1 Martingales

Definition 1.1. Let probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\{\mathcal{F}_t\}_{t\in\mathscr{T}}$. Then the quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathscr{T}}, \mathbb{P})$ is also called a **filtered probability space**.

In this course, $\mathscr{T}$ will typically be the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ or $\mathbb{R}^+ = [0, \infty)$ the non-negative numbers. A martingale is a stochastic process defined with respect to a filtered probability space. Loosely speaking, it represents the total payout of a fair game. That is, the expected value in the future is equal to its current value.

Definition 1.2. Let $X = \{X_t\}_{t\in\mathscr{T}}$ be a stochastic process satisfies the following two conditions.

- X is **adapted** to $\{\mathcal{F}_t\}_{t\in\mathscr{T}}$, i.e., X_t is $\mathcal{F}_t$ measurable for all $t \in \mathscr{T}$.
- $\mathbb{E}[|X_t|] < \infty$ for all $t \in \mathscr{T}$.

X is called a **martingale** (with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathscr{T}}\}$) if

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \qquad \text{for all } s, t \in \mathscr{T} \text{ with } s \le t. \tag{1}$$

If we say that $X = \{X_t\}_{t\in\mathscr{T}}$ is a martingale without specifying the filtration, we mean that $X = \{X_t\}_{t\in\mathscr{T}}$ is a martingale w.r.t. its natural filtration $\mathcal{F}_t^X = \sigma(X_s | s \in \mathscr{T},\, s \le t)$.

Remark 1.3. The condition (1) is equivalent to

$$\mathbb{E}[X_t - X_s|\mathcal{F}_s] = 0 \qquad \text{for all } s, t \in \mathscr{T} \text{ with } s \le t. \tag{2}$$

If we let X_t denote the total payouts of a game at time t, then $X_t - X_s$ represents the gain (or loss) accumulated between times t and s. Condition (2) implies that based on all the information available at time s, the expected value of this gain (or loss) is zero. In this sense, a martingale can be understood the mathematical formalization of a fair game.

Remark 1.4. In discrete time, $\mathscr{T} = \mathbb{N}$, the condition (1) is equivalent to

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n \qquad \text{for all } n \ge 0. \tag{3}$$

The proof is a direct application of the tower property.

Example 1.5 (Simple Random Walk)**.** Let $Y_1, Y_2, \dots$ be i.i.d. Rademacher random variables, i.e. $\mathbb{P}(Y = \pm 1) = \frac{1}{2}$. Then $\{X_n\}_{n\ge 0}$ defined through

$$X_0 = 0 \qquad \text{and} \qquad X_n = \sum_{k=1}^{n} Y_k \tag{4}$$

is a martingale in discrete time with respect to the natural filtration $\mathcal{F}_n^X$. Indeed, since Y_{n+1} is independent of $\mathcal{F}_n^X$,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n + Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n|\mathcal{F}_n] + \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = X_n + \mathbb{E}[Y_{n+1}] = X_n$$

which satisfies condition (3).

1.1 Properties

Naturally, the expected value of the earnings of a fair game is equal to zero.

Proposition 1.6

If $\{X_t\}_{t\in\mathscr{T}}$ is a martingale, then

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] \qquad \text{for all } t \in \mathscr{T}.$$

We have the following formula for the second moment of the earnings between time s and t.

Proposition 1.7

Let $\{X_t\}_{t\in\mathscr{T}}$ be a martingale with $\mathbb{E}[(X_t)^2] < \infty$ for all $t \in \mathscr{T}$. Then, for $s, t \in \mathscr{T}$ with $s \le t$,

$$\mathbb{E}[(X_t - X_s)^2 | \mathcal{F}_s] = \mathbb{E}[X_t^2 | \mathcal{F}_s] - X_s^2.$$

In particular,

$$\mathbb{E}[(X_t - X_s)^2] = \mathbb{E}[X_t^2] - \mathbb{E}[X_s^2].$$

Example 1.8 (Martingale Betting Strategy). Let X_t be a simple random walk defined Example 4. We now take an *adapted* stochastic process $\{\xi_n\}_{n\ge 0}$ where $\xi_0 = 1$ and, for $n \ge 1$,

$$\xi_n = \begin{cases} 2^n, & \text{if } Y_1 = \cdots = Y_n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

This represents a betting strategy where we double our bet until we win. Then the gambler's total return at time $n \ge 1$ is

$$\begin{aligned} V_n &= \sum_{k=0}^{n-1} \xi_k (X_{k+1} - X_k) \\ &= \xi_0 Y_1 + \cdots + \xi_{n-1} Y_n \\ &= \begin{cases} -1 - 2 - \cdots - 2^{n-1} = -(2^n - 1), & \text{if } Y_1 = \cdots = Y_n = -1 \\ +1, & \text{otherwise.} \end{cases} \end{aligned}$$

One can show that with probability one there will eventually be some (random) integer n such that $Y_n = 1$, in which case the gambler will have won \$1.

