Week 9

Problem 1. Use the method of separation of variables to solve the following PDE:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L \quad t > 0 \\ u(0,x) = \phi(x) \\ u_t(0,x) = \psi(x) \\ u_x(t,0) = 0 = u(t,L) \end{cases}$$

Solution 1. This is a homogeneous problem with 0 boundary conditions.

Step 1 — Separation of Variables: We look for a separated solution $u(t,x) = T(t)X(x)$ to our PDE. Plugging this into our PDE gives

$$T''(t)X(x) - kT(t)X''(x) = 0 \implies \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This gives the following ODEs

$$X''(x) + \lambda X(x) = 0 \text{ and } T''(t) + c^2 \lambda T(t) = 0,$$

with boundary conditions

$$T(t)X'(0) = 0 = T(t)X(L) \implies X'(0) = X(L) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < L \\ X'(0) = X(L) = 0. \end{cases}$$

We consider the 3 cases corresponding to the different forms of the ODE:

1. $\lambda = \beta^2 > 0$: The solution is of the form

   $$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

   From the boundary conditions we get

   $$\begin{aligned} \beta B &= 0 \\ A\cos(\beta L) + B\sin(\beta L) &= 0. \end{aligned}$$

   We can write this system of equations in matrix form

   $$\begin{bmatrix} 0 & \beta \\ \cos(\beta L) & \sin(\beta L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

   which has a non-trivial solution when

   $$\begin{vmatrix} 0 & \beta \\ \cos(\beta L) & \sin(\beta L) \end{vmatrix} = 0 \implies -\beta\cos(\beta L) = 0 \implies \beta = \frac{(2n-1)\pi}{2L} \text{ for } n = 1, 2, \ldots$$

   since $\beta > 0$. The first boundary condition also implies $B = 0$, which means the corresponding eigenfunction to the eigenvalue $\lambda_n = (\frac{(2n-1)\pi}{2L})^2$ is $X_n(x) = \cos(\frac{(2n-1)\pi}{2L}x)$.

2. $\lambda = 0$: The solution is of the form
$$X(x) = A + Bx.$$
From the boundary conditions we get
$$\begin{aligned} B &= 0 \\ A + BL &= 0. \end{aligned}$$
We can write this system of equations in matrix form
$$\begin{bmatrix} 0 & 1 \\ 1 & L \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
which has has no non-trivial solutions because the first matrix is invertible. Therefore, there are no 0 eigenvalues.

3. $\lambda = -\beta^2 < 0$: The solution is of the form
$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$
From the boundary conditions we get
$$\begin{aligned} \beta B &= 0 \\ A\cosh(\beta L) + B\sinh(\beta L) &= 0. \end{aligned}$$
We can write this system of equations in matrix form
$$\begin{bmatrix} 0 & \beta \\ \cosh(\beta L) & \sinh(\beta L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
which has a non-trivial solution when
$$\begin{vmatrix} 0 & \beta \\ \cosh(\beta L) & \sinh(\beta L) \end{vmatrix} = 0 \implies -\beta\cosh(\beta L) = 0$$
which has no positive roots since $-\beta < 0$ and $\cosh(\beta L) > 0$. Therefore, there are no negative eigenvalues.

Therefore, the solution to the eigenvalue problem is

Eigenvalues:
$$\lambda_n = \Big(\frac{(2n-1)\pi}{2L}\Big)^2 \text{ for } n = 1, 2, 3, \dots.$$

Eigenfunctions:
$$X_n(x) = \cos\Big(\frac{(2n-1)\pi}{2L}x\Big).$$

Step 3 — Time Problem: The time problem related to the eigenvalues λ_n is
$$T_n''(t) + c^2\Big(\frac{(2n-1)\pi}{2L}\Big)^2 T_n(t) = 0 \text{ for } n = 1, 2, \dots$$
which has solution
$$T_n(t) = A_n\cos\Big(\frac{c(2n-1)\pi}{2L}t\Big) + B_n\sin\Big(\frac{c(2n-1)\pi}{2L}t\Big).$$

