Week 6: Reflections

Introduction

Suppose we have the heat equation or the wave equation on the half line. We want to reduce this problem to a PDE on the entire line by finding an appropriate extension of the initial conditions that satisfies the given boundary conditions. The restriction of the solution to the extended problem to the half line is the unique solution to the PDE on the half line.

The choice of the extension only depends on the boundary conditions:

1. Dirichlet ($u(0,t) = 0$): We take an odd extension of the initial conditions

$$\phi_{odd} = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \\ 0 & x = 0 \end{cases}.$$

2. Neumann ($u_x(0,t) = 0$): We take an even extension of the initial conditions

$$\phi_{even} = \begin{cases} \phi(x) & x \geq 0 \\ \phi(-x) & x \leq 0 \end{cases}.$$

We will go over several examples of different types of extensions for various boundary conditions.

Problems

Problem 1. Solve the following problem

$$\begin{cases} u_t = ku_{xx} & 0 < x < \infty, \quad t > 0 \\ u(x,0) = \phi(x) & 0 < x < \infty, \\ u_x(0,t) - hu(0,t) = 0 & t > 0 \end{cases}$$

where h is a constant. Solve this problem first in general, then in the case where $\phi(x) = x$.

Solution 1. Suppose we find an extension of the initial conditions such that $u_x(0,t) - hu(0,t) = 0$ for all $t > 0$. Then our solution will be of the form

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi_{ext}(y)\, dy = \frac{1}{\sqrt{4kt}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4\pi kt}} \phi_{ext}(x-y)\, dy,$$

using the change of variables $y \to x - y$. Plugging this into our boundary condition, we must have

$$u(0,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}} \phi_{ext}(-y)\, dy$$

and

$$u_x(x,t)\Big|_{x=0} = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}} \phi'_{ext}(x-y)\, dy\Big|_{x=0} = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}} \phi'_{ext}(-y)\, dy.$$

Since we must have $u_x(0,t) - hu(0,t) = 0$ for all $t > 0$, we can therefore conclude that

$$\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}} (\phi'_{ext}(-y) - h\phi_{ext}(-y))\, dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}} (\phi'_{ext}(y) - h\phi_{ext}(y))\, dy = 0.$$

A sufficient condition for this to hold is for the integrand to be odd, and in particular this means that

$$\phi'_{ext}(y) - h\phi_{ext}(y) = \begin{cases} \phi'(y) - h\phi(y) & y > 0 \\ -\phi'(-y) + h\phi(-y) & y < 0. \end{cases}$$

Therefore, for $y < 0$, our extended function must satisfy the first order linear ODE

$$\phi'_{ext}(y) - h\phi_{ext}(y) = -\phi'(-y) + h\phi(-y)$$

which can be solved using an integrating factor of the form e^{-hy}, giving us

$$(\phi_{ext}(y)e^{-hy})' = e^{-hy}(h\phi(-y) - \phi'(-y)) \implies \phi_{ext}(y) = Ce^{hy} + e^{hy}\int_0^y e^{-hs}(h\phi(-s) - \phi'(-s))\,ds$$

To make our solution continuous, we will require that $\phi_{ext}(0) = \phi(0^+) \implies C = \phi(0^+)$. Therefore, we have the extended function is given by

$$\phi_{ext}(y) = \begin{cases} \phi(y) & y > 0 \\ e^{hy}\phi(0^+) + e^{hy}\int_0^y e^{-hs}(h\phi(-s) - \phi'(-s))\,ds & y \leq 0 \end{cases}$$

Notice that if $h = 0$, then the above is just the even extension of ϕ as expected. The corresponding particular solution is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\phi_{ext}(y)\,dy.$$

