1 Poisson processes

1.1 Basic Definitions

We now consider stochastic processes in continuous time. By definition, this is a collection $\{X(t)\}_{t\geq 0}$ of random variables $X(t)$ on some given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We begin by reviewing the Poisson process.

The Poisson process satisfies two key properties.

Definition 1.1.

(1) A stochastic process $\{X(t)\}_{t\geq 0}$ is said to has **independent increments** if, for any $0 \leq t_0 < t_1 < \cdots < t_n$, the increments

$$X(t_1) - X(t_0),\ X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are all independent.

(2) A stochastic process $\{X(t)\}_{t\geq 0}$ is said to has **stationary increments**, if for any $t \geq 0$ and $h > 0$, the increment $X(t+h) - X(t)$ has the same distribution (or law) as $X(h) - X(0)$.

Remark 1.2. These conditions seem similar to the Markov property and time homogeneous, but they are different concepts. There are examples of processes with stationary increments that are not homogeneous and vice versa. Furthermore, one can show that independent increments implies the Markov property, but the converse is not true. This will be elaborated when we introduce continuous time Markov chains in the next section.

When the stochastic processes represents the counts of the occurrence of events, we call it a counting process.

Definition 1.3. A stochastic process $\{N(t)\}_{t\geq 0}$ is called a **counting process** if

(1) $N(t)$ takes values in $\{0, 1, 2, \ldots\}$.

(2) $N(t) \geq N(s)$ if $t \geq s$.

The Poisson process is a special counting process with independent and stationary increments given in terms of a Poisson distribution. It is used to model random events that happen at a consistent rate.

Definition 1.4. A stochastic process $\{N(t)\}_{t\geq 0}$ is called a **Poisson process** with intensity $\lambda > 0$ if

(1) $N(0) = 0$.

(2) It has independent increments.

(3) For any $t \geq 0$ and $h > 0$, the increment $N(t+h) - N(t)$ has a Poisson distribution with parameter λh, i.e.,

$$\mathbb{P}\left(N(t+h) - N(t) = n\right) = e^{-\lambda h}\frac{(\lambda h)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

Remark 1.5. Note that the stationarity of the increments follows from (3). Moreover, a Poisson process must necessarily be a counting process, because (3) and (1) imply that $N(t) = N(t) - N(0)$ takes values in the nonnegative integers and (3) also implies that $N(t+h) - N(t) \geq 0$ for all t and h.

1.2 Properties of the Poisson process

Let $\{T_n\}_{n=0,1,2,...}$ be the **arrival times** of a Poisson process with

$$T_0 = 0 \text{ and } T_n = \min\{t \geq 0 \,:\, N(t) = n\}.$$

The **interarrival times** or **waiting times** are defined as $\{\tau_n\}_{n=1,2,...}$ by

$$\tau_n := T_n - T_{n-1}, \qquad n = 1, 2, , ...$$

Remark 1.6. Conversely, the interarrival times completely encode the distribution of a counting process. That is, given a sequence $\{\widetilde{\tau}_n\}_{n=1,2,...}$ of random variables, we can define

$$\begin{aligned} \widetilde{T}_n &= \widetilde{\tau}_1 + \cdots + \widetilde{\tau}_n, \qquad n = 1, 2, \ldots \\ \widetilde{N}_t &= \max\{n \,:\, \widetilde{T}_n \leq t\}. \end{aligned} \tag{1}$$

Then $\{\widetilde{N}(t)\}_{t\geq 0}$ is counting process. In particular, there is a one-to-one correspondence between a counting process and its interarrival times.

We are interested in the joint distribution of interarrival times $\{\tau_n\}_{n=1,2,...}$ and arrival times $\{T_n\}_{n=0,1,2,...}$ for a Poisson process. The next result implies that the waiting times are exponential with rate λ.

Theorem 1.7

Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity λ. Then the interarrival times $\{\tau_n\}_{n=1,2,...}$ form an i.i.d. sequence of exponential random variables with parameter λ.