Example 1.9 (General Betting Strategies). In general, let $\{X_n\}_{n\ge 0}$ be a martingale denoting the outcomes of a fair game. We let the process $\{\xi_n\}_{n\ge 0}$ be an adapted process denoting a betting strategy. This means that the ξ_n bet is a function of the information up to the nth game.

Suppose we are at game k, if we bet ξ_k on the kth game, then we earn $\xi_k(X_{k+1} - X_k)$ on the kth game. Our earnings associated with this betting strategy is therefore

$$V_0 = 0, \qquad V_n = \sum_{k=0}^{n-1} \xi_k (X_{k+1} - X_k). \tag{5}$$

A natural question is if one can come up with a smart betting strategy such that $\mathbb{E}[V_n] > 0 = \mathbb{E}[V_0]$ for some n? The answer to that question is no, and it is demonstrated in the following theorem. That is, no betting strategy that can turn a martingale into a favorable game.

Theorem 1.10

Suppose $\{\xi_n\}_{n\ge 0}$ is an adapted process such that for every n there exists a constant C_n such that $|\xi_n(\omega)| \le C_n$ for all $\omega \in \Omega$. If $\{X_n\}_{n\ge 0}$ is a martingale, then $\{V_n\}_{n\ge 0}$ defined in (5) is again a martingale. In particular, we have $\mathbb{E}[V_n] = 0$ for all n.

1.2 Example Problems

1.2.1 Proofs of Results

Problem 1.1. Prove Proposition 1.6.

Solution 1.1. It follows from (1.2) that

$$\mathbb{E}[X_t] = \mathbb{E}\big[\mathbb{E}[X_t|\mathcal{F}_0]\big] = \mathbb{E}[X_0].$$

Problem 1.2. Prove Proposition 1.7.

Solution 1.2. We have

$$\begin{aligned}
\mathbb{E}[(X_t - X_s)^2|\mathcal{F}_s] &= \mathbb{E}[X_t^2 - 2X_tX_s + X_s^2|\mathcal{F}_s] \\
&= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2\mathbb{E}[X_tX_s|\mathcal{F}_s] + \mathbb{E}[X_s^2|\mathcal{F}_s] \\
&= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2X_s\mathbb{E}[X_t|\mathcal{F}_s] + X_s^2 \\
&= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2X_sX_s + X_s^2 \\
&= \mathbb{E}[X_t^2|\mathcal{F}_s] - X_s^2.
\end{aligned}$$

The second identity follows from the first by taking expectations.

Problem 1.3. Prove Theorem 1.10.

Solution 1.3. We check the properties of a martingale.

(i) Clearly, $\{V_n\}_{n\geq 0}$ is adapted.

(ii) Since $|\xi_k| \leq C_k$, we define $C := \max\{C_1, \dots, C_{n-1}\}$ so that

$$\begin{aligned}
\mathbb{E}[|V_n|] = \mathbb{E}\bigg[\bigg|\sum_{k=1}^{n} \xi_{k-1}(X_k - X_{k-1})\bigg|\bigg] &\leq \sum_{k=1}^{n} \mathbb{E}[|\xi_{k-1}(X_k - X_{k-1})|] \\
&\leq \sum_{k=1}^{n} C_{k-1}\mathbb{E}[|X_k - X_{k-1}|] \leq C\sum_{k=1}^{n}\big(\mathbb{E}[|X_k|] + \mathbb{E}[|X_{k-1}|]\big) < \infty.
\end{aligned}$$

(iii) Next, we have

$$\begin{aligned}
\mathbb{E}[V_{n+1} - V_n|\,\mathcal{F}_n\,] &= \mathbb{E}[\,\xi_n(X_{n+1} - X_n)\,|\,\mathcal{F}_n\,] \\
&= \xi_n\mathbb{E}[\,X_{n+1} - X_n\,|\,\mathcal{F}_n\,] = 0
\end{aligned}$$

so $\{V_n\}_{n\geq 0}$ is a martingale. Finally, the martingale property Proposition 1.6 implies that

$$\mathbb{E}[V_n] = \mathbb{E}[V_0] = 0, \qquad \text{for all } n.$$

1.2.2 Definitions and Properties of Martingales

Problem 1.4. Let $Y_1, Y_2, \dots$ be independent (though not necessarily identically distributed) random variables with common expectation $\mathbb{E}[Y_k] = 0$ for all k. Show that $\{X_n\}_{n=0,1,2,\dots}$ defined by

$$X_0 = 0 \qquad \text{and} \qquad X_n = \sum_{k=1}^{n} Y_k$$

is a martingale in discrete time with respect to its natural filtration $\mathcal{F}_n^X$.

