1 Birth and death processes

In this section, we present a commonly used model for **population dynamics**. These processes are a special case of CTMC where the only transitions are up one unit (birth) or down one unit (death).

Let $X(t)$ be the number of individuals in a population at time t. The state space in this setting is

$$S = \{0, 1, 2, \cdots\}.$$

If there are i individuals in the population, a new individual will be born with intensity $\lambda_i \geq 0$, in which case the population will increase by 1. Similarly, the intensity for a death is $\beta_i \geq 0$, in which case the population size decrease by 1. The infinitesimal generator matrix can be expressed as

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \\ \beta_1 & -(\lambda_1 + \beta_1) & \lambda_1 & 0 & \\ 0 & \beta_2 & -(\lambda_2 + \beta_2) & \lambda_2 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

which implies that

$$\begin{aligned} \mathbb{P}(X(t + \triangle t) = n | X(t) = n) &= 1 - (\beta_n + \lambda_n)\triangle t + o(\triangle t) \\ \mathbb{P}(X(t + \triangle t) = n + 1 | X(t) = n) &= \lambda_n \triangle t + o(\triangle t) \\ \mathbb{P}(X(t + \triangle t) = n - 1 | X(t) = n) &= \beta_n \triangle t + o(\triangle t). \end{aligned}$$

The transition matrix of the embedded DTMC is given by

$$\tilde{\boldsymbol{P}} = \begin{pmatrix} 0 & 1 & & & \\ \frac{\beta_1}{\lambda_1 + \beta_1} & 0 & \frac{\lambda_1}{\lambda_1 + \beta_1} & & \\ & \frac{\beta_2}{\lambda_2 + \beta_2} & 0 & \frac{\lambda_2}{\lambda_2 + \beta_2} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

Remark 1.1. We can interpret the rates λ_i and β_i as the rates of births and deaths. When the system is in state i, the next potential birth happens at the random time $T_i^{(b)}$, which is an exponential time with intensity λ_i. The next potential death happens after $T_i^{(d)}$, which is an exponential time with intensity β_i. These two times compete with each other, and at their minimum $T_i^{(b)} \wedge T_i^{(d)}$, the system changes to a new state. Thus, the sojourn time at state i is given by $T_i^{(b)} \wedge T_i^{(d)} \sim \mathrm{Exp}(\lambda_i + \beta_i)$ with the survival function

$$\mathbb{P}\left(T_i^{(b)} \wedge T_i^{(d)} > t\right) = e^{-(\lambda_i + \beta_i)t}.$$

This is consistent with the rate of the sojourn time that one can read off directly from the Q-matrix.

1.0.1 Stationary Distribution

In general, the stationary distribution of a birth and death process may not exist. For example, for the Poisson process, it is clear that $N(t) \to \infty$ and $t \to \infty$, so the Poisson process does not have a stationary distribution. This is because in the Poisson process there are no deaths. If the birth and death rates are balanced, then it might be possible that there might be an equilibrium.

Proposition 1.2

A birth and death process is positive recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_1 \lambda_0}{\beta_n \cdots \beta_2 \beta_1} < \infty.$$

This condition states the birth rates cannot be too large relative to the death rates. If we know that the birth and death process is positive recurrent, then we can solve for the stationary distribution.

Proposition 1.3

Suppose that

$$\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_1 \lambda_0}{\beta_n \cdots \beta_2 \beta_1} < \infty.$$

Then a stationary distribution exists, and it is given by

$$\pi_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_1 \lambda_0}{\beta_n \cdots \beta_2 \beta_1}}.$$

and

$$\pi_n = \frac{\lambda_{n-1} \cdots \lambda_1 \lambda_0}{\beta_n \cdots \beta_2 \beta_1} \pi_0 \qquad \text{for } n = 1, 2, 3, \ldots$$

Remark 1.4. Combining the results in Proposition 1.2 and Proposition 1.3, implies that if

$$\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_1 \lambda_0}{\beta_n \cdots \beta_2 \beta_1} < \infty,$$

then the limiting distribution exists and is equal to the unique stationary distribution by the fundamental limit theorem for CTMCs.

1.1 Special Cases

Here are a few important special cases of birth and death processes.

