1 Stochastic Differential Equations

Stochastic differential equations is the stochastic calculus counterpart of ordinary differential equations. In these types of problems we are given a differential equations involving stochastic processes, and we want to find the stochastic process that satisfies these differential equations. In the following subsections, we go over some applications of Itô's lemma to solve some classical SDEs.

1.1 Geometric Brownian Motion

We want to find a stochastic process that models the behavior of a stock price. Notice that W_t is not a realistic model of a stock price since it can take both positive and negative values. At the very least, a model for a stock price should always be non-negative, and its movement should be proportional to its current value.

Definition 1.1. Given a standard Brownian motion $\{W_t\}_{t\geq 0}$, a stochastic process $\{S_t\}_{t\geq 0}$ is called a **geometric Brownian motion** if it satisfies the stochastic differential equation

$$\mathrm{d}S_t = \sigma S_t \,\mathrm{d}W_t + \mu S_t \,\mathrm{d}t \tag{1}$$

for some constants $\sigma \geq 0$ and $\alpha \in \mathbb{R}$.

In (1), the term $\mu S_t \,\mathrm{d}t$ induces a proportionally constant rate of average growth. The term $\sigma S_t \,\mathrm{d}W_t$ induces proportionally constant random fluctuations. This is a reasonable model for a stock price for two reasons:

- The microscopic fluctuations of an asset price over a very short time interval $[t, t+\varepsilon]$ are approximately proportional to S_t,
- the microscopic fluctuations of the value of an investment of x units of cash made at time t will only depend on x and not on S_t.

This implies that for an x amount of cash, one can buy $\xi = x/S_t$ shares. The fluctuation of the value will thus be

$$\text{instantaneous change in value} = \xi \,\mathrm{d}S_t = x \cdot \frac{\mathrm{d}S_t}{S_t} = x \cdot \sigma \,\mathrm{d}W_t + x \cdot \mu \,\mathrm{d}t.$$

The SDE in (1) can be explicitly solved (Problem 1.1).

Proposition 1.2

The solution to (1) *is*

$$S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}.$$

Remark 1.3. This equation should remind us of the deterministic growth equation. Indeed, the ODE

$$df(t) = kf(t)\,dt \iff \frac{df}{dt} = kf(t)$$

is solved by separating variables, which gives us $f(t) = f(0)e^{kf(t)}$ which agrees with the solution of geometric Brownian motion up to a correction term.

We often call μ the drift and σ the volatility. By taking logarithms, we have

$$\log S_t = \log S_0 + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t,$$

where the right-hand side is normally distributed with mean $\log S_0 + (\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$. Hence, S_t has a **log-normal distribution**. Furthermore, we expect to see fluctuations that are constant in time and a linear trend in log stock prices. We have shown several times that

$$Z_t = S_0 e^{\sigma W_t - \frac{\sigma^2}{2} t}$$

is a martingale. This means that the $S_0 e^{\sigma W_t - \frac{\sigma^2}{2} t}$ has no tendency to go up or down so

$$S_t = S_0 e^{\sigma W_t - \frac{\sigma^2}{2} t} e^{\mu t}$$

has a mean rate of return of the stock is μ. A generalization of geometric Brownian motion is given in Problem 1.1.

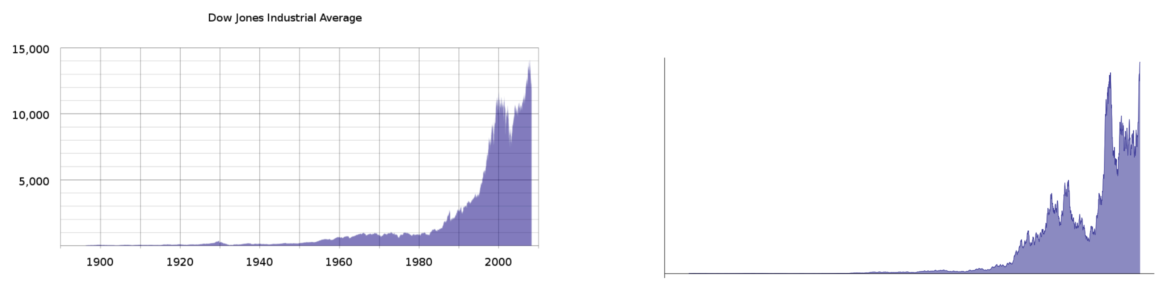

Figure 1: DJIA in absolute units vs. $S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$. For the DJIA, fluctuations are much stronger when the value of the index is high

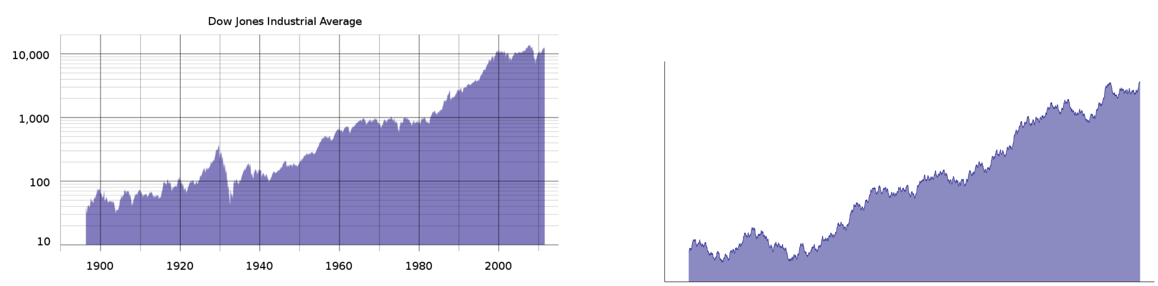

Figure 2: DJIA in logarithmic units vs. $Y_t = Y_0 + \sigma W_t + (\mu - \frac{\sigma^2}{2})t$. The historic development of the DJIA on a logarithmic scale has a linear trend.

