1 Classification of states

In this section, we show several properties of DTMCs and the corresponding terminology.

1.1 Irreducibility

Informally, we say that the state j is accessible from state i if it is possible to get to state j from state i. Similarly, states i and j communicate if it is possible to go from state i to state j and vice versa.

Definition 1.1. A state j is called **accessible** from state i, denoted as $i \to j$, if there exists $n \in \{0, 1, \dots\}$ such that

$$\mathbb{P}(X_n = j \mid X_0 = i) = p_{ij}^{(n)} > 0.$$

We say states i and j **communicate**, denoted as $i \leftrightarrow j$, if there exist $m, n \in \{0, 1, \dots\}$ such that

$$\mathbb{P}(X_n = j \mid X_0 = i) = p_{ij}^{(n)} > 0 \quad \text{and} \quad \mathbb{P}(X_n = i \mid X_0 = j) = p_{ji}^{(m)} > 0.$$

Or equivalently, we have $i \leftrightarrow j$ means that $i \to j$ and $j \to i$.

Remark 1.2. In the above definition, we allow for $n = 0$ or $m = 0$. Since

$$\mathbb{P}(X_0 = j \mid X_0 = i) = p_{ij}^{(0)} = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

this implies that $i \leftrightarrow i$ for all $i \in S$. The symbol δ_{ij} is called the **Kronecker delta**.

The relation of communication divides the state space S into a partition of different **equivalence classes**. That is, all the states in one class communicate with each other, but not with any states from any other class.

Proposition 1.3 (*Equivalence Relation of Communication*)

The relation of communication is an **equivalence relation**. *That is, the following three conditions are satisfied:*

(1) Reflexivity: $i \leftrightarrow i$

(2) Symmetry: If $i \leftrightarrow j$*, then* $j \leftrightarrow i$

(3) Transitivity: If $i \leftrightarrow j$ *and* $j \leftrightarrow k$*, then* $i \leftrightarrow k$

Let

$$S_i = \{j \in S : j \leftrightarrow i\}$$

denote the set of states communicating with state i. If $i \leftrightarrow j$, then we have $S_i = S_j$ (see Problem 1.2). This implies that the communicating states can be divided into disjoint clusters or **communicating classes**.

Definition 1.4. A DTMC that has only one communication class is called **irreducible**.

In an irreducible DTMC, each state can be reached from any other state.

Definition 1.5. A state i is called **absorbing** if $p_{ii} = 1$

Since $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$, we must have $p_{ij} = 0$ for $j \neq i$ if state i is absorbing. Thus, once the DTMC enters state i, it will be trapped there forever.

Definition 1.6. A set of states C is called **closed**, if $p_{ij} = 0$ for all $i \in C$ and $j \in S \backslash C$.

If the DTMC is in a closed set C, then it will never leave this set again.

1.2 Periodicity

We now define a notion that measures how regularly a DTMC can return to a particular state.

Definition 1.7. The **period** of state i is defined as

$$d(i) = \gcd\left\{n \in \mathbb{N} : p_{ii}^{(n)} > 0\right\},$$

where gcd is the *greatest common divisor* of times at which return to state i is possible. If $d(i) = k$ for all $i \in S$, then we say that the DTMC has period k.

Remark 1.8. It is possible that the DTMC will never return to its current state, i.e. $p_{ii}^{(n)} = 0$ for all $n \in \mathbb{N}$. In this case, we let $d(i) = \infty$.

The period can be interpreted as the regular time interval in which is it possible to return to its current state.

Example 1.9. The simple random walk has period 2 because

$$\begin{cases} p_{ii}^{(n)} = 0, \text{ if } n \text{ is odd} \\ p_{ii}^{(n)} > 0, \text{ if } n \text{ is even} \end{cases}$$

However, the period does not mean that it is always possible to return to the state every period.

Remark 1.10. Note that $d(i) = k$ does **not** imply that $p_{ii}^{(k)} > 0$. For example, consider the DTMC with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$$

with $p_{11}^{(2)} = p_{11}^{(3)} = 0.5$ and, hence, $d(1) = 1$. However, $p_{11}^{(1)} = p_{11} = 0$.

