v(a)=0$. Consider the function

$$T[u+\epsilon v] = \int_0^a \frac{1}{\sqrt{2gu+2g\epsilon v}}\sqrt{1+(u'+\epsilon v')^2}\,dx.$$

Taking the derivative, we have

$$\begin{aligned}\frac{d}{d\epsilon}T[u+\epsilon v]\Big|_{\epsilon=0} &= \int_0^a \bigg(-\frac{1}{2}\frac{2gv\sqrt{1+(u'+\epsilon v')^2}}{(2g(u+\epsilon v))^{3/2}} + \frac{1}{2}\frac{2(u'+\epsilon v')v'}{\sqrt{2gu+2g\epsilon v}\sqrt{1+(u'+\epsilon v')^2}}\bigg)\Big|_{\epsilon=0}\,dx \\ &= \int_0^a -\frac{1}{2}\frac{2gv\sqrt{1+(u')^2}}{(2gu)^{3/2}} + \frac{1}{2}\frac{2u'v'}{\sqrt{2gu}\sqrt{1+(u')^2}}\,dx.\end{aligned}$$

We simplify this by integrating the second term by parts to remove the v' term

$$\begin{aligned}\frac{d}{d\epsilon}T[u+\epsilon v]\Big|_{\epsilon=0} &= -\int_0^a \frac{1}{2}\bigg(\frac{2g\sqrt{1+(u')^2}}{(2gu)^{3/2}} + \frac{d}{dx}\frac{2u'}{\sqrt{2gu}\sqrt{1+(u')^2}}\bigg)v\,dx + \frac{2u'v}{\sqrt{2gu}\sqrt{1+(u')^2}}\bigg|_0^a \\ \scriptstyle v(0)=v(a)=0 \quad &= -\int_0^a \bigg(\frac{g\sqrt{1+(u')^2}}{(2gu)^{3/2}} + \frac{d}{dx}\frac{u'}{\sqrt{(2gu)(1+(u')^2)}}\bigg)v\,dx.\end{aligned}$$

If this were equal to 0 for all v, then we must have

$$\frac{g\sqrt{1+(u')^2}}{(2gu)^{3/2}} + \frac{d}{dx}\frac{u'}{\sqrt{(2gu)(1+(u')^2)}} = 0,$$

which is the Euler–Lagrange equations associated with the Lagrangian,

$$L(u,u') = (1+(u')^2)^{1/2}(2gu)^{-1/2}.$$

Solving the Euler–Lagrange equation: We now solve the ODE

$$\frac{g\sqrt{1+(u')^2}}{(2gu)^{3/2}} + \frac{d}{dx}\frac{u'}{\sqrt{(2gu)(1+(u')^2)}} = 0,$$

subject to the constraints $u(0) = 0$ and $u(a) = h$ to find an equation for the minimizer. To simplify computations, notice

$$\frac{d}{dx}\frac{(1+(u')^2)^{1/2}}{[2gu]^{1/2}} = -u'\frac{g(1+(u')^2)^{1/2}}{[2gu]^{3/2}} + \frac{u'u''}{[(2gu)(1+(u')^2)]^{1/2}} \tag{1}$$

so we can multiply both sides of our Euler–Lagrange equation by u',

$$u'\frac{g(1+(u')^2)^{1/2}}{(2gu)^{3/2}} + u'\frac{d}{dx}\frac{u'}{[(2gu)(1+(u')^2)]^{1/2}} = 0,$$

and substitute (1) to conclude

$$\begin{aligned}
0 &= -\frac{d}{dx}\frac{(1+(u')^2)^{1/2}}{[2gu]^{1/2}} + \frac{u'u''}{[(2gu)(1+(u')^2)]^{1/2}} + u'\cdot\frac{d}{dx}\frac{u'}{[(2gu)(1+(u')^2)]^{1/2}} \\
&= -\frac{d}{dx}\frac{(1+(u')^2)^{1/2}}{[2gu]^{1/2}} + \frac{d}{dx}u'\cdot\frac{u'}{[(2gu)(1+(u')^2)]^{1/2}} + u'\cdot\frac{d}{dx}\frac{u'}{[(2gu)(1+(u')^2)]^{1/2}} \\
&= -\frac{d}{dx}\frac{(1+(u')^2)^{1/2}}{[2gu]^{1/2}} + \frac{d}{dx}\Big(\frac{(u')^2}{[(2gu)(1+(u')^2)]^{1/2}}\Big).
\end{aligned}$$

Integrating both sides with respect to x, we can conclude that

$$-\frac{(1+(u')^2)^{1/2}}{[2gu]^{1/2}} + \frac{(u')^2}{[(2gu)(1+(u')^2)]^{1/2}} = C$$

for some constant C. Simplifying, we see that

$$-\frac{1}{[(2gu)(1+(u')^2)]^{1/2}} = C \implies u\bigg(1+\Big(\frac{du}{dx}\Big)^2\bigg) = \frac{1}{2gC^2} =: D.$$

