1 Continuous-time Markov chains

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{X(t)\}_{t\geq 0}$ be a stochastic process taking values in a state space S and

$$\mathcal{F}_t = \mathcal{F}_t^X = \sigma(X(s) : 0 \leq s \leq t)$$

be its natural filtration.

Definition 1.1. $\{X(t)\}_{t\geq 0}$ is called a **continuous-time Markov chain (CTMC)** if

(1) the state space S is at most countable.

(2) The process satisfies the **Markov property**: for $s, t \geq 0$ and $i \in S$,

$$\mathbb{P}(X(t+s) = i \,|\, \mathcal{F}_t) = \mathbb{P}(X(t+s) = i \,|\, X(t))$$

To mirror the notion of the Markov property for DTMC, we see that Condition (2) is equivalent to

(2') for any $s, t \geq 0$, $n \in \mathbb{N}$, $0 \leq r_0 < \cdots < r_n \leq t$, and $i, j, x_0, \ldots, x_n \in S$,

$$\mathbb{P}\left(X(t+s) = j \,|\, X(t) = i, X(r_n) = x_n, \ldots, X(r_0) = x_0 \right) = \mathbb{P}\left(X(t+s) = j \,|\, X(t) = i \right).$$

Remark 1.2. The Poisson process is a CTMC. More generally, every continuous-time process that has independent increments and takes values in $\mathbb{Z}$ is a CTMC. However, recall that the Markov property does not necessarily imply independent increments (See Problem 1.3)

The key difference between a CTMC and a DTMC is that in discrete time, the Markov chain moves to a new state at times $t = 1, 2, 3, \ldots$ while in a CTMC, the Markov chain can move to a new state at any $t \geq 0$. Just like for the Poisson process, the times of the jumps are the **arrival** times.

Definition 1.3. A CTMC is called **(time-)homogeneous** if, for any $s, t \geq 0$ and $i, j \in S$,

$$\mathbb{P}\left(X(t+s) = j \,|\, X(t) = i \right) = \mathbb{P}\left(X(s) = j \,|\, X(0) = i \right).$$

Example 1.4. The non-homogeneous Poisson process is a non-homogeneous CTMC. On the other hand, the (homogeneous) Poisson process becomes a homogeneous CTMC, if we define a **Poisson process with start in** $i \in \{0, 1, \ldots\}$ as

$$\widetilde{N}(t) := i + N(t),$$

where $N(t)$ is a Poisson process with start in $N(0) = 0$.

As in the case of a DTMC, from now on, we only consider homogeneous CTMCs unless otherwise stated.

1.1 The transition semigroup of a CTMC

Definition 1.5. The **transition probabilities** of a **homogeneous CTMC** are defined as

$$p_{ij}(t) = \mathbb{P}\left(X(t) = j \,|\, X(0) = i \right) = \mathbb{P}\left(X(t+s) = j \,|\, X(s) = i \right).$$

The **transition semigroup** is defined as

$$\boldsymbol{P}(t) = \left(p_{ij}(t) \right)_{i,j \in S}, \qquad t \geq 0,$$

with $P(0) = I$, the identity matrix. We assume from now on that

$$\boldsymbol{P}(h) \longrightarrow \boldsymbol{P}(0) = \boldsymbol{I},$$

that is, $p_{ij}(h) \to \delta_{ij}$ as $h \downarrow 0$.

Just like for the transition matrix, for each $t \geq 0$, the matrix $\boldsymbol{P}(t) = (p_{ij}(t))_{i,j\in S}$ satisfies

$$\begin{cases} p_{ij}(t) \geq 0, & \text{for all } i, j \in S, \\ \sum_{j\in S} p_{ij}(t) = 1, & \text{for all } i \in S. \end{cases}$$

The following result explains the term "semigroup".

