1 Indicator Functions

The indicator functions provide a fundamental link between probability and expected values. Everything in this section is not unique to discrete random variables and will hold more generally.

Definition 1 (Indicator Function)**.** Let $A \subset \Omega$ be an event. We say that $\mathbb{1}_A$ is the *indicator* random variable of the event A. $\mathbb{1}_A$ is defined by:

$$\mathbb{1}(\omega \in A) = \mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A, \\ 0 & \omega \in A^c \end{cases}.$$

Remark 1. The random variable $\mathbb{1}_A(\omega)$ is a Bernoulli random variable where a success is the occurrence of the event A.

1.1 Link Between Probabilities and Expected Values

The indicators link the concepts of expected values with the probability measure,

$$\mathbb{E}[1_A] = \mathbb{P}(A),$$

which follows from the simple fact that $\mathbb{1}_A \sim \text{Bern}(\mathbb{P}(A))$. This means that we can use indicator functions to write the theory of probability as the theory of integration, since the the probability of an event is precisely the integral of the indicator of the event against its probability distribution.

Naturally, the indicator functions behave quite similarly to probabilities and can be used as an alternative proof of the basic probability identities,

1. Complements: $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$ which implies that $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
2. Intersections: $\mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B$, so if A and B are independent, then

$$\mathbb{P}(A \cap B) = \mathbb{E}[\mathbb{1}_{A\cap B}] = \mathbb{E}[1_A \mathbb{1}_B] = \mathbb{E}[\mathbb{1}_A]\,\mathbb{E}[\mathbb{1}_B] = \mathbb{P}(A)\,\mathbb{P}(B)$$

3. Inclusion − Exclusion: $\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B}$ which implies that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
4. Union Bound: $\mathbb{1}_{A\cup B} \le \mathbb{1}_A + \mathbb{1}_B$. which implies that $\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$ by monotonicity of the expected value.

1.2 The Expected Value of Counts

Whenever a random variable N takes values in $\{0, 1, 2, \ldots, n\}$, we can use the linearity of expectation to compute the expected values in another possibly simpler way. Suppose that N counts the number of events $A_1, A_2, \ldots, A_n$ (that are not necessarily independent) that occurred, then

$$N = N(\omega) = \sum_{i=1}^{n} \mathbb{1}(\omega \in A_i) = \sum_{i=1}^{n} \mathbb{1}_{A_i}.$$

Therefore, by the linearity of expectation

$$\mathbb{E}[N] = \sum_{i=1}^{n} \mathbb{E}[\mathbb{1}_{A_i}] = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

This trick is especially useful if the joint distribution is tricky to compute, but its marginals are relatively simpler.

1.3 Example Problems

Problem 1.1. N passengers board a plane with N seats, where $N > 1$. Despite every passenger having an assigned seat, when they board the plane they sit in one of the remaining available seats at random. Show that the mean and variance of the number of people sitting in the correct seat once everyone is on board are both 1 (independent of the number N of passengers, weirdly enough).

Solution 1.1. This is called the matching problem. Let N denote the number of people sitting in the correct of seat once everyone is on board, and let A_i be the event that the ith passenger is in the correct seat. We have

$$\mathbb{1}_{A_i} = \begin{cases} 1 & \text{the } i\text{th passenger is in the correct seat} \\ 0 & \text{the } i\text{th passenger is not in the correct seat} \end{cases}.$$

Clearly, $N = \sum_{i=1}^n \mathbb{1}_{A_i}$. We can now compute the mean and variance.