Solving for $\frac{du}{dx}$, we arrive at the separable ODE,

$$\frac{du}{dx} = \Big(\frac{D-u}{u}\Big)^{1/2} \implies x = \int\Big(\frac{u}{D-u}\Big)^{1/2}\,du,$$

where $D > 0$. To compute this integral, we can use the change of variables $u = D\sin^2(\theta)$ for $0 < \theta < \pi/2$ to conclude

$$x(\theta) = \int\Big(\frac{\sin^2(\theta)}{1-\sin^2(\theta)}\Big)^{1/2} 2D\sin(\theta)\cos(\theta)\,d\theta = 2D\int\sin^2(\theta)d\theta = D\Big(\theta - \frac{1}{2}\sin(2\theta)\Big) + E.$$

Therefore, our parameterized general solution is given by

$$x(\theta) = D\Big(\theta - \frac{1}{2}\sin(2\theta)\Big) + E, \qquad y(\theta) = D\sin^2(\theta) = D\Big(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\Big) \qquad 0 \le \theta \le \pi/2.$$

Since $(0,0)$ must lie on the curve, we can set $E = 0$, which implies that $x(0) = y(0) = 0$. Similarly, since (a,h) lies on our curve, we can also solve for D by first finding the θ^* that solves the ratio $\frac{y(\theta^*)}{x(\theta^*)} = \frac{h}{a}$, then setting D such that $y(\theta^*) = h$,

$$\frac{h}{a} = \frac{\frac{1}{2} - \frac{1}{2}\cos(2\theta^*)}{\theta^* - \frac{1}{2}\sin(2\theta^*)} \implies D = \frac{h}{\frac{1}{2} - \frac{1}{2}\cos(2\theta^*)}.$$

Problem 2. The area of the surface $z = z(x,y)$ is

$$S[u] = \iint_D \sqrt{1 + u_x^2 + u_y^2}\, dxdy$$

where $(x,y) \in D$ is an projection of the surface. Write Euler–Lagrange equation of the surface of the minimal area (with boundary conditions $u(x,y) = \varphi(x,y)$ for $(x,y) \in \partial D$).

Solution 2. We want to find the minimizer of

$$S[u] = \iint_D \sqrt{1 + u_x^2 + u_y^2}\, dxdy,$$

over the space of smooth functions

$$\Omega = \{u \in C^\infty(D) : u(x,y) = \varphi(x,y)\ \forall (x,y) \in \partial D\}.$$

The Euler–Lagrange Equations: We begin by finding some critical point conditions for the minimizer of $S[u]$. If $v \in C_c^\infty(D)$ is a smooth function such that $v(x,y) = 0$ on ∂D, then

$$u + \epsilon v \in \Omega$$

since $u(x,y) + \epsilon v(x,y) = \varphi(x,y)$ on ∂D. If u is a minimizer then,

$$\frac{d}{d\epsilon} S[u + \epsilon v]\Big|_{\epsilon=0} = 0$$

for all $v \in C_c^\infty(D)$. Consider the function

$$\begin{aligned} S[u + \epsilon v] &= \iint_D \sqrt{1 + (u + \epsilon v)_x^2 + (u + \epsilon v)_y^2}\, dxdy \\ &= \iint_D (1 + u_x^2 + 2\epsilon u_x v_x + \epsilon^2 v_x^2 + u_y^2 + 2\epsilon u_y v_y + \epsilon^2 v_y^2)^{1/2}\, dxdy. \end{aligned}$$

Taking the derivative, we have

$$\begin{aligned} \frac{d}{d\epsilon} S[u + \epsilon v]\Big|_{\epsilon=0} &= \iint_D \frac{1}{2} \frac{2u_x v_x + 2\epsilon v_x^2 + 2u_y v_y + 2\epsilon v_y^2}{(1 + u_x^2 + 2\epsilon u_x v_x + \epsilon^2 v_x^2 + u_y^2 + 2\epsilon u_y v_y + \epsilon^2 v_y^2)^{1/2}}\Big|_{\epsilon=0} dxdy \\ &= \iint_D \frac{u_x v_x + u_y v_y}{(1 + u_x^2 + u_y^2)^{1/2}}\, dxdy. \end{aligned}$$

We simplify this by integrating by parts to remove the v_x and v_y terms,

$$\begin{aligned} \frac{d}{d\epsilon} S[u + \epsilon v]\Big|_{\epsilon=0} &= -\iint_D \left(\frac{d}{dx} \frac{u_x}{(1 + u_x^2 + u_y^2)^{1/2}} + \frac{d}{dy} \frac{u_y}{(1 + u_x^2 + u_y^2)^{1/2}} \right) v\, dxdy \\ &\quad + \int_{\partial D} \left(\frac{u_x \nu_1 + u_y \nu_2}{(1 + u_x^2 + u_y^2)^{1/2}} \right) v\, dS \\ v\big|_{\partial D} = 0 \qquad &= -\iint_D \left(\frac{d}{dx} \frac{u_x}{(1 + u_x^2 + u_y^2)^{1/2}} + \frac{d}{dy} \frac{u_y}{(1 + u_x^2 + u_y^2)^{1/2}} \right) v\, dxdy. \end{aligned}$$

If this were equal to 0 for all v, then we must have

$$\frac{d}{dx} \frac{u_x}{(1 + u_x^2 + u_y^2)^{1/2}} + \frac{d}{dy} \frac{u_y}{(1 + u_x^2 + u_y^2)^{1/2}} = 0,$$

which is the Euler–Lagrange equation associated with the Lagrangian,

$$L(u, \nabla u) = \sqrt{1 + \|\nabla u\|^2}.$$