Definition 1.11. We call state i is **aperiodic** if $d(i) = 1$. The DTMC is called aperiodic if all states are aperiodic.

An aperiodic state has no pattern for return times. Naturally, if two states communicate with each other, then they share the same period.

Proposition 1.12 (*Class Property I*)

If $i \leftrightarrow j$, then $d(i) = d(j)$. In particular, if a DTMC is irreducible, then all its states have the same period.

Remark 1.13. There will be several results called **class properties** of a DTMC. These are properties shared by all states in the same communication class.

1.3 Recurrence and transience

For $n \in \mathbb{N}$, we define

$$f_{ij}^{(n)} = \mathbb{P}\left(X_n = j, X_{n-1} \neq j, \cdots, X_1 \neq j \,|\, X_0 = i\right).$$

Then $f_{ij}^{(n)}$ is the probability that X visits state j for the first time at time n, given that $X_0 = i$. Clearly, if $X_0 = i$, then probability that X visits j for the first time is smaller than the probability X visits j starting at i

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} = f_{ij}^{(n)} + \sum_{k=1}^{n-1} f_{ij}^{(k)} p_{jj}^{(n-k)} \geq f_{ij}^{(n)}.$$

We also define

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

which is the probability that X ever visits state j, given that $X_0 = i$.

Definition 1.14.

1. A state i is called **recurrent** if $f_{ii} = 1$, i.e., if started at i, the DTMC will return to i at some time in the future with probability one.
2. A state i is called **transient** if $f_{ii} < 1$, i.e., there is a positive probability that the DTMC will never return to state i.

Another way to characterize recurrence and transience is through the total number of visits. Define

$$M_i = \sum_{n=1}^{\infty} \mathbb{1}\{X_n = i\}$$

which denotes the total number of times X visits state i after time 0. There is a direct relationship between recurrence, the number of total visits, and the transition probabilities.

Proposition 1.15 (*Equivalent Notions of Recurrence and Transience*)

$$\text{state } i \text{ is recurrent} \iff \mathbb{E}\left[M_i \middle| X_0 = i\right] = \infty \iff \sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$$

$$\text{state } i \text{ is transient} \iff \mathbb{E}\left[M_i \middle| X_0 = i\right] < \infty \iff \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$$

The next proposition shows that recurrence and transience are class properties.

Proposition 1.16 (*Class Property II*)

Suppose that $i \leftrightarrow j$.

(i) If i is recurrent, then j is recurrent.

(ii) If i is transient, then j is transient.

As a result, if a DTMC is irreducible, then either all states are recurrent, or they are all transient. The latter is not possible if the state space is finite (see Proposition **??**).

1.3.1 Mean Recurrence

We define the first time at which state i is visited after time 0 by:

$$T_i = \min\{n \in \mathbb{N} : X_n = i\}.$$

Note that T_i is a random stopping time. Clearly,

$$\mathbb{P}(T_i = k \mid X_0 = i) = \mathbb{P}\left(X_k = i, X_{k-1} \neq i, \cdots, X_1 \neq i \middle| X_0 = i\right) = f_{ii}^{(k)}. \tag{1}$$

Define the **mean recurrence time** of state i by

$$\mu_i = \mathbb{E}\left[T_i \middle| X_0 = i\right] = \sum_{k=1}^{\infty} k\mathbb{P}\left(T_i = k \middle| X_0 = i\right) = \sum_{k=1}^{\infty} k f_{ii}^{(k)}. \tag{2}$$

which denotes the average time to returns to state i.

Definition 1.17. A recurrent state i is called **positive recurrent** if $\mu_i < \infty$, and **null recurrent** if $\mu_i = \infty$. We call the DTMC is positive (null, resp.) recurrent if all states are positive (null, resp.) recurrent.

Remark 1.18. If state i is recurrent, we have $f_{ii} = \sum_{k=1}^{\infty} f_{ii}^{(k)} = 1$, which implies

$$\lim_{k\uparrow\infty} f_{ii}^{(k)} = \mathbb{P}\left(T_i = \infty \middle| X_0 = i\right) = 0.$$

However, this does not imply the finiteness of μ_i since we need $(kf_{ii}^{(k)})_{k\geq 1}$ so be summable.

