Week 7

Problem 1. Solve the following eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < L \\ X'(0) - X(0) = 0 \\ X'(L) + X(L) = 0 \end{cases}$$

Solution 1. We want to find non-trivial solutions to the eigenvalue problem i.e. $X(x) \neq 0$. Notice that the general solution to the ODE

$$X'' + \lambda X = 0 \qquad 0 < x < L,$$

has different forms depending on λ. We must solve each of the cases separately.

Positive Eigenvalues $\lambda = \beta^2 > 0$:

Step 1: The second order constant coefficient ODE

$$X'' + \beta^2 X = 0$$

has characteristic equation with roots $r = \pm\beta i$, which corresponds to the solution

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

Step 2: We now solve for the values of β that satisfy both the boundary conditions. From the first boundary condition ($X'(0) - X(0) = 0$), we must have

$$\beta B - A = 0$$

and from the second boundary condition ($X'(L) + X(L) = 0$),

$$\beta B\cos(\beta L) - \beta A\sin(\beta L) + A\cos(\beta L) + B\sin(\beta L) = 0.$$

From the first equation, we have $A = B\beta$, which implies

$$(\beta B + B\beta)\cos(\beta L) + (B - B\beta^2)\sin(\beta L) = 0.$$

We can assume that $B \neq 0$, because if $B = 0$, then $A = 0$ which corresponds to the trivial solution. Therefore, we have the equation

$$2\beta\cos(\beta L) + (1 - \beta^2)\sin(\beta L) = 0$$

Let $\beta^* > 0$ be a positive root of the above equation. The corresponding eigenvalue is $\lambda^* = (\beta^*)^2$ and eigenfunction is

$$X(x) = \beta^* B\cos(\beta^* x) + B\sin(\beta^* x),$$

since $A = B\beta$. Since any constant multiple of an eigenfunction is an eigenfunction, we may take $B = 1$.

Remark: Depending on the value of L, we can simplify the above equation slightly. For example, suppose we could find a β such that $\cos(\beta L) = 0$. Then the term with the sin implies that $1 - \beta^2 = 0 \implies \beta = 1$. For example, if L is such that $\cos(L) = 0$ then $\beta^2 = 1$ is an eigenvalue.

In all other cases, we can divide by $\cos(\beta L)$ and rearrange to give

$$\tan(\beta L) = \frac{2\beta}{\beta^2 - 1}.$$

The squares of the positive roots of this equation will give us the corresponding eigenvalues.

Zero Eigenvalue $\lambda = \beta = 0$:

Step 1: The second order constant coefficient ODE

$$X'' = 0$$

has characteristic equation with repeated roots $r = 0$, which corresponds to the solution

$$X(x) = A + Bx.$$

Step 2: We now attempt to find a zero eigenvalue. Plugging the general solution into the boundary conditions gives

$$\begin{aligned} B - A &= 0 \\ B + A + BL &= 0. \end{aligned}$$

From the first equation, we must have $A = B$. Plugging this into the second equation gives

$$2B + BL = 0 \implies L = -2,$$

which is impossible since $L > 0$, so there is no 0 eigenvalue.

Negative Eigenvalue $\lambda = -\beta^2 < 0$:

Step 1: $\lambda = \beta = 0$: The second order constant coefficient ODE

$$X'' - \beta^2 X = 0$$

has characteristic equation with roots $r = \pm\beta$, which corresponds to the solution

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

Step 2: We now attempt to find a negative eigenvalue. Plugging the general solution into the boundary conditions gives

$$\begin{aligned} \beta B - A &= 0 \\ A\beta\sinh(\beta L) + B\beta\cosh(\beta L) + A\cosh(\beta L) + B\sinh(\beta L) &= 0. \end{aligned}$$

From the first equation, we must have $A = \beta B$ which implies

$$(B\beta + B\beta)\cosh(\beta L) + (\beta^2 B + B)\sinh(\beta L) = 0 \implies 2\beta\cosh(\beta L) + (\beta^2 + 1)\sinh(\beta L) = 0.$$

Again, we could divide by B because $B = 0$ corresponds to the trivial solution. Since $\cosh(x) > 0$ for all $x \in \mathbb{R}$, we can simplify the above to give

$$\tanh(\beta L) = \frac{-2\beta}{\beta^2 + 1}.$$

The only solution to the above equation is $\beta = 0$, which is not strictly positive, so there are no negative eigenvalues.

