

# 1 Conditional Expectation

We saw that conditional probabilities represent how our beliefs for probabilities of events given more information. The conditional expectation is how the belief of our expected value of a random variable changes given more information, often in the form of information from a related random quantity.

## 1.1 Conditional Distributions Review

Recall the following concepts we defined in Week 8,

- the **marginal mass or density function of  $X$**

$$p_X(x) = \sum_y p_{X,Y}(x, y) \quad \text{or} \quad f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x, y) dy.$$

- the **marginal mass or density function of  $Y$**

$$p_Y(y) = \sum_x p_{X,Y}(x, y) \quad \text{or} \quad f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y) dx.$$

- the **conditional mass or density function of  $X$  given  $Y = y$**

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)} \quad \text{or} \quad f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}. \quad (1)$$

provided that we are not dividing by zero.

## 1.2 Conditional Expectation w.r.t. Random Variables

Throughout this section, we assume that  $X$  given  $Y = y$  is a continuous random variable with density function  $f_{X|Y}(\cdot|y)$  (if  $X|Y$  is discrete, replace all the integral signs by summation signs). The conditional expectation of  $X$  given  $Y = y$  is given by the expected value with respect to the conditional density function

$$\mathbb{E}[X|Y = y] = \int_{\mathbb{R}} x f_{X|Y}(x|y) dx.$$

This motivates the following definition:

**Definition 1** (Conditional Expectation). The *conditional expectation* of  $X$  given  $Y$  is the random variable

$$\mathbb{E}[X|Y] = \int_{\mathbb{R}} x f_{X|Y}(x|Y) dx.$$

**Remark 1.** The conditional expectation is a random variable since it takes elements in the range of  $Y$  and assigns it to a number. In other words, if we define the function  $g$  through

$$g(y) = \mathbb{E}[X|Y = y] = \int_{\mathbb{R}} x f_{X|Y}(x|y) dx,$$

then, with a slight abuse of notation, we define

$$\mathbb{E}[X|Y] = g(Y).$$

We can interpret the conditional expected value as the “best” estimate for the value of  $X$  given a realization of  $Y$  (see Problem 1.8). This also leads to a nice geometric interpretation of the conditional expectation (see Problem 1.2)

The conditional expectation can be used to compute complicated expected values by breaking it into more manageable pieces. This is because of the following properties.

**Proposition 1**

The conditional expectation has the following properties:

1. Independence: If  $X$  and  $Y$  are independent, then

$$\mathbb{E}[X | Y] = \mathbb{E}[X] \quad \text{and} \quad \mathbb{E}[Y | X] = \mathbb{E}[Y].$$

2. Linearity: For any random variables  $X_1$  and  $X_2$ ,

$$\mathbb{E}[X_1 + X_2 | Y] = \mathbb{E}[X_1 | Y] + \mathbb{E}[X_2 | Y]$$

3. Pulling out known factors: If  $h$  is a function, then

$$\mathbb{E}[h(Y)X | Y] = h(Y)\mathbb{E}[X | Y]$$

4. Law of total expectation:  $\mathbb{E}[\mathbb{E}[X | Y]] = \mathbb{E}[X]$

**Remark 2.** Recall that we can only pull constants  $c \in \mathbb{R}$  out of the expected values in general  $\mathbb{E}[cX] = c\mathbb{E}[X]$ . The second property allows us to pull random variables out if we can condition on it.

Likewise, one can define the conditional variance in the obvious way.

**Definition 2** (Conditional Variance). The *conditional variance* of  $X$  given  $Y$  is defined as

$$\text{Var}(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]$$

The conditional variance can also be used to compute complicated variances values by breaking it into more manageable pieces. This is because of the following useful properties.

**Proposition 2**

We have

1.  $\text{Var}(X|Y) = \mathbb{E}[X^2 | Y] - (\mathbb{E}[X | Y])^2$
2. Law of total variance:  $\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y])$

**1.3 Example Problems**

**Problem 1.1.** Show that the converse of the independence property in Proposition 1 is false. That is, if  $\mathbb{E}[X | Y] = \mathbb{E}[X]$ , then  $X$  and  $Y$  may not necessarily be independent.

