1 Brownian Motion

In this section, we will introduce the best known continuous time stochastic process called a Brownian motion. Unlike the DTMC and CTMC we have seen before, which required discrete state spaces, this process will have continuous sample paths.

Definition 1.1. A stochastic process $\{W_t\}_{t\geq 0}$ is called a (standard) **Brownian motion** if it satisfies the following conditions:

1. $W_0 = 0$.
2. For each $\omega \in \Omega$, the sample path $t \longmapsto W_t(\omega)$ is continuous.
3. It has independent increments, i.e., for $0 = t_0 < t_1 < \cdots < t_m$,

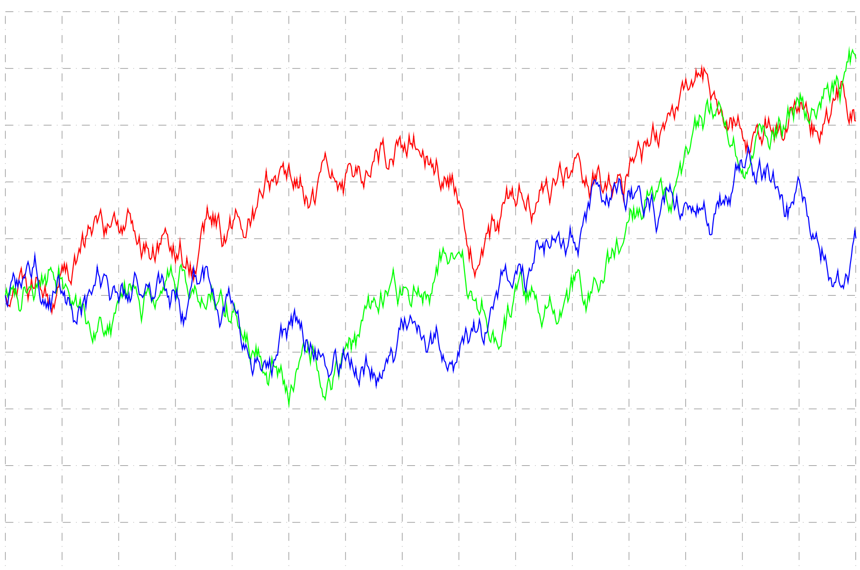
$$W_{t_1} - W_{t_0},\ W_{t_2} - W_{t_1}, \dots,\ W_{t_m} - W_{t_{m-1}}$$

   are independent.
4. $W_t - W_s$ has law $N(0, t-s)$ for $0 \leq s < t$.

Remark 1.2. We have the following direct consequences,

1. Since $W_0 = 0$, (3) implies in particular that W_s and $W_t - W_s$ are independent for $0 < s < t$.
2. Condition (4) implies that W has stationary increments. In particular $W_t = W_t - W_0$ has distribution $N(0,t)$.

Sample paths of three realizations of Brownian motion is displayed below:



Remark 1.3. It is possible to construct Brownian motion from a rescaled simple random walk $\{X_n\}_{n\geq 0}$ (i.e., the increments $X_1 - X_0, \dots, X_n - X_{n-1}$ are independent Rademacher random variables with $\mathbb{P}(X_n - X_{n-1} = \pm 1) = 1/2$). To this end, let

$$W_t^{(n)} := \frac{1}{\sqrt{n}} X_{\lfloor nt \rfloor}$$

Then one can show that $W_t^{(n)}$ converges to Brownian motion as $n \to \infty$.

This definition is quite similar to the Poisson process, in the sense that they both independent increments with distribution given by a well known probability distribution. There are two main differences are

- continuity: Brownian motion has continuous sample paths while Poisson processes are a counting process with discontinuities at the jumps;
- distribution of increments: the increments of a Brownian motion are normally distributed with variance depending on the length of the interval while the increments of a Poisson process are Poisson distributed with variance depending on the length of the interval.

Recall that Gaussian random vectors are completely determined by its mean and covariance matrix. Naturally, Gaussian processes are also characterized by its mean and covariance functions.