Theorem 1.6 (Chapman–Kolmogorov equations)

For $s, t \geq 0$,

$$\boldsymbol{P}(t+s) = \boldsymbol{P}(t)\boldsymbol{P}(s) = \boldsymbol{P}(s)\boldsymbol{P}(t).$$

The transition mechanism of a CTMC has two components:

1. The time spent at a given state i before leaving it.
2. The jump mechanism by which the next state is chosen.

1.2 The sojourn times of a CTMC

We begin by defining a notion of how long a CTMC stays at a state before leaving it.

Definition 1.7. Given that $X(0) = i$, the **sojourn time** of the CTMC at state i is the random time U_i defined as

$$U_i = \min\{t \geq 0 : X(t) \neq i\}.$$

The sojourn time for the Poisson process is exponential. We will see that this holds more generally.

Remark 1.8. Note that $U_i > 0$ can only happen if $X(0) = i$. Therefore, we will consider U_i only under the probability measure $\mathbb{P}(\,\cdot\,|X(0) = i)$.

Proposition 1.9

Under $\mathbb{P}(\,\cdot\,|X(0) = i)$, the sojourn time U_i has an exponential distribution.

This result says that the time to leave a state is an exponential random variable. Two natural follow-up questions are:

- Q1: What is the value of the parameter in the exponential distribution of U_i?
- Q2: What states does the CTMC move to?

Knowing the answers to these questions gives us a procedure to simulate a CTMC on a computer, first by generating the time the CTMC jumps to the next state, then generating the state the CTMC jumps to. We will answer these questions in the next sections, but we start by giving some brief motivation.

Notice that

$$\begin{aligned} \mathbb{P}(U_i > \triangle t) = e^{-\alpha_i \triangle t} &= 1 - \alpha_i \triangle t + \frac{1}{2}\alpha_i(\triangle t)^2 - \frac{1}{3!}\alpha_i(\triangle t)^3 + \cdots \\ &= 1 - \alpha_i \triangle t + o(\triangle t) \end{aligned}$$

where the error term $o(\triangle t)$ is much smaller than $\triangle t$ for $\triangle t$ small,

$$\lim_{\triangle t \to 0} \frac{o(\triangle t)}{\triangle t} = 0.$$

That is,

$$\mathbb{P}(\text{no transition occurred by time } \triangle t) = 1 - \alpha_i \triangle t + o(\triangle t).$$

By taking complements of this set, we see that

$$\mathbb{P}(\text{one transition occurred by time } \triangle t) = \alpha_i \triangle t + o(\triangle t),$$
$$\mathbb{P}(\text{at least two transitions occurred by time } \triangle t) = o(\triangle t).$$

Thus one might expect that the small time behavior of a CTMC is the key to this information.

1.2.1 Infinitesimal generator matrix

We answer the first question in Section 1.2, namely what is the rate of the exponential clock that determines when the CTMC leaves its current state. From now on, we assume that

$$t \longmapsto p_{ij}(t) \quad \text{is differentiable at } t = 0 \text{ for all } i, j \in S.$$

This allows us to define a matrix that encodes the instantaneous rate at which a Markov chain transitions between states.

Definition 1.10. The matrix $\boldsymbol{Q} := (q_{ij})_{i,j \in S}$ with entries

$$q_{ij} = \frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t) \Big|_{t=0} = \lim_{h \downarrow 0} \frac{p_{ij}(h) - \delta_{ij}}{h} = p'_{ij}, (0). \tag{1}$$

is called the **infinitesimal generator** of the CTMC.

Remark 1.11. In matrix form, the matrix $\boldsymbol{Q}$ can be written as

$$\boldsymbol{Q} := (q_{ij})_{i,j \in S} = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{P}(t) \Big|_{t=0} = \lim_{h \downarrow 0} \frac{\boldsymbol{P}(h) - \boldsymbol{I}}{h} = \boldsymbol{P}'(0).$$

since $\boldsymbol{P}(0) = \boldsymbol{I}$ the identity matrix.