Conversely, for a given i.i.d. sequence $\{\widetilde{\tau}_n\}_{n=1,2,...}$ of exponential random variables with parameter λ, the $\{\widetilde{N}(t)\}_{t\geq 0}$ defined in (1) is a Poisson process with intensity λ. let $\widetilde{T}_0 = 0$ and

$$\begin{aligned} \widetilde{T}_n &= \widetilde{\tau}_1 + \cdots + \widetilde{\tau}_n, \qquad n = 1, 2, \ldots \\ \widetilde{N}_t &= \max\{n \,:\, \widetilde{T}_n \leq t\}. \end{aligned} \tag{2}$$

Then $\{\widetilde{N}(t)\}_{t\geq 0}$ is a Poisson process with intensity λ.

Remark 1.8. Theorem 1.7 implies that

$$\mathbb{P}(\tau_n = 0 \text{ for some } n) = 0.$$

Therefore, with probability one, a Poisson process jumps only one step at a time.

Remark 1.9. The second part of Theorem 1.7 provides a method for simulating a Poisson process on a computer: Simulate an i.i.d. sequence $\{\widetilde{\tau}_n\}_{n=1,2,...}$ of exponential random variables with parameter λ, and then define the corresponding counting process via (2).

Next we discuss result that allow us to combine and split Poisson processes. The first result allows us to combine the two independent Poisson processes.

Proposition 1.10 (*Superposition Theorem*)

*Let $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ be two **independent** Poisson processes with intensity λ_1 and λ_2, respectively. Then $N_1(t) + N_2(t)$ is also a Poisson process with intensity $\lambda_1 + \lambda_2$.*

Remark 1.11. The independence is a crucial condition. For example, if we consider $N_1(t) = N_2(t)$, which are not independent, then $N(t) = N_1(t) + N_2(t) = 2N_1(t)$ is not a Poisson process because this process jumps twice every step.

The next result allows us to decompose two independent Poisson processes.

Proposition 1.12 (*Splitting Theorem*)

Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity λ. Suppose that at each arrival time, we mark it with "1" with probability p and with "2" with probability $1-p$. Let $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ be the counting processes of the points with mark "1" and mark "2", respectively. Then $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}$ are two ***independent*** *Poisson processes with intensity $p\lambda$ and $(1-p)\lambda$, respectively.*

Example 1.13. Suppose an insurance company issues two different types of insurance policies. If claims for the i^{th} policy arrive according to a Poisson process with intensity λ_i, then the combined claims process is a Poisson process with intensity $\lambda_1 + \lambda_2$. Conversely, splitting the combined claims process as in Proposition 1.12 yields two Poisson processes.

1.3 Nonhomogeneous Poisson process and compound Poisson process

We introduce two extensions of the Poisson process: non-homogeneous Poisson processes and compound Poisson processes. The non-homogeneous Poisson process allows for rates $\lambda(t)$ that are not constant. The corresponding rate the Poisson process will be proportional to the length of the interval times the average rate along that interval.

Definition 1.14. Let $\lambda(t)$ be a positive and deterministic function of $t \geq 0$. A stochastic process $\{N(t)\}_{t\geq 0}$ is called a **non-homogeneous Poisson process** with intensity function $\lambda(t)$, $t \geq 0$ if

(1) $N(0) = 0$.

(2) It has independent increments.

(3) For any $t \geq 0$ and $h > 0$, the increment $N(t+h) - N(t)$ follows a Poisson distribution with parameter

$$h\left(\fint_t^{t+h} \lambda(u)\,\mathrm{d}u\right) = h\left(\frac{1}{h}\int_t^{t+h} \lambda(u)\,\mathrm{d}u\right) = \int_t^{t+h} \lambda(u)\,\mathrm{d}u,$$

that is,

$$\mathbb{P}\left(N(t+h) - N(t) = n\right) = e^{-\int_t^{t+h}\lambda(u)\mathrm{d}u}\frac{\left(\int_t^{t+h}\lambda(u)\mathrm{d}u\right)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

Definition 1.15. A stochastic process $\{X(t)\}_{t\geq 0}$ is called a **compound Poisson process** if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where $\{N(t)\}_{t\geq 0}$ is a Poisson process and $\{Y_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables which are also independent of $\{N(t)\}_{t\geq 0}$.