Problem 2. Solve the following eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < L \\ X(0) = 0 \\ X'(L) + X(L) = 0 \end{cases}$$

Solution 2. We want to find non-trivial solutions to the eigenvalue problem i.e. $X(x) \neq 0$. Notice that the general solution to the ODE

$$X'' + \lambda X = 0 \qquad 0 < x < L,$$

has different forms depending on λ. We must solve each of the cases separately.

Positive Eigenvalues $\lambda = \beta^2 > 0$:

Step 1: The second order constant coefficient ODE

$$X'' + \beta^2 X = 0$$

has characteristic equation with roots $r = \pm\beta i$, which corresponds to the solution

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

Step 2: We attempt to find a positive eigenvalue. Plugging the general solution into the boundary conditions gives

$$\begin{aligned} A &= 0 \\ \beta B\cos(\beta L) - \beta A\sin(\beta L) + A\cos(\beta L) + B\sin(\beta L) &= 0. \end{aligned}$$

From the first equation, we have $A = 0$, which implies

$$\beta B\cos(\beta L) + B\sin(\beta L) = 0.$$

We could assume that $B \neq 0$ because the case $B = 0$ which corresponds to the trivial solution. Therefore, we have the equation

$$\beta\cos(\beta L) + \sin(\beta L) = 0.$$

If $\cos(\beta L) = 0$, then $\sin(\beta L) \neq 0$ which means there are no solutions such that $\cos(\beta L) = 0$. Therefore, we can divide out by $\cos(\beta L)$ giving us the equation

$$\tan(\beta L) = -\beta.$$

Let $\beta^* > 0$ be a positive root of the above equation. The corresponding eigenvalue is $\lambda^* = (\beta^*)^2$ and eigenfunction is

$$X(x) = B\sin(\beta^* x).$$

Since any constant multiple of an eigenfunction is an eigenfunction, we can take $B = 1$.

Zero Eigenvalue $\lambda = \beta = 0$:

Step 1: The second order constant coefficient ODE

$$X'' = 0$$

has characteristic equation with repeated roots $r = 0$, which corresponds to the solution

$$X(x) = A + Bx.$$

Step 2: $\lambda = \beta = 0$: We now attempt to find a zero eigenvalue. Plugging the general solution into the boundary conditions gives

$$\begin{aligned} A &= 0 \\ B + A + BL &= 0. \end{aligned}$$

From the first equation, we must have $A = 0$. Plugging this into the second equation gives

$$B + BL = 0 \implies L = -1,$$

which is impossible since $L > 0$, so there is no 0 eigenvalue.

Negative Eigenvalue $\lambda = -\beta^2 < 0$:

Step 1: $\lambda = \beta = 0$: The second order constant coefficient ODE

$$X'' - \beta^2 X = 0$$

has characteristic equation with roots $r = \pm\beta$, which corresponds to the solution

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

Step 2: We now attempt to find a negative eigenvalue. Plugging the general solution into the boundary conditions gives

$$\begin{aligned} A &= 0 \\ A\beta\sinh(\beta L) + B\beta\cosh(\beta L) + A\cosh(\beta L) + B\sinh(\beta L) &= 0. \end{aligned}$$

From the first equation, we must have $A = 0$ which implies

$$B\beta\cosh(\beta L) + B\sinh(\beta L) = 0 \implies \tanh(\beta L) = -\beta.$$

We could assume that $B \neq 0$ because the case $B = 0$ which corresponds to the trivial solution. The only solution to the above equation is $\beta = 0$, which is not strictly positive, so there are no negative eigenvalues.

Problem 3. (Strauss 4.3.16) Find the positive eigenvalues and eigenfunctions of

$$\begin{cases} X^{(4)} = \lambda X & 0 < x < L \\ X(0) = X(L) = X''(0) = X''(L) = 0 \end{cases}$$

Solution 3. We want to find non-trivial solutions to the eigenvalue problem i.e. $X(x) \neq 0$.