If a state i is recurrent, then the DTMC returns to i with probability one. However, the time T_i until the first return can have such a heavy tail that $\mu_i = \mathbb{E}[T_i|X_0 = i]$ is infinite. In this case, the state i is called null recurrent.

The follow proposition shows that positive recurrence and null recurrence are also class properties.

Proposition 1.19 (*Class Property III*)

Suppose $i \leftrightarrow j$.

(i) If i is positive recurrent, then j is positive recurrent.

(ii) If i is null recurrent, then j is null recurrent.

To summarize, every state $i \in S$ can either be transient or recurrent. If it is recurrent, it must either be positive recurrent or null recurrent. Any of these properties is shared by all other states in the same communication class. This is called the **classification of states**.

Theorem 1.20 (*Decomposition theorem*)

The state space S of a DTMC can be partitioned uniquely as follows:

$$S = T \cup C_1 \cup C_2 \cup \cdots$$

where T is the set of all transient states, and C_i are irreducible closed sets of recurrent states.

These sets of states have the following interpretations:

- If the process is in any C_i, it never leaves it.
- If the process is in T, it may stay there forever but it can also move to one of the C_i, where it subsequently remains.

As a consequence, we can **reorder** the states such that the transition matrix can be rewritten as

$$\boldsymbol{P} = \begin{array}{c c} & \begin{array}{ccccc} C_1 & C_2 & C_3 & \cdots & T \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ \vdots \\ T \end{array} & \begin{pmatrix} \boldsymbol{P}_1 & & & & \\ & \boldsymbol{P}_2 & & & \\ & & \boldsymbol{P}_3 & & \\ & & & \ddots & \\ \boldsymbol{Q}_1 & \boldsymbol{Q}_2 & \boldsymbol{Q}_3 & \cdots & \cdots \end{pmatrix} \end{array} \tag{3}$$

where $\boldsymbol{P}_i$ is the transition matrix of the DTMC *restriced to the closed set C_i*.

1.4 Example Problems

1.4.1 Proofs of Results

Problem 1.1. Prove that the communication is an equivalence relation (Proposition 1.3).

Solution 1.1. Both reflexivity and symmetry are clear from the definition. Thus, we only need to show the transitivity. Suppose that $i \leftrightarrow j$ and $j \leftrightarrow k$, i.e. there exists $m, n \in \{0, 1, \dots\}$ such that $p_{ij}^{(m)} > 0$ and $p_{jk}^{(n)} > 0$. Then, by the Chapman–Kolmogorov equations,

$$p_{ik}^{(m+n)} = \sum_{l \in S} p_{il}^{(m)} p_{lk}^{(n)} \geq p_{ij}^{(m)} p_{jk}^{(n)} > 0,$$

which implies $i \to k$. An indentical argument shows that $k \to i$.

Problem 1.2. Let

$$C_i := \{j \in S : j \leftrightarrow i\}.$$

Show that

1. $i \in C_i$
2. $j \in C_i$ if and only if $i \in C_j$
3. if $i \leftrightarrow j$ then $C_i = C_j$

Furthermore, show that all distinct communication classes create a partition of S.

Solution 1.2. The first three properties follow from the fact that communication is an equivalence class:

1. Since $i \leftrightarrow i$ by reflexivity, we have that $i \in C_i$.
2. Since $i \leftrightarrow j$ and $j \leftrightarrow i$ by symmetry, we have $j \in C_i$ if and only if $i \in C_j$.
3. Suppose that $i \leftrightarrow j$. If $k \in C_i$ then $k \to i$, so $k \to j$ by transitivity. In particular, we have

$$C_i \subseteq C_j.$$

   An identical argument shows that then $C_j \subseteq C_i$ so $C_i = C_j$.