Definition 1.4. The **covariance function** of a stochastic process X_s is given by

$$K(s,t) = \mathrm{Cov}(X_s, X_t) = \mathbb{E}[(X_s - \mathbb{E}[X_s])(X_t - \mathbb{E}[X_t])]$$

The computation for Brownian motion is simple (see Problem 1.1). This gives us the following alternative definition of Brownian motion.

Definition 1.5. Brownian motion is a centered Gaussian process on $\mathbb{R}^+$ with continuous sample paths and covariance function

$$K(s,t) = \min(s,t).$$

Remark 1.6. Recall that a **Gaussian process** $(X_t)_{t\geq 0}$ is a stochastic process with jointly Gaussian points, i.e. for every finite set of indices $t_1, \dots, t_n$, we have

$$(X_{t_1}, \dots, X_{t_n})$$

is a multivariate Gaussian.

To conclude this section, we introduce a more general notion of Brownian motion, which takes into consideration different variances and means.

Definition 1.7. If W is a standard Brownian motion as in Definition 1.1, $x_0, \mu \in \mathbb{R}$ and $\sigma > 0$, then

$$x_0 + \sigma W_t + \mu t$$

is called a **Brownian motion with start in x_0, volatility σ, and drift μ**. Changing the volatility of Brownian motion is similar to changing the intensity of a Poisson process.

1.1 Properties of Brownian Motion

In the following, we let

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma(W_s : s \leq t), \qquad t \geq 0,$$

be the natural filtration of the Brownian motion W. The fact that W has independent increments and Remark 1.2 1 imply that for $u > t$ the increment $W_u - W_t$ is independent of $\mathcal{F}_t$. The independent increments implies that Brownain motion is a Markov process.

Proposition 1.8

Brownian motion satisfies the Markov property:

$$\mathbb{P}(W_t \in A \mid \mathcal{F}_s) = \mathbb{P}(W_t \in A \mid W_s) \qquad \textit{for all } A \subset \mathbb{R} \textit{ and } s \leq t.$$

The increments are also centered, so we have that Brownian motion is also a martingale.

Proposition 1.9

Brownian motion is a martingale.

We will now state several interesting properties about the behavior of Brownian motion. The proofs are a bit technical and require background that is slightly beyond the scope of this course, so they will be skipped.

The first result is a law of large numbers results that describes the long run behavior of Brownian motion. We have that Brownian motion grows sub-linearly.

Proposition 1.10

Let W_t be a Brownian motion. We have

$$\lim_{t\to\infty}\frac{1}{t}W_t = 0$$

almost surely.

Remark 1.11. The precise asymptotics is given by a result called the law of iterated logarithms. We have

$$\limsup_{t\to\infty}\frac{W_t}{\sqrt{2t\log\log t}} = 1$$

almost surely.

The oscillations of Brownian motion are so rapid that although it's sample paths are continuous, it is not a smooth function.

Theorem 1.12

With probability one, the function $t\mapsto W_t$ is nowhere differentiable.

A natural question is if there is another way to make precise how rapid the oscillations of Brownian motion is. A natural way to measure the oscillations is a quantity called the total variation.

Definition 1.13. Let $\Pi = \{t_0,\dots,t_n\}$ be a partition of $[0,T]$ such that

$$0 = t_0 < t_1 < \cdots < t_n = T.$$

If we let $\|\Pi\| = \max_i(t_{i+1}-t_i)$ denote maximal step size partition. The **total variation** of a function f on the interval $[0,T]$ is given by

$$V_1(f;0,T) = \lim_{\|\Pi\|\to 0}\sum_{i=0}^{n-1}|f(t_{i+1})-f(t_i)|$$

where the limit is taken as the number of points in the partition tend to infinity and the maximal step size goes to 0. A function is of **bounded variation** if its total variation is finite.

Remark 1.14. If f has a continuous derivative, then the total variation is simply the "arc length" of a 1 dimensional curve,

$$V_1(f;0,T) = \int_0^T |f'(t)|\,dt.$$

In particular, differentiable functions with bounded derivatives have finite total variation.

One can show that Brownian motion has infinite total variation.

Proposition 1.15

For $T > 0$,

$$V_1(W_t; 0, T) = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}| = \infty.$$

That is, W has infinite total variation.