Step 1: We are interested in positive eigenvalues, so we can set $\lambda = \beta^4 > 0$, where $\beta > 0$. We first find the general solution to the ODE

$$X^{(4)} = \beta^4 X \qquad 0 < x < L.$$

This is a fourth order constant coefficient ODE with roots $r = \pm\beta, \pm\beta i$, which corresponds to the solution

$$X(x) = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x).$$

Step 2: We now solve for the values of β that satisfy the boundary conditions. Plugging the solution into the initial conditions gives

$$\begin{aligned} A + C &= 0 \\ A\cos(\beta L) + B\sin(\beta L) + C\cosh(\beta L) + D\sinh(\beta L) &= 0 \\ -\beta^2 A + \beta^2 C &= 0 \\ -A\beta^2\cos(\beta L) - B\beta^2\sin(\beta L) + C\beta^2\cosh(\beta L) + D\beta^2\sinh(\beta L) &= 0. \end{aligned}$$

Since $\beta > 0$, the first and third equation implies $A + C = 0$ and $-A + C = 0$ which can only happen when $A = C = 0$. We now have to solve the system

$$\begin{aligned} B\sin(\beta L) + D\sinh(\beta L) &= 0 \\ -B\beta^2\sin(\beta L) + D\beta^2\sinh(\beta L) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2\sin(\beta L) & \beta^2\sinh(\beta L) \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has a non-trivial solution when

$$\det\left(\begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2\sin(\beta L) & \beta^2\sinh(\beta L) \end{bmatrix}\right) = 0 \implies 2\beta^2\sinh(\beta L)\sin(\beta L) = 0.$$

Since $\beta > 0$ and $\sinh(\beta L) > 0$, this simplifies to $\sin(\beta L) = 0$, which occurs precisely when

$$\beta_n = \frac{n\pi}{L}, \quad n \geq 1.$$

Therefore, the corresponding eigenvalues are $\lambda_n = \frac{n^4\pi^4}{L^4}$. Furthermore, notice that from the equation

$$B\sin(\beta_n L) + D\sinh(\beta_n L) = 0,$$

we must have $D = 0$ since $\sinh(\beta_n L) > 0$ and $\sin(\beta_n L) = 0$. Finally, for $n \geq 1$, the corresponding eigenfunction for the eigenvalue λ_n is

$$X_n(x) = B\sin\Big(\frac{n\pi x}{L}\Big).$$

We may take $B = 1$.

Problem 4. (Strauss 4.3.12) Consider the unusual eigenvalue problem

$$\begin{cases} X'' = -\lambda X & 0 < x < L \\ X'(0) = X'(L) = \frac{X(L)-X(0)}{L}. \end{cases}$$

1. Show that $\lambda = 0$ is a double eigenvalue.
2. Get an equation for the positive eigenvalues $\lambda > 0$.
3. Letting $\gamma = \frac{1}{2}L\sqrt{\lambda}$, reduce the equation to

$$\gamma\sin\gamma\cos\gamma = \sin^2\gamma.$$

4. Assuming all the eigenvalues are non-negative, make a list of all the eigenfunctions.

Solution 4.

1. We first consider the case $\lambda = 0$. The ODE

$$X'' = 0$$

has the solution

$$X(x) = A + Bx.$$

Plugging this into the boundary conditions gives

$$B = \frac{A + BL - A}{L} = B.$$

Strangely, any solution of the form $X(x) = A + Bx$ automatically satisfies the boundary conditions. That is, there are 2 eigenfunctions, namely

$$X_1(x) = 1 \text{ and } X_2(x) = x$$

corresponding to the eigenvalue 0. In particular, $\lambda = 0$ is a double eigenvalue.