Example 1.16. The example of the Poisson process $2N(t)$ in Remark 1.11 is an example of a compound Poisson process. Indeed, we have

$$2N(t) = \sum_{i=1}^{N(t)} 2$$

so it is a compound Poisson process where $Y_i = 2$ for all i.

Proposition 1.17

Consider a compound Poisson process defined above. We have

$$\begin{aligned}\mathbb{E}\left[X(t)\right] &= \mathbb{E}\left[Y_1\right]\mathbb{E}[N(t)] = \lambda t\mathbb{E}\left[Y_1\right] \\ \mathrm{Var}\left[X(t)\right] &= \mathrm{Var}(Y_1)\mathbb{E}[N(t)] + \mathbb{E}\left[Y_1\right]^2\mathrm{Var}(N(t)) = \lambda t\mathbb{E}\left[Y_1^2\right].\end{aligned}$$

1.4 Normal Approximation

In the following, we introduce a technique to approximate a compound Poisson process defined in Definition 1.15 using the CLT.
Idea: Approximate the distribution of

$$\frac{X(t) - \mathbb{E}[X(t)]}{\sqrt{\mathrm{Var}(X(t))}}$$

by the standard normal distribution $Z_{0,1} \sim \mathcal{N}(0,1)$ by the CLT.
Application:

1. Calculate $\mathbb{E}[X(t)]$ and $\mathrm{Var}(X(t))$.
2. For $x \geq 0$,

$$\begin{aligned}\mathbb{P}(X(t) \leq x) &= \mathbb{P}\left(\frac{X(t) - \mathbb{E}[X(t)]}{\sqrt{\mathrm{Var}(X(t))}} \leq \frac{x - \mathbb{E}[X(t)]}{\sqrt{\mathrm{Var}(X(t))}}\right)\\ &\approx \Pr\left(Z_{0,1} \leq \frac{x - \mathbb{E}[X(t)]}{\sqrt{\mathrm{Var}(X(t))}}\right) = \Phi\left(\frac{x - \mathbb{E}[X(t)]}{\sqrt{\mathrm{Var}(X(t))}}\right),\end{aligned}$$

   where $\Phi(x)$ is the c.d.f. of the standard normal distribution.

Remark 1.18. When $\mathbb{E}[N(t)]$ is large, the approximated value $\Phi\left(\frac{x-\mathbb{E}[X(t)]}{\sqrt{\mathrm{Var}(X(t))}}\right)$ will be close to the true value $\mathbb{P}(X(t) \leq x)$. Otherwise, these two values could be much different.

1.5 Example Problems

1.5.1 Proofs of Main Results

Problem 1.1. Prove Theorem 1.7.

Solution 1.1. For τ_1,

$$\mathbb{P}(\tau_1 > t) = \mathbb{P}(N(t) = 0) = e^{-\lambda t},$$

which shows that τ_1 is an exponential random variable with mean $1/\lambda$. For τ_2, we have

$$\begin{aligned}\mathbb{P}(\tau_2 > t \mid \tau_1 = s) &= \mathbb{P}(N(s+t) - N(s) = 0 \mid \tau_1 = s)\\ &= \mathbb{P}(N(s+t) - N(s) = 0) = \mathbb{P}(N(t) = 0) = e^{-\lambda t},\end{aligned}$$

i.e., τ_2 is independent of τ_1 and has the same distribution. Iterating this argument completes the proof.

For the second part of the assertion, it is enough to observe that (1) yields a one-to-one correspondence between a counting process and its interarrival times.

Problem 1.2. Prove the superposition Theorem (Proposition 1.10).

Solution 1.2. We check the three defining properties of a Poisson process.

1. $N(0) = N_1(0) + N_2(0) = 0$.
2. N has independent increments because $N_1(t)$ both $N_2(t)$ have independent increments, so for any $0 \leq t_0 < \cdots < t_n$,

$$N(t_{k+1}) - N(t_k) = N_1(t_{k+1}) - N_1(t_k) + N_2(t_{k+1}) - N_2(t_k)$$

   are all independent for $0 \leq k \leq n$ since each increment depends on independent random variables.