This is somewhat expected. Brownian motion is not as nice as a smooth function, so its oscillations are more rapid. The right notion to measure how rapid these oscillations are is quadratic variation.

Definition 1.16. The **quadratic variation** of a function f on the interval $[0, T]$ is given by

$$V_2(f; 0, T) = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|^2$$

where the limit is taken as the number of points in the partition tend to infinity and the maximal step size goes to 0. We sometimes use the notation $[f, f]_T = [f]_T$ to denote the quadratic variation of a function f on the interval $[0, T]$.

Remark 1.17. If f has a continuous derivative, then the quadratic variation is always zero

$$V_2(f; 0, T) = 0.$$

The quadratic variation measures the variation at a different scale than total variation, and assigns much smaller values to small variations since we are summing the squares of small numbers. We can observe that although Brownian motion has infinite total variation, it has finite quadratic variation.

Theorem 1.18

For $T > 0$,

$$V_2(W_t; 0, T) = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}|^2 = T.$$

That is, W has **quadratic variation** *t, i.e. $[W, W]_T = [W]_T = T$.*

The proof is a bit difficult, but some intuition can be gained by looking at the symmetric random walk (Problem 1.7). The fact that Brownian motion has non-zero quadratic motion is a crucial result. Its behavior is very different than the differentiable functions we have seen before, so the usual notions of calculus no longer make sense, and a new generalization of calculus has to be developed.

1.2 Example Problems

1.2.1 Proofs of Main Results

Problem 1.1. Find the covariance function of Brownian motion.

Solution 1.1. We want to find

$$K(s, t) = \mathrm{Cov}(W_s, W_t).$$

Without loss of generality, suppose that $t > s$. In this case, we have

$$\mathrm{Cov}(W_s, W_t) = \mathrm{Cov}(W_s, W_t - W_s + W_s) = \mathrm{Cov}(W_s, W_t - W_s) + \mathrm{Cov}(W_s, W_s) = s = \min(s, t)$$

since $\mathrm{Cov}(W_s, W_t - W_s) = 0$ because of independent increments. The exact same argument show that if $s > t$, then

$$\mathrm{Cov}(W_s, W_t) = t = \min(s, t).$$

Problem 1.2. Show that Brownian motion satisfies the Markov property (Proposition 1.8).

Solution 1.2. The following proof uses only the fact that Brownian motion has independent increments. We need to show that for any set $A \subset \mathbb{R}$ and $t > s$,

$$\mathbb{P}(W_t \in A \mid \mathcal{F}_s) = \mathbb{P}(W_t \in A \mid W_s).$$

We have

$$\mathbb{P}(W_t \in A \mid \mathcal{F}_s) = \mathbb{P}((W_t - W_s) + W_s \in A \mid \mathcal{F}_s).$$

Notice that $W_t + W_s$ is independent of $\mathcal{F}_s$ and W_s is $\mathcal{F}_s$ measurable. This implies that the conditional probability only depends on $\mathcal{F}_s$ through its most recent information $\sigma(W_s)$. We conclude that the conditional probability only depends on W_s,

$$\mathbb{P}((W_t - W_s) + W_s \in A \mid \mathcal{F}_s) = \mathbb{P}((W_t - W_s) - W_s \in A \mid W_t) = \mathbb{P}(W_t \in A \mid W_s).$$

Remark 1.19. To make the dependence on W_t more precise, we can use the fact that conditionally on $\mathcal{F}_s$, that $W_{t+s} + W_t$ is Gaussian with mean W_s and variance $(t-s)$, so its conditional probability is explicit

$$\mathbb{P}((W_t - W_s) + W_s \in A \mid \mathcal{F}_s) = \int_A \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-W_s)}{2(t-s)}}\, dx = \mathbb{P}((W_t - W_s) + W_s \in A \mid W_s)$$

Problem 1.3. Show that Brownian motion is a martingale (Proposition 1.9).