Step 4 — General Solution: By the principle of superposition, the general form of our solution is
$$u(t,x) = \sum_{n=1}^{\infty} T_n(t)X_n(t) = \sum_{n=1}^{\infty}\bigg(A_n\cos\Big(\frac{c(2n-1)\pi}{2L}t\Big) + B_n\sin\Big(\frac{c(2n-1)\pi}{2L}t\Big)\bigg)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution by solving for the constants A_n and B_n. The initial conditions imply

$$u(0,x) = \phi(x) \implies \sum_{n=1}^{\infty} A_n \cos\Big(\frac{(2n-1)\pi}{2L}x\Big) = \phi(x) \tag{1}$$

and

$$u_t(0,x) = \psi(x) \implies \sum_{n=1}^{\infty} B_n \frac{c(2n-1)\pi}{2L} \cos\Big(\frac{(2n-1)\pi}{2L}x\Big) = \psi(x).$$

The eigenfunction corresponding to symmetric boundary conditions are orthogonal so the coefficients are given by

$$A_n = \frac{\langle \phi(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} = \frac{\int_0^L \phi(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx}{\int_0^L \cos^2\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx} = \frac{2}{L}\cdot\int_0^L \phi(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx$$

and

$$\begin{aligned} B_n = \Big(\frac{c(2n-1)\pi}{2L}\Big)^{-1}\frac{\langle \psi(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} &= \Big(\frac{c(2n-1)\pi}{2L}\Big)^{-1}\cdot\frac{\int_0^L \psi(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx}{\int_0^L \cos^2\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx} \\ &= \Big(\frac{c(2n-1)\pi}{2L}\Big)^{-1}\cdot\frac{2}{L}\cdot\int_0^L \psi(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx. \end{aligned}$$

Remark: We got the formulas for the coefficients by using othogonality of the eigenfunctions. Namely, $\langle X_n(x), X_m(x)\rangle = 0$ whenever $m \neq n$. For example, to recover the coefficient of A_k, we can take the inner product of both sides of (1) with respect to $X_k(x)$ and notice

$$\sum_{n=1}^{\infty}\langle A_n X_n(x), X_k(x)\rangle = A_n\langle X_n(x), X_k(x)\rangle = \langle \phi(x), X_k(x)\rangle \implies A_k(x) = \frac{\langle \phi(x), X_k(x)\rangle}{\langle X_k(x), X_k(x)\rangle}.$$

Remark: It is easy to check that these mixed boundary conditions satisfy the symmetry condition. For example, if X_1 and X_2 satisfy the boundary conditions $X_1'(0) = 0$, $X_1(L) = 0$ and $X_2'(0) = 0$, $X_2(L) = 0$ then they satisfy the symmetric condition

$$X_1'(x)X_2(x) - X_1(x)X_2'(x)\Big|_0^L = X_1'(L)X_2(L) - X_1(L)X_2'(L) - X_1'(0)X_2(0) + X_1(0)X_2'(0) = 0,$$

so the eigenfunctions of distinct eigenvalues are orthogonal.

Problem 2.

Use the method of separation of variables to solve the following PDE:

$$\begin{cases} u_t = k u_{xx} & 0 < x < 1 \qquad t > 0 \\ u(0,x) = x \\ u_x(t,0) = 0, \quad u_x(t,1) + u(t,1) = 0 \end{cases}$$

Solution 2. This is a homogeneous problem with 0 boundary conditions.

Step 1 — Separation of Variables: We look for a separated solution $u(t,x) = T(t)X(x)$ to our PDE. Plugging this into our PDE gives

$$T'(t)X(x) - kT(t)X''(x) = 0 \implies \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This implies the following ODEs

$$X''(x) + \lambda X(x) = 0 \text{ and } T'(t) + k\lambda T(t) = 0,$$

with boundary conditions

$$T(t)X'(0) = 0 \text{ and } T(t)X'(1) + T(t)X(1) = 0 \implies X'(0) = X'(1) + X(1) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < 1 \\ X'(0) = X'(1) + X(1) = 0. \end{cases}$$