We now solve the case when $\phi(x) = x$ for $x > 0$. In this case, we require the extended function to satisfy

$$\phi_{ext}(y) = \begin{cases} y & y > 0 \\ e^{-hy}\int_0^y e^{-hs}(-hs - 1)\,ds & y \leq 0 \end{cases} = \begin{cases} y & y > 0 \\ y + \frac{2}{h} - \frac{2}{h}e^{hy} & y \leq 0. \end{cases}$$

And the solution in this case is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}}y\,dy + \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}}\left(y + \frac{2}{h} - \frac{2}{h}e^{hy}\right)dy.$$

Problem 2. Solve the PDE

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < \infty \quad t \geq 0 \\ u(x,0) = 0 \\ u_t(x,0) = V \\ u_t(0,t) + au_x(0,t) = 0 & t > 0 \end{cases}$$

where $V, a,$ and c are positive constants and $a > c$.

Solution 2. Suppose we find an extension of the initial conditions such that $u_t(0,t) + au_x(0,t) = 0$ for all $t > 0$. Then by D'Alembert's formula our solution will be of the form

$$u(x,t) = \frac{1}{2}\Big(\phi_{ext}(x+ct) + \phi_{ext}(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{ext}(s)\,ds.$$

Since $\phi(z) = 0$ for $z > 0$ it is clear that extending this to be $\phi(z) = 0$ on $\mathbb{R}^-$ will imply the first term satisfies the boundary condition. We are left with finding the appropriate extension for the second term. Plugging this into our boundary condition, by the fundamental theorem of calculus

$$u_t(x,t)\Big|_{x=0} = \frac{1}{2c}\Big(c\psi_{ext}(x+ct) + c\psi_{ext}(x-ct)\Big)\Big|_{x=0} = \frac{1}{2}\Big(\psi_{ext}(ct) + \psi_{ext}(-ct)\Big)$$

and

$$u_x(x,t)\Big|_{x=0} = \frac{1}{2c}\Big(\psi_{ext}(x+ct) - \psi_{ext}(x-ct)\Big)\Big|_{x=0} = \frac{1}{2c}\Big(\psi_{ext}(ct) - \psi_{ext}(-ct)\Big).$$

Using the fact that $\psi_{ext}(z) = \psi(z) = V$ for $z \geq 0$ we have

$$0 = u_t(0,t) + au_x(0,t) = \frac{1}{2}\Big(V + \psi_{ext}(-ct)\Big) + \frac{a}{2c}\Big(V - \psi_{ext}(-ct)\Big) = \Big(\frac{1}{2} - \frac{a}{2c}\Big)\psi_{ext}(-ct) + \Big(\frac{1}{2} + \frac{a}{2c}\Big)V.$$

Setting $z = -ct$ and solving for $\psi_{ext}(-ct) = \psi_{ext}(z)$ and using the fact $a > c$, we arrive at

$$\psi_{ext}(z) = -\Big(\frac{1}{2} - \frac{a}{2c}\Big)^{-1}\Big(\frac{1}{2} + \frac{a}{2c}\Big)V = \frac{V(a+c)}{(a-c)} \text{ for } z \leq 0.$$

Therefore, we have found our extension

$$\psi_{ext}(z) = \begin{cases} V & z > 0 \\ \frac{V(a+c)}{(a-c)} & z < 0. \end{cases}$$

Since this function is constant piecewise, so it is easy to simplify the integral. For instance, if $x - ct > 0$ then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} V\,ds = Vt.$$

If $x - ct < 0$ then we have

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{0} \psi_{ext}(s)\,ds + \frac{1}{2c}\int_{0}^{x+ct} \psi_{ext}(s)\,ds \\ &= \frac{1}{2c}\int_{x-ct}^{0} \frac{V(a+c)}{(a-c)}\,ds + \frac{1}{2c}\int_{0}^{x+ct} V\,ds \\ &= \frac{1}{2c}(ct-x)\frac{V(a+c)}{(a-c)} + \frac{1}{2c}(x+ct)V \\ &= \frac{V(at-x)}{a-c}. \end{aligned}$$

Therefore,

$$u(x,t) = \begin{cases} Vt & 0 < ct < x \\ \frac{V(at-x)}{a-c} & 0 < x < ct. \end{cases}$$

Problem 3. Solve the following problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < \infty, \quad t > 0 \\ u(x,0) = \phi(x) & 0 < x < \infty, \\ u_t(x,0) = 0 & 0 < x < \infty \\ u_x(at,t) = 0 & t > 0 \end{cases}$$

where $\phi(0) = 0$ and $0 \leq a < c$.