**Solution 1.1.** Let  $Z \sim N(0, 1)$  and let  $Y = Z^2$ . Writing down the conditional PDF is possible, but the intuition is very clear, and we don't have to compute it. We have that  $\mathbb{E}[Z | Y] = \mathbb{E}[Z] = 0$ , since knowing  $Z^2$  tells you the magnitude of  $Z$ , but not its sign. If we know that  $Y = Z^2 = y$ , then  $Z$  can only take two values  $\pm\sqrt{y}$ , with probability 0.5 by symmetry. We don't know which one it takes, so our best guess is the average of these two numbers, which is 0.

This is a counterexample, because although  $\mathbb{E}[Z | Y] = \mathbb{E}[Z]$ ,  $Y$  and  $Z$  are not independent, since knowing  $Z$  completely determines  $Y$ .

**Remark 3.** It is possible to show that

$$Z | Y = y \sim \begin{cases} +\sqrt{y}, & \text{with probability } \frac{1}{2}, \\ -\sqrt{y}, & \text{with probability } \frac{1}{2}, \end{cases}$$

however, it will use techniques we have not learned in this course since the joint distribution is neither discrete nor continuous.

**Problem 1.2.** For any function  $h$ , show that

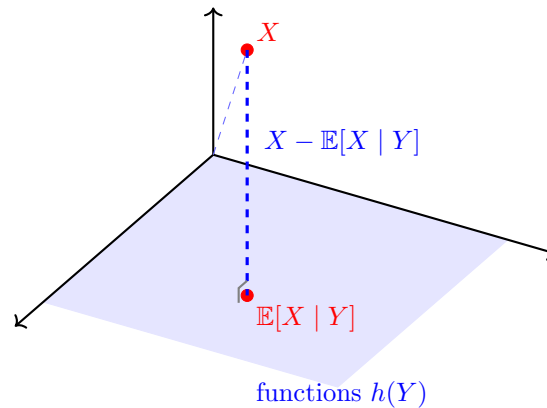
$$\mathbb{E}[(X - \mathbb{E}[X | Y])h(Y)] = 0.$$

That is, the random variable  $Z = X - \mathbb{E}[X | Y]$  is uncorrelated with  $h(Y)$ .

**Solution 1.2.** We can explicitly compute this using the properties of conditional expectation,

$$\begin{aligned} \mathbb{E}[(X - \mathbb{E}[X | Y])h(Y)] &= \mathbb{E}[Xh(Y)] - \mathbb{E}[\mathbb{E}[X | Y]h(Y)] && \text{linearity} \\ &= \mathbb{E}[Xh(Y)] - \mathbb{E}[\mathbb{E}[Xh(Y) | Y]] && \text{pulling out known factors} \\ &= \mathbb{E}[Xh(Y)] - \mathbb{E}[Xh(Y)] = 0. && \text{law of total expectation} \end{aligned}$$

This can be visualized as the projection, i.e. the line residual  $Y - \mathbb{E}[Y | X]$  is orthogonal to the space of functions of  $X$ , when we treat the expected value  $\mathbb{E}[X \cdot Y]$  as the inner product  $\langle X, Y \rangle$  between two random variables  $X$  and  $Y$ . The conditional expected value  $\mathbb{E}[Y | X]$  is the closest point on the space of functions  $h(Y)$  to  $X$ , which is proved in Problem 1.8.



**Problem 1.3.** Suppose that  $X$  and  $\Theta$  are two random variables such that  $X$  given  $\Theta = \theta$  is Poisson distributed with mean  $\theta$ , i.e.,

$$f_{X|\Theta}(k|\theta) = e^{-\theta} \frac{\theta^k}{k!}, \quad k = 0, 1, 2, \dots$$

and  $\Theta$  is Gamma distributed with parameters  $\alpha, \beta > 0$ . That is,  $\Theta$  has the density function

$$f_{\Theta}(\theta) = \frac{\beta^{\alpha} \theta^{\alpha-1} e^{-\beta\theta}}{\Gamma(\alpha)}, \quad \theta > 0,$$

where  $\Gamma$  denotes the Gamma function,

$$\Gamma(\alpha) = \int_0^{\infty} \theta^{\alpha-1} e^{-\theta} d\theta.$$

Compute the marginal mass function of  $X$ .