Solution 1.4. Since Y_{n+1} is independent of $\mathcal{F}_n^X$,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n + Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n|\mathcal{F}_n] + \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = X_n + \mathbb{E}[Y_{n+1}] = X_n.$$

Problem 1.5. Let $Y_1, Y_2, \dots$ be independent and nonnegative (though not necessarily identically distributed) random variables with common expectation $\mathbb{E}[Y_k] = 1$ for all k.

1. Show that $\{X_n\}_{n\geq 0}$ defined through

$$X_0 = 1 \qquad \text{and} \qquad X_n = \prod_{k=1}^{n} Y_k$$

   is a martingale.

2. Let Y_k be of the form $Y_k = e^{Z_k - c_k}$ for independent random variables Z_k with distribution $N(0, \sigma_k^2)$ and certain constants c_k. That is, determine c_k such that $\{X_n\}_{n\geq 0}$ is a martingale.

Solution 1.5.

Part 1: The fact that X is adapted and integrable is clear. To show (3), notice that by independence,

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[Y_{n+1} \prod_{k=1}^{n} Y_k \,\middle|\, \mathcal{F}_n\right] = \prod_{k=1}^{n} Y_k \,\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \prod_{k=1}^{n} Y_k \,\mathbb{E}[Y_{n+1}] = \prod_{k=1}^{n} Y_k = X_n.$$

Part 2: From part 1, it suffices to find a constant so that $\mathbb{E}[Y_k] = 1$. We have by the moment generating function formula for the Gaussian,

$$\mathbb{E}[Y_k] = e^{-c_k}\,\mathbb{E}[e^{Z_k}] = e^{-c_k} e^{\frac{\sigma^2}{2}} = 1 \iff c_k = \frac{\sigma^2}{2}.$$

Problem 1.6. Let X be a random variable such that $\mathbb{E}[|X|] < \infty$ and $\mathscr{T}$ either $\{0, 1, 2, \dots\}$ or $[0, \infty)$. Show that

$$X_t := \mathbb{E}[X|\mathcal{F}_t], \qquad t \in \mathscr{T},$$

is a martingale.

Solution 1.6. Clearly X_t is $\mathcal{F}_t$ measurable because the conditional expected value. Furthermore, by Jensen's inequality and the law of total expectation

$$\mathbb{E}[|X_t|] = \mathbb{E}[|\,\mathbb{E}[X|\mathcal{F}_t]|] \leq \mathbb{E}[\mathbb{E}[|X||\mathcal{F}_t]] = \mathbb{E}[|X|] < \infty.$$

Next, so show property (1) we have by the tower property that

$$\mathbb{E}\left[X_t|\mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_t]|\mathcal{F}_s\right] = \mathbb{E}[X|\mathcal{F}_s] = X_s.$$

2 Stopping time

For this section, we focus on discrete time martingales, but similar statements can be made in continuous time. A stopping time is a random variable that depends on the historical information up to time n. We can think of a stopping time as a rule that tells use when to stop playing a game, which naturally can only depends on past historical information.

Definition 2.1. A random time $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a **stopping time** if $\{\tau \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$.

Example 2.2. Let $\{Y_n\}_{n\geq 0}$ be any adapted process and define

$$\tau = \min\{n \,:\, Y_n \geq c\}.$$

Then τ is a stopping time, which is sometimes called the **first passage time** of the level c.

Given a stopping time, we can define σ-algebra

$$\mathscr{F}_\tau = \{A \in \mathcal{F} : \{\tau \leq n\} \cap A \in \mathcal{F}_n \text{ for all n}\}$$

which consists of the events that depend on the information up to a random stopping time τ. If X_n is $\mathcal{F}_n$ measurable, then the random variable X_τ is $\mathcal{F}_\tau$ measurable.

A stopping time can be interpreted as a strategy to stop a game based only on current and historical information. The next theorem states that we cannot come up with a clever stopping strategy that can turn a martingale into a favorable game.

Theorem 2.3 (*Optional stopping theorem*)

Let $\{X_n\}_{n\geq 0}$ be a martingale and τ be a stopping time. Suppose that

$$\mathbb{E}\,|X_\tau| < \infty, \qquad \lim_{n\to\infty} \mathbb{E}\,|X_n| \mathbb{1}(n \leq \tau) = 0$$

then

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0].$$

Remark 2.4. Notice that this condition is satisfied if there exists a constant C such that $\tau \leq C$ almost surely or $|X_{n\wedge\tau}| \leq C$ for all n almost surely.

The integrability conditions are essential as demonstrated by the following example.

