1 Fourier Transforms

Let $f : \mathbb{R} \to \mathbb{C}$. Its Fourier transform $\mathcal{F}f = \hat{f}$ is given by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

and its inverse Fourier transform $\mathcal{F}^{-1} f = \check{f}$ is given by

$$\check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{ikx}\, dk.$$

It follows that $\mathcal{F}^{-1}[\mathcal{F}f] = f$ and $\mathcal{F}[\mathcal{F}^{-1} f] = f$, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk \quad \text{and} \quad f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{f}(x) e^{-ikx}\, dx.$$

List of Important Transformations:

$$FT : \begin{cases} e^{-a|x|} \\ e^{-\frac{x^2}{2}} \end{cases} \mapsto \begin{cases} \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{a^2+k^2} \right) & (\mathrm{Re}(a) > 0) \\ e^{-\frac{k^2}{2}} \end{cases}$$

Properties of Fourier Transform:

$$FT : \begin{cases} f(x-a) \\ f(x)e^{ibx} \\ f'(x) \\ xf(x) \\ f(\lambda x) \\ (f*g)(x) \\ f(x)g(x) \end{cases} \mapsto \begin{cases} e^{-ika}\hat{f}(k) \\ \hat{f}(k-b) \\ ik\hat{f}(k) \\ i\hat{f}'(k) \\ |\lambda|^{-1}\hat{f}(\lambda^{-1}k) \\ \sqrt{2\pi}\hat{f}(k)\hat{g}(k) \\ \sqrt{2\pi}(\hat{f}*\hat{g})(k) \end{cases}$$

It is also useful to notice that a change of variables implies

$$\mathcal{F}^{-1}[f(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{ikx}\, dk \overset{k=-y}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-y) e^{-ixy}\, dy = \mathcal{F}[f(-s)]. \tag{1}$$

1.1 Finding Fourier Transforms

Problem 1.1. Find the Fourier transform of

$$f(x) = e^{-a|x|} \qquad a > 0.$$

Solution 1.1. This can be computed directly. We split the region of integration,

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-a|x|}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{-ikx+ax}\, dx + \int_{0}^{\infty} e^{-ikx-ax}\, dx \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ikx+ax}}{a-ik} \Big|_{x=-\infty}^{x=0} + \frac{e^{-ikx-ax}}{-a-ik} \Big|_{x=0}^{x=\infty} \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a-ik} + \frac{1}{a+ik} \right) \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+k^2}. \end{aligned}$$

Problem 1.2. Find the Fourier transform of

$$f(x) = xe^{-a\frac{x^2}{2}} \qquad a > 0.$$

Solution 1.2. Instead of computing it directly, we start from the Fourier transform of the Gaussian,

$$g(x) = e^{-\frac{x^2}{2}} \implies \hat{g}(k) = e^{-\frac{k^2}{2}}.$$

Since

$$f(x) = xe^{-a\frac{x^2}{2}} = x \cdot g(\sqrt{a}x),$$

the properties of the Fourier transform implies that

$$\begin{aligned}
\hat{f}(k) &= i\frac{d}{dk}\mathcal{F}[g(\sqrt{a}x)](k) && xf(x) \mapsto i\hat{f}'(k) \\
&= i\frac{d}{dk}\left(\frac{1}{\sqrt{a}}e^{-\frac{k^2}{2a}}\right) && f(\lambda x) \mapsto |\lambda|^{-1}\hat{f}(\lambda^{-1}k) \\
&= -ika^{-\frac{3}{2}}e^{-\frac{k^2}{2a}}.
\end{aligned}$$

Problem 1.3. Find the Fourier transform of

$$f(x) = (x^2 + a^2)^{-1}\sin(bx) \qquad a > 0,\ b > 0.$$

Solution 1.3. Instead of computing it directly, we start from the Fourier transform of the exponential,

$$g(x) = e^{-a|x|} \implies \hat{g}(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{a^2 + k^2}\right).$$

Unfortunately, the inverse of this transformation is what appears in the problem. Using a change of variables (1), we see that

$$\mathcal{F}[\mathcal{F}[g(s)]] = \mathcal{F}[\mathcal{F}^{-1}[g(-s)]] = g(-s),$$

so we can conclude that

$$h(x) = \hat{g}(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{a^2 + x^2}\right) \implies \hat{h}(k) = \mathcal{F}[\mathcal{F}[g(k)]] = g(-k) = e^{-a|k|}.$$