By rearranging the definition of $\boldsymbol{Q}$, we see that for $h = \triangle t$, we have that for $i \neq j$

$$p_{ij}(\triangle t) = (\triangle t) \times q_{ij}$$

so the probability that the chain moves from i to j in a short time is proportional to its rate q_{ij}. The values Q-matrix will encode the rates at which the Markov chain move from i to j. This is made precise with the following result.

Proposition 1.12

1. *The diagonal elements of* $\boldsymbol{Q}$ *are*

$$q_{ii} = -\alpha_i, \qquad i \in S,$$

   where α_i *is the parameter of the exponential distribution of the sojourn time at state* i.

2. *The off-diagonal elements of* $\boldsymbol{Q}$ *satisfy*

$$q_{ij} \geq 0 \qquad \text{for } i \neq j,$$

   and

$$\sum_{j \neq i} q_{ij} = \alpha_i \qquad \text{for all } i \in S.$$

This proposition allows us to read the parameters of the exponential distributions fo the sojourn times from the diagonal elements of the Q-matrix.

1.3 Embedded DTMC

We answer the second question in Section 1.2, namely when the CTMC leaves its current state, what is the probability it goes to each state. One might expect that this information will be encoded by a DTMC.

Definition 1.13. Consider the transition matrix $\widetilde{\boldsymbol{P}} := (\tilde{p}_{ij})_{i,j\in S}$ with entries

$$\widetilde{p}_{ij} := \begin{cases} \dfrac{q_{ij}}{-q_{ii}} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

The DTMC with transition matrix $\widetilde{\boldsymbol{P}}$ is called the **embedded DTMC** of the CTMC.

Remark 1.14. The second part of Proposition 1.12 implies that

$$\begin{aligned} \widetilde{p}_{ij} &\geq 0 \qquad \text{for } i \neq j, \\ \sum_j \widetilde{p}_{ij} &= 1 \qquad \text{for all } i \in S, \end{aligned}$$

so $\widetilde{\boldsymbol{P}}$ is a stochastic matrix, and therefore a valid transition matrix.

To see why $\widetilde{\boldsymbol{P}}$ is the transition matrix of a DTMC, we can simply define $X_n = X(T_n)$, where T_n denotes the time of the nth jump, to be the location of the DTMC at the time nth jump. Since the Markov chain is homogeneous, we will see in the following result that the values $\tilde{p}_{ij}$ is the probability that the Markov chain jumps to state j given that it started at state i and a jump just occurred.

Proposition 1.15

For $i \neq j$, we have

$$\mathbb{P}(X(U_i) = j | X(0) = i) = \tilde{p}_{ij}, \qquad j \neq i.$$

The intuition of the above description of the CTMC is that $X(t)$ stays at a state i for a random time period U_i with distribution $\text{Exp}(\alpha_i)$ and after that moves on to the next state j, which is chosen according to the transition matrix $\widetilde{\boldsymbol{P}}$.

1.4 Example Problems

1.4.1 Proofs of Results

Problem 1.1. Prove the Chapman–Kolmogorov Equations.

Solution 1.1. The proof is identical to the discrete case. By Markov property and homogeneity, we have by the chain rule for conditional probabilities

$$\begin{aligned} p_{ij}(t+s) &= \mathbb{P}\left(X(t+s) = j \middle| X(0) = i \right) \\ &= \sum_{k\in S} \mathbb{P}\left(X(t+s) = j \middle| X(t) = k, X(0) = i \right) \mathbb{P}\left(X(t) = k \middle| X(0) = i \right) \\ &= \sum_{k\in S} \mathbb{P}\left(X(t+s) = j \middle| X(t) = k \right) \mathbb{P}\left(X(t) = k \middle| X(0) = i \right) \\ &= \sum_{k\in S} \mathbb{P}\left(X(s) = j \middle| X(0) = k \right) \mathbb{P}\left(X(t) = k \middle| X(0) = i \right) \\ &= \sum_{k\in S} p_{ik}(t) p_{kj}(s). \end{aligned}$$

Writing this identity in matrix notation finishes the proof.

