1 Solving Semilinear First Order PDEs

Problem 1.1. Find the general solutions to the following equations

$$-4u_x + u_y + u = 0, \tag{1}$$
$$-2u_x + 4u_y = e^{x+3y} - 5u. \tag{2}$$

Solution 1.1.

(1) We have the system of equations

$$\frac{dx}{-4} = \frac{dy}{1} = \frac{du}{-u}.$$

Characteristic Curve: We start by solving the equation involving the first and second term,

$$\frac{dx}{-4} = \frac{dy}{1} \Rightarrow \frac{dx}{dy} = -4 \Rightarrow C = x + 4y.$$

General Solution: We now solve the equation involving the second and third term,

$$\frac{dy}{1} = \frac{du}{-u} \Rightarrow \frac{du}{dy} = -u.$$

This is a separable ODE, which has solution

$$\log|u| = -y + f(C) \Rightarrow u = \pm e^{f(C)} e^{-y}.$$

Since $f(C)$ is an arbitrary function and $u \equiv 0$ is a solution, we might can redefine $\pm e^{f(C)} =: g(C)$. Since $C = x + 4y$, we have our general solution is

$$u(x, y) = g(x + 4y)e^{-y}.$$

(2) We have the system of equations

$$\frac{dx}{-2} = \frac{dy}{4} = \frac{du}{e^{x+3y} - 5u}.$$

Characteristic Curve: We start by solving the equation involving the first and second term,

$$\frac{dx}{-2} = \frac{dy}{4} \Rightarrow \frac{dy}{dx} = -2 \Rightarrow C = y + 2x.$$

General Solution: We now solve the equation involving the first and second term,

$$\frac{dx}{-2} = \frac{du}{e^{x+3y} - 5u} \Rightarrow \frac{du}{dx} = -\frac{1}{2}(e^{x+3y} - 5u) \Rightarrow \frac{du}{dx} - \frac{5}{2}u = -\frac{1}{2}e^{x+3y}.$$

There is a y variable appearing in this ODE that we must eliminate first. Since $y = C - 2x$, we need to solve

$$\frac{du}{dx} - \frac{5}{2}u = -\frac{1}{2}e^{-5x+3C}.$$

This is a linear ODE, which can be solved using an integrating factor of the form $\phi(x) = e^{-\frac{5}{2}x}$, which gives us

$$u = e^{\frac{5}{2}x}\left(-\frac{1}{2}\int e^{-5x+3C}e^{-\frac{5}{2}x}\,dx\right) = -\frac{1}{2}e^{\frac{5}{2}x}\left(\frac{2e^{-\frac{15}{2}x+3C}}{-15} + f(C)\right) \Rightarrow u = \frac{1}{15}e^{-5x+3C} - \frac{1}{2}f(C)e^{\frac{5}{2}x}.$$

Since $C = y + 2x$, if we set $g(z) = -\frac{1}{2}f(z)$ then we get the general solution

$$u(x, y) = \frac{1}{15}e^{x+3y} + g(y + 2x)e^{\frac{5}{2}x}.$$

Problem 1.2. Solve the initial value problem

$$u_x + xu_y = 0, \quad u(x,0) = \sin(x^2).$$

In which region of the xy plane is the solution uniquely determined by the initial condition?

Solution 1.2. We have the system of equations

$$\frac{dx}{1} = \frac{dy}{x} = \frac{du}{0}.$$

Characteristic Curves: We start by solving the equation involving the first and second term,

$$\frac{dx}{1} = \frac{dy}{x} \Rightarrow \frac{dx}{dy} = \frac{1}{x}.$$

This is a separable ODE, with solution

$$\frac{x^2}{2} = y + C \Rightarrow C = \frac{x^2}{2} - y.$$

General Solution: We now solve the equation involving the first and third term,

$$\frac{dx}{1} = \frac{du}{0} \Rightarrow \frac{du}{dx} = 0 \Rightarrow u = f(C).$$

Since $C = \frac{x^2}{2} - y$, we get the general solution

$$u(x,y) = f\Big(\frac{x^2}{2} - y\Big).$$

Particular Solution: We now use the initial value to solve for f. Since $u(x,0) = \sin(x^2)$ we have

$$\sin(x^2) = u(x,0) = f\Big(\frac{x^2}{2}\Big).$$

If we set $s = \frac{x^2}{2}$, then we have

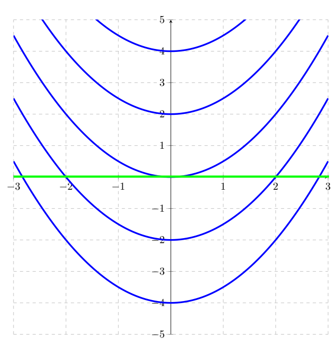
$$f(s) = \sin(2s).$$

Therefore, a particular solution is of the form

$$u(x,y) = f\Big(\frac{x^2}{2} - y\Big) = \sin(x^2 - 2y).$$

However, since $s = \frac{x^2}{2} \geq 0$, this initial condition only specified the values of $f(s)$ for $s \geq 0$. Therefore, our solution is only uniquely determined for $\frac{x^2}{2} - y \geq 0 \Rightarrow y \leq \frac{x^2}{2}$.

Remark. Since the general solution implies that $u(x,0) = f(\frac{x^2}{2})$, we need the initial condition to be an even function in x for a solution to exist. We can visualize this by plotting the characteristic curves. The characteristic curves intersect the initial condition (the green line) when $y \leq \frac{x^2}{2}$. Furthermore, since the solution is constant along the characteristic curves the initial condition must be equal when the parabolas intersect initial condition, i.e. the initial condition must be an even function in x.