3. The distribution of $N(t+h)-N(t) = N_1(t+h)-N_1(t)+N_2(t+h)-N_2(t)$ is Poisson distributed with parameter $(\lambda_1+\lambda_2)h$ since it is the sum of independent Poisson random variables with parameters $\lambda_1 h$ and $\lambda_2 h$ respectively (see Remark 1.19 for a quick proof).

Remark 1.19. Recall that the sums of independent Poisson distributed random variables are Poisson distributed. To see this recall that if $X \sim \mathrm{Poi}(\lambda)$ then its moment generating function is

$$M_X(t) = e^{\lambda(e^t-1)} \text{ for } t \in \mathbb{R}$$

Therefore, if $X \sim \mathrm{Poi}(\lambda_1)$ and $Y \sim \mathrm{Poi}(\lambda_2)$ are independent, then

$$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{(\lambda_1+\lambda_2)(e^t-1)}$$

so by the inversion theorem, $X+Y \sim \mathrm{Poi}(\lambda_1+\lambda_2)$.

Problem 1.3. Prove the splitting theorem (Proposition 1.12)

Solution 1.3. We first compute the marginal distributions of N_1 and N_2 and show that they are Poisson processes, then show that they are independent. Let $\{Y_i\}_{i\geq 1}$ be iid random variables that take values 1 with probability p and 2 with probability $q = 1-p$.

Poisson Process: Clearly $N_1(0) = 0$ and the independent increments property is inherited by the fact that $N(t)$ is a Poisson process. To find the distribution of the increments, by time homogeneity, it suffices to find the marginal distribution of $N_1(t)$. Notice that

$$N_1(t) = \sum_{i=1}^{N(t)} \mathbb{1}(Y_i = 1).$$

Recall the the probability generating function of $X \sim Poi(\lambda)$ satisfies

$$\mathbb{E}[z^X] = e^{\lambda(z-1)}.$$

The probability generating function of $N_1(t)$ satisfies

$$\mathbb{E}[z^{N_1(t)}] = \mathbb{E}[\mathbb{E}[z^{\sum_{i=1}^{N(t)} \mathbb{1}(Y_i=1)}] \mid N(t)] = \mathbb{E}[\mathbb{E}[(q+pz)^{N(t)} \mid N(t)]] = e^{\lambda t(q+pz-1)} = e^{\lambda pt(z-1)}$$

so the increments are Poisson with parameter λpt. Therefore, $N_1(t)$ is a Poisson process with rate λp. An identical argument shows that $N_2(t)$ is a Poisson process with rate λq.

Independence: To show independence, we will compute the conditional distribution of $N_1(t)$ given $N_2(t)$. We have that

$$N(t) = \sum_{i=1}^{N(t)} \mathbb{1}(Y_i = 1) + \sum_{i=1}^{N(t)} \mathbb{1}(Y_i = 2) = N_1(t) + N_2(t).$$

We compute for arbitrary $n, m \geq 0$,

$$\begin{aligned}
\mathbb{P}(N_1(t) = n \mid N_2(t) = m) &= \mathbb{P}(N(t) = n+m \mid N_2(t) = m) \\
&= \frac{\mathbb{P}(N(t) = n+m, N_2(t) = m)}{\mathbb{P}(N_2(t) = m)} \\
&= \frac{\mathbb{P}(N_2(t) = m \mid N(t) = n+m)\,\mathbb{P}(N(t) = n+m)}{\mathbb{P}(N_2(t) = m)}
\end{aligned}$$

Conditionally on $N(t) = n+m$, we have that $N_2(t) = \sum_{i=1}^{n+m} \mathbb{1}(Y_i = 2)$ is $Bin(n+m, q)$. Therefore,

$$\begin{aligned}\frac{\mathbb{P}(N_2(t) = m \mid N(t) = n+m)\,\mathbb{P}(N(t) = n+m)}{\mathbb{P}(N_2(t) = m)} &= \frac{\binom{n+m}{n} q^m p^n e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!}}{e^{-\lambda q t} \frac{(\lambda q t)^m}{m!}} \\ &= e^{-\lambda p t} \frac{(\lambda p t)^n}{n!} = \mathbb{P}(N_1(t) = n)\end{aligned}$$

so N_2 and N_1 are independent.

Problem 1.4. Prove the mean and variance formula in Proposition 1.17.

