1 Solving the Heat Equation

Consider the heat equation on the whole line

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}. \end{cases}$$

The particular solution to this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} g(y)\, dy.$$

It is usually impossible to compute $u(x,t)$ except for some very specific initial data. We will go over a few of these cases in the following examples.

Problem 1.1. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = \mathbb{1}_{[a,\infty)}(x) & x \in \mathbb{R}, \end{cases}$$

where $a \in \mathbb{R}$ and

$$\mathbb{1}_{[a,\infty)}(x) = \begin{cases} 1 & x \geq a \\ 0 & x < a. \end{cases}$$

Write your answer in terms of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp.$$

Solution 1.1. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_a^{\infty} e^{-\frac{(y-x)^2}{4kt}}\, dy.$$

We can use a change of variables to write this function in terms of error functions. Using the change of variables

$$p = \frac{y-x}{\sqrt{4kt}}, \qquad \sqrt{4kt}\, dp = dy, \qquad y = a \implies p = \frac{a-x}{\sqrt{4kt}}, \qquad y = \infty \implies p = \infty$$

we get

$$\begin{aligned} u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{a-x}{\sqrt{4kt}}}^{\infty} e^{-p^2}\, dp &= \frac{1}{\sqrt{\pi}} \int_{\frac{a-x}{\sqrt{4kt}}}^{0} e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-p^2}\, dp \\ &= -\frac{1}{\sqrt{\pi}} \int_0^{\frac{a-x}{\sqrt{4kt}}} e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-p^2}\, dp \\ &= -\frac{1}{2}\operatorname{erf}\left(\frac{a-x}{\sqrt{4kt}}\right) + \frac{1}{2}. \end{aligned}$$

Remark. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_a^{\infty} e^{-\frac{(y-x)^2}{4kt}}\, dy.$$

If we set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{P}(X \geq a)$. This can be written in terms of the error function as

$$u(x,t) = \mathbb{P}(X \geq a) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{a-x}{\sqrt{4kt}}\right).$$

Problem 1.2. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = e^{ax} & x \in \mathbb{R}, \end{cases}$$

where $a \in \mathbb{R}$.

Solution 1.2. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{ay} \, dy.$$

Since the initial data is an exponential, we can complete the square to simplify the integral. Completing the square in y, we have

$$\begin{aligned} -\frac{(y-x)^2}{4kt} + ay &= -\frac{y^2 - 2xy + x^2 - 4ktay}{4kt} \\ &= -\frac{y^2 - (2x + 4kta)y + x^2}{4kt} \\ &= -\frac{(y - (x + 2kta))^2 - (x + 2kta)^2 + x^2}{4kt} \\ &= -\frac{(y - (x + 2kta))^2}{4kt} + ax + a^2kt. \end{aligned}$$

Therefore, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+2kta))^2}{4kt} + ax + a^2kt} \, dy = e^{ax+a^2kt} \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+2kta))^2}{4kt}} \, dy}_{=1} = e^{ax+a^2kt}.$$

We used the fact that this Gaussian integral corresponds to the p.d.f. of a Gaussian random variable with mean $(x + 2kta)$ and variance $2kt$, so its integral over $\mathbb{R}$ is equal to 1.

Remark. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{ay} \, dy.$$

If set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, e^{aX}$. In particular, using the formula for the moment generating function of the Gaussian random variable, we have

$$u(x,t) = \mathbb{E}\, e^{aX} = e^{a\mu + \frac{1}{2}a^2\sigma^2} = e^{ax+a^2kt}.$$

Problem 1.3. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = \mathbb{1}_{[-1,1]}(x) & x \in \mathbb{R}, \end{cases}$$

where

$$\mathbb{1}_{[-1,1]}(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & x > 1. \end{cases}$$

Write your answer in terms of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} \, dp.$$

Solution 1.3. We can use a change of variables to write this function in terms of error functions like in Problem 1.1. Instead, we can use linearity and our result in the last problem to simplify this problem. It is easy to check that

$$\mathbb{1}_{[-1,1]}(x) = \mathbb{1}_{[-1,\infty)}(x) - \mathbb{1}_{(1,\infty)}(x).$$

In particular, if we let u_1 be a solution to the heat equation with initial condition $\mathbb{1}_{[-1,\infty)}(x)$ and u_2 be a solution to the heat equation with initial condition $-\mathbb{1}_{(1,\infty)}(x)$, then it is easy to check that $u = u_1 + u_2$ is a solution to the PDE given in this problem. Therefore, by linearity and our result from Problem 1.1, we have

$$\begin{aligned} u(x,t) = u_1(x,t) + u_2(x,t) &= -\frac{1}{2}\operatorname{erf}\left(\frac{-1-x}{\sqrt{4kt}}\right) + \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) - \frac{1}{2} \\ &= -\frac{1}{2}\operatorname{erf}\left(\frac{-1-x}{\sqrt{4kt}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) \\ &= \frac{1}{2}\operatorname{erf}\left(\frac{1+x}{\sqrt{4kt}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{x-1}{\sqrt{4kt}}\right) \end{aligned}$$

since the error function is odd.