Solution 1.3. Since $W_t \sim N(0,t)$ is satisfies the integrability conditions. For $s \leq t$, we have again by the independence of the increments,

$$\begin{aligned}
\mathbb{E}[W_t|\mathcal{F}_s] &= \mathbb{E}[W_t - W_s + W_s|\mathcal{F}_s] \\
&= \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \mathbb{E}[W_s|\mathcal{F}_s] \\
&= \mathbb{E}[W_t - W_s] + W_s \\
&= 0 + W_s.
\end{aligned}$$

1.2.2 Applications

Problem 1.4. Let W be a standard Brownian motion. Show that the process $\widehat{W}$ defined through $\widehat{W}_0 := 0$ and $\widehat{W}_t := tW_{1/t}$ for $t > 0$ is again a standard Brownian motion.

Solution 1.4. There are several ways to check this. Perhaps it is easiest to show Definition 1.5. Clearly, $\widehat{W}_t$ is a centered Gaussian process because it is the scalar multiple of a Gaussian process under a chance of time. It is continuous because by Proposition 1.10

$$\lim_{t\to 0} \widehat{W}_t = \lim_{t\to 0} tW_{1/t} = \lim_{s\to\infty} \frac{1}{s} W_s = 0.$$

To compute the covariance function, notice that for $t > s$,

$$K(s,t) = \mathrm{Cov}(\widehat{W}_s, \widehat{W}_t) = \mathrm{Cov}(sW_{1/s}, tW_{1/t}) = st \min\left(\frac{1}{s}, \frac{1}{t}\right) = s = \min(s,t).$$

The same result holds for $s > t$, so we can conclude that $\widehat{W}_t$.

Alternative Solution: We can also check the conditions in Definition 1.1:

1. By definition $\widehat{W}_0 = 0$
2. The sample paths are continuous at 0 because
$$\lim_{t\to 0} \widehat{W}_t = \lim_{t\to 0} tW_{1/t} = \lim_{s\to\infty} \frac{1}{s} W_s = 0.$$
The continuity elsewhere is clear because it is rescaling of a continuous function.
3. For any $t_1 < \cdots < t_n$, we have $\widehat{W}_{t_1} - \widehat{W}_{t_0}, \ldots, \widehat{W}_{t_n} - \widehat{W}_{t_{n-1}}$ are independent because for any pairs of
$$\widehat{W}_{t_k} - \widehat{W}_{t_{k-1}} = t_k W_{\frac{1}{t_k}} - t_{k-1} W_{\frac{1}{t_{k-1}}}$$
we have (without loss of generality we may assume that $t_{k-1} < t_k \le t_{k'-1} < t_{k'}$)
$$\begin{aligned}
&\mathrm{Cov}(\widehat{W}_{t_k} - \widehat{W}_{t_{k-1}}, \widehat{W}_{t'_k} - \widehat{W}_{t_{k'-1}}) \\
&= t_k t_{k'} \min(t_k^{-1}, t_{k'}^{-1}) - t_k t_{k'-1} \min(t_k^{-1}, t_{k'-1}^{-1}) - t_{k-1} t_{k'} \min(t_{k-1}^{-1}, t_{k'}^{-1}) + t_{k-1} t_{k'-1} \min(t_{k-1}^{-1}, t_{k'-1}^{-1}) \\
&= t_k - t_k - t_{k-1} + t_{k-1} \\
&= 0.
\end{aligned}$$
Since the increments are Gaussian, we have that they are uncorrelated and therefore independent.
4. We have
$$\mathbb{E}[\widehat{W}_t - \widehat{W}_s] = t\,\mathbb{E}[W_{1/t}] - s\,\mathbb{E}[W_{1/s}] = 0$$
since W_t is centered. Likewise, we have for $t > s$,
$$\begin{aligned}
\mathrm{Var}(\widehat{W}_t - \widehat{W}_s) &= \mathrm{Var}(\widehat{W}_t) + \mathrm{Var}(\widehat{W}_s) - 2\mathrm{Cov}(\widehat{W}_s, \widehat{W}_t) \\
&= t^2\,\mathrm{Var}(W_{\frac{1}{t}}) + s^2\,\mathrm{Var}(W_{\frac{1}{s}}) - 2st\mathrm{Cov}(W_{\frac{1}{s}}, W_{\frac{1}{t}}) \qquad W_{\frac{1}{t}} \sim N(0, t^{-1}) \\
&= t + s - 2st \min\left(\frac{1}{s}, \frac{1}{t}\right) = t - s.
\end{aligned}$$

Problem 1.5. For a constant σ let
$$Z_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$
Show that Z is a martingale.

