Week 10

Problem 1. (Strauss 6.1.2) Find the solutions that depend only on r of the equation $u_{xx}+u_{yy}+u_{zz} = k^2u$, where k is a positive constant. (*Hint:* Substitute $u = v/r$.)

Solution 1. Recall that in $\mathbb{R}^3$, if we do a change of variables to spherical coordinates,

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right).$$

If we are looking for solutions that only depend on r, that is $u(r,\phi,\psi) = u(r)$ then we can safely ignore the terms on the right, so $u_{xx} + u_{yy} + u_{zz} = k^2u$ can be expressed in spherical coordinates as

$$u_{rr} + \frac{2}{r}u_r = k^2u.$$

This is a second order ODE, which we can solve using the substitution $u = v/r$. Notice

$$u_r = \frac{v_r}{r} - \frac{v}{r^2}, \quad u_{rr} = \frac{v_{rr}}{r} - 2\frac{v_r}{r^2} + 2\frac{v}{r^3}$$

so under this change of variables, we have

$$u_{rr} + \frac{2}{r}u_r = k^2u \implies \frac{v_{rr}}{r} = k^2\frac{v}{r} \implies v_{rr} - k^2v = 0.$$

This is a second order constant coefficient ODE with roots $r = \pm k$, so

$$v = Ae^{kr} + Be^{-kr} \implies u = A\frac{e^{kr}}{r} + B\frac{e^{-kr}}{r},$$

is the general solution.

Problem 2. (Strauss 6.1.5) Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x,y)$ vanishing on $r = a$.

Solution 2. Since we are on the disk, and neither our source or initial conditions depend on the angle θ we can use rotational invariance to solve this problem. Recall that in $\mathbb{R}^2$, if we do a change of variables to polar form,

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

If we are looking for solutions that only depend on r, that is $u(r,\theta) = u(r)$, then we can safely ignore the terms on the right, so $u_{xx} + u_{yy} = 1$ can be expressed in spherical coordinates as

$$u_{rr} + \frac{1}{r}u_r = 1 \implies ru_{rr} + u_r = r \implies (u_r r)' = r.$$

This ODE can be solved by directly integrating, which implies

$$u_r r = \frac{r^2}{2} + C_1 \implies u_r = r + \frac{C_1}{r} \implies u(r) = \frac{r^2}{4} + C_1\log r + C_2.$$

We now use the boundary conditions to solve for the coefficients. We have the implicit condition $\lim_{r\to 0} u(r) < \infty$ and the boundary condition $u(a) = 0$. Therefore, we must have

$$\lim_{r\to 0} u(r) = \lim_{r\to 0}\frac{r^2}{4} + C_1\log r + C_2 < \infty \text{ and } 0 = u(a) = \frac{a^2}{4} + C_1\log a + C_2.$$

The first condition implies that $C_1 = 0$ and the second condition implies $C_2 = -\frac{a^2}{4}$. Therefore,

$$u(r) = \frac{r^2}{4} - \frac{a^2}{4},$$

is the particular solution.

Problem 3. (Strauss 6.1.10) Prove the uniqueness of the Dirichlet problem $\Delta u = f$ in D, $u = g$ on the boundary of D by the energy method. That is, after subtracting two solution $w = u - v$, multiply the Laplace equation for w by w itself and use the divergence theorem.

Solution 3. Assume that u and v are both solutions to the $\Delta u = f$ in D and $u = g$ on ∂D. If we define $w = u - v$ then $\Delta w = 0$ in D and $w = 0$ on ∂D. Therefore, by integration by parts

$$0 = -\int_D w \Delta w \, dx = \int_D |\nabla w|^2 \, dx - \int_{\partial D} w \frac{\partial w}{\partial \nu} \, dS = \int_D |\nabla w|^2 \, dx$$

which implies that $\nabla w \equiv 0$ in D (in other words, all partials of w are 0 on D). Since $w = 0$ on ∂D we must have $w \equiv 0$ which implies $u = v$ on $\bar{D}$.

Problem 4. (Strauss 6.1.12) Check the validity of the maximum principle for the harmonic function $(1 - x^2 - y^2)/(1 - 2x + x^2 + y^2)$ in the disk $\bar{D} = \{x^2 + y^2 \leq 1\}$. Explain.