**Solution 1.3.** The marginal mass function of  $X$  is given by

$$\begin{aligned}
 \mathbb{P}(X = k) &= \int_0^\infty f_{X|\Theta}(k|\theta) f_\Theta(\theta) d\theta \\
 &= \int_0^\infty \frac{\theta^k e^{-\theta}}{k!} \cdot \frac{\beta^\alpha \theta^{\alpha-1} e^{-\beta\theta}}{\Gamma(\alpha)} d\theta \\
 &= \frac{\beta^\alpha}{k! \Gamma(\alpha)} \int_0^\infty \theta^{k+\alpha-1} e^{-(\beta+1)\theta} d\theta \\
 &= \frac{\beta^\alpha}{k! \Gamma(\alpha)} \cdot \frac{1}{(\beta+1)^{k+\alpha}} \int_0^\infty x^{k+\alpha-1} e^{-x} dx \\
 &= \frac{1}{k! \Gamma(\alpha)} \left( \frac{\beta}{\beta+1} \right)^\alpha \left( \frac{1}{\beta+1} \right)^k \Gamma(k+\alpha) \\
 &= \frac{(k+\alpha-1)(k+\alpha-2)\cdots(\alpha+1)\alpha}{k!} \left( 1 - \frac{1}{\beta+1} \right)^\alpha \left( \frac{1}{\beta+1} \right)^k \\
 &= \binom{k+\alpha-1}{k} \left( 1 - \frac{1}{\beta+1} \right)^\alpha \left( \frac{1}{\beta+1} \right)^k.
 \end{aligned}$$

Therefore,  $X$  follows a negative binomial distribution with parameters  $\alpha$  and  $\frac{1}{\beta+1}$ .

**Problem 1.4.** Suppose that  $X$  given  $\Theta = \theta$  is Poisson distributed with mean  $\theta$  and  $\Theta$  is Gamma distributed with density function

$$f_\Theta(\theta) = \frac{\beta^\alpha \theta^{\alpha-1} e^{-\beta\theta}}{\Gamma(\alpha)}, \quad \theta > 0.$$

1. Compute  $\mathbb{E}[X]$ .
2. Compute  $\text{Var}[X]$ .

Recall that if  $X \sim \text{Poi}(\lambda)$  then  $\mathbb{E}[X] = \lambda$  and  $\text{Var}(X) = \lambda$  and if  $X \sim \Gamma(\alpha, \beta)$  then  $\mathbb{E}[X] = \frac{\alpha}{\beta}$  and  $\text{Var}(X) = \frac{\alpha}{\beta^2}$ .

**Solution 1.4.**

(a) Using the law of total expectation,

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\Theta]] = \mathbb{E}[\Theta] = \frac{\alpha}{\beta}.$$

We used the fact that the expected value of a Poisson distributed random variable is equal to its mean parameter.

(b) By the law of total variance

$$\begin{aligned}
 \text{Var}(X) &= \mathbb{E}[\text{Var}(X|\Theta)] + \text{Var}(\mathbb{E}[X|\Theta]) \\
 &= \mathbb{E}[\Theta] + \text{Var}(\Theta) \\
 &= \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} = \frac{\alpha(\beta+1)}{\beta^2}.
 \end{aligned}$$

We used the fact that the expected value of a Poisson distributed random variable is equal to its mean parameter and its variance is equal to its mean parameter.

**Problem 1.5.** Suppose that

$$X = \begin{cases} \sum_{i=1}^N Y_i, & \text{if } N > 0, \\ 0, & \text{if } N = 0, \end{cases}$$

where  $N$  is Poisson distributed with mean  $\lambda$  and  $Y_1, Y_2, \dots$  is a sequence of iid random variables with mean  $\mu$  and variance  $\sigma^2$  that is independent of  $N$ . We say that  $X$  is a **compound Poisson random variable**.

1. Compute  $\mathbb{E}[X]$ .
2. Compute  $\text{Var}[X]$ .

Recall that if  $X \sim \text{Poi}(\lambda)$  then  $\mathbb{E}[X] = \lambda$  and  $\text{Var}(X) = \lambda$ .