1. **Poisson process**: Here, $\lambda_i = \lambda > 0$ for $i = 0, 1, 2, \ldots$ and $\beta_i = 0$ for $i = 1, 2, \ldots$.
2. **Linear growth model**: Let $\lambda_i = i\lambda$ for $\lambda > 0$ and $\beta_i = i\beta$ for $\beta > 0$ and $i = 1, 2, \ldots$. Intuitively, all individuals in the population reproduce independently at the same rate λ and die independently at the same rate β.
   In particular, if $\beta_i = 0$ for $i = 1, 2, \ldots$, we call it as a **simple birth process**, i.e., all individuals in the population reproduce independently at the same rate λ, and there are no deaths.
3. **Population model with immigration**: Let $\lambda_i = i\lambda + \nu$ for $\lambda > 0$ and $\beta_i = n\beta$ for $\beta > 0$ and $i = 1, 2, \ldots$ Intuitively, individuals dies and reproduce with rates β and λ, respectively, as in the Linear growth model. At the same time, new individuals arrive at a constant rate $\nu > 0$.
4. **Markovian queueing models**: Suppose X_t denotes the number of people on line for some service. We assume that people arrive at a rate λ, i.e., the arrival number follows a Poisson process with rate λ. Customers are serviced at an exponential rate β.
   (a) **M/M/1 queue**: There is one server and the first person in line is being serviced. Then $\lambda_n = \lambda$ and $\beta_n = \beta$.
   (b) **M/M/k queue**: There are k servers and anyone in the first k positions in the line can be served. Then $\lambda_n = \lambda$ and
   $$\beta_n = \begin{cases} n\beta, & \text{if } n \le k \\ k\beta, & \text{if } n \ge k. \end{cases}$$

1.2 Example Problems

1.2.1 Proofs of Main Results

Problem 1.1. Find the transition matrix of the embedded DTMC of a birth and death process.

Solution 1.1. Recall that the formula for the transition matrix is

$$p_{ii=0} \qquad p_{ij} = \frac{q_{ij}}{-q_{ii}}$$

so

$$\tilde{\boldsymbol{P}} = \begin{pmatrix} 0 & 1 & & & \\ \frac{\beta_1}{\lambda_1+\beta_1} & 0 & \frac{\lambda_1}{\lambda_1+\beta_1} & & \\ & \frac{\beta_2}{\lambda_2+\beta_2} & 0 & \frac{\lambda_2}{\lambda_2+\beta_2} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

Problem 1.2. Find the stationary distribution of the birth and death process (Proposition 1.3)

Solution 1.2. If a stationary distribution $\boldsymbol{\pi}$ exists, it must satisfy the balance equation $\boldsymbol{\pi}Q = \boldsymbol{0}$, i.e.,

$$\begin{cases} \lambda_0\pi_0 - \beta_1\pi_1 = 0 \\ \lambda_0\pi_0 - (\lambda_1+\beta_1)\pi_1 + \beta_2\pi_2 = 0 \\ \vdots \\ \lambda_{n-1}\pi_{n-1} - (\lambda_n+\beta_n)\pi_n + \beta_{n+1}\pi_{n+1} = 0, \text{ for } n = 1,2,3,\ldots \end{cases}$$

We can solve these systems inductively. The first equation implies that

$$\pi_1 = \frac{\lambda_0}{\beta_1}\pi_0.$$

Substituting this into the second equation implies that

$$\pi_2 = \frac{1}{\beta_2}\Big((\lambda_1+\beta_1)\pi_1 - \lambda_0\pi_0\Big) = \frac{\lambda_1}{\beta_2}\pi_1 = \frac{\lambda_1\lambda_0}{\beta_2\beta_1}\pi_0.$$

Continuing this argument inductively using the induction hypothesis that $\beta_n\pi_n = \lambda_{n-1}\pi_{n-1}$ implies

$$\pi_{n+1} = \frac{1}{\beta_{n+1}}\big((\lambda_n+\beta_n)\pi_n - \lambda_{n-1}\pi_{n-1}\big) = \frac{1}{\beta_{n+1}}\big((\lambda_n+\beta_n)\pi_n - \beta_n\pi_n\big) = \frac{\lambda_n}{\beta_{n+1}}\pi_n = \cdots = \frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1}\pi_0.$$

Since $\sum_{n=0}^{\infty}\pi_n = 1$, we must have

$$\pi_0 = \frac{1}{1+\sum_{n=1}^{\infty}\frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1}}.$$

and

$$\pi_n = \frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1}\cdot\frac{1}{1+\sum_{n=1}^{\infty}\frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1}} = \frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1}\pi_0 \qquad \text{, for } n = 1,2,3,\ldots$$

We see that this exists, precisely when

$$\sum_{n=1}^{\infty}\frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1} < \infty.$$

Remark 1.5. In conclusion, the necessary and sufficient condition for the existence of the stationary distribution, or equivalently the birth-death process is positive recurrent, is that

$$\sum_{n=1}^{\infty}\frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1} < \infty.$$

This proves Proposition 1.2.

