Week 8: Boundary Value Problems

Problem 1. Solve the following PDE:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L \quad t > 0 \\ u(0,x) = \phi(x) \\ u_t(0,x) = \psi(x) \\ u_x(t,0) = 0 = u(t,L) \end{cases}$$

Solution 1. This is a homogeneous PDE with vanishing boundary conditions.

Step 1 — Separation of Variables: We look for a separated solution $u(t,x) = T(t)X(x)$ to our PDE. Plugging this into our PDE gives

$$T''(t)X(x) - c^2 T(t)X''(x) = 0 \implies \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This gives the following ODEs

$$X''(x) + \lambda X(x) = 0 \text{ and } T''(t) + c^2 \lambda T(t) = 0,$$

with boundary conditions

$$T(t)X'(0) = 0 = T(t)X(L) \implies X'(0) = X(L) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < L \\ X'(0) = X(L) = 0. \end{cases}$$

We consider the 3 cases corresponding to the different forms of the ODE:

1. $\lambda = \beta^2 > 0$: The solution is of the form

   $$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

   From the boundary conditions we get

   $$\begin{aligned} \beta B &= 0 \\ A\cos(\beta L) + B\sin(\beta L) &= 0. \end{aligned}$$

   We can write this system of equations in matrix form

   $$\begin{bmatrix} 0 & \beta \\ \cos(\beta L) & \sin(\beta L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

   which has a non-trivial solution when

   $$\begin{vmatrix} 0 & \beta \\ \cos(\beta L) & \sin(\beta L) \end{vmatrix} = 0 \implies -\beta\cos(\beta L) = 0 \implies \beta = \frac{(2n-1)\pi}{2L} \text{ for } n = 1, 2, \ldots$$

   since $\beta > 0$. The first boundary condition also implies $B = 0$, which means the corresponding eigenfunction to the eigenvalue $\lambda_n = (\frac{(2n-1)\pi}{2L})^2$ is $X_n(x) = \cos(\frac{(2n-1)\pi}{2L} x)$.

2. $\lambda = 0$: The solution is of the form
$$X(x) = A + Bx.$$
From the boundary conditions we get
$$\begin{aligned} B &= 0 \\ A + BL &= 0. \end{aligned}$$
We can write this system of equations in matrix form
$$\begin{bmatrix} 0 & 1 \\ 1 & L \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
which has has no non-trivial solutions because the first matrix is invertible. Therefore, there are no 0 eigenvalues.

3. $\lambda = -\beta^2 < 0$: The solution is of the form
$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$
From the boundary conditions we get
$$\begin{aligned} \beta B &= 0 \\ A\cosh(\beta L) + B\sinh(\beta L) &= 0. \end{aligned}$$
We can write this system of equations in matrix form
$$\begin{bmatrix} 0 & \beta \\ \cosh(\beta L) & \sinh(\beta L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
which has a non-trivial solution when
$$\begin{vmatrix} 0 & \beta \\ \cosh(\beta L) & \sinh(\beta L) \end{vmatrix} = 0 \implies -\beta\cosh(\beta L) = 0$$
which has no positive roots since $-\beta < 0$ and $\cosh(\beta L) > 0$. Therefore, there are no negative eigenvalues.

Therefore, the solution to the eigenvalue problem is

Eigenvalues:
$$\lambda_n = \Big(\frac{(2n-1)\pi}{2L}\Big)^2 \text{ for } n = 1, 2, 3, \dots.$$

Eigenfunctions:
$$X_n(x) = \cos\Big(\frac{(2n-1)\pi}{2L}x\Big).$$

Step 3 — Time Problem: The time problem related to the eigenvalues λ_n is
$$T_n''(t) + c^2\Big(\frac{(2n-1)\pi}{2L}\Big)^2 T_n(t) = 0 \text{ for } n = 1, 2, \dots$$
which has solution
$$T_n(t) = A_n\cos\Big(\frac{c(2n-1)\pi}{2L}t\Big) + B_n\sin\Big(\frac{c(2n-1)\pi}{2L}t\Big).$$