2. We now consider the case $\lambda = \beta^2 > 0$. The second order constant coefficient ODE

$$X'' + \beta^2 X = 0$$

has characteristic equation with roots $r = \pm\beta i$, which corresponds to the solution

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

Plugging this into the boundary conditions gives

$$\beta B = -\beta A \sin(\beta L) + \beta B \cos(\beta L) = \frac{A\cos(\beta L) + B\sin(\beta L) - A}{L}.$$

From the first and second equation, we have

$$-\beta\sin(\beta L)A + (\beta\cos(\beta L) - \beta)B = 0$$

and the first and third equation, we have

$$(\cos(\beta L) - 1)A + (\sin(\beta L) - \beta L)B = 0.$$

We can write this system of equations in matrix form

$$\begin{bmatrix} -\beta\sin(\beta L) & (\beta\cos(\beta L) - \beta) \\ (\cos(\beta L) - 1) & (\sin(\beta L) - \beta L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has a non-trivial solution when

$$\det\left(\begin{bmatrix} -\beta\sin(\beta L) & (\beta\cos(\beta L) - \beta) \\ (\cos(\beta L) - 1) & (\sin(\beta L) - \beta L) \end{bmatrix}\right) = 0 \implies -\beta\sin(\beta L)(\sin(\beta L) - \beta L) = (\beta\cos(\beta L) - \beta)(\cos(\beta L) - 1).$$

Since $\beta > 0$, we can divide by β giving

$$-\sin(\beta L)(\sin(\beta L) - \beta L) = (\cos(\beta L) - 1)^2.$$

3. If we let $\gamma = \frac{1}{2}L\beta \implies L\beta = 2\gamma$, then our equation simplifies to

$$-\sin(2\gamma)(\sin(2\gamma) - 2\gamma) = (\cos(2\gamma) - 1)^2.$$

Using some elementary trig identities, we have

$$4\sin\gamma\cos\gamma(\sin\gamma\cos\gamma - \gamma) = 4\sin^4(\gamma)$$

which simplifies to

$$\gamma\sin\gamma\cos\gamma = \sin^4(\gamma) + \sin^2(\gamma)\cos^2(\gamma) = \sin^2(\gamma)(\sin^2\gamma + \cos^2\gamma) = \sin^2\gamma.$$

4. We can solve the above equation explicitly. One solution corresponds to the case when $\cos(\gamma) = 0$. In this case, we must have

$$\sin(\gamma) = 0 \implies \gamma = \pi n \text{ for } n \geq 1.$$

In this case, since $\beta = 2\gamma/L$ the corresponding eigenfunction is

$$X(x) = A\cos\big(\frac{2\pi n}{L}x\big) \text{ for } n \geq 1.$$

To find the second solution, we assume that $\cos(\gamma) \neq 0$. Dividing by $\cos(\gamma)$ gives us

$$\tan(\gamma) = \gamma$$

which we have to solve numerically. Let γ^* be a solution to $\tan(\gamma) = \gamma$. The corresponding $\beta^* = 2\gamma^*/L$ which implies the corresponding eigenvalue is given by $\lambda^* = (\beta^*)^2 = (\frac{2\gamma^*}{L})^2$.

From the boundary condition $X'(0) = X'(L)$, we have

$$\beta^* B = -\beta^* A \sin(\beta^* L) + \beta^* B\cos(\beta^* L) \implies A = B\frac{\cos(2\gamma^*) - 1}{\sin(2\gamma^*)} = -B\tan(\gamma^*).$$

Notice that $\gamma^* = \tan(\gamma^*)$, so we simply have $A = B\gamma^*$. Therefore the corresponding eigenfunction is given by

$$X(x) = -\gamma^* \cos\Big(\frac{2\gamma^*}{L}x\Big) + \sin\Big(\frac{2\gamma^*}{L}x\Big),$$

since $A = -B\gamma^*$. We took $B = 1$ above. The eigenfunction can be written in terms of $\sqrt{\lambda} = \beta^*$ as

$$X(x) = -\frac{1}{2}L\beta^* \cos\big(\beta^* x\big) + \sin\big(\beta^* x\big).$$

Remark: Using the determinant to find a non-trivial solution is an efficient way to find conditions on β. If we use this method, we have to remember to write one of the coefficients A or B in terms of the other coefficient when solving for the eigenfunction. If solving for β without using determinants, we usually solve for A or B in terms of the other coefficient first.