Solution 4. One can easily check that

$$\frac{\partial^2}{\partial x^2} \frac{(1 - x^2 - y^2)}{(1 - 2x + x^2 + y^2)} = \frac{4(x-1)(x^2 - 2x - 3y^2 + 1)}{(x^2 - 2x + y^2 + 1)^3} = \frac{\partial^2}{\partial y^2} \frac{(1 - x^2 - y^2)}{(1 - 2x + x^2 + y^2)}$$

so $u(x, y) = \frac{(1-x^2-y^2)}{(1-2x+x^2+y^2)}$ is a solution to $u_{xx} + u_{yy} = 0$. If we factor our solution, notice

$$u(x, y) = \frac{(1 - x^2 - y^2)}{(1 - 2x + x^2 + y^2)} = \frac{1 - (x^2 + y^2)}{(x-1)^2 + y^2}.$$

Notice that on the interior $D = \{x^2 + y^2 < 1\}$, the numerator is positive so

$$\max_{(x,y) \in D} u(x, y) > 0$$

while on the boundary $\partial D = \{x^2 + y^2 = 1\}$ the numerator is 0, so

$$\max_{(x,y) \in \partial D \setminus (1,0)} u(x, y) = 0,$$

(our function is not defined at $(1, 0)$ so we ignore this point). In particular, for this example we have

$$\max_{(x,y) \in \partial D \setminus (1,0)} u(x, y) < \max_{(x,y) \in D} u(x, y),$$

which appears to contradict the maximum principle. However, this is not a counterexample because the maximum principle does not apply to this case, because $u(x, y)$ is not continuous on $\bar{D} = \{x^2 + y^2 \leq 1\}$ since there is a discontinuity at the point $(1, 0)$.

Extra Practice

Problem 1. Use separation of variables to solve the PDE

$$\begin{cases} u_t - k u_{xx} = 0 & & 0 < x < 1 & t > 0 \\ u(0, x) = x & & 0 < x < 1 & \\ u(t, 0) = \sin(t), & u_x(t, 1) + u(t, 1) = 2 & t > 0. & \end{cases}$$

Solution 1. This is an inhomogeneous PDE with time dependent boundary conditions.

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with homogeneous boundary conditions. We set

$$u(t,x) = v(t,x) + w(t,x)$$

where $w(t,x)$ is chosen to satisfy the inhomogeneous boundary conditions. For second order PDEs, we can choose $w(t,x)$ to be a polynomial in x of the form

$$w(t,x) = (Ax^2 + Bx + C)\sin(t) + (Dx^2 + Ex + F)2,$$

for some constants $A, B, \dots F$. Substituting $w(t,x)$ into the boundary conditions gives

$$\begin{aligned} C\sin(t) + 2F &= \sin(t) = w(t,0) \\ (3A + 2B + C)\sin(t) + (3D + 2E + F)2 &= 2 = w(t,\pi). \end{aligned}$$

By inspection it is clear that $C = 1$, $B = \frac{-1}{2}$, and $E = \frac{1}{2}$ with the rest of the coefficients zero works. Therefore,

$$w(t,x) = (-2^{-1}x + 1)\sin(t) + (2^{-1}x)2 = \frac{2 - \sin(t)}{2}x + \sin(t).$$

Step 2 — Separation of Variables: Since $v(t,x) = u(t,x) - w(t,x)$, our choice of $w(t,x)$ implies

$$\begin{cases} v_t - kv_{xx} = \frac{\cos(t)}{2}x - \cos(t) & 0 < x < \pi \quad t > 0 \\ v(0,x) = 0 & 0 < x < \pi \\ v(t,0) = v_x(t,1) + v(t,1) = 0 & t > 0. \end{cases}$$

This is an inhomogeneous PDE with homogeneous boundary conditions. We begin by using separation of variables to solve the homoegenous PDE. We look for a solution of the form $v(t,x) = T(t)X(x)$. For such a solution, the PDE implies

$$T'X - kTX'' = 0 \implies \frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

This results in the ODE

$$X''(x) + \lambda X(x) = 0,$$

with boundary conditions

$$T(t)X(0) = T(t)X'(1) + T(t)X(1) = 0$$

For non-trivial solutions, we can require $T(t) \not\equiv 0$, $X(0) = X'(1) + X(1) = 0$.