Problem 1.2. Prove Proposition 1.9.

Solution 1.2. We consider the conditional probability

$$\begin{aligned}
\mathbb{P}\left(U_i > t+s \middle| U_i > s,\, X(0)=i \right) &= \mathbb{P}\left(X(u)=i \text{ for } u\in[0,t+s] \middle| X(u)=i \text{ for } u\in[0,s] \right) \\
&= \mathbb{P}\left(X(u)=i \text{ for } u\in(s,t+s] \middle| X(u)=i \text{ for } u\in[0,s] \right) \\
&= \mathbb{P}\left(X(u)=i \text{ for } u\in(s,t+s] \middle| X(s)=i \right), \text{ Markov property} \\
&= \mathbb{P}\left(X(u)=i \text{ for } u\in(0,t] \middle| X(0)=i \right), \text{ homogeneous} \\
&= \mathbb{P}\left(U_i > t \middle| X(0)=i \right), \text{ definition}
\end{aligned}$$

Therefore, the distribution of the sojourn time U_i is **memoryless**. Since the exponential distribution is the only distribution with the memoryless property, U_i follows an exponential distribution with some parameter α_i.

Problem 1.3. Give an example of stochastic process that satisfies the Markov property, but does not have independent increments.

Solution 1.3. We first provide a discrete time example. Let $(\xi_n)_{n\geq 1}$ be independent Rademacher random variables,

$$\mathbb{P}(\xi = \pm 1) = \frac{1}{2}.$$

Notice that $(\xi_n)_{n\geq 0}$ is Markov because

$$\mathbb{P}(\xi_n = x_n \mid \xi_{n-1} = x_{n-1}, \dots, \xi_0 = x_0) = \mathbb{P}(\xi_n = x_n) = \mathbb{P}(\xi_n = x_n \mid \xi_{n-1} = x_{n-1})$$

but clearly the increments $X_n = \xi_n - \xi_{n-1}$ and $X_{n-1} = \xi_{n-1} - \xi_{n-2}$ are not independent because they depend on the same random variables. A continuous time version of this example can be constructed by consider the same process $(\xi_t)_{t\geq 0}$ but indexed by time.

Problem 1.4. Prove Proposition 1.12.

Solution 1.4. This proof formalizes the computations at the end of Section 1.2.

Part 1: We have as $h \downarrow 0$, the law of total probability implies that

$$p_{ii}(h) = \mathbb{P}\left(X(h)=i \middle| X(0)=i \right) = \mathbb{P}\left(U_i > h \middle| X(0)=i \right) + \mathbb{P}\left(X(h)=i, U_i < h \middle| X(0)=i \right).$$

Notice that

$$\begin{aligned}
\mathbb{P}\left(X(h)=i, U_i < h \middle| X(0)=i \right) &\leq \mathbb{P}\left(\text{at least two transitions occurred by time } h\right) \\
&= o(h).
\end{aligned}$$

Therefore,

$$p_{ii}(h) = \mathbb{P}\left(U_i > h \middle| X(0)=i \right) + o(h) = e^{-\alpha_i h} + o(h)$$

which implies

$$q_{ii} = \lim_{h\downarrow 0} \frac{p_{ii}(h) - \delta_{ii}}{h} = \lim_{h\downarrow 0} \frac{e^{-\alpha_i h} + o(h) - 1}{h} = -\alpha_i.$$

Part 2: For $i \neq j$, we have

$$q_{ij} = \lim_{h \downarrow 0} \frac{p_{ij}(h) - 0}{h}$$

Since $p_{ij}(h)/h \geq 0$ for all $h > 0$, so must be q_{ij}. Next, we have

$$\sum_{j \neq i} p_{ij}(h) = 1 - p_{ii}(h),$$

and therefore

$$\sum_{j \neq i} q_{ij} = -p'_{ii}(0) = \alpha_i.$$

(The preceding argument is correct if S is finite and needs some additional care if S is infinite, because then the interchange of limit and an infinite sum needs extra justification).

