Fourier Series

The (full) Fourier series of $f : [-L, L] \to \mathbb{R}$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big),$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

Problem 1. Decompose the following functions into its Fourier series on the interval $[-1, 1]$ and sketch the graph of the sum of the first three nonzero terms of its Fourier series.

(a) $f(x) = x$

(b) $f(x) = |x|$

Solution 1.

(a) Since x is odd, only the b_n coefficients are non-zero. The Fourier coefficients are given explicitly by:

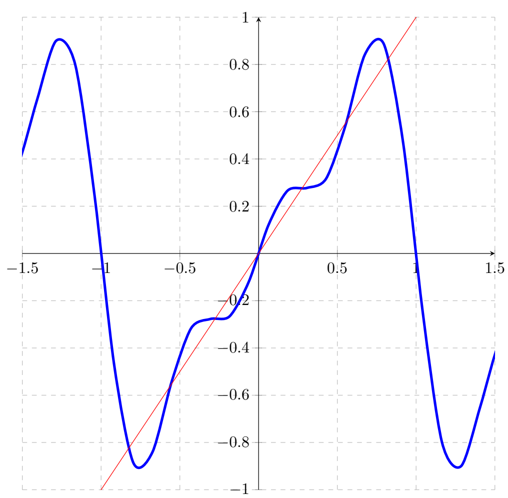
b_n**:** Using integration by parts,

$$b_n = \int_{-1}^{1} x \sin(n\pi x)\,dx = 2\int_{0}^{1} x\sin(n\pi x)\,dx = -\frac{2(-1)^n}{\pi n}.$$

Therefore, the Fourier sine series of x is given by

$$x = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi n}\sin(n\pi x).$$

The plot of the first 3 non-zero terms is below:



(b) Since $|x|$ is even, only the a_n coefficients are non-zero. The Fourier coefficients are given explicitly by:

a_0**:** A simple computation shows

$$a_0 = \int_{-1}^{1} |x| \, dx = 2 \int_0^1 x \, dx = 1.$$

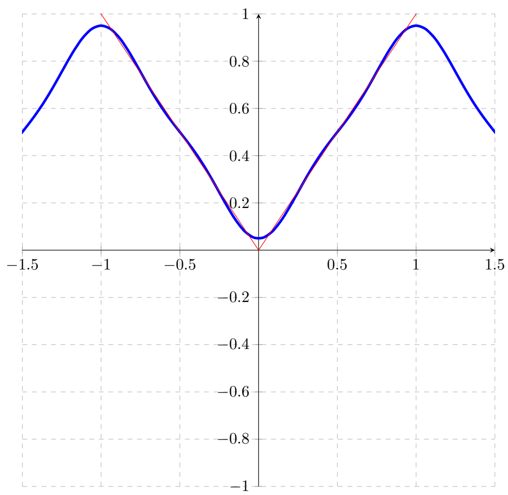
a_n**:** Using integration by parts,

$$a_n = \int_{-1}^{1} |x| \cos(n\pi x) \, dx = 2 \int_0^1 x \cos(n\pi x) \, dx = \frac{2((-1)^n - 1)}{\pi^2 n^2}.$$

Therefore, the corresponding Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{\pi^2 n^2} \cos(n\pi x).$$

The plot of the first 3 non-zero terms is below:



Problem 2. Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < \pi, \ 0 < t < \infty \\ u(x,0) = 0 & 0 < x < \pi \\ u_t(x,0) = x & 0 < x < \pi \\ u_x(0,t) = u_x(\pi,t) = 0 & 0 < t < \infty. \end{cases}$$

Solution 2. This is a homogeneous PDE with vanishing Neumann boundary conditions.

Step 1 — Separation of Variables: We first find a solution to the homogeneous equation.

$$T''(t)X(x) - c^2 T(t)X''(x) = 0 \implies \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

with boundary conditions

$$T(t)X'(0) = T(t)X'(\pi) = 0 \implies X'(0) = X'(\pi) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < \pi \\ X'(0) = X'(\pi) = 0. \end{cases}$$

This is a standard eigenvalue problem with solution

Eigenvalues: $\lambda_n = n^2$ for $n = 0, 1, 2, \ldots$

Eigenfunctions: $X_n = \cos(nx)$ and $X_0 = 1$.