Step 3 — Eigenvalue Problem: We solve the spatial eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < \pi \\ X(0) = X'(1) + X(1) = 0. \end{cases}$$

This eigenvalue problem was solved in (Tutorial 7 Q2), and its eigenvalues and corresponding eigenfunctions are given by

$$\lambda_n = \beta_n^2, \ X_n(x) = \sin(\beta_n x), \quad n = 1, 2, 3, \dots$$

where β_n are the ordered positive roots of

$$\tan(\beta) = -\beta.$$

Since the boundary conditions are symmetric, we have that the eigenfunctions $\sin(\beta_n x)$ are orthogonal.

Step 4 — Time Problem: We now use the method of eigenfunction expansion to find $T_n(t)$ that satisfies the inhomogeneous equation. By the principle of superposition, the general solution to the homogeneous PDE is of the form

$$v(t,x) = \sum_{n=1}^{\infty} T_n(t) \sin(\beta_n x).$$

Differentiating term by term (valid since the boundary conditions are homogeneous) and plugging this into our inhomogeneous PDE gives

$$v_t - kv_{xx} = \sum_{n=1}^{\infty} T_n'(t) \sin(\beta_n x) + k \sum_{n=1}^{\infty} T_n(t) \beta_n^2 \sin(\beta_n x) = -\frac{\cos(t)}{2} x + \cos(t).$$

We fix t and write the right hand side of the above equation as the generalized Fourier sine series

$$-\frac{\cos(t)}{2} x + \cos(t) = \sum_{n=1}^{\infty} b_n(t) \sin(\beta_n x)$$

where

$$b_n(t) = \frac{\int_0^1 \left(\frac{\cos(t)}{2} x - \cos(t)\right) \sin(\beta_n x)\, dx}{\int_0^1 \sin^2(\beta_n x)\, dx}.$$

Equating coefficients, we have for $n \geq 1$,

$$T_n'(t) + k\beta_n^2 T_n(t) = b_n(t).$$

This is a first order linear ODE. Its solution can be found using an integrating factor of the form $e^{kn^2 t}$, leading to the general solution

$$T_n(t) = C_n e^{-k\beta_n^2 t} + \int_0^t b_n(s) \exp(-k\beta_n^2 (t-s))\, ds.$$

where C_n is a yet to be determined constant.

Step 5 — Particular Solution: We now use the initial conditions to determine C_n. The initial conditions imply

$$v(0,x) = \sum_{n=1}^{\infty} C_n \sin(\beta_n x) = 0.$$

Clearly we must have $C_n = 0$ for all n.

Step 6 — Final Answer: We now summarize our solution. Recalling that $u = v + w$, we have

$$u(t,x) = \sum_{n=1}^{\infty} \left(\int_0^t b_n(s) \exp(-k\beta_n^2 (t-s))\, ds \right) \sin(\beta_n x) + \frac{2 - \sin(t)}{2} x + \sin(t),$$

where β_n are the ordered positive roots of $\tan(\beta) = -\beta$ and

$$b_n(s) = \frac{\int_0^1 \left(\frac{\cos(s)}{2} x - \cos(s)\right) \sin(\beta_n x)\, dx}{\int_0^1 \sin^2(\beta_n x)\, dx}.$$

Problem 2. Use separation of variables to solve the PDE

$$\begin{cases} u_{tt} - 9u_{xx} = 0 & 0 < x < 1 \quad t > 0 \\ u(0,x) = \phi(x) & 0 < x < 1 \\ u_t(0,x) = \psi(x) & 0 < x < 1 \\ u(t,0) = 0, \quad u_x(t,1) = \frac{1}{2} & t > 0 \end{cases}$$

Solution 2. This is an inhomogeneous PDE with inhomogeneous Neumann boundary conditions.

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with homogeneous boundary conditions. We set

$$u(t,x) = v(t,x) + w(x)$$

where $w(x)$ is chosen to satisfy the inhomogeneous boundary conditions. For second order PDEs, we can choose $w(x)$ to be a polynomial in x of the form

$$w(t,x) = Ax^2 + Bx + C,$$

for some constants A, B, C. Substituting $w(x)$ into the boundary conditions gives

$$C = 0 = w(t,0)$$
$$2A + B = \frac{1}{2} = w(t,1).$$

By inspection it is clear that $B = \frac{1}{2}$, $A = 0$, and $C = 0$ works. Therefore,

$$w(x) = \frac{1}{2}x.$$

Step 2 — Separation of Variables: For this choice of $w(x)$ we have the following homogeneous PDE with homogeneous boundary conditions

$$\begin{cases} u_{tt} - 9u_{xx} = 0 & 0 < x < 1 \quad t > 0 \\ u(0,x) = \phi(x) - \frac{1}{2}x & 0 < x < 1 \\ u_t(0,x) = \psi(x) & 0 < x < 1 \\ u(t,0) = 0, \quad u_x(t,1) = 0 & t > 0 \end{cases}$$

