1 Inhomogeneous Wave Equation on the Half Line

Suppose we have the wave equation on the half line. We want to reduce this problem to a PDE on the entire line by finding an appropriate extension of the initial conditions that satisfies the given boundary conditions. The choice of the extension only depends on the boundary conditions:

1. Dirichlet ($u|_{x=0} = 0$): We take an odd extension of the initial conditions

$$\phi_{odd} = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \\ 0 & x = 0 \end{cases}.$$

2. Neumann ($u_x|_{x=0} = 0$): We take an even extension of the initial conditions

$$\phi_{even} = \begin{cases} \phi(x) & x \geq 0 \\ \phi(-x) & x \leq 0 \end{cases}.$$

This extension reduces the wave equation on the half line to the full line, so we can apply D'Alembert's formula

$$u(x,t) = \frac{g_{ext}(x+ct) + g_{ext}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h_{ext}(s)\,ds. \tag{1}$$

where the initial conditions are replaced by the respective odd or even extensions. The restriction of the solution is the unique solution to the PDE on the half line. The general formulas are derived in the problems below.

Problem 1.1. Solve the following PDE

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \infty, \quad t > 0 \\ u|_{t=0} = g(x) & 0 < x < \infty \\ u_t|_{t=0} = h(x) & 0 < x < \infty \\ u|_{x=0} = 0 & t > 0 \end{cases}.$$

Solution 1.1. Since we have Dirichlet boundary conditions, we can find a solution using an odd-extension. Define

$$g_{odd}(x) = \begin{cases} g(x) & x > 0 \\ 0 & x = 0 \\ -g(-x) & x < 0 \end{cases} \quad \text{and} \quad h_{odd}(x) = \begin{cases} h(x) & x > 0 \\ 0 & x = 0 \\ -h(-x) & x < 0. \end{cases}$$

By (1), the general solution for $x > 0$ is given by

$$u(x,t) = \frac{g_{odd}(x+ct) + g_{odd}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h_{odd}(s)\,ds.$$

We now examine the cases depending on the sign of $x - ct$:

1. For $x - ct > 0$, we have $g_{odd}(x \pm ct) = g(x \pm ct)$ and $h_{odd}(x \pm ct) = h(x \pm ct)$ so the solution is given by

$$u(x,t) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds.$$

2. For $x - ct < 0$, we have $g_{odd}(x - ct) = -g(ct - x)$ and for $x - ct < s < 0$, $h_{odd}(s) = -h(-s)$, so

$$\begin{aligned} u(x,t) &= \frac{g(x+ct) - g(ct-x)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} h(s)\,ds - \int_{x-ct}^0 h(-s)\,ds \right) \\ &= \frac{g(x+ct) - g(ct-x)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} h(s)\,ds + \int_{ct-x}^0 h(\tilde{s})\,d\tilde{s} \right) && \tilde{s} = -s \\ &= \frac{g(x+ct) - g(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} h(s)\,ds. \end{aligned}$$

3. For $x - ct = 0$, we have $g_{odd}(x - ct) = 0$, so

$$u(x,t) = \frac{g(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} h(s)\,ds.$$

Problem 1.2. Solve the following PDE

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \infty, \quad t > 0 \\ u|_{t=0} = g(x) & 0 < x < \infty \\ u_t|_{t=0} = h(x) & 0 < x < \infty \\ u_x|_{x=0} = 0 & t > 0 \end{cases}.$$

Solution 1.2. Since we have Neumann boundary conditions, we can find a solution using an even-extension. Define

$$g_{even}(x) = \begin{cases} g(x) & x \geq 0 \\ g(-x) & x \leq 0 \end{cases} \quad \text{and} \quad h_{even}(x) = \begin{cases} h(x) & x \geq 0 \\ h(-x) & x \leq 0. \end{cases}$$

By (1), the general solution for $x > 0$ is given by

$$u(x,t) = \frac{g_{even}(x+ct) + g_{even}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h_{even}(s)\,ds.$$

