Week 9: Laplace's Equation

Problem 1. (Strauss 6.1.2) Find the solutions that depend only on r of the equation $u_{xx}+u_{yy}+u_{zz} = k^2u$, where k is a positive constant. (*Hint:* Substitute $u = v/r$.)

Solution 1. Recall that in $\mathbb{R}^3$, if we do a change of variables to spherical coordinates,

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right).$$

If we are looking for solutions that only depend on r, that is $u(r, \phi, \psi) = u(r)$ then we can safely ignore the terms on the right, so $u_{xx} + u_{yy} + u_{zz} = k^2u$ can be expressed in spherical coordinates as

$$u_{rr} + \frac{2}{r}u_r = k^2u.$$

This is a second order ODE, which we can solve using the substitution $u = v/r$. Notice

$$u_r = \frac{v_r}{r} - \frac{v}{r^2}, \quad u_{rr} = \frac{v_{rr}}{r} - 2\frac{v_r}{r^2} + 2\frac{v}{r^3}$$

so under this change of variables, we have

$$u_{rr} + \frac{2}{r}u_r = k^2u \implies \frac{v_{rr}}{r} = k^2\frac{v}{r} \implies v_{rr} - k^2v = 0.$$

This is a second order constant coefficient ODE with roots $r = \pm k$, so

$$v = Ae^{kr} + Be^{-kr} \implies u = A\frac{e^{kr}}{r} + B\frac{e^{-kr}}{r},$$

is the general solution.

Problem 2. (Strauss 6.1.5) Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x, y)$ vanishing on $r = a$.

Solution 2. Since we are on the disk, and neither our source or initial conditions depend on the angle θ we can use rotational invariance to solve this problem. Recall that in $\mathbb{R}^2$, if we do a change of variables to polar form,

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

If we are looking for solutions that only depend on r, that is $u(r, \theta) = u(r)$, then we can safely ignore the terms on the right, so $u_{xx} + u_{yy} = 1$ can be expressed in spherical coordinates as

$$u_{rr} + \frac{1}{r}u_r = 1 \implies ru_{rr} + u_r = r \implies (u_r r)' = r.$$

This ODE can be solved by directly integrating, which implies

$$u_r r = \frac{r^2}{2} + C_1 \implies u_r = r + \frac{C_1}{r} \implies u(r) = \frac{r^2}{4} + C_1\log r + C_2.$$

We now use the boundary conditions to solve for the coefficients. We have the implicit condition $\lim_{r\to 0} u(r) < \infty$ and the boundary condition $u(a) = 0$. Therefore, we must have

$$\lim_{r\to 0} u(r) = \lim_{r\to 0}\frac{r^2}{4} + C_1\log r + C_2 < \infty \text{ and } 0 = u(a) = \frac{a^2}{4} + C_1\log a + C_2.$$

The first condition implies that $C_1 = 0$ and the second condition implies $C_2 = -\frac{a^2}{4}$. Therefore,

$$u(r) = \frac{r^2}{4} - \frac{a^2}{4},$$

is the particular solution.

Problem 3. (Strauss 6.1.13) A function $u(x,y)$ is subharmonic if $u_{xx} + u_{yy} \geq 0$ in D. Prove that its maximum value is attained on ∂D. [Note that this is not true for the minimum value.]

Solution 3.

Maximum Principle: The maximum principle holds for subharmonic functions. The same proof for harmonic functions applies in this case. Let u be a continuous subharmonic function on $\bar{D}$. Let $\epsilon > 0$ and define $v^\epsilon(x,y) = u(x,y) + \epsilon(x^2 + y^2)$. The following interior point condition also holds for subharmonic functions,

$$\Delta v^\epsilon(x,y) = \Delta u(x,y) + \epsilon \Delta(x^2 + y^2) \geq 0 + 4\epsilon > 0 \text{ in } D.$$

The rest of the proof is identical to the harmonic case. By the second derivative test, any interior maximum must satisfy the critical point condition $v^\epsilon_{xx} + v^\epsilon_{yy} \leq 0$, which is impossible because it contradicts the interior point condition $\Delta v^\epsilon(x,y) > 0$. Therefore, $v^\epsilon(x,y)$ does not attain an interior maximum.

Since $v^\epsilon(x,y)$ is a continuous function and $\bar{D}$ is compact, we must have $v^\epsilon(x,y)$ attains a maximum at some point $(\tilde{x},\tilde{y}) \in \partial D$. We are on a bounded domain, so there exists a M such that $x^2 + y^2 \leq M$ for all $(x,y) \in \bar{D}$. Since $0 \leq \epsilon(x^2 + y^2) \leq \epsilon M$, we have

$$\max_{(x,y)\in\bar{D}} u(x,y) \leq \max_{(x,y)\in\bar{D}} v^\epsilon(x,y) \leq v^\epsilon(\tilde{x},\tilde{y}) \leq u(\tilde{x},\tilde{y}) + \epsilon(\tilde{x}^2 + \tilde{y}^2) \leq \max_{(x,y)\in\partial D} u(x,y) + \epsilon M.$$

The upperbound holds for all $\epsilon > 0$, so taking $\epsilon \to 0$ implies

$$\max_{(x,y)\in\bar{D}} u(x,y) \leq \max_{(x,y)\in\partial D} u(x,y)$$

as required.

Minimum Principle: The minimum principle fails. For example, consider the continuous function $u(x,y) = x^2 + y^2$ on the disc $\bar{D} = \{(x,y) : x^2 + y^2 \leq 1\}$. We have $u_{xx} + u_{yy} = 4 \geq 0$ in D, so our function is subharmonic. Since $u(0,0) = 0$ and our function is strictly positive whenever $(x,y) \neq (0,0)$ the maximum is attained on the interior of our set and is strictly less than all values of u on the boundary, disproving the minimum principle.