Expected Value: By linearity of expectation

$$\mathbb{E}[N] = \sum_{i=1}^n \mathbb{E}[\mathbb{1}_{A_i}] = \sum_{i=1}^n \mathbb{P}(A_i).$$

By symmetry, we have that the probability that the ith passenger is in the correct seat is

$$\mathbb{P}(A_i) = \frac{1}{n}$$

since the seat the ith passenger sits in is uniform over the n possible seats. Therefore,

$$\mathbb{E}[N] = \sum_{i=1}^n \mathbb{E}[\mathbb{1}_{A_i}] = \sum_{i=1}^n \mathbb{P}(A_i) = \sum_{i=1}^n \frac{1}{n} = 1.$$

Variance: By the linearity of expectation

$$\mathbb{E}[N^2] = \mathbb{E}\left[\left(\sum_{i=1}^n \mathbb{1}_{A_i}\right)^2\right] = \sum_{i,j=1}^n \mathbb{E}[\mathbb{1}_{A_i}\mathbb{1}_{A_j}].$$

We have two cases

1. $i = j$: Suppose that $i = j$. Since $\mathbb{1}(A_i)\mathbb{1}(A_i) = 1$ if an only if A_i happens, so we have

$$\mathbb{E}[\mathbb{1}_{A_i}\mathbb{1}_{A_i}] = \mathbb{E}[\mathbb{1}_{A_i}] = \mathbb{P}(A_i) = \frac{1}{n}$$

as we computed before.

2. $i \neq j$: Suppose that $i \neq j$. Since $\mathbb{1}_{A_i}\mathbb{1}_{A_j} = 1$ if an only if A_i and A_j happens

$$\mathbb{E}[\mathbb{1}_{A_i}\mathbb{1}_{A_j}] = \mathbb{P}(A_i \cap A_j) = \frac{1}{n(n-1)}.$$

Note that the events A_i and A_j are not independent, so we can't simply multiply the probabilities. Instead, we can use the fact that sets the i and j passengers sit in are uniform over the $n(n-1)$ possible seats for two passengers.

Since there are $n(n-1)$ ways to pick indices $i \neq j$ and n ways to pick indices $i = j$, we have

$$\mathbb{E}[N^2] = \sum_{i,j=1}^{n} \mathbb{E}[\mathbb{1}_{A_i}\mathbb{1}_{A_j}] = \sum_{i=j} \mathbb{E}[\mathbb{1}_{A_i}\mathbb{1}_{A_i}] + \sum_{i\neq j} \mathbb{E}[\mathbb{1}_{A_i}\mathbb{1}_{A_j}] = \frac{n}{n} + \frac{n(n-1)}{n(n-1)} = 2.$$

Therefore,

$$\mathrm{Var}(N) = \mathbb{E}[N^2] - (\mathbb{E}[N])^2 = 2 - 1 = 1.$$

Remark 2. Notice that the events $A_1, \ldots A_n$ are clearly not independent. For example, if $A_1, \ldots, A_{n-1}$ were to happen then A_n must be true too since the seat left is the one assigned to the last passenger. The linearity of expectation allowed us to decompose the random variable into a sum of possibly dependent events. However, by symmetry we only needed to compute the probability of a single event A_1 in isolation without worrying about the other events $A_2, \ldots, A_n$.

Remark 3. Instead of using the uniform distribution and symmetry, we could argue that

$$\mathbb{P}(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

since there are $(n-1)!$ seating patterns where the ith passenger is in the right seat and $n!$ total seating patterns (all of which are equally likely). Likewise, we have

$$\mathbb{P}(A_i \cap A_j) = \frac{1}{n(n-1)} = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$

since there are $(n-2)!$ seating patterns where the ith and jth passenger is in the right seat and $n!$ total seating patterns (all of which are equally likely).

Yet another way to compute the probability is to argue sequentially using the chain rule,

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i \mid A_j)\,\mathbb{P}(A_j) = \frac{1}{n-1} \cdot \frac{1}{n} = \frac{1}{n(n-1)},$$

since the probability the jth passenger sits in the right seat is $\frac{1}{n}$ and the probability the ith passenger sits in the right seat given that the jth passenger is in the right seat is $\frac{1}{n-1}$ since the jth passenger is already in the correct seat so there are $n-1$ seats left.