Solution 1.5. We use the fact that $W_t \sim N(0,t)$. Recall that if $Z \sim N(0,\sigma^2)$ then the formula for the moment generating function of a standard Gaussian says that
$$\mathbb{E}[e^{tZ}] = e^{\frac{1}{2}t^2\sigma^2}.$$
The integrability assumptions follow from the fact that the MGF of the Gaussian random variable is finite.
$$\begin{aligned}
\mathbb{E}\left[Z_t | \mathcal{F}_s\right] &= \mathbb{E}\left[e^{\sigma W_t - \frac{1}{2}\sigma^2 t} \,\middle|\, \mathcal{F}_s \right] \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 t} \mathbb{E}\left[e^{\sigma(W_t - W_s)} \,\middle|\, \mathcal{F}_s \right] \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 t} \mathbb{E}\left[e^{\sigma(W_t - W_s)} \right] \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 t} e^{\frac{1}{2}\sigma^2(t-s)} \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 s} \\
&= Z_s.
\end{aligned}$$

Problem 1.6. For $\mu > 0$, let

$$Z_t^\mu := e^{\sigma W_t + \mu t},$$

which is called **geometric Brownian motion** with **drift** μ and **volatility** σ.

(a) Compute $\mathbb{E}[Z_t^\mu]$.

(b) Determine the asymptotic rate of growth:

$$\lim_{t \uparrow \infty} \frac{1}{t} \log Z_t^\mu.$$

Solution 1.6. We will see later in this course that geometric Brownian motion is used to model stock prices in the Black–Scholes model.

Part (a): Using the fact that $W_t \sim N(0,t)$ and the MGF of a Gaussian random variable, we have that

$$\mathbb{E}[Z_t^\mu] = \mathbb{E}[e^{\sigma W_t + \mu t}] = e^{\mu t} e^{\frac{1}{2}\sigma^2 t} = e^{\frac{1}{2}\sigma^2 t + \mu t}.$$

Part (b): We have by Proposition 1.10 that

$$\lim_{t \to \infty} \frac{1}{t} \log Z_t^\mu = \lim_{t \to \infty} \left(\frac{\sigma W_t}{t} + \mu \right) = \mu.$$

This implies that at the logarithmic scale, $\log Z_t^\mu$ grows like $t\mu$.This is the the growth rate of a deterministic process that satisfies $\frac{d}{dt} S_t = \mu S_t$.

Remark 1.20. Notice that

$$\mathbb{E}\, \frac{1}{t} \log Z_t^\mu = \mu \le \frac{1}{t} \log \mathbb{E}\, Z_t^\mu = \frac{1}{2}\sigma^2 + \mu$$

by Jensen's inequality. So the fact that both (a) and (b) do not have the same exponential scaling is not surprising.

Problem 1.7. We define the quadratic variation of a random walk X_n to be

$$[X]_n = \sum_{i=1}^n (X_i - X_{i-1})^2.$$

Show that the quadratic variation of X_n is n.

Solution 1.7. Since $X_i - X_{i-1} = \xi_i$ where ξ_i is an independent Rademacher random variable, which takes values in $\{\pm 1\}$ we have that

$$(X_i - X_{i-1})^2 = \xi_i^2 = 1,$$

so

$$[X]_n = \sum_{i=1}^n (X_i - X_{i-1})^2 = n.$$

Remark 1.21. Since Brownian motion can be though of as the limit of the scaled random walk $W_t^{(n)} := \frac{1}{\sqrt{n}} X_{\lfloor nt \rfloor}$, one might expect that the quadratic variation of Brownian motion is

$$\Big[\frac{1}{\sqrt{n}} X_{\lfloor nt \rfloor}\Big]_n = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} (X_i - X_{i-1})^2 \approx t.$$