We consider the 3 cases corresponding to the different forms of the ODE:

1. $\lambda = \beta^2 > 0$: The solution is of the form

   $$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

   From the boundary conditions we get

   $$\begin{aligned} \beta B &= 0 \\ -\beta A\sin(\beta) + \beta B\cos(\beta) + A\cos(\beta) + B\sin(\beta) &= 0. \end{aligned}$$

   We can write this system of equations in matrix form

   $$\begin{bmatrix} 0 & \beta \\ \cos(\beta) - \beta\sin(\beta) & \beta\cos(\beta) + \sin(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

   which has a non-trivial solution when

   $$\begin{vmatrix} 0 & \beta \\ \cos(\beta) - \beta\sin(\beta) & \beta\cos(\beta) + \sin(\beta) \end{vmatrix} = 0 \implies \beta\cos(\beta) - \beta^2\sin(\beta) = 0.$$

   If β_n is chosen such that $\cos(\beta_n) = 0$, then $\beta_n \neq 0$ and $\sin(\beta_n) \neq 0$ which means there are no solutions such that $\cos(\beta_n) = 0$. Therefore, we can rearrange terms to recover the condition

   $$\beta\cos(\beta) - \beta^2\sin(\beta) \implies \tan(\beta) = \frac{1}{\beta}.$$

   The eigenvalues β_n are the positive roots of $\tan(\beta) = \frac{1}{\beta}$ for which there are infinitely many of them. The first boundary condition also implies $B = 0$, which means the corresponding eigenfunction of the eigenvalue $\lambda_n = \beta_n^2$ is $X_n = \cos\big(\beta_n x\big)$.

2. $\lambda = 0$: The solution is of the form

   $$X(x) = A + Bx.$$

   From the boundary conditions we get

   $$\begin{aligned} B &= 0 \\ A + 2B &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has has no non-trivial solutions because the first matrix is invertible. Therefore, there are no 0 eigenvalues.

3. $\lambda = -\beta^2 < 0$: The solution is of the form

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

From the boundary conditions we get

$$\begin{aligned} \beta B &= 0 \\ \beta A\sinh(\beta) + \beta B\cosh(\beta) + A\cosh(\beta) + B\sinh(\beta) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 0 & \beta \\ \cosh(\beta) + \beta\sinh(\beta) & \beta\cosh(\beta) + \sinh(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has a non-trivial solution when

$$\begin{vmatrix} 0 & \beta \\ \cosh(\beta) + \beta\sinh(\beta) & \beta\cosh(\beta) + \sinh(\beta) \end{vmatrix} = 0 \implies \beta\cosh(\beta) + \beta^2\sinh(\beta) = 0.$$

Since $\cosh(\beta) > 0$ and $\beta > 0$, we can write the above as

$$\tanh(\beta) = -\frac{1}{\beta}$$

which has no positive roots. Therefore, there are no negative eigenvalues.

Therefore, the solution to the eigenvalue problem is

Eigenvalues: $\lambda_n = \beta_n^2$ for $n = 1, 2, \dots$ where β_n are the ordered positive roots of $\tan(\beta) = \frac{1}{\beta}$

Eigenfunctions: $X_n = \cos\big(\beta_n x\big)$.

Step 3 — Time Problem: The time problem related to the eigenvalues λ_n is

$$T_n'(t) + k(\beta_n)^2 T_n(t) = 0 \text{ for } n = 1, 2, \dots$$

which has solution

$$T_n(t) = A_n e^{-k\beta_n^2 t}.$$

Step 4 — General Solution: By the principle of superposition, the general form of our solution is

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(t) = \sum_{n=1}^{\infty} A_n e^{-k\beta_n^2 t}\cos(\beta_n x).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution by solving for the constants A_n. The initial conditions imply

$$u(0,x) = x \implies \sum_{n=1}^{\infty} A_n\cos(\beta_n x) = x.$$

The eigenfunction corresponding to Robin boundary conditions are also symmetric boundary conditions, so the eigenfunctions are orthogonal. Therefore, the coefficients are given by

$$A_n = \frac{\langle x, X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} = \frac{\int_0^1 x\cos(\beta_n x)\,dx}{\int_0^1 \cos^2(\beta_n x)\,dx}.$$

Problem 3.