Step 3 — Time Problem: When $n = 0$, the time problem is

$$T_0''(t) = 0 \text{ for } n = 1, 2, \ldots$$

which has solution

$$T_0(t) = A_0 + B_0 t.$$

The time problem related to the eigenvalues λ_n for $n \geq 1$ is

$$T_n''(t) + c^2 n^2 T_n(t) = 0 \text{ for } n = 1, 2, \ldots$$

which has solution

$$T_n(t) = A_n \cos(cnt) + B_n \sin(cnt).$$

Step 4 — General Solution: By the principle of superposition, the general form of our solution is

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = A_0 + B_0 t + \sum_{n=1}^{\infty} \Big(A_n \cos(cnt) + B_n \sin(cnt) \Big) \cos(nx).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution by solving for the constants A_n and B_n. The initial conditions imply

$$u(x,0) = \phi(x) \implies A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) = 0$$

and

$$u_t(x,0) = \psi(x) \implies B_0 + \sum_{n=1}^{\infty} B_n cn \cos(nx) = x.$$

Clearly the first initial condition implies that $A_n = 0$ for all $n \geq 0$. To find the B_n coefficients, we decompose x into its Fourier cosine series (or equivalently, decomposing $|x|$ into its full Fourier series on $[-\pi, \pi]$)

$$x = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{\pi n^2} \cos(nx)$$

and equate coefficients to conclude

$$B_0 = \frac{\pi}{2}, \qquad B_n cn = \frac{2((-1)^n - 1)}{\pi n^2} \implies B_n = \frac{2((-1)^n - 1)}{c\pi n^3}.$$

Therefore, our particular solution is

$$u(x,t) = B_0 t + \sum_{n=1}^{\infty} B_n \sin(cnt) \cos(nx)$$

where $B_0 = \frac{\pi}{2}$ and $B_n = \frac{2((-1)^n - 1)}{c\pi n^3}$.

Fourier Transforms

The Fourier transform of $f : \mathbb{R} \to \mathbb{C}$ is given by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

and the corresponding inverse Fourier transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk.$$

We will also use the following two properties of Fourier Transforms,

1. $\widehat{\frac{df}{dx}}(k) = ik\hat{f}(k)$.
2. $(\widehat{f * g})(k) = \sqrt{2\pi}\hat{f}(k)\hat{g}(k)$.

Problem 3. Find the Fourier Transform of

$$f(x) = e^{-a|x|} \qquad a > 0.$$

Solution 3. We split the region of integration,

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-a|x|}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{-ikx+ax}\, dx + \int_{0}^{\infty} e^{-ikx-ax}\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ikx+ax}}{a-ik} \Big|_{x=-\infty}^{x=0} + \frac{e^{-ikx-ax}}{-a-ik} \Big|_{x=0}^{x=\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a-ik} + \frac{1}{a+ik} \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+k^2}.
\end{aligned}$$

Problem 4. Using the properties of the Fourier transform, recover the general formula for the solution $u(x,t)$ of Laplace's equation

$$u_{xx} + u_{yy} = 0,\ x \in \mathbb{R},\ y \geq 0 \qquad u(x,0) = \phi(x).$$

Solution 4.

Step 1 — Transform the Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to Laplace's equation, and consider its Fourier transform

$$\hat{u}(k,y) = \int_{-\infty}^{\infty} e^{-ikx} u(x,y)\, dx.$$

Since $u_{xx} + u_{yy} = 0$, taking the Fourier transform of both sides implies that

$$-k^2\hat{u}(k,y) + \hat{u}_{yy}(k,y) = 0 \qquad y > 0.$$

The solution to this ODE (in y) is given by

$$\hat{u}(k,y) = A(k)e^{-ky} + B(k)e^{ky},$$

where $A(k)$ and $B(k)$ are some yet to be determined functions of k.

Step 2 — Find the Particular Solution: Since our solution should be bounded for $y \geq 0$, we have $B(k) = 0$ for $k > 0$ and $A(k) = 0$ for $k < 0$. The general solution can be simplified as

$$\hat{u}(k,y) = C(k)e^{-|k|y}, \qquad C(k) = \begin{cases} A(k) & k > 0 \\ A(0) + B(0) & k = 0 \\ B(k) & k < 0 \end{cases}$$

We can find $C(k)$ by using our initial condition,

$$u(x,0) = \phi(x) \implies \hat{u}(k,0) = \hat{\phi}(k) \implies C(k) = \hat{\phi}(k).$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,y) = \hat{\phi}(k)e^{-|k|y}.$$

Step 3 — Recover the Solution: We take the inverse Fourier transform of both sides to recover our original function. Let $S(x,y)$ be the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}}e^{-|k|y}$,

$$\begin{aligned} S(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-y|k|}\,dk &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{ikx+yk}\,dk + \frac{1}{2\pi}\int_{0}^{\infty} e^{ikx-yk}\,dk \\ &= \frac{1}{2\pi}\frac{e^{ikx+yk}}{ix+y}\Big|_{k=-\infty}^{k=0} + \frac{1}{2\pi}\frac{e^{ikx-yk}}{ix-y}\Big|_{k=0}^{k=\infty} \\ &= \frac{1}{2\pi}\Big(\frac{1}{ix+y} - \frac{1}{ix-y}\Big) \\ &= \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}. \end{aligned}$$