1.2.2 Applications

Problem 1.3. Customers arrive at a single-server queue in accordance with a Poisson process with intensity $\lambda > 0$. However, an arrival that finds n customers already in the system (including the customer being served) will only join the system with probability $\frac{1}{1+n}$. That is, with probability $\frac{n}{n+1}$ such an arrival will not join the system. The service times of customers are mutually independent with common exponential distribution with mean $1/\mu$. Let $X(t)$ be the number of customers in the system at time t with $X(0) = 0$. Show that the limiting distribution of the number of customers in the system is Poisson and then identity its Poisson parameter.

Solution 1.3. First note that this is a birth and death process. In general, the generator matrix can be expressed as

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \\ \beta_1 & -(\lambda_1 + \beta_1) & \lambda_1 & 0 & \\ 0 & \beta_2 & -(\lambda_2 + \beta_2) & \lambda_2 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Since the service time of each customer follows the exponential distribution with mean $1/\mu$, the corresponding rate is $\beta_n = \mu$ for all n. On the other hand, we know that the waiting time for the next person arrived follows the exponential distribution with rate λ. Since there is a probability $1/(1+n)$ this person will join the queue, the customers will arrive at an average rate of

$$\lambda \frac{\text{customers arrive}}{\text{minute}} \implies \lambda \frac{\text{customers arrive}}{\text{minute}} \times \frac{1}{n+1} \frac{\text{join queue}}{\text{customers arrive}} = \frac{\lambda}{n+1} \frac{\text{join queue}}{\text{minute}}$$

This implies that the rate of arrivals is $\lambda_n = \frac{\lambda}{1+n}$ for all n. The generator matrix is

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \\ \mu & -\left(\frac{\lambda}{2} + \mu\right) & \frac{\lambda}{2} & 0 & \\ 0 & \mu & -\left(\frac{\lambda}{3} + \mu\right) & \frac{\lambda}{3} & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

By Theorem 1.3, the limiting distribution is given by

$$\pi_0 = \left(1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n n!}\right)^{-1} = \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{\mu^n n!}\right)^{-1} = e^{-\lambda/\mu} \qquad \text{and} \qquad \pi_n = \frac{\lambda^n}{\mu^n \cdot n!} e^{-\lambda/\mu}, \qquad n = 1, 2, \ldots.$$

That is, the limiting distribution of the number of customers in the system is Poisson with mean λ/μ.

Problem 1.4. Suppose $X(t)$ denotes the number of customers on line for some service. We assume that customers arrive at a rate $\lambda > 0$, i.e., the arrival number follows a Poisson with rate λ. Customers are serviced at an exponential rate $\beta > 0$. In a $M/M/\infty$ queueing model, we assume there are infinitely many servers and anyone in the line can be served.

1. Determine the infinitesimal generator matrix for the $M/M/\infty$ queueing model.
2. Determine the 1-step transition matrix of the embedded DTMC of the $M/M/\infty$ queueing model.
3. Determine a stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ for the $M/M/\infty$ queueing model.

Solution 1.4.

Part 1: For a birth and death processes, the generator matrix is

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \\ \beta_1 & -(\lambda_1+\beta_1) & \lambda_1 & 0 & \\ 0 & \beta_2 & -(\lambda_2+\beta_2) & \lambda_2 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

For $M/M/\infty$ queueing model, it is easy to see $\lambda_n = \lambda$ and $\beta_n = n\beta$. Therefor, the generator is

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \\ \beta & -(\lambda+\beta) & \lambda & 0 & \\ 0 & 2\beta & -(\lambda+2\beta) & \lambda & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Part 2: From the generator matrix A, the 1-step transition matrix of the embedded DTMC is

$$\widetilde{\boldsymbol{P}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ \beta/(\lambda+\beta) & 0 & \lambda/(\lambda+\beta) & 0 & \\ 0 & 2\beta/(\lambda+2\beta) & 0 & \lambda/(\lambda+2\beta) & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Part 3: To find the stationary distribution, we have that

$$\sum_{n=1}^{\infty} \frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1} = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!\beta^n} = e^{\frac{\lambda}{\beta}} - 1 < \infty$$

Therefore, Proposition 1.3 implies that

$$\pi_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1}} = e^{-\frac{\lambda}{\beta}}$$

and

$$\pi_n = \frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\beta_n\cdots\beta_2\beta_1}\pi_0 = \frac{\lambda^n}{n!\beta^n}e^{-\frac{\lambda}{\beta}} \qquad \text{for } n = 1, 2, 3, \ldots$$

The stationary distribution is a Poisson distribution with mean $\frac{\lambda}{\beta}$.