For the sake of contradiction, suppose that $i \in S$ belongs to distinct equivalence classes. That is, suppose that $i \in C_j$ and $i \in C_k$ but $C_i \neq C_j$. However, the second property implies that $j \in C_i$ and $k \in C_i$, so $j \leftrightarrow k$ by transitivity. The third property implies that $C_i = C_j$, which is a contradiction. We conclude that every $i \in S$ belongs to exactly one equivalence class, so all distinct communcation classes form a partition of S.

Problem 1.3. Prove the Class Property I (Proposition 1.12).

Solution 1.3. We only need to consider the case that $i \neq j$, since the case that $i = j$ is trivial. It suffices to show that $d(i)$ divides $d(j)$. Since symmetry will imply that $d(j)$ also divides $d(i)$, which implies that $d(i) = d(j)$.

We we define $N(i) = \{n : p_{ii}^{(n)} > 0\}$ then $d(i) = \gcd(N(i))$. We can define $N(j)$ in a similar way. Since $i \leftrightarrow j$, there exists m, n such that

$$p_{ij}^{(m)} > 0 \text{ and } p_{ji}^{(n)} > 0.$$

This implies by the Chapman–Kolmogorov equations that

$$p_{ii}^{(m+n)} \geq p_{ij}^{(m)} p_{ji}^{(n)} > 0$$

and in turn $m + n \in N(i)$. Next, for any $l \in N(j)$, we have $p_{jj}^{l} > 0$, so

$$p_{ii}^{(l+m+n)} \geq p_{ij}^{(m)} p_{jj}^{(l)} p_{ji}^{(n)} > 0.$$

Therefore, $m + n + l \in N(i)$ for all $l \in N(j)$. We conclude that $d(i)$ divides both $m + n$ and $m + n + l$ for any $l \in N(j)$. Therefore, $d(i)$ is a divisor of $N(j)$ and, hence, in particular divides $d(j)$.

Problem 1.4. Prove the equivalence of the different notions of recurrence and transcience (Proposition 1.15).

Solution 1.4. We first show the recurrence and transicience in terms of the expected visits. Note that if $f_{ii} < 1$,

$$\mathbb{P}\left(M_i = k \middle| X_0 = i\right) = \underbrace{f_{ii} \times f_{ii} \times \cdots \times f_{ii}}_{\text{return to state } i \text{ for exactly } k \text{ visits}} \times \underbrace{(1 - f_{ii})}_{\text{never visit } i \text{ ever again}}$$

In other words, conditionally on $X_0 = i$, M_i follows a geometric distribution with parameter $1 - f_{ii}$, so by the properties of the geometric distribution

$$\mathbb{E}\left[M_i \middle| X_0 = i\right] = \frac{f_{ii}}{1 - f_{ii}} < \infty.$$

By taking limits,

$$\lim_{f_{ii} \uparrow 1} \mathbb{E}\left[M_i \middle| X_0 = i\right] = \lim_{f_{ii} \uparrow 1} \frac{f_{ii}}{1 - f_{ii}} = \infty$$

and so if i is recurrent, we have

$$\mathbb{E}\left[M_i \middle| X_0 = i\right] = \infty.$$

To show that last equivalence in terms of the transition probabilities, notice that by linearity

$$\mathbb{E}\left[M_i \middle| X_0 = i\right] = \sum_{n=1}^{\infty} \mathbb{E}\left[\mathbb{1}\{X_n = i\} \middle| X_0 = i\right] = \sum_{n=1}^{\infty} \mathbb{P}\left[X_n = i \,\middle|\, X_0 = i\right] = \sum_{n=1}^{\infty} p_{ii}^{(n)}.$$

1.4.2 Applications

Problem 1.5. Is the random walk irreducible?

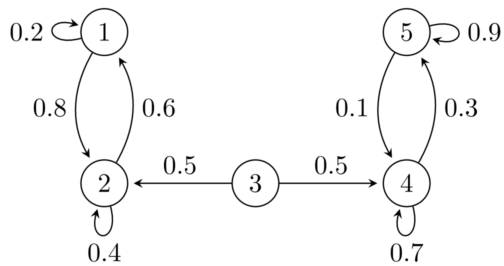
Solution 1.5. The random walk is irreducable because given any state $i \in \mathbb{Z}$, we have $i + 1 \leftrightarrow i$ and $i - 1 \leftrightarrow j$. Therefore, by repetitive applications of the transitivity property, we can conclude that $i \leftrightarrow j$ for any $i, j \in \mathbb{Z}$, so the random walk is irreducible.