**Solution 1.5.**

(a) By the law of total expectation

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X|N]] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^N Y_i \mid N\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{\infty} Y_i \mathbb{1}(i \leq N) \mid N\right]\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbb{1}(i \leq N) \mathbb{E}[Y_i \mid N]\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbb{1}(i \leq N) \mathbb{E}[Y_i]\right] \\ &= \mathbb{E}[N\mu] = \lambda\mu. \end{aligned}$$

(b) By the law of total variance, and a similar computation as above

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}[\text{Var}(X|N)] + \text{Var}(\mathbb{E}[X|N]) \\ &= \mathbb{E}[N\sigma^2] + \text{Var}(N\mu) \\ &= \sigma^2 \mathbb{E}[N] + \mu^2 \text{Var}(N) \\ &= \lambda(\sigma^2 + \mu^2). \end{aligned}$$

We used the fact that the expected value of a Poisson distributed random variable is equal to its mean parameter and its variance is equal to its mean parameter.

## 1.4 Proofs of Key Results

**Problem 1.6.** Prove the properties of conditional expectations in Proposition 1.

**Solution 1.6.** The properties follow directly from the definition.

**Part 1:** If  $X$  and  $Y$  are independent, then  $f_{X|Y}(x|y) = f_X(x)$ , so by the definition,

$$\mathbb{E}[X|Y] = \int_{\mathbb{R}} x f_{X|Y}(x|Y) dx = \int_{\mathbb{R}} x f_X(x) dx = \mathbb{E}[X].$$

**Part 2:** Linearity follows from the fact that integrals are linear and  $f_{X_1+X_2|Y}(x_1+x_2|y)$  is a PDF for each fixed  $y$ .

**Part 3:** For any  $y$  in the support of  $Y$ ,

$$g(y) = \mathbb{E}[h(Y)X|Y = y] = \int_{\mathbb{R}} h(y)xf_{X|Y}(x|y) dx = h(y) \int_{\mathbb{R}} xf_{X|Y}(x|y) dx = h(y)\mathbb{E}[X|Y = y].$$

Therefore,

$$\mathbb{E}[h(Y)X|Y] = g(Y) = h(Y)\mathbb{E}[X|Y].$$

**Part 4:** We define  $g(y) = \mathbb{E}[X|Y = y] = \int_{\mathbb{R}} xf_{X|Y}(x|y) dx$ . By the definition of the expected value,

$$\begin{aligned} \mathbb{E}[\mathbb{E}[X|Y]] &= \mathbb{E}[g(Y)] = \int_{\mathbb{R}} g(y)f_Y(y) dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} xf_{X|Y}(x|y) dx \right) f_Y(y) dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} xf_{X|Y}(x|y) f_Y(y) dx dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} xf_{X,Y}(x, y) dx dy \\ &= \int_{\mathbb{R}} x \left( \int_{\mathbb{R}} f_{X,Y}(x, y) dy \right) dx \\ &= \int_{\mathbb{R}} xf_X(x) dx = \mathbb{E}[X]. \end{aligned}$$

**Problem 1.7.** For any constant  $c$ , show that

$$\mathbb{E}[(X - c)^2] \geq \mathbb{E}[(X - \mathbb{E}[X])^2].$$

In particular, the expected value is the constant that minimizes the mean squared error.

**Solution 1.7.** This proof follows directly from the properties of the expected value. By adding and subtracting  $\mathbb{E}[X]$ , we see that

$$\begin{aligned} \mathbb{E}[(X - c)^2] &= \mathbb{E}[(X - \mathbb{E}[X] + \mathbb{E}[X] - c)^2] \\ &= \mathbb{E}[(X - \mathbb{E}[X])^2] + \mathbb{E}[(\mathbb{E}[X] - c)^2] + 2\mathbb{E}[(X - \mathbb{E}[X])(\mathbb{E}[X] - c)] \end{aligned}$$

Since  $\mathbb{E}[X] - c$  is not random, we see that the cross terms vanish

$$\mathbb{E}[(X - \mathbb{E}[X])(\mathbb{E}[X] - c)] = (\mathbb{E}[X] - c)\mathbb{E}[(X - \mathbb{E}[X])] = (\mathbb{E}[X] - c)(\mathbb{E}[X] - \mathbb{E}[X]) = 0.$$

Since  $\mathbb{E}[(\mathbb{E}[X] - c)^2] \geq 0$ , we conclude that

$$\mathbb{E}[(X - c)^2] = \mathbb{E}[(X - \mathbb{E}[X])^2] + \mathbb{E}[(\mathbb{E}[X] - c)^2] \geq \mathbb{E}[(X - \mathbb{E}[X])^2]$$

as required.