We look for a solution of the form $v(t,x) = T(t)X(x)$. For such a solution, the PDE implies

$$T'X - kTX'' = 0 \implies \frac{T''}{3^2 T} = \frac{X''}{X} = -\lambda.$$

This results in the ODEs

$$X''(x) + \lambda X(x) = 0, \quad T''(t) + 3^2\lambda T(t) = 0$$

with boundary conditions

$$T(t)X(0) = 0 = T(t)X'(1).$$

For non-trivial solutions, we can require $T(t) \not\equiv 0$, $X(0) = X'(1) = 0$.

Step 3 — Eigenvalue Problem: We solve the spatial eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < \pi \\ X(0) = X'(1) = 0. \end{cases}$$

1. $\lambda = \beta^2 > 0$: The solution is of the form

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

From the boundary conditions we get

$$\begin{aligned} A &= 0 \\ -A\sin(\beta) + B\cos(\beta) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 1 & 0 \\ -\sin(\beta) & \cos(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has a non-trivial solution when

$$\begin{vmatrix} 1 & 0 \\ -\sin(\beta) & \cos(\beta) \end{vmatrix} = 0 \implies \cos(\beta) = 0 \implies \beta = \frac{(2n-1)\pi}{2} \text{ for } n = 1, 2, \ldots$$

since $\beta > 0$. The first boundary condition also implies $A = 0$, which means the corresponding eigenfunction to the eigenvalue $\lambda_n = (\frac{(2n-1)\pi}{2L})^2$ is $X_n(x) = \sin(\frac{(2n-1)\pi}{2}x)$.

2. $\lambda = 0$: The solution is of the form

$$X(x) = A + Bx.$$

From the boundary conditions we get

$$\begin{aligned} A &= 0 \\ B &= 0 \end{aligned}$$

which is only satisfied by the trivial solution, so there are no 0 eigenvalues.

3. $\lambda = -\beta^2 < 0$: The solution is of the form

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

From the boundary conditions we get

$$\begin{aligned} A &= 0 \\ A\sinh(\beta) + B\cosh(\beta) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 1 & 0 \\ \sinh(\beta) & \cosh(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has a non-trivial solution when

$$\begin{vmatrix} 1 & 0 \\ \sinh(\beta) & \cosh(\beta) \end{vmatrix} = 0 \implies \cosh(\beta L) = 0$$

which has no positive roots since $-\beta < 0$ and $\cosh(\beta L) > 0$. Therefore, there are no negative eigenvalues.

Therefore, the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \Big(\frac{(2n-1)\pi}{2}\Big)^2, \ X_n(x) = \sin\Big(\frac{(2n-1)\pi}{2}x\Big), \quad n = 1, 2, 3, \ldots$$

Step 4 — Time Problem: The time problem related to the eigenvalues λ_n is

$$T_n''(t) + 3^2\Big(\frac{(2n-1)\pi}{2}\Big)^2 T_n(t) = 0 \text{ for } n = 1, 2, \ldots$$

which has solution

$$T_n(t) = A_n \cos\Big(\frac{3(2n-1)\pi}{2}t\Big) + B_n \sin\Big(\frac{3(2n-1)\pi}{2}t\Big).$$

where A_n and B_n are yet to be determined constants. Taking the linear combination of T_n with the eigenfunctions imply our general solution is of the form,

$$v(t,x) = \sum_{n=1}^{\infty} \bigg(A_n \cos\Big(\frac{3(2n-1)\pi}{2}t\Big) + B_n \sin\Big(\frac{3(2n-1)\pi}{2}t\Big)\bigg) \sin\Big(\frac{(2n-1)\pi}{2}x\Big)$$