Problem 1.5. Prove Proposition 1.15

Solution 1.5. We provide an intuitive derivation of this result. The technical details are on the homework assignment. If the CTMC is at state i, then it jumps to state $j \neq i$ at rate q_{ij}. That is, there are independent random variables τ_{ij}, which are exponentially distributed with respective parameters q_{ij} so that the CTMC jumps to that state j for which τ_{ij} comes first.

The sojourn time at state i is thus

$$U_i := \min\{t \geq 0 : X(t) \neq i\} = \min_{j \neq i} \tau_{ij},$$

which is an exponentially distributed with parameter

$$\alpha_i = -q_{ii} = \sum_{j \neq i} q_{ij}$$

by part 2 of Proposition 1.12. The probability that $\min_{j \neq i} \tau_{ij}$ is achieved at τ_{ij} is proportional to its rate

$$\mathbb{P}\left(\min_{j \neq i} \tau_{ij} = \tau_{ij}\right) = \frac{q_{ij}}{\sum_{j \neq i} q_{ij}} = \frac{q_{ij}}{\alpha_i}.$$

This means that

$$\mathbb{P}(X(U_i) = j | X(0) = i) = \frac{q_{ij}}{\alpha_i} = \frac{q_{ij}}{-q_{ii}}, \qquad j \neq i.$$

1.4.2 Applications

Problem 1.6. What is the transition semigroup of a Poisson process with intensity $\lambda > 0$?

Solution 1.6. Recall that a Poisson process with intensity $\lambda > 0$ satisfies

$$\mathbb{P}(N(t+h) - N(t) = n) = e^{-\lambda h} \frac{(\lambda h)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

Therefore,

$$p_{ij}(t) = \mathbb{P}(N(t) = j \mid N(0) = i) = \begin{cases} e^{-\lambda h} \frac{(\lambda h)^{j-i}}{(j-i)!} & j \geq i \\ 0 & j < i. \end{cases}$$

Problem 1.7. Find the infinitesimal generator of a Poisson process with intensity λ.

Solution 1.7. For the Poisson process with intensity $\lambda > 0$, we have

$$p_{ij}(t) = \begin{cases} e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!} & \text{if } j \geq i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $j = i$,

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ii}(t)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} e^{-\lambda t}\bigg|_{t=0} = -\lambda,$$

for $j = i + 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{i,i+1}(t)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} e^{-\lambda t}\lambda t\bigg|_{t=0} = \lambda,$$

and for $j \geq i + 2$,

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}\bigg|_{t=0} = 0$$

Therefore, condition (1) holds and the matrix $(q_{ij})_{i,j=0,1,\ldots}$ looks like this:

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & \ddots \\ 0 & 0 & 0 & -\lambda & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Problem 1.8. Find the transition matrix of the embedded DTMC associated with the Poisson process with intensity λ.

Solution 1.8. The entries can be computed using the formula and the formula for the Q-matrix in Problem 1.7

$$\tilde{p}_{n,n+1} = \frac{q_{n,n+1}}{-q_{nn}} = \frac{\lambda}{-(-\lambda)} = 1.$$

Since the sums along the rows must be 1, we have that all other entries are zero (which can be seen by also applying the formulas for to the other entries).

Remark 1.16. This is a very intuitive result. For instance, if $T_1, T_2, \ldots,$ are the arrival times, then we define $X_n = X(T_n)$, then we have that

$$\tilde{p}_{n,n+1} = 1$$

and $\tilde{p}_{n,j} = 0$ for all $j \neq n + 1$, since we know that the Poisson process increases by 1 at each arrival time. The matrix is given below

$$\widetilde{\boldsymbol{P}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$