Since $\hat{u}(k,y) = \hat{\phi}(k)e^{-|k|y} = \sqrt{2\pi}\hat{\phi}(k)\cdot\frac{e^{-|k|y}}{\sqrt{2\pi}}$, taking the inverse Fourier transform of both sides implies

$$u(x,y) = (\phi(\cdot) * S(\cdot,y))(x) = \int_{-\infty}^{\infty} S(x-\tau,y)\phi(\tau)\,d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\tau)^2+y^2}\phi(\tau)\,d\tau.$$

Problem 5. Using the properties of the Fourier transform, recover the general formula for the solution $u(x,t)$ of the heat equation

$$u_t - u_{xx} = 0,\ x \in \mathbb{R},\ t \geq 0 \qquad u(x,0) = \phi(x).$$

Solution 5.

Step 1 — Transform the Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to the heat equation, and consider its Fourier transform

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} e^{-ikx}u(x,t)\,dx.$$

Since $u_t - u_{xx} = 0$, taking the Fourier transform of both sides implies that

$$\hat{u}_t(k,t) + k^2\hat{u}(k,t) = 0 \qquad y > 0.$$

The solution to this ODE (in t) is given by

$$\hat{u}(k,t) = A(k)e^{-k^2 t},$$

where $A(k)$ is some yet to be determined function of k.

Step 2 — Find the Particular Solution: We can find $A(k)$ by using our initial condition,

$$u(x,0) = \phi(x) \implies \hat{u}(k,0) = \hat{\phi}(k) \implies A(k) = \hat{\phi}(k).$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,t) = \hat{\phi}(k)e^{-k^2 t}.$$

Step 3 — Recover the Solution: We take the inverse Fourier transform of both sides to recover our original function. Let $S(x,t)$ be the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}}e^{-k^2 t}$,

$$\begin{aligned} S(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-k^2 t}\,dk &= \frac{1}{2\pi}e^{-\frac{x^2}{4t}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\,dk \\ &= \frac{1}{2\pi}e^{-\frac{x^2}{4t}}\cdot\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-z^2}\,dz && \text{(See the Remark)} \\ &= \frac{1}{\sqrt{4\pi t}}\cdot e^{-\frac{x^2}{4t}}. && \int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}. \end{aligned}$$

Since $\hat{u}(k,y) = \hat{\phi}(k)e^{-k^2 t} = \sqrt{2\pi}\hat{\phi}(k)\cdot\frac{e^{-k^2 t}}{\sqrt{2\pi}}$, taking the inverse Fourier transform of both sides implies

$$u(x,t) = (\phi(\cdot) * S(\cdot,t))(x) = \int_{-\infty}^{\infty} S(x-\tau,t)\phi(\tau)\,d\tau = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{4t}}\phi(\tau)\,d\tau.$$

Remark: The imaginary change of variables $z = \sqrt{t}k - i\frac{x}{2\sqrt{t}}$ can be justified using complex analysis.

$$\int_{\mathbb{R}} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\,dk = \frac{1}{\sqrt{t}}\int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz.$$

Consider the contour integral over the closed rectangular path oriented counter clockwise,

(1) $k - i\frac{x}{2\sqrt{t}}$ for k from $-M$ to M

(2) $M + iy$ for y from $-\frac{x}{2\sqrt{t}}$ to 0

(3) k for k from M to $-M$

(4) $M + iy$ for y from 0 to $-\frac{x}{2\sqrt{t}}$.

Since e^{-z^2} is analytic, the integral over this closed path is 0, so

$$\int_{(1)} e^{-z^2}\,dz + \int_{(2)} e^{-z^2}\,dz + \int_{(3)} e^{-z^2}\,dz + \int_{(4)} e^{-z^2}\,dz = 0.$$

Since e^{-z^2} is small when the $\mathrm{Re}(z) = \pm M$, if we take $M \to \infty$, the integrals over the regions (2) and (4) vanish leaving us with

$$\int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\, dz + \int_{\infty}^{-\infty} e^{-z^2}\, dz = 0 \implies \int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\, dz = \int_{-\infty}^{\infty} e^{-z^2}\, dz.$$

Therefore,

$$\int_{\mathbb{R}} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\, dk = \frac{1}{\sqrt{t}} \int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\, dz = \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-z^2}\, dz.$$