Problem 1.4. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = x & x \in \mathbb{R}. \end{cases}$$

Solution 1.4. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} y \, dy.$$

If we set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, X$. In particular, since the mean of this normal distribution is x, we have

$$u(x,t) = \mathbb{E}\, X = x.$$

Problem 1.5. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = x^2 & x \in \mathbb{R}. \end{cases}$$

Solution 1.5. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} y^2 \, dy.$$

If we set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, X^2$. In particular, since the variance of this normal distribution is $2kt$ and the mean is x, we have

$$u(x,t) = \mathbb{E}\, X^2 = \operatorname{Var}(X) + (\mathbb{E}\, X)^2 = 2kt + x^2,$$

where we used the fact $\operatorname{Var}(X) = \mathbb{E}\, X^2 - (\mathbb{E}\, X)^2$.

2 Inhomogeneous Heat Equation on the Half Line

Suppose we have the heat equation on the half line. We want to reduce this problem to a PDE on the entire line by finding an appropriate extension of the initial conditions that satisfies the given boundary conditions. The choice of the extension only depends on the boundary conditions:

1. Dirichlet ($u|_{x=0} = 0$): We take an odd extension of the initial conditions.
2. Neumann ($u_x|_{x=0} = 0$): We take an even extension of the initial conditions.

This extension reduces the heat equation on the half line to the full line, so we can apply (1) where the initial conditions are replaced by the respective odd or even extensions.

Problem 2.1. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & 0 < x < \infty,\ t > 0, \\ u|_{t=0} = g(x) & 0 < x < \infty, \\ u|_{x=0} = h(t) & t > 0 \end{cases}$$

where

$$g(x) = \mathbb{1}_{(-\infty,1)}(x) = \begin{cases} 1 & x < 1 \\ 0 & x \geq 1 \end{cases}, \qquad h(t) = 0.$$

Solution 2.1. We want to solve the heat equation on the half line with Dirichlet boundary conditions. We can use an odd reflection to extend the initial condition,

$$g_{odd}(x) = \begin{cases} -g(-x) & x < 0 \\ 0 & x = 0 \\ g(x) & x > 0 \end{cases} = \begin{cases} 0 & x \leq 1 \\ -1 & -1 < x < 0 \\ 0 & x = 0 \\ 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}.$$

The particular solution to the extended PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} g_{odd}(y)\, dy.$$

We can compute this by splitting the region of integration,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-1}^{0} e^{-\frac{(y-x)^2}{4kt}} (-1)\, dy + \frac{1}{\sqrt{4\pi kt}} \int_{0}^{1} e^{-\frac{(y-x)^2}{4kt}} 1\, dy.$$

Using the change of variables $z = \frac{y-x}{\sqrt{4kt}}$ to simplify the fist integral, we see

$$-\frac{1}{\sqrt{4\pi kt}} \int_{-1}^{0} e^{-\frac{(y-x)^2}{4kt}}\, dy = -\frac{1}{\sqrt{\pi}} \int_{\frac{-1-x}{\sqrt{4kt}}}^{\frac{-x}{\sqrt{4kt}}} e^{-z^2}\, dz = \frac{1}{2}\left(\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{1+x}{\sqrt{4kt}}\right)\right),$$

and using the change of variables $z = \frac{y-x}{\sqrt{4kt}}$ to simplify the second integral, we see

$$\frac{1}{\sqrt{4\pi kt}} \int_{0}^{1} e^{-\frac{(y-x)^2}{4kt}}\, dy = \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4kt}}}^{\frac{1-x}{\sqrt{4kt}}} e^{-z^2}\, dz = \frac{1}{2}\left(\operatorname{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right).$$

Therefore, for $x > 0$ and $t > 0$, the solution to the PDE is given by

$$u(x,t) = \frac{1}{2}\left(\operatorname{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) + 2 \cdot \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{1+x}{\sqrt{4kt}}\right)\right).$$