Example 2.5. Consider the situation of Example 1.8, where $\{X_n\}_{n\geq 0}$ is a simple random walk and $\{\xi_n\}_{n\geq 0}$ is the martingale betting strategy. We have seen that $\{V_n\}_{n\geq 0}$ is a martingale with $V_0 = 0$. We let

$$\tau(\omega) = \min\{n \,:\, Y_n(\omega) = +1\},$$

where $Y_n = X_n - X_{n-1}$, denote the first time we win a game. Then τ is a stopping time with $\mathbb{P}(\tau < \infty) = 1$ and $V_\tau = 1$. Therefore,

$$\mathbb{E}[V_\tau] = 1 \neq 0 = \mathbb{E}[V_0].$$

This does not contradict Theorem 2.3 because

$$\mathbb{P}(\tau = n) = 2^{-n} \qquad \mathbb{P}(\tau > n) = \sum_{k\geq n+1} 2^{-k} = 2^{-n}$$

so

$$\mathbb{E}\,|V_n| \mathbb{1}(n \leq \tau) = \mathbb{P}(\tau = n) + (2^n - 1)\,\mathbb{P}(\tau > n) = 1$$

which does not go to zero.

2.1 Example Problems

2.1.1 Proofs of Results

Problem 2.1. Prove the Optional Stopping Theorem (Theorem 2.3) under the assumption that τ is a bounded stopping time.

Solution 2.1. We first show that $X_{n\wedge\tau}$ is a martingale. We only consider the case $\mathscr{T} = \{0, 1, 2, \dots\}$. We have

$$X_{n\wedge\tau} - X_{(n-1)\wedge\tau} = \mathbb{1}\{\tau > n-1\}(X_n - X_{n-1}).$$

Thus, stopping the process is the same as using the betting strategy $\xi_n = \mathbb{1}\{\tau > n\}$, which is adapted since τ is a stopping time. More precisely,

$$X_{n\wedge\tau} = X_0 + \sum_{k=1}^{n} \xi_{k-1}(X_k - X_{k-1})$$

Therefore Theorem 1.10 implies that $X_{n\wedge\tau}$ is a martingale.

If τ is almost surely bounded by some constant C, then $X_{N\wedge\tau} = X_\tau$ for all $N > C$. Hence, by Proposition 1.6,

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_{N\wedge\tau}] = \mathbb{E}[X_{0\wedge\tau}] = \mathbb{E}[X_0].$$

Remark 2.6. The general proof of Theorem 2.3 uses the dominated convergence theorem to interchange the limit and expected value.

Problem 2.2. Show that for a random time $\tau : \Omega \to \{0, 1, \dots\} \cup \{\infty\}$, the following conditions are equivalent:

(a) τ is a stopping time.

(b) For every $n \geq 0$, we have $\{\tau \leq n\} \in \mathcal{F}_n$.

(c) For every $n \geq 0$, we have $\{\tau > n\} \in \mathcal{F}_n$.

(d) For every $n \geq 0$, we have $\{\tau = n\} \in \mathcal{F}_n$.

Solution 2.2. The equivalence of (a) and (b) is immediate by the definition. To see that (b) and (c) are equivalent, recall that

$$A \in \mathcal{F}_n \iff A^c \in \mathcal{F}_n.$$

Since $\{\tau \leq n\}^c = \{\tau > n\}$, it follows that $\{\tau \leq n\} \in \mathcal{F}_n \iff \{\tau > n\} \in \mathcal{F}_n$ so (b) and (c) are equivalent. To see that (b) and (c) are equivalent, notice that

$$\{\tau = n\} = \{\tau \leq n\} \setminus \{\tau \leq n-1\}$$

so $\{\tau = n\}$ if (b) holds since the σ-algebra is closed under unions and complements.

2.1.2 Applications

Problem 2.3. Let $\{X_n\}_{n\geq 0}$ is a simple random walk, $a, b \in \mathbb{N}$, and

$$\tau = \min\{n \,|\, X_n = -a \text{ or } X_n = b\}.$$

Find

$$\mathbb{P}[X_\tau = b].$$

Solution 2.3. Recall that $\{X_n\}_{n\geq 0}$ is a martingale and τ is a stopping time. We can interpret $X_{n\wedge\tau}$ as the balance in a fair coin-tossing game between two players with respective capital a and b. We are interested in the probability that the player with capital b goes bankrupt before the other player, i.e., $\mathbb{P}(X_\tau = b)$.

We have that the stopping time τ satisfies $\mathbb{P}(\tau < \infty) = 1$ and $|X_{n\wedge\tau}| \leq a \vee b$ for all n almost surely. Therefore, using Theorem 2.3 with uniformly bounded stopped martingales implies that

$$0 = \mathbb{E}[X_0] = \mathbb{E}[X_\tau] = b\mathbb{P}(X_\tau = b) - a\mathbb{P}(X_\tau = -a) = b\mathbb{P}(X_\tau = b) - a(1 - \mathbb{P}(X_\tau = b))$$

which gives

$$\mathbb{P}[X_\tau = b] = \frac{a}{a+b}.$$