We now examine the cases depending on the sign of $x - ct$:

1. For $x - ct \geq 0$, we have $g_{even}(x \pm ct) = g(x \pm ct)$ and $h_{even}(x \pm ct) = h(x \pm ct)$ so the solution is given by

$$u(x,t) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds.$$

2. For $x - ct < 0$, we have $g_{even}(x - ct) = g(ct - x)$ and for $x - ct < s < 0$, $h_{even}(s) = h(-s)$, so

$$\begin{aligned} u(x,t) &= \frac{g(x+ct) + g(ct-x)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} h(s)\,ds + \int_{x-ct}^0 h(-s)\,ds \right) \\ &= \frac{g(x+ct) + g(ct-x)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} h(s)\,ds - \int_{ct-x}^0 h(\tilde{s})\,d\tilde{s} \right) && \tilde{s} = -s \\ &= \frac{g(x+ct) + g(ct-x)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} h(s)\,ds + \int_0^{ct-x} h(s)\,ds \right). \end{aligned}$$

2 General Boundary Conditions

Problem 2.1. Solve the following PDE

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \infty, \quad t > 0 \\ u|_{t=0} = g(x) & 0 < x < \infty \\ u_t|_{t=0} = cg'(x) & 0 < x < \infty \\ u_x + \alpha u|_{x=0} = 0 & t > 0 \end{cases}.$$

Solution 2.1. Recall that the general solution of $u_{tt} - c^2 u_{xx} = 0$ is given by

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \text{ for } x,t > 0,$$

for some yet to be determined functions ϕ and ψ. Using the initial and boundary conditions, we can recover the specific form of ϕ and ψ.

Initial Condition: For $x > 0$, the first initial condition implies that

$$\phi(x+ct) + \psi(x-ct)\big|_{t=0} = g(x) \implies \phi(s) + \psi(s) = g(s) \text{ for } s > 0 \tag{2}$$

and the second boundary condition initial that

$$c\phi'(x+ct) - c\psi'(x-ct)\big|_{t=0} = cg'(x) \implies \phi'(s) - \psi'(s) = g'(s) \text{ for } s > 0. \tag{3}$$

Differentiating (2) and adding it to (3) implies that

$$2\phi'(s) = 2g'(s) \implies \phi(s) = g(s) + C \text{ for } s > 0. \tag{4}$$

Substituting this into (2) implies

$$g(s) + C + \psi(s) = g(s) \implies \psi(s) = -C \text{ for } s > 0. \tag{5}$$

Boundary Condition: Since $x + ct > 0$ on the domain, it remains to find ψ for $s < 0$. For $t > 0$, the boundary condition implies

$$\phi'(x+ct) + \psi'(x-ct) + \alpha\phi(x+ct) + \alpha\psi(x-ct)\big|_{x=0} = \phi'(ct) + \psi'(-ct) + \alpha\phi(ct) + \alpha\psi(-ct) = 0.$$

Since $\phi(ct) = g(ct) + C$ for $ct > 0$ from our computations above,

$$g'(ct) + \psi'(-ct) + \alpha g(ct) + \alpha C + \alpha\psi(-ct) = 0 \overset{s=-ct}{\implies} \psi'(s) + \alpha\psi(s) = -g'(-s) - \alpha g(-s) - \alpha C,$$

where $s < 0$. This is a linear first order ODE, so we can solve it using an integrating factor $e^{\alpha s}$,

$$\frac{d}{ds}\Big(e^{\alpha s}\psi(s)\Big) = -e^{\alpha s}g'(-s) - \alpha e^{\alpha s}g(-s) - \alpha C e^{\alpha s} \tag{6}$$

$$\implies \qquad \psi(s) = e^{-\alpha s}\int_0^s -e^{\alpha r}g'(-r) - \alpha e^{\alpha r}g(-r)\,dr - C + De^{-\alpha s} \tag{7}$$

$$= e^{-\alpha s}\int_0^{-s} e^{-\alpha r}g'(r) + \alpha e^{-\alpha r}g(r)\,dr - C + De^{-\alpha s} \text{ for } s < 0. \tag{8}$$