Problem 2.2. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & 0 < x < \infty,\ t > 0, \\ u|_{t=0} = g(x) & 0 < x < \infty, \\ u_x|_{x=0} = h(t) & t > 0 \end{cases}$$

where

$$g(x) = xe^{-x}, \qquad h(t) = 0.$$

Solution 2.2. We want to solve the heat equation on the half line with Neumann boundary conditions. We can use an even reflection to extend the initial condition,

$$g_{even}(x) = \begin{cases} g(-x) & x < 0 \\ g(x) & x \geq 0 \end{cases} = \begin{cases} -xe^{x} & x < 0 \\ xe^{-x} & x \geq 0 \end{cases}.$$

The particular solution to the extended PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} g_{even}(y)\, dy.$$

We can compute this by splitting the region of integration,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\frac{(y-x)^2}{4kt}} (-ye^{y})\, dy + \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(y-x)^2}{4kt}} ye^{-y}\, dy.$$

Completing the square and integrating by parts, we see that

$$\begin{aligned}
&- \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\frac{(y-x)^2}{4kt}} ye^{y}\, dy \\
&= -\frac{e^{x+kt}}{\sqrt{4\pi kt}} \int_{-\infty}^{0} ye^{-\frac{(y-(x+2kt))^2}{4kt}}\, dy \\
&= -\frac{e^{x+kt}}{\sqrt{4\pi kt}} \int_{-\infty}^{0} (y-(x+2kt))e^{-\frac{(y-(x+2kt))^2}{4kt}}\, dy - (x+2kt) \cdot \frac{e^{x+kt}}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\frac{(y-(x+2kt))^2}{4kt}}\, dy \\
&= \frac{e^{x+kt}}{\sqrt{\pi}} \cdot \left(\frac{e^{-\frac{(y-(x+2kt))^2}{4kt}}}{2} \right)\Bigg|_{y=-\infty}^{y=0} - (x+2kt) \cdot \frac{e^{x+kt}}{\sqrt{\pi}} \cdot \int_{-\infty}^{\frac{-(x+2kt)}{\sqrt{4kt}}} e^{-z^2}\, dz \\
&= \frac{e^{x+kt}}{\sqrt{\pi}} \cdot \frac{e^{-\frac{(x+2kt)^2}{4kt}}}{2} - (x+2kt) \cdot e^{x+kt} \cdot \frac{1}{2}\left(-\operatorname{erf}\left(\frac{x+2kt}{\sqrt{4kt}} \right) + 1 \right),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(y-x)^2}{4kt}} ye^{-y}\, dy \\
&= \frac{e^{-x+kt}}{\sqrt{4\pi kt}} \int_{0}^{\infty} ye^{-\frac{(y-(x-2kt))^2}{4kt}}\, dy \\
&= \frac{e^{-x+kt}}{\sqrt{4\pi kt}} \int_{0}^{\infty} (y-(x-2kt))e^{-\frac{(y-(x-2kt))^2}{4kt}}\, dy + (x-2kt) \cdot \frac{e^{-x+kt}}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(y-(x-2kt))^2}{4kt}}\, dy \\
&= -\frac{e^{-x+kt}}{\sqrt{\pi}} \cdot \left(\frac{e^{-\frac{(y-(x-2kt))^2}{4kt}}}{2} \right)\Bigg|_{y=0}^{y=\infty} + (x-2kt) \cdot \frac{e^{-x+kt}}{\sqrt{\pi}} \cdot \int_{\frac{-(x-2kt)}{\sqrt{4kt}}}^{\infty} e^{-z^2}\, dz \\
&= \frac{e^{-x+kt}}{\sqrt{\pi}} \cdot \frac{e^{-\frac{(x-2kt)^2}{4kt}}}{2} + (x-2kt) \cdot e^{-x+kt} \cdot \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x-2kt}{\sqrt{4kt}} \right) \right).
\end{aligned}$$

Therefore, for $x > 0$ and $t > 0$, the solution to the PDE is given by

$$
\begin{aligned}
u(x,t) = \frac{e^{-x+kt}}{\sqrt{\pi}} \cdot \frac{e^{-\frac{(x-2kt)^2}{4kt}}}{2} + \frac{e^{x+kt}}{\sqrt{\pi}} \cdot \frac{e^{-\frac{(x+2kt)^2}{4kt}}}{2} \\
+ (x-2kt) \cdot e^{-x+kt} \cdot \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x-2kt}{\sqrt{4kt}}\right)\right) - (x+2kt) \cdot e^{x+kt} \cdot \frac{1}{2}\left(-\operatorname{erf}\left(\frac{x+2kt}{\sqrt{4kt}}\right) + 1\right).
\end{aligned}
$$