Since

$$f(x) = (x^2 + a^2)^{-1}\sin(bx) = \frac{\sqrt{2\pi}}{2a}h(x)\sin(bx) = \frac{\sqrt{2\pi}}{2a}\left(\frac{h(x)e^{ibx} - h(x)e^{-ibx}}{2i}\right),$$

the properties of the Fourier transform implies that

$$\begin{aligned}
\hat{f}(k) &= \frac{\sqrt{2\pi}}{4ai}\big(\hat{h}(k-b) - \hat{h}(k+b)\big) && f(x)e^{ibx} \mapsto \hat{f}(k-b) \\
&= \frac{\sqrt{2\pi}}{4ai}\big(e^{-a|k-b|} - e^{-a|k+b|}\big).
\end{aligned}$$

1.2 Solving PDEs on Infinite Regions

Problem 1.4. Using the properties of the Fourier transform, recover the general formula for the solution $u(x,t)$ of Laplace's equation

$$u_{xx} + u_{yy} = 0,\ x \in \mathbb{R},\ y \geq 0 \qquad u(x,0) = \phi(x).$$

Solution 1.4.

Step 1 — Transform the Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to Laplace's equation, and consider its Fourier transform

$$\hat{u}(k,y) = \int_{-\infty}^{\infty} e^{-ikx} u(x,y)\, dx.$$

Since $u_{xx} + u_{yy} = 0$, taking the Fourier transform of both sides implies that

$$-k^2 \hat{u}(k,y) + \hat{u}_{yy}(k,y) = 0 \qquad y > 0.$$

The solution to this ODE (in y) is given by

$$\hat{u}(k,y) = A(k)e^{-ky} + B(k)e^{ky},$$

where $A(k)$ and $B(k)$ are some yet to be determined functions of k.

Step 2 — Find the Particular Solution: Since our solution should be bounded for $y \geq 0$, we have $B(k) = 0$ for $k > 0$ and $A(k) = 0$ for $k < 0$. The general solution can be simplified as

$$\hat{u}(k,y) = C(k)e^{-|k|y}, \qquad C(k) = \begin{cases} A(k) & k > 0 \\ A(0) + B(0) & k = 0 \\ B(k) & k < 0 \end{cases}$$

We can find $C(k)$ by using our initial condition,

$$u(x,0) = \phi(x) \implies \hat{u}(k,0) = \hat{\phi}(k) \implies C(k) = \hat{\phi}(k).$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,y) = \hat{\phi}(k)e^{-|k|y}.$$

Step 3 — Recover the Solution: We take the inverse Fourier transform of both sides to recover our original function. Let $S(x,y)$ be the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}}e^{-|k|y}$,

$$\begin{aligned} S(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-y|k|}\, dk &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{ikx+yk}\, dk + \frac{1}{2\pi}\int_{0}^{\infty} e^{ikx-yk}\, dk \\ &= \frac{1}{2\pi}\frac{e^{ikx+yk}}{ix+y}\Big|_{k=-\infty}^{k=0} + \frac{1}{2\pi}\frac{e^{ikx-yk}}{ix-y}\Big|_{k=0}^{k=\infty} \\ &= \frac{1}{2\pi}\Big(\frac{1}{ix+y} - \frac{1}{ix-y}\Big) \\ &= \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}. \end{aligned}$$

Since $\hat{u}(k,y) = \hat{\phi}(k)e^{-|k|y} = \sqrt{2\pi}\hat{\phi}(k)\cdot\frac{e^{-|k|y}}{\sqrt{2\pi}}$, taking the inverse Fourier transform of both sides implies

$$u(x,y) = (\phi(\cdot) * S(\cdot,y))(x) = \int_{-\infty}^{\infty} S(x-\tau,y)\phi(\tau)\, d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\tau)^2+y^2}\phi(\tau)\, d\tau.$$

Remark. Instead of computing the Fourier transform directly, we could use property (1)

$$\mathcal{F}^{-1}[\mathcal{F}^{-1}[f(s)]] = \mathcal{F}^{-1}[\mathcal{F}[f(-s)]] = f(-s),$$

and the fact

$$\mathcal{F}[e^{-a|x|}] = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{a^2+k^2}\right)$$

to conclude that (treating k as the variable and y as a constant)

$$\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}e^{-|k|y}\right] = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left[\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}\frac{2y}{y^2+x^2}\right]\right] = \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}.$$

Problem 1.5. Using the properties of the Fourier transform, recover the general formula for the solution $u(x,t)$ of the heat equation

$$u_t - u_{xx} = 0,\ x\in\mathbb{R},\ t\geq 0 \qquad u(x,0) = \phi(x).$$

Solution 1.5.