Solution: For $x, t > 0$ the formulas in (4), (5) and (6) implies

$$u(x,t) = \begin{cases} g(x+ct) & x - ct > 0 \\ g(x+ct) + e^{\alpha(ct-x)}\displaystyle\int_0^{ct-x} e^{-\alpha r}g'(r) + \alpha e^{-\alpha r}g(r)\,dr + De^{\alpha(ct-x)} & x - ct < 0. \end{cases}$$

This formula is well defined because $g(s)$ is defined for $s > 0$. If we require our solution to be continuous, then as $x \to ct$ from the left we also require

$$g(2ct) = g(2ct) + D \implies D = 0.$$

Problem 2.2. Solve the following PDE

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & x > -t, \quad t > 0 \\ u|_{t=0} = 0 & 0 < x < \infty \\ u_t|_{t=0} = 0 & 0 < x < \infty \\ u_x|_{x=-t} = \sin(t) & t > 0 \end{cases}.$$

Solution 2.2. Recall that the general solution of $u_{tt} - 4u_{xx} = 0$ is given by

$$u(x,t) = \phi(x+2t) + \psi(x-2t) \text{ for } x > -t,\ t > 0,$$

for some yet to be determined functions ϕ and ψ. Using the initial and boundary conditions, we can recover the specific form of ϕ and ψ.

Initial Condition: For $x > 0$, the first initial condition implies that

$$\phi(x+2t) + \psi(x-2t)\big|_{t=0} = 0 \implies \phi(s) + \psi(s) = 0 \text{ for } s > 0 \tag{9}$$

and the second boundary condition initial that

$$2\phi'(x+2t) - 2\psi'(x-2t)\big|_{t=0} = 0 \implies \phi'(s) - \psi'(s) = 0 \text{ for } s > 0. \tag{10}$$

Differentiating (9) and adding it to (10) implies that

$$2\phi'(s) = 0 \implies \phi(s) = C \text{ for } s > 0. \tag{11}$$

Substituting this into (9) implies

$$C + \psi(s) = 0 \implies \psi(s) = -C \text{ for } s > 0. \tag{12}$$

Boundary Condition: Since $x + 2t > 0$ It remains to find ψ for $s < 0$. For $t > 0$, the boundary condition implies

$$\phi'(x+2t) + \psi'(x-2t)\big|_{x=-t} = \phi'(t) + \psi'(-3t) = \sin(t).$$

Since $\phi(t) = C$ for $t > 0$ from our computations above,

$$\psi'(-3t) = \sin(t) \overset{s=-3t}{\implies} \psi'(s) = \sin\Big(-\frac{s}{3}\Big),$$

where $s < 0$. Integrating this, we see that

$$\psi(s) = 3\cos\Big(-\frac{s}{3}\Big) + D \text{ for } s < 0. \tag{13}$$

Solution: For $x > -t$, $t > 0$ the formulas in (11), (12) and (13) implies

$$u(x,t) = \begin{cases} 0 & x > 2t \\ C + 3\cos(\frac{2t-x}{3}) + D & -t < x < 2t. \end{cases}$$

If we require our solution to be continuous, then as $x \to 2t$ from the left we also require

$$0 = C + 3\cos(0) + D \implies C + D = -3,$$

that is,

$$u(x,t) = \begin{cases} 0 & x > 2t \\ 3\cos(\frac{2t-x}{3}) - 3 & -t < x < 2t. \end{cases}$$

Remark. The odd and even reflection formulas in Problems 1.1 and 1.2 can also be derived directly from the general formulas using the approaches in Problems 2.1 and 2.2.