2 Solving Semilinear First Order PDEs in Higher Dimensions

Problem 2.1. Find the general solution to the equation

$$u_x + 3u_y - 2u_z = u.$$

Find the particular solution when $u(0, y, z) = f(y, z)$.

Solution 2.1. We have the system of equations,

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{-2} = \frac{du}{u}.$$

Characteristic Curves Part 1: We start by solving the equation involving the first and second term,

$$\frac{dx}{1} = \frac{dy}{3} \implies C = 3x - y.$$

Characteristic Curves Part 2: We now solve the equation involving the first and third term,

$$\frac{dx}{1} = \frac{dz}{-2} \implies D = 2x + z.$$

General Solution: We now solve the equation involving the first and fourth term,

$$\frac{dx}{1} = \frac{du}{u} \implies x = \log|u| + f(C, D) \implies u(x, y, z) = g(C, D)e^x = g(3x - y, 2x + z)e^x,$$

for some function g of two variables.

Particular Solution: Plugging in our initial conditions, we have

$$u(0, y, z) = g(-y, z) = f(y, z).$$

We set $s = -y$ and $t = z$ to conclude that

$$g(s, t) = f(-s, t).$$

Therefore, our particular solution is given by

$$u(x, y, z) = g(3x - y, 2x + z)e^x = f(y - 3x, 2x + z)e^x.$$

Remark. The general form of the solution may be expressed differently depending on which characteristic curves we solve, but the particular solution will be the same. For example, if we solved for the equation involving the second and third term in the second step, we would get

$$C = 3x - y \quad \text{and} \quad D = 3z + 2y.$$

The general solution in this case after solving the equation involving the first and fourth term will result in

$$u(x, y, z) = g(C, D)e^x = g(3x - y, 3z + 2y)e^x.$$

We can plug in the initial conditions to conclude that

$$u(0, y, z) = g(-y, 3z + 2y) = f(y, z).$$

We set $s = -y$ and $t = 3z + 2y$. We write y and z as function of s and t,

$$y = -s \quad \text{and} \quad z = \frac{t - 2y}{3} = \frac{t + 2s}{3} \implies g(s, t) = f(y, z) = f\Big(-s, \frac{t + 2s}{3}\Big).$$

Therefore, our particular solution is given by

$$u(x, y, z) = g(3x - y, 3z + 2y)e^x = f\left(y - 3x, \frac{3z + 2y + 2(3x - y)}{3}\right)e^x = f(y - 3x, 2x + z)e^x,$$

which is the same as above.

Problem 2.2. Find the general solution to the equation

$$yu_x + xu_y + u_z = 0.$$

Find the particular solution when $u(x,y,0) = f(x,y)$.

Solution 2.2. We have the system of equations,

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{1} = \frac{du}{0}.$$

Characteristic Curves Part 1: We start by solving the equation involving the first and second term,

$$\frac{dx}{y} = \frac{dy}{x} \implies x^2 = y^2 + C \implies C = x^2 - y^2.$$

Characteristic Curves Part 2: We now solve the equation involving the first and third term using the fact $y = \sqrt{x^2 - C}$,

$$\frac{dz}{1} = \frac{dx}{y} = \frac{dx}{\sqrt{x^2 - C}} \implies z = \log|\sqrt{x^2 - C} + x| + D = \log|y + x| + D \implies D = \frac{(x+y)}{e^z}.$$

General Solution: We now solve the equation involving the third and fourth term,

$$\frac{dz}{1} = \frac{du}{0} \implies u(x,y,z) = g(C,D) = g\Big(x^2 - y^2, \frac{(x+y)}{e^z}\Big).$$

Particular Solution: Plugging in our initial conditions, we have

$$u(x,y,0) = g(x^2 - y^2, x + y) = f(x,y).$$

We set $s = x^2 - y^2$ and $t = x + y$. Our goal is to write x and y as some functions of s and t. We see that

$$s = x^2 - y^2 = (x-y)(x+y) = (x-y)t \implies x - y = \frac{s}{t}.$$

Since $x + y = t$ and $x - y = \frac{s}{t}$, we can add and subtract our answers to conclude

$$x = \frac{1}{2}\Big(t + \frac{s}{t}\Big) \quad \text{and} \quad y = \frac{1}{2}\Big(t - \frac{s}{t}\Big),$$

so

$$g(s,t) = f(x,y) = f\Big(\frac{1}{2}\Big(t + \frac{s}{t}\Big), \frac{1}{2}\Big(t - \frac{s}{t}\Big)\Big).$$

Therefore, our particular solution is given by

$$\begin{aligned}
u(x,y,z) &= g\Big(x^2 - y^2, \frac{(x+y)}{e^z}\Big) \\
&= f\Big(\frac{1}{2}\Big(\frac{x+y}{e^z} + \frac{x^2 - y^2}{\frac{(x+y)}{e^z}}\Big), \frac{1}{2}\Big(\frac{x+y}{e^z} - \frac{x^2 - y^2}{\frac{(x+y)}{e^z}}\Big)\Big) \\
&= f\Big(\frac{1}{2}\Big(\frac{x+y}{e^z} + \frac{x-y}{e^{-z}}\Big), \frac{1}{2}\Big(\frac{x+y}{e^z} - \frac{x-y}{e^{-z}}\Big)\Big).
\end{aligned}$$

Remark. We were a bit sloppy with the constants and domains of our functions above. The constants C and D changed each line and writing $x^2 - y^2 = C$ implicitly in terms of y depends on the value of C. We should check our general solution to ensure that it is a solution to our PDE by differentiating.