Step 1 — Transform the Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to the heat equation, and consider its Fourier transform

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} e^{-ikx}u(x,t)\,dx.$$

Since $u_t - u_{xx} = 0$, taking the Fourier transform of both sides implies that

$$\hat{u}_t(k,t) + k^2\hat{u}(k,t) = 0 \qquad y > 0.$$

The solution to this ODE (in t) is given by

$$\hat{u}(k,t) = A(k)e^{-k^2t},$$

where $A(k)$ is some yet to be determined function of k.

Step 2 — Find the Particular Solution: We can find $A(k)$ by using our initial condition,

$$u(x,0) = \phi(x) \implies \hat{u}(k,0) = \hat{\phi}(k) \implies A(k) = \hat{\phi}(k).$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,t) = \hat{\phi}(k)e^{-k^2t}.$$

Step 3 — Recover the Solution: We take the inverse Fourier transform of both sides to recover our original function. Let $S(x,t)$ be the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}}e^{-k^2t}$,

$$\begin{aligned} S(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-k^2t}\,dk &= \frac{1}{2\pi}e^{-\frac{x^2}{4t}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\,dk \\ &= \frac{1}{2\pi}e^{-\frac{x^2}{4t}}\cdot\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-z^2}\,dz && \text{(See the Remark)} \\ &= \frac{1}{\sqrt{4\pi t}}\cdot e^{-\frac{x^2}{4t}}. && \int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}. \end{aligned}$$

Since $\hat{u}(k,y) = \hat{\phi}(k)e^{-k^2t} = \sqrt{2\pi}\hat{\phi}(k)\cdot\frac{e^{-k^2t}}{\sqrt{2\pi}}$, taking the inverse Fourier transform of both sides implies

$$u(x,t) = (\phi(\cdot) * S(\cdot,t))(x) = \int_{-\infty}^{\infty} S(x-\tau,t)\phi(\tau)\,d\tau = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{4t}}\phi(\tau)\,d\tau.$$

Remark. The imaginary change of variables $z = \sqrt{t}k - i\frac{x}{2\sqrt{t}}$ can be justified using complex analysis.

$$\int_{\mathbb{R}} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2} \, dk = \frac{1}{\sqrt{t}} \int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2} \, dz.$$

Consider the contour integral over the closed rectangular path oriented counter clockwise,

(1) $k - i\frac{x}{2\sqrt{t}}$ for k from $-M$ to M

(2) $M + iy$ for y from $-\frac{x}{2\sqrt{t}}$ to 0

(3) k for k from M to $-M$

(4) $M + iy$ for y from 0 to $-\frac{x}{2\sqrt{t}}$.

Since e^{-z^2} is analytic, the integral over this closed path is 0, so

$$\int_{(1)} e^{-z^2} \, dz + \int_{(2)} e^{-z^2} \, dz + \int_{(3)} e^{-z^2} \, dz + \int_{(4)} e^{-z^2} \, dz = 0.$$

Since e^{-z^2} is small when the $\mathrm{Re}(z) = \pm M$, if we take $M \to \infty$, the integrals over the regions (2) and (4) vanish leaving us with

$$\int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2} \, dz + \int_{\infty}^{-\infty} e^{-z^2} \, dz = 0 \implies \int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2} \, dz = \int_{-\infty}^{\infty} e^{-z^2} \, dz.$$

Therefore,

$$\int_{\mathbb{R}} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2} \, dk = \frac{1}{\sqrt{t}} \int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2} \, dz = \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-z^2} \, dz.$$

Remark. Instead of computing the Fourier transform directly, we could use property (1) and the fact

$$\mathcal{F}[e^{-\frac{x}{2}^2}] = e^{\frac{-k^2}{2}},$$

to conclude that (treating k as the variable and t as a constant)

$$\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}} e^{-k^2 t}\right] = \mathcal{F}\left[\frac{1}{\sqrt{2\pi}} e^{-k^2 t}\right] = \frac{1}{\sqrt{2\pi}} \mathcal{F}\left[e^{-\frac{(\sqrt{2t}k)^2}{2}}\right] = \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}.$$