Step 4 — General Solution: By the principle of superposition, the general form of our solution is
$$u(t,x) = \sum_{n=1}^{\infty} T_n(t)X_n(t) = \sum_{n=1}^{\infty}\bigg(A_n\cos\Big(\frac{c(2n-1)\pi}{2L}t\Big) + B_n\sin\Big(\frac{c(2n-1)\pi}{2L}t\Big)\bigg)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution by solving for the constants A_n and B_n. The initial conditions imply

$$u(0,x) = \phi(x) \implies \sum_{n=1}^{\infty} A_n \cos\Big(\frac{(2n-1)\pi}{2L}x\Big) = \phi(x) \tag{1}$$

and

$$u_t(0,x) = \psi(x) \implies \sum_{n=1}^{\infty} B_n \frac{c(2n-1)\pi}{2L} \cos\Big(\frac{(2n-1)\pi}{2L}x\Big) = \psi(x).$$

The eigenfunction corresponding to symmetric boundary conditions are orthogonal so the coefficients are given by

$$A_n = \frac{\langle \phi(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} = \frac{\int_0^L \phi(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx}{\int_0^L \cos^2\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx} = \frac{2}{L}\cdot\int_0^L \phi(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx$$

and

$$\begin{aligned} B_n = \Big(\frac{c(2n-1)\pi}{2L}\Big)^{-1}\frac{\langle \psi(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} &= \Big(\frac{c(2n-1)\pi}{2L}\Big)^{-1}\cdot\frac{\int_0^L \psi(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx}{\int_0^L \cos^2\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx} \\ &= \Big(\frac{c(2n-1)\pi}{2L}\Big)^{-1}\cdot\frac{2}{L}\cdot\int_0^L \psi(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx. \end{aligned}$$

Remark: We got the formulas for the coefficients by using othogonality of the eigenfunctions. Namely, $\langle X_n(x), X_m(x)\rangle = 0$ whenever $m \neq n$. For example, to recover the coefficient of A_k, we can take the inner product of both sides of (1) with respect to $X_k(x)$ and notice

$$\sum_{n=1}^{\infty}\langle A_n X_n(x), X_k(x)\rangle = A_k\langle X_k(x), X_k(x)\rangle = \langle \phi(x), X_k(x)\rangle \implies A_k(x) = \frac{\langle \phi(x), X_k(x)\rangle}{\langle X_k(x), X_k(x)\rangle}.$$

Remark: It is easy to check that these mixed boundary conditions satisfy the symmetry condition. For example, if X_1 and X_2 satisfy the boundary conditions $X_1'(0) = 0$, $X_1(L) = 0$ and $X_2'(0) = 0$, $X_2(L) = 0$ then they satisfy the symmetric condition

$$X_1'(x)X_2(x) - X_1(x)X_2'(x)\Big|_0^L = X_1'(L)X_2(L) - X_1(L)X_2'(L) - X_1'(0)X_2(0) + X_1(0)X_2'(0) = 0,$$

so the eigenfunctions of distinct eigenvalues are orthogonal.

Problem 2.

Solve the following PDE:

$$\begin{cases} u_t = ku_{xx} & 0 < x < 1 \qquad t > 0 \\ u(0,x) = x \\ u_x(t,0) = 0, \quad u_x(t,1) + u(t,1) = 0 \end{cases}$$

Solution 2. This is a homogeneous PDE with vanishing boundary conditions.

Step 1 — Separation of Variables: We look for a separated solution $u(t,x) = T(t)X(x)$ to our PDE. Plugging this into our PDE gives

$$T'(t)X(x) - kT(t)X''(x) = 0 \implies \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This implies the following ODEs

$$X''(x) + \lambda X(x) = 0 \text{ and } T'(t) + k\lambda T(t) = 0,$$

with boundary conditions

$$T(t)X'(0) = 0 \text{ and } T(t)X'(1) + T(t)X(1) = 0 \implies X'(0) = X'(1) + X(1) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < 1 \\ X'(0) = X'(1) + X(1) = 0. \end{cases}$$

We consider the 3 cases corresponding to the different forms of the ODE:

1. $\lambda = \beta^2 > 0$: The solution is of the form

   $$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

   From the boundary conditions we get

   $$\begin{aligned} \beta B &= 0 \\ -\beta A\sin(\beta) + \beta B\cos(\beta) + A\cos(\beta) + B\sin(\beta) &= 0. \end{aligned}$$

   We can write this system of equations in matrix form

   $$\begin{bmatrix} 0 & \beta \\ \cos(\beta) - \beta\sin(\beta) & \beta\cos(\beta) + \sin(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

   which has a non-trivial solution when

   $$\begin{vmatrix} 0 & \beta \\ \cos(\beta) - \beta\sin(\beta) & \beta\cos(\beta) + \sin(\beta) \end{vmatrix} = 0 \implies \beta\cos(\beta) - \beta^2\sin(\beta) = 0.$$

   If β_n is chosen such that $\cos(\beta_n) = 0$, then $\beta_n \neq 0$ and $\sin(\beta_n) \neq 0$ which means there are no solutions such that $\cos(\beta_n) = 0$. Therefore, we can rearrange terms to recover the condition

   $$\beta\cos(\beta) - \beta^2\sin(\beta) \implies \tan(\beta) = \frac{1}{\beta}.$$

   The eigenvalues β_n are the positive roots of $\tan(\beta) = \frac{1}{\beta}$ for which there are infinitely many of them. The first boundary condition also implies $B = 0$, which means the corresponding eigenfunction of the eigenvalue $\lambda_n = \beta_n^2$ is $X_n = \cos\big(\beta_n x\big)$.

2. $\lambda = 0$: The solution is of the form

   $$X(x) = A + Bx.$$

   From the boundary conditions we get

   $$\begin{aligned} B &= 0 \\ A + 2B &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has has no non-trivial solutions because the first matrix is invertible. Therefore, there are no 0 eigenvalues.

3. $\lambda = -\beta^2 < 0$: The solution is of the form

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

From the boundary conditions we get

$$\begin{aligned} \beta B &= 0 \\ \beta A\sinh(\beta) + \beta B\cosh(\beta) + A\cosh(\beta) + B\sinh(\beta) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 0 & \beta \\ \cosh(\beta) + \beta\sinh(\beta) & \beta\cosh(\beta) + \sinh(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has a non-trivial solution when

$$\begin{vmatrix} 0 & \beta \\ \cosh(\beta) + \beta\sinh(\beta) & \beta\cosh(\beta) + \sinh(\beta) \end{vmatrix} = 0 \implies \beta\cosh(\beta) + \beta^2\sinh(\beta) = 0.$$

Since $\cosh(\beta) > 0$ and $\beta > 0$, we can write the above as

$$\tanh(\beta) = -\frac{1}{\beta}$$

which has no positive roots. Therefore, there are no negative eigenvalues.

Therefore, the solution to the eigenvalue problem is

Eigenvalues: $\lambda_n = \beta_n^2$ for $n = 1, 2, \dots$ where β_n are the ordered positive roots of $\tan(\beta) = \frac{1}{\beta}$

Eigenfunctions: $X_n = \cos\big(\beta_n x\big)$.

Step 3 — Time Problem: The time problem related to the eigenvalues λ_n is

$$T_n'(t) + k(\beta_n)^2 T_n(t) = 0 \text{ for } n = 1, 2, \dots$$

which has solution

$$T_n(t) = A_n e^{-k\beta_n^2 t}.$$

Step 4 — General Solution: By the principle of superposition, the general form of our solution is

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(t) = \sum_{n=1}^{\infty} A_n e^{-k\beta_n^2 t}\cos(\beta_n x).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution by solving for the constants A_n. The initial conditions imply

$$u(0,x) = x \implies \sum_{n=1}^{\infty} A_n \cos(\beta_n x) = x.$$

The eigenfunction corresponding to Robin boundary conditions are also symmetric boundary conditions, so the eigenfunctions are orthogonal. Therefore, the coefficients are given by

$$A_n = \frac{\langle x, X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} = \frac{\int_0^1 x\cos(\beta_n x)\,dx}{\int_0^1 \cos^2(\beta_n x)\,dx}.$$