Step 5 — Particular Solution: The initial conditions imply

$$v(0,x) = \phi(x) - \frac{x}{2} \implies \sum_{n=1}^{\infty} A_n \sin\Big(\frac{(2n-1)\pi}{2}x\Big) = \phi(x) - \frac{x}{2}$$

and

$$v_t(0,x) = \psi(x) \implies \sum_{n=1}^{\infty} B_n \frac{3(2n-1)\pi}{2} \sin\Big(\frac{(2n-1)\pi}{2L}x\Big) = \psi(x).$$

The eigenfunction corresponding to symmetric boundary conditions are orthogonal so the coefficients are given by

$$\begin{aligned} A_n = \frac{\langle \phi(x) - \frac{x}{2}, X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} &= \frac{\int_0^1 \Big(\phi(x) - \frac{x}{2}\Big)\sin\Big(\frac{(2n-1)\pi}{2}x\Big)\,dx}{\int_0^1 \sin^2\Big(\frac{(2n-1)\pi}{2}x\Big)\,dx} \\ &= 2 \cdot \int_0^1 \Big(\phi(x) - \frac{x}{2}\Big)\sin\Big(\frac{(2n-1)\pi}{2}x\Big)\,dx \end{aligned}$$

and

$$\begin{aligned} B_n = \Big(\frac{3(2n-1)\pi}{2}\Big)^{-1} \frac{\langle \psi(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} &= \Big(\frac{3(2n-1)\pi}{2}\Big)^{-1} \cdot \frac{\int_0^1 \psi(x)\sin\Big(\frac{(2n-1)\pi}{2}x\Big)\,dx}{\int_0^1 \sin^2\Big(\frac{(2n-1)\pi}{2}x\Big)\,dx} \\ &= \frac{4}{3(2n-1)\pi} \cdot \int_0^1 \psi(x)\sin\Big(\frac{(2n-1)\pi}{2}x\Big)\,dx. \end{aligned}$$

Step 6 — Final Answer: We now summarize our solution. Recalling that $u = v + w$, we have

$$u(t,x) = \sum_{n=1}^{\infty} \bigg(A_n \cos\Big(\frac{3(2n-1)\pi}{2}t\Big) + B_n \sin\Big(\frac{3(2n-1)\pi}{2}t\Big)\bigg) \sin\Big(\frac{(2n-1)\pi}{2}x\Big) + \frac{1}{2}x$$

where the coefficients are given by

$$A_n = 2 \cdot \int_0^1 \Big(\phi(x) - \frac{x}{2}\Big)\sin\Big(\frac{(2n-1)\pi}{2}x\Big)\,dx, \quad B_n = \frac{4}{3(2n-1)\pi} \cdot \int_0^1 \psi(x)\sin\Big(\frac{(2n-1)\pi}{2}x\Big)\,dx.$$

Problem 3. Use separation of variables to solve the PDE

$$\begin{cases} u_t - ku_{xx} = e^{-x} & 0 < x < \pi \qquad t > 0 \\ u(0,x) = \phi(x) & 0 < x < \pi \\ u_x(t,0) = 1, \qquad u_x(t,\pi) = 0 & t > 0. \end{cases}$$

Solution 3. This is an inhomogeneous PDE with inhomogeneous Neumann boundary conditions.

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with homogeneous boundary conditions. We set

$$u(t,x) = v(t,x) + w(x)$$

where $w(x)$ is chosen to satisfy the inhomogeneous boundary conditions. For second order PDEs, we can choose $w(x)$ to be a polynomial in x of the form

$$w(t,x) = Ax^2 + Bx + C,$$

for some constants A, B, C. Substituting $w(x)$ into the boundary conditions gives

$$\begin{aligned} B = 1 &= w(t,0) \\ 2\pi A + B = 0 &= w(t,\pi). \end{aligned}$$

By inspection it is clear that $B = 1$, $C = -\frac{1}{2\pi}$, and $C = 0$ works. Therefore,

$$w(x) = -\frac{1}{2\pi}x^2 + x.$$

Step 2 — Separation of Variables: For this choice of $w(x)$ we have the following inhomogeneous PDE with homogeneous boundary conditions

$$\begin{cases} v_t - kv_{xx} = e^{-x} - k\frac{1}{\pi} & 0 < x < \pi \quad t > 0 \\ v(0,x) = \phi(x) + \frac{1}{2\pi}x^2 - x & 0 < x < \pi \\ v_x(t,0) = v_x(t,\pi) = 0 & t > 0. \end{cases}$$

We look for a solution of the form $v(t,x) = T(t)X(x)$. For such a solution, the PDE implies

$$T'X - kTX'' = 0 \implies \frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

This results in the ODEs

$$X''(x) + \lambda X(x) = 0,$$

with boundary conditions

$$T(t)X'(0) = 0 = T(t)X'(\pi).$$

For non-trivial solutions, we can require $T(t) \not\equiv 0$, $X'(0) = X'(\pi) = 0$.