Solution 1.4. These computations follow from manipulations of conditional expectations and variances. To compute the expected value, we apply the law of total expectation

$$\begin{aligned}\mathbb{E}[X(t)] &= \mathbb{E}[\mathbb{E}[X(t) \mid N(t)]] && \text{law of total expectation} \\ &= \mathbb{E}[Y_1 N(t)] && \text{linearity} \\ &= \mathbb{E}[Y_1]\,\mathbb{E}[N(t)] && \text{independence} \\ &= \lambda t \mathbb{E}[Y_1]. && \mathbb{E}[N(t)] = \lambda t\end{aligned}$$

To compute the variance, we apply the law of total variance,

$$\begin{aligned}\operatorname{Var}[X(t)] &= \mathbb{E}[\operatorname{Var}(X(t) \mid N(t))] + \operatorname{Var}(\mathbb{E}[X(t) \mid N(t)]) && \text{law of total variance} \\ &= \mathbb{E}[\operatorname{Var}(Y_1) N(t)] + \operatorname{Var}(\mathbb{E}[Y_1] N(t))) && \text{linearity} \\ &= \operatorname{Var}(Y_1)\,\mathbb{E}[N(t)] + \mathbb{E}[Y_1]^2 \operatorname{Var}(N(t))) && \mathbb{E}[aX] = a\,\mathbb{E}[X], \operatorname{Var}(aX) = a^2 \operatorname{Var}(X) \\ &= \lambda t\,\mathbb{E}[Y_1^2] && \mathbb{E}[N(t)] = \operatorname{Var}(N(t)) = \lambda t\end{aligned}$$

and the fact that $\operatorname{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

Remark 1.20. It is almost obvious that by linearity and independence,

$$\mathbb{E}[X(t) \mid N(t)] = \mathbb{E}\left[\sum_{i=1}^{N(t)} Y_i \mid N(t)\right] = \mathbb{E}[Y_1]\,\mathbb{E}[N(t)].$$

A rigorous proof of this fact in general follows from Wald's Identity, but for this problem we can do the computation explicitly. Let $n \geq 0$, we have

$$\begin{aligned}\mathbb{E}[X(t) \mid N(t) = n] &= \iiint \left(\sum_{i=1}^n y_i\right) f_{Y_1,\dots,Y_n \mid N(t)}(y_1, \dots, y_n \mid n)\, dy_1 \dots dy_n \\ &= \iiint \left(\sum_{i=1}^n y_i\right) f_{Y_1}(y_1) \cdots f_{Y_n}(y_n)\, dy_1 \dots dy_n && \text{independence} \\ &= \sum_{i=1}^n \int y_i f_{Y_i}(y_i)\, dy_i && \text{linearity} \\ &= n\,\mathbb{E}[Y_1]. && \text{definition}\end{aligned}$$

The computation above makes it very clear how independence plays its role.

1.5.2 Applications

Problem 1.5. Suppose that the number of calls per hour arriving at an answering service follows a Poisson process with intensity $\lambda = 4$.

1. What is the probability that fewer than two calls come in the first hour?
2. Suppose that six calls arrive in the first hour. What is the probability that at least two calls will arrive in the second hour?
3. Suppose it is known that exactly eight calls arrived in the first two hours. What is the probability that exactly five of them arrived in the first hour?

Solution 1.5.

Part (a):

$$\mathbb{P}(N(1) < 2) = \mathbb{P}(N(1) = 0) + \mathbb{P}(N(1) = 1) = e^{-\lambda} + \lambda e^{-\lambda} = 5e^{-4}.$$

Part (a):

$$\begin{aligned}\mathbb{P}(N(2) - N(1) \geq 2 | N(1) = 6) &= \mathbb{P}(N(2) - N(1) \geq 2) \\ &= \mathbb{P}(N(1) \geq 2) \\ &= 1 - 5e^{-4}\end{aligned}$$