1.2 Vasicek Interest Rate Model

We now introduce a stochastic process that models the behavior of interest rates.

Definition 1.4. Given a standard Brownian motion $\{W_t\}_{t \geq 0}$, a stochastic process $\{R_t\}_{t \geq 0}$ is called an **interest rate process** if it satisfies the stochastic differential equation

$$\mathrm{d}R_t = \sigma \,\mathrm{d}W_t + (\alpha - \beta R_t)\,\mathrm{d}t \tag{2}$$

for some constants $\sigma, \alpha, \beta > 0$.

This is a mean reverting model for the interest rate. In particular, if $R(t) > \frac{\alpha}{\beta}$ then the drift is negative so $R(t)$ is pushed down towards $\frac{\alpha}{\beta}$. The opposite happens if $R(t) < \frac{\alpha}{\beta}$. One downside of this model is that the interest rates can be negative. More complicated interest rate models exist, but they do not necessarily admit a closed form. The SDE in (2) can be solved explicitly (Problem 1.2)

Proposition 1.5

The solution to (2) is

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\, dW_s$$

Remark 1.6. This equation should remind us of the deterministic linear equation. Indeed, the ODE

$$df(t) = (\alpha - \beta f(t))dt \iff \frac{df}{dt} - \beta f = \alpha$$

is solved using the integrating factor $e^{-\beta t}$, which gives us $f(t) = e^{-\beta t}f(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t})$ which agrees with the solution of interest model up to a correction term.

1.3 Example Problems

1.3.1 Proofs of Main Results

Problem 1.1. Solve the SDE

$$\mathrm{d}S_t = \sigma S_t\, \mathrm{d}W_t + \mu S_t\, \mathrm{d}t.$$

Solution 1.1. Recall that the solution to the ODE

$$\frac{dy}{dt} = ky \tag{3}$$

can be solved by separating the vairables,

$$\frac{dy}{y} = kdt \implies d\ln(y) = kdt \implies \ln(y(t)) - \ln(y(0)) = kt \implies y(t) = y(0)e^{kt}.$$

The SDE looks very similar to (3). We can apply Itô's lemma (using the shorthand in differentials) to see that

$$\begin{aligned} d\log(S_t) &= \frac{1}{S_t}\, dS_t - \frac{1}{2}\frac{1}{S_t^2}dS_t dS_t \\ &= \sigma\, \mathrm{d}W_t + \mu\, \mathrm{d}t - \frac{1}{2}\sigma^2\, \mathrm{d}t \end{aligned}$$

because

$$\mathrm{d}S_t = \sigma S_t\, \mathrm{d}W_t + \mu S_t\, \mathrm{d}t$$

and

$$dS_t dS_t = \sigma^2 S_t^2\, \mathrm{d}W_t \mathrm{d}W_t + 2\mu\sigma S_t^2\, \mathrm{d}W_t \mathrm{d}t + \mu^2 S_t^2\, \mathrm{d}t\mathrm{d}t = \sigma^2 S_t^2\, \mathrm{d}t.$$

Therefore, we can integrate to see that

$$\log(S_t) - \log(S_0) = \sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t \implies S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}.$$

Problem 1.2. Solve the SDE

$$\mathrm{d}R_t = \sigma\,\mathrm{d}W_t + (\alpha - \beta R_t)\,\mathrm{d}t$$

Solution 1.2. Recall that the solution to the linear ODE

$$\frac{dy}{dt} + \beta y = \alpha \tag{4}$$

can be solved using the integrating factor $e^{\beta t}$

$$\frac{dy}{dt}e^{\beta t} + \beta y e^{\beta t} = \alpha e^{\beta t} \implies d(ye^{\beta t}) = \alpha e^{\beta t} \implies ye^{\beta t} - y(0) = \frac{\alpha}{\beta}(e^{\beta t} - 1)$$

so

$$y(t) = e^{-\beta t}y(0) + \frac{\alpha}{\beta}(1 - e^{\beta t})$$

The SDE looks very similar to (4). We can apply Itô's lemma (using the shorthand in differentials) to $f(x,t) = xe^{\beta t}$

$$\begin{aligned}
d(R_t e^{\beta t}) &= \beta R_t e^{\beta t}\,dt + e^{\beta t}\,dR_t + \frac{1}{2}\cdot 0\,dR_t dR_t \\
&= \beta R_t e^{\beta t}\,dt + \sigma e^{\beta t}\,dW_t + (\alpha - \beta R_t)e^{\beta t}\,dt \\
&= \sigma e^{\beta t}\,dW_t + \alpha e^{\beta t}\,dt
\end{aligned}$$

because

$$\mathrm{d}R_t = \sigma\,\mathrm{d}W_t + (\alpha - \beta R_t)\,\mathrm{d}t$$

Therefore, we can integrate to see that

$$R_t e^{\beta t} - R_0 = \sigma \int_0^t e^{\beta s}\,dW_s + \frac{\alpha}{\beta}(e^{\beta t} - 1)$$

which can be rearranged to give

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dW_s.$$