Use the method of separation of variables to solve the following PDE:

$$\begin{cases} u_t - ku_{xx} + hu = hu_0 & -\pi < x < \pi \quad t > 0 \quad h \text{ and } u_0 \text{ are constants} \\ u(0,x) = \phi(x) & -\pi < x < \pi \\ u_x(t,-\pi) = u_x(t,\pi), \qquad u(t,-\pi) = u(t,\pi) & t > 0 \end{cases}$$

Solution 3. This is a inhomogeneous problem with 0 boundary conditions. We will use the method of eigenfunction expansion to solve the problem with source.

Step 1 — Separation of Variables: We first find a solution to the homogeneous equation.

$$T'(t)X(x) - kT(t)X''(x) + hX(x)T(t) = 0 \implies \frac{T'(t) + hT(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

with boundary conditions

$$T(t)X'(-\pi) - T(t)X'(\pi) = 0, T(t)X(-\pi) - T(t)X(\pi) = 0 \implies X'(-\pi) - X'(\pi) = X(-\pi) - X(\pi) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < 1 \\ X'(-\pi) - X'(\pi) = X(-\pi) - X(\pi) = 0. \end{cases}$$

We consider the 3 cases corresponding to the different forms of the ODE:

1. $\lambda = \beta^2 > 0$: The solution is of the form

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

From the boundary conditions we get

$$\begin{aligned} -\beta A\sin(-\beta\pi) + \beta B\cos(-\beta\pi) + \beta A\sin(\beta\pi) - \beta B\cos(\beta\pi) &= 0 \\ A\cos(-\beta\pi) + B\sin(-\beta\pi) - A\cos(\beta\pi) - B\sin(\beta\pi) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 2\beta\sin(\beta\pi) & 0 \\ 0 & -2\sin(\beta\pi) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has a non-trivial solution when

$$\begin{vmatrix} 2\beta\sin(\beta\pi) & 0 \\ 0 & -2\sin(\beta\pi) \end{vmatrix} = 0 \implies -4\beta\sin^2(\beta\pi) = 0 \implies \beta = n \text{ for } n = 1, 2, \dots.$$

For $\beta_n = n$, our matrix has rank 0, so we get two eigenfunctions, which means the two eigenfunctions corresponding to $\lambda_n = n^2$ are given by $X_n = \cos\big(nx\big)$ and $Y_n = \sin\big(nx\big)$.

2. $\lambda = 0$: The solution is of the form

$$X(x) = A + Bx.$$

From the boundary conditions we get

$$\begin{aligned} 0 &= 0 \\ A - B\pi - A - B\pi &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 0 & 0 \\ 0 & -2\pi \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This always has a non-trivial solution since our matrix has rank 1. The second condition implies that $B = 0$ and the first condition that A can be arbitrary. Therefore, there is a 0 eigenvalue, and the corresponding eigenfunction is $X_0(x) = 1$.

3. $\lambda = -\beta^2 < 0$: The solution is of the form

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

The solution is of the form

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

From the boundary conditions we get

$$\begin{aligned} \beta A\sinh(-\beta\pi) + \beta B\cosh(-\beta\pi) - \beta A\sinh(\beta\pi) - \beta B\cosh(\beta\pi) &= 0 \\ A\cosh(-\beta\pi) + B\sinh(-\beta\pi) - A\cosh(\beta\pi) - B\sinh(\beta\pi) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} -2\beta\sinh(\beta\pi) & 0 \\ 0 & -2\sinh(\beta\pi) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has a non-trivial solution when

$$\begin{vmatrix} -2\beta\sinh(\beta\pi) & 0 \\ 0 & -2\sinh(\beta\pi) \end{vmatrix} = 0 \implies 4\beta\sinh^2(\beta\pi) = 0.$$

Which has no positive roots, so there are no negative eigenvalues.