Problem 1.6. Consider a DTMC with states $\{1, 2, 3, 4, 5\}$ and transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0.2 & 0.8 & & & \\ 0.6 & 0.4 & & & \\ & 0.5 & 0 & 0.5 & \\ & & & 0.7 & 0.3 \\ & & & 0.1 & 0.9 \end{pmatrix}$$

What are the communication classes of this DTMC.

Solution 1.6. We draw the corresponding state transition diagram associated with this DTMC:



Clearly, the communication classes are $\{1, 2\}$, $\{3\}$, and $\{4, 5\}$.

Remark 1.21. Notice that the values of the transition is not important to figure out the communication classes.

Problem 1.7. Consider a DTMC with transition matrix

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} p & 0 & 1-p & 0 \\ 0 & r & 0 & 1-r \\ q & 0 & 1-q & 0 \\ 0 & s & 0 & 1-s \end{pmatrix} \end{array}$$

where $p, q, r, s \in (0, 1)$. What are the communication classes? Which ones are recurrent and which are transient? Show that we can $\boldsymbol{P}$ in the form of (3).

Solution 1.7. This DTMC is reducible with two closed recurrent classes $\{1, 3\}$ and $\{2, 4\}$. We can reorder the states to $\{1, 3, 2, 4\}$ so that the transition matrix becomes

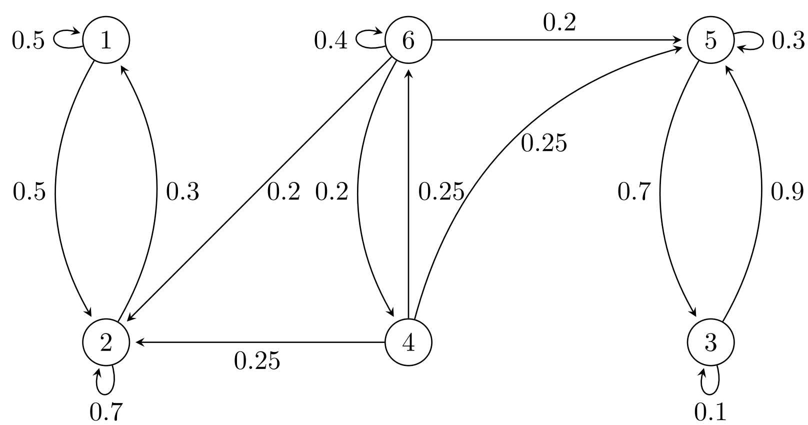
$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 3 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 2 & 4 \end{array} \\ \begin{pmatrix} p & 1-p & 0 & 0 \\ q & 1-q & 0 & 0 \\ 0 & 0 & r & 1-r \\ 0 & 0 & s & 1-s \end{pmatrix} \end{array}$$

Problem 1.8. Consider the Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 3/10 & 7/10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/10 & 0 & 9/10 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 7/10 & 0 & 3/10 & 0 \\ 0 & 1/5 & 0 & 1/5 & 1/5 & 2/5 \end{pmatrix}.$$

What are the communication classes? Which ones are recurrent and which are transient? Show that we can $\boldsymbol{P}$ in the form of (3).

Solution 1.8. We draw the corresponding state transition diagram associated with this DTMC:



The communication classes are $\{1,2\}$, $\{3,5\}$, and $\{4,6\}$. The classes $\{1,2\}$ and $\{3,5\}$ are recurrent and the class $\{4,6\}$ is transient. After relabeling the states to $\{1,2,3,5,4,6\}$, the transition matrix becomes

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 3/10 & 7/10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/10 & 9/10 & 0 & 0 \\ 0 & 0 & 7/10 & 3/10 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 1/5 & 0 & 1/5 & 1/5 & 2/5 \end{pmatrix}.$$