**Problem 1.8.** For any measurable function  $f$ , show that

$$\mathbb{E}[(X - f(Y))^2] \geq \mathbb{E}[(X - \mathbb{E}[X|Y])^2].$$

In particular, the conditional expectation minimizes the mean squared error.

**Solution 1.8.** This proof follows directly from the properties of the conditional expected value. By adding and subtracting  $\mathbb{E}[X | Y]$ , we see that

$$\begin{aligned}\mathbb{E}[(X - f(Y))^2] &= \mathbb{E}[(X - \mathbb{E}[X | Y] + \mathbb{E}[X | Y] - f(Y))^2] \\ &= \mathbb{E}[(X - \mathbb{E}[X | Y])^2] + \mathbb{E}[(\mathbb{E}[X | Y] - f(Y))^2] + 2\mathbb{E}[(X - \mathbb{E}[X | Y])(\mathbb{E}[X | Y] - f(Y))]\end{aligned}$$

Applying the law of total expectation and using the fact that  $\mathbb{E}[X | Y]$  and  $f(Y)$  are measurable functions of  $Y$ , we see that the cross terms vanish

$$\begin{aligned}\mathbb{E}[(X - \mathbb{E}[X | Y])(\mathbb{E}[X | Y] - f(Y))] &= \mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X | Y])(\mathbb{E}[X | Y] - f(Y)) | Y]] \\ &= \mathbb{E}[(\mathbb{E}[X | Y] - f(Y)) \mathbb{E}[(X - \mathbb{E}[X | Y]) | Y]] \\ &= \mathbb{E}[(\mathbb{E}[X | Y] - f(Y))(\mathbb{E}[X | Y] - \mathbb{E}[X | Y])] \\ &= 0.\end{aligned}$$

Since  $\mathbb{E}[(\mathbb{E}[X | Y] - f(Y))^2] \geq 0$ , we conclude that

$$\mathbb{E}[(X - f(Y))^2] = \mathbb{E}[(X - \mathbb{E}[X | Y])^2] + \mathbb{E}[(\mathbb{E}[X | Y] - f(Y))^2] \geq \mathbb{E}[(X - \mathbb{E}[X | Y])^2]$$

as required.

**Remark 4.** Notice that this proof only uses the law of total expectation and the trick that allows us to pull known factors.

**Remark 5.** Problem 1.8 is a special case when we look at estimators that do not depend on  $Y$ .

**Problem 1.9.** Prove the properties of conditional variance in Proposition 2.

**Solution 1.9.**

**Part 1:** With  $g(Y) = \mathbb{E}[X|Y]$  we have from Proposition 1 (b) that

$$\begin{aligned}\text{Var}(X|Y) &= \mathbb{E}[X^2 - 2X\mathbb{E}[X|Y] + (\mathbb{E}[X|Y])^2 | Y] \\ &= \mathbb{E}[X^2 | Y] - 2\mathbb{E}[X\mathbb{E}[X|Y] | Y] + \mathbb{E}[(\mathbb{E}[X|Y])^2 | Y] \\ &= \mathbb{E}[X^2 | Y] - 2\mathbb{E}[Xg(Y) | Y] + \mathbb{E}[(g(Y))^2 | Y] \\ &= \mathbb{E}[X^2 | Y] - 2g(Y) \cdot \mathbb{E}[X | Y] + (g(Y))^2 \mathbb{E}[1 | Y] \quad (\text{by Proposition 1 (b)}) \\ &= \mathbb{E}[X^2 | Y] - 2\mathbb{E}[X | Y] \cdot \mathbb{E}[X | Y] + (\mathbb{E}[X|Y])^2 \\ &= \mathbb{E}[X^2 | Y] - (\mathbb{E}[X | Y])^2\end{aligned}$$

**Part 2:** It follows from Part 1 and Proposition 1 Part 4 that

$$\begin{aligned}\mathbb{E}[\text{Var}(X|Y)] &= \mathbb{E}[\mathbb{E}[X^2 | Y]] - \mathbb{E}[(\mathbb{E}[X|Y])^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2].\end{aligned}$$

On the other hand,

$$\begin{aligned}\text{Var}(\mathbb{E}[X|Y]) &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[\mathbb{E}[X|Y]])^2 \\ &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2.\end{aligned}$$

Combining the preceding two relations implies

$$\mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y]) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \text{Var}(X).$$