Therefore, the solution to the eigenvalue problem is

Eigenvalues: $\lambda_n = n^2$ for $n = 0, 1, 2, \ldots$

Eigenfunctions: $X_n = \cos\big(nx\big)$ and $Y_n = \sin\big(nx\big)$ and $X_0 = 1$.

Step 3 — General Homogeneous Solution: By the principle of superposition, our general solution is of the form

$$u(t,x) = T_0(t)X_0(x) + \sum_{n=1}^{\infty}\Big(T_n(t)X_n(x) + S_n(t)Y_n(x)\Big) = T_0(t) + \sum_{n=1}^{\infty}\Big(T_n(t)\cos(nx) + S_n(t)\sin(nx)\Big).$$

Step 4 — General Inhomogeneous Solution: Our goal is to now solve for $T_n(t)$ and $S_n(t)$ using the fact

$$u_t - ku_{xx} + hu = hu_0.$$

Differentiating our general solution, we have

$$(T_0' + hT_0) + \sum_{n=1}^{\infty}\Big((T_n' - n^2T_n + hT_n)\cos(nx) + (S_n' - n^2S_n + hS_n)\sin(nx)\Big) = hu_0.$$

To compute T_n and S_n notice that these functions play the role of the Fourier coefficients. That is, $a_0(t) = (T_0' + hT_0)$, $a_n(t) = (T_n' - n^2T_n + hT_n)$ and $b_n(t) = (S_n' - n^2S_n + hS_n)$. If we compute the

Fourier series of hu_0 we get only the constant term remains that is $a_0 = hu_0$, $a_n = 0$ and $b_n = 0$. Equating coefficients implies

$$T_0' + hT_0 = hu_0, \quad (T_n' - n^2 T_n + hT_n) = 0, \quad (S_n' - n^2 S_n + hS_n) = 0.$$

Notice that the first ODE is a linear first order equation, so its solution is

$$T_0 = u_0 + A_0 e^{-ht}$$

and the other ODES are separable with solutions

$$T_n = A_n e^{(n^2-h)t}, \quad S_n = B_n e^{(n^2-h)t}.$$

Putting this all together, we have our solution general solution to the inhomogeneous PDE is given by

$$u(t,x) = u_0 + A_0 e^{-ht} + \sum_{n=1}^{\infty} \Big(A_n e^{(n^2-h)t} \cos(nx) + B_n e^{(n^2-h)t} \sin(nx) \Big).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution. The initial conditions imply

$$u(0,x) = \phi(x) \implies u_0 + A_0 + \sum_{n=1}^{\infty} \Big(A_n \cos(nx) + B_n \sin(nx) \Big) = \phi(x).$$

Writing $\phi(x)$ in terms of its full Fourier series and equating coefficients implies

$$A_0 = -u_0 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(x)\, dx,$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \cos(nx)\, dx,$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \sin(nx)\, dx.$$

Remark: We could have also gotten these coefficients using the orthogonality of the eigenfunctions like in the previous examples. There are repeated repeated eigenvalues in this case, but we know that the standard Fourier basis is orthogonal so the same technique applies. For example,

$$A_0 = -u_0 + \frac{\langle \phi(x), X_0(x) \rangle}{\langle X_0(x), X_0(x) \rangle} = -u_0 + \frac{\int_{-\pi}^{\pi} \phi(x)\, dx}{\int_{-\pi}^{\pi} 1\, dx} = -u_0 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(x)\, dx,$$

$$A_n = \frac{\langle \phi(x), X_n(x) \rangle}{\langle X_n(x), X_n(x) \rangle} = \frac{\int_{-\pi}^{\pi} \phi(x) \cos(nx)\, dx}{\int_{-\pi}^{\pi} \cos^2(nx)\, dx} = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \cos(nx)\, dx,$$

$$B_n = \frac{\langle \phi(x), Y_n(x) \rangle}{\langle Y_n(x), Y_n(x) \rangle} = \frac{\int_{-\pi}^{\pi} \phi(x) \sin(nx)\, dx}{\int_{-\pi}^{\pi} \sin^2(nx)\, dx} = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \sin(nx)\, dx.$$