Solution 3. Suppose we find an extension of the initial conditions such that $u_x(at,t) = 0$ for all $t > 0$. Then by D'Alembert's formula our solution will be of the form

$$u(x,t) = \frac{1}{2}\Big(\phi_{ext}(x+ct) + \phi_{ext}(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(s)\, ds.$$

Since $\psi(z) = 0$ for $z > 0$ it is clear that extending this to be $\psi(z) = 0$ on $\mathbb{R}^-$ will imply the second term satisfies the boundary condition. We are left with finding the appropriate extension for the first term. Plugging this into our boundary condition, we see

$$u_x(at,t) = \frac{1}{2}\Big(\phi'_{ext}(x+ct) + \phi'_{ext}(x-ct)\Big)\Big|_{x=at} = \frac{1}{2}\Big(\phi'_{ext}((a+c)t) + \phi'_{ext}((a-c)t)\Big) = 0.$$

Therefore, we must have $\phi'_{ext}((a+c)t) = -\phi'_{ext}((a-c)t)$ for all $t > 0$. Integrating this gives us

$$\int_0^t \phi'_{ext}((a+c)s)\, ds = -\int_0^t \phi'_{ext}((a-c)t)\, ds \implies \frac{\phi_{ext}((a+c)t) - \phi(0)}{(a+c)} = -\frac{\phi_{ext}((a-c)t) - \phi(0)}{(a-c)}.$$

Since $\phi(0) = 0$, our extension must satisfy

$$\frac{\phi_{ext}((a+c)t)}{(a+c)} = -\frac{\phi_{ext}((a-c)t)}{(a-c)}.$$

Since $0 \le a < c$, we have $a+c > 0$ and $a-c < 0$. Therefore, using the change of variables $\tilde{t} = (c-a)t > 0$ we have

$$\frac{\phi_{ext}\Big(\frac{(a+c)}{c-a}\tilde{t}\Big)}{(a+c)} = -\frac{\phi_{ext}(-\tilde{t})}{(a-c)} \implies \phi_{ext}(-\tilde{t}) = \frac{c-a}{a+c}\phi_{ext}\Big(\frac{(a+c)}{c-a}\tilde{t}\Big).$$

Therefore, since $\phi_{ext}(x) = \phi(x)$ for $x > 0$, our extension must satisfy

$$\phi_{ext}(x) = \begin{cases} \phi(x) & x > 0 \\ 0 & x = 0 \\ \frac{c-a}{a+c}\phi\Big(\frac{-(a+c)}{c-a}x\Big) & x < 0. \end{cases}$$

Notice that if $a = 0$, then this is the even reflection of $\phi(x)$ as expected. Therefore, our solution is of the form

$$u(x,t) = \frac{1}{2}\Big(\phi_{ext}(x+ct) + \phi_{ext}(x-ct)\Big)$$

where $\phi_{ext}(x)$ is given above. Our solution changes depending on whether $x - ct < 0$ or $x - ct > 0$,

$$u(x,t) = \begin{cases} \frac{1}{2}\Big(\phi(x+ct) + \phi(x-ct)\Big) & x > ct \\ \frac{1}{2}\Big(\phi(x+ct) + \frac{c-a}{a+c}\phi\Big(\frac{a+c}{c-a}(ct-x)\Big)\Big) & x < ct. \end{cases}$$

One can easily verify that this solution satisfies our PDE.