Step 3 — Eigenvalue Problem: We solve the spatial eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < \pi \\ X'(0) = X'(\pi) = 0. \end{cases}$$

This is a standard eigenvalue problem and the eigenvalues and corresponding eigenfunctions are

$$\lambda_0 = 0, \ X_0 = 1 \quad \lambda_n = n^2, \ X_n(x) = \cos(nx), \quad n = 1, 2, 3, \ldots$$

Step 4 — Time Problem: We now use the method of eigenfunction expansion to find $T_n(t)$ that satisfies the inhomogeneous equation. By the principle of superposition, the general solution to the homogeneous PDE is of the form

$$v(t,x) = T_0 + \sum_{n=1}^{\infty} T_n(t)\cos(nx).$$

Differentiating term by term (valid since the boundary conditions are homogeneous) and plugging this into our inhomogeneous PDE gives

$$v_t - kv_{xx} = T_0' + \sum_{n=1}^{\infty} T_n'(t)\cos(nx) + k\sum_{n=1}^{\infty} T_n(t)n^2\cos(nx) = e^{-x} - k\frac{1}{\pi}.$$

We write the right hand side of the above equation as the generalized Fourier sine series

$$e^{-x} - k\frac{1}{\pi} = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx)$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi}\Big(e^{-x} - k\frac{1}{\pi}\Big)\cos(nx)\,dx = \frac{2}{\pi}\cdot\frac{(-1)^{n+1}e^{-\pi}+1}{n^2+1}$$

and

$$a_0 = \frac{1}{\pi}\int_0^{\pi}\Big(e^{-x} - k\frac{1}{\pi}\Big)\,dx = \frac{-k+1+\sinh(\pi)-\cosh(\pi)}{\pi}.$$

Equating coefficients, we have for $n \geq 1$,

$$T_0'(t) = a_0 \quad T_n'(t) + kn^2T_n(t) = a_n.$$

This is a first order linear ODE. Its solution can be found using an integrating factor of the form e^{kn^2t}, leading to the general solution

$$T_0(t) = a_0t + A_0, \quad T_n(t) = A_ne^{-kn^2t} + \frac{a_n}{kn^2}.$$

where A_0, A_n are yet to be determined constants.

Step 5 — Particular Solution: We now use the initial conditions to determine A_0, and A_n. The initial conditions imply

$$v(0,x) = A_0 + \sum_{n=1}^{\infty} A_n\cos(nx) = \frac{1}{2\pi}x^2.$$

The coefficients A_0, A_n are the coefficients of the Fourier cosine series of $\phi(x) + \frac{1}{2\pi}x^2 - x$, which is given explicitly by

$$A_n = \frac{2}{\pi}\int_0^{\pi}\Big(\phi(x) + \frac{1}{2\pi}x^2 - x\Big)\cos(nx)\,dx \quad A_0 = \frac{1}{\pi}\int_0^{\pi}\Big(\phi(x) + \frac{1}{2\pi}x^2 - x\Big)\,dx.$$

Step 6 — Final Answer: We now summarize our solution. Recalling that $u = v + w$, we have

$$\begin{aligned} u(t,x) &= \frac{-k+1+\sinh(\pi)-\cosh(\pi)}{\pi}\cdot t + A_0 \\ &\quad + \sum_{n=1}^{\infty}\Big(A_n\cdot e^{-kn^2t} + \frac{2}{\pi kn^2}\cdot\frac{(-1)^{n+1}e^{-\pi}+1}{n^2+1}\Big)\cos(nx) - \frac{1}{2\pi}x^2 + x \end{aligned}$$

where A_0 and A_n are given by

$$A_n = \frac{2}{\pi}\int_0^{\pi}\Big(\phi(x) + \frac{1}{2\pi}x^2 - x\Big)\cos(nx)\,dx \quad A_0 = \frac{1}{\pi}\int_0^{\pi}\Big(\phi(x) + \frac{1}{2\pi}x^2 - x\Big)\,dx.$$