Problem 1.2. Show that

1. $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$
2. $\mathrm{Var}(\mathbb{1}_A) = \mathbb{P}(A)(1 - \mathbb{P}(A))$
3. $\mathrm{Cov}(\mathbb{1}_A, \mathbb{1}_B) = \mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B)$

Solution 1.2. The proof is somewhat straightforward, and it relies on the observation that

$$\mathbb{1}_A\mathbb{1}_B = \begin{cases} 1 & \omega \in A \cap B, \\ 0 & \omega \in (A \cap B)^c \end{cases}$$

We can now compute the required objects

1. $$\mathbb{E}(\mathbb{1}_A) = 1 \cdot \mathbb{P}(\mathbb{1}_A = 1) + 0 \cdot P(\mathbb{1}_A = 0) = \mathbb{P}(A)$$

2. We have $\mathbb{1}_A^2 = 1$ if and only if $\omega \in A$, so

$$\mathbb{E}(\mathbb{1}_A^2) = 1 \cdot \mathbb{P}(\mathbb{1}_A^2 = 1) + 0 \cdot \mathbb{P}(\mathbb{1}_A^2 = 0) = 1 \cdot \mathbb{P}(A) + 0 \cdot \mathbb{P}(A^c) = \mathbb{P}(A)$$

so

$$\mathrm{Var}(\mathbb{1}_A) = \mathbb{E}(\mathbb{1}_A^2) - \mathbb{E}(\mathbb{1}_A)^2 = \mathbb{P}(A) - \mathbb{P}(A)^2 = \mathbb{P}(A)(1 - \mathbb{P}(A))$$

3. Similarly, we have $\mathbb{1}_A \mathbb{1}_B = 1$ if and only if $\omega \in A \cap B$, so

$$\mathbb{E}(\mathbb{1}_A \cdot \mathbb{1}_B) = 1 \cdot \mathbb{P}(\mathbb{1}_A \mathbb{1}_B = 1) + 0 \cdot \mathbb{P}(\mathbb{1}_A \mathbb{1}_B = 0) = 1 \cdot \mathbb{P}(A \cap B) + 0 \cdot \mathbb{P}((A \cap B)^c) = \mathbb{P}(A \cap B)$$

giving us

$$\mathrm{Cov}(\mathbb{1}_A, \mathbb{1}_B) = \mathbb{E}(\mathbb{1}_A \cdot \mathbb{1}_B) - \mathbb{E}(\mathbb{1}_A)\,\mathbb{E}(\mathbb{1}_B) = \mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B).$$

1.4 Proofs of Key Results

Problem 1.3. Show the following properties of an indicator function

1. Complements: $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$
2. Intersections: $\mathbb{1}_{A \cap B} = \mathbb{1}_A \mathbb{1}_B$
3. Inclusion – Exclusion: $\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B}$
4. Union Bound: $\mathbb{1}_{A \cup B} \leq \mathbb{1}_A + \mathbb{1}_B$.

Solution 1.3. The proofs are quite straightforward and essentially follow from the facts that $1 - 0 = 1$ and $1 \cdot 1 = 1$.

1. Complements: On one side we have

$$\mathbb{1}_{A^c} = \begin{cases} 1 & x \in A^C \\ 0 & x \in A \end{cases}.$$

On the other hand, we have

$$1 - \mathbb{1}_A = \begin{cases} 1 - 1 & x \in A \\ 1 - 0 & x \in A^c \end{cases} = \begin{cases} 1 & x \in A^C \\ 0 & x \in A \end{cases},$$

so both sides are equivalent.

2. Intersections: On one side, we have

$$\mathbb{1}_{A \cap B} = \begin{cases} 1 & x \in A \text{ and } x \in B \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, we have

$$\mathbb{1}_{A \cap B} = \begin{cases} 1 \cdot 1 & x \in A \text{ and } x \in B \\ 0 & \text{otherwise} \end{cases}$$

so both sides are equivalent.

The rest of the identities are verified similarly.