Part (c):

$$\begin{aligned}\mathbb{P}(N(1) = 5 | N(2) = 8) &= \frac{\mathbb{P}(N(1) = 5, N(2) = 8)}{\mathbb{P}(N(2) = 8)} \\ &= \frac{\mathbb{P}(N(1) = 5, N(2) - N(1) = 3)}{\mathbb{P}(N(2) = 8)} \\ &= \frac{\mathbb{P}(N(1) = 5)\,\mathbb{P}(N(2) - N(1) = 3)}{\mathbb{P}(N(2) = 8)} \\ &= \frac{\mathbb{P}(N(1) = 5)\,\mathbb{P}(N(1) = 3)}{\mathbb{P}(N(2) = 8)} \\ &= \frac{\frac{e^{-\lambda}\lambda^5}{5!}\frac{e^{-\lambda}\lambda^3}{3!}}{\frac{e^{-2\lambda}(2\lambda)^8}{8!}} \\ &= \frac{7}{32}.\end{aligned}$$

Problem 1.6. Consider the situation of Problem 1.5, where calls arrive at a desk according to a Poisson process with intensity $\lambda = 4$. What is distribution of the time it takes until 15 calls have arrived, and what is the corresponding expectation?

Solution 1.6. Write $T_{15} = \tau_1 + \cdots + \tau_{15}$. Since the τ_i are i.i.d. with $\text{Exp}(\lambda)$, the random time T_{15} has a Gamma (or Erlang) distribution with density

$$f_{T_{15}}(x) = \frac{\lambda^{15}}{14!}x^{14}e^{-\lambda x}.$$

The corresponding mean is

$$\mathbb{E}[T_{15}] = \sum_{i=1}^{15}\mathbb{E}[\tau_i] = \frac{15}{\lambda} = \frac{15}{4}.$$

Problem 1.7. A store opens at 8 A.M. From 8 until 10 customers arrive at a Poisson rate of four an hour. Between 10 and 12 they arrive at a Poisson rate of eight an hour. From 12 to 2 the arrival rate increases steadily from eight per hour at 12 to ten per hour at 2; and from 2 to 5 the arrival rate drops steadily from ten per hour at 2 to four per hour at 5. Determine the probability distribution of the number of customers that enter the store on a given day.

Solution 1.7. The arrival rate can be written as

$$\lambda(t) = \begin{cases} 0, & 0 \le t < 8, \\ 4, & 8 \le t < 10, \\ 8, & 10 \le t < 12, \\ t-4, & 12 \le t < 14, \\ 38-2t, & 14 \le t < 17, \\ 0, & 17 \le t < 24. \end{cases}$$

Since $\int_0^{24} \lambda(t) \mathrm{d}t = 63$, the number of customers that enter the store on a given day follows a Poisson distribution with parameter 63.

Problem 1.8. Customers arrive at an automatic teller machine (ATM) according to a Poisson process with rate 12 per hour. The amount of money withdrawn on each transaction is a random variable with mean \$30 and standard deviation \$50. (A negative withdrawal means that money was deposited.) Suppose that the machine is in use 15 hours per day. Apply normal approximation to calculate the probability that the total daily withdraw is less than \$6000.

Solution 1.8. Let $X(t)$ be the total amount withdrawn during the interval $[0, t]$, where time t is measured in hours. Assuming that the successive withdrawals are i.i.d. random variables, then $\{X(t)\}_{t\ge 0}$ can be modeled as a compound Poisson process:

$$X(t) = \sum_{i=1}^{N(t)} Y_i,$$

where $\{N(t)\}_{t\ge 0}$ is a Poisson process with intensity 12 and $\{Y_i\}_{i=1,2,\ldots}$ is a sequence of i.i.d. random variables with mean 30 and variance 50^2. By the last proposition,

$$\mathbb{E}[X(15)] = 15 \cdot 12 \cdot 30 = 5400 \text{ and } \mathrm{Var}[X(t)] = 15 \cdot 12 \cdot \left(50^2 + 30^2\right) = 612000.$$

Hence,

$$\begin{aligned} \mathbb{P}(X(15) \le 6000) &= \mathbb{P}\left(\frac{X(15) - 5400}{\sqrt{612000}} \le \frac{6000-5400}{\sqrt{612000}}\right) \\ &\approx \mathbb{P}(Z \le 0.767) \text{ where } Z \sim N(0,1) \text{ according to the CLT} \\ &\approx 0.78 \end{aligned}$$