1 Fourier Series

The Fourier series of f is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big),$$

There are 3 main types of coefficients:

1. The coefficients of the (full) Fourier series of $f : [-L, L] \to \mathbb{R}$ is given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

2. The coefficients of the Fourier cosine series of $f : [0, L] \to \mathbb{R}$ is given by the coefficients of the full Fourier series of the even extension of f:

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{even}(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f_{even}(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx = 0.$$

3. The coefficients of the Fourier sine series of $f : [0, L] \to \mathbb{R}$ is given by the coefficients of the full Fourier series of the odd extension of f:

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx = 0$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

Problem 1.1. Decompose the following functions into its Fourier series on the interval $[-1, 1]$ and sketch the graph of the sum of the first three nonzero terms of its Fourier series.

(a) $f(x) = x$

(b) $f(x) = |x|$

Solution 1.1.

(a) We find the Fourier coefficients:

a_n: Since $f(x) = x$ is odd, the a_n coefficients are zero.

b_n: Using integration by parts,

$$b_n = \int_{-1}^{1} x\sin(n\pi x)\,dx = 2\int_{0}^{1} x\sin(n\pi x)\,dx = -\frac{2(-1)^n}{\pi n}.$$

The corresponding Fourier series of x is given by

$$x = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi n}\sin(n\pi x).$$

(b) We find the Fourier coefficients:

a_0: A simple computation shows

$$a_0 = \int_{-1}^{1} |x| \, dx = 2\int_0^1 x \, dx = 1.$$

a_n: For $n \geq 1$, we can integration by parts,

$$a_n = \int_{-1}^{1} |x| \cos(n\pi x) \, dx = 2\int_0^1 x\cos(n\pi x) \, dx = \frac{2((-1)^n - 1)}{\pi^2 n^2}.$$

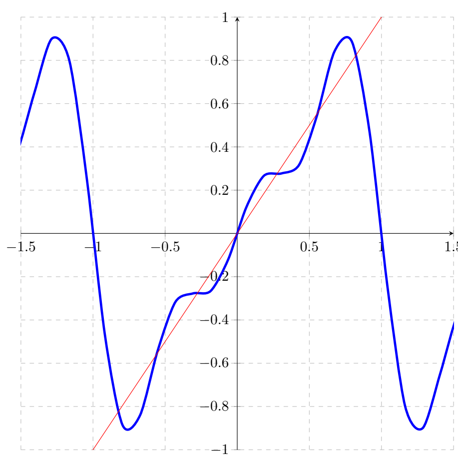
We had to treat the a_0 case separately, because we would've divided by 0 in the computation above if $n = 0$.

b_n: Since $f(x) = |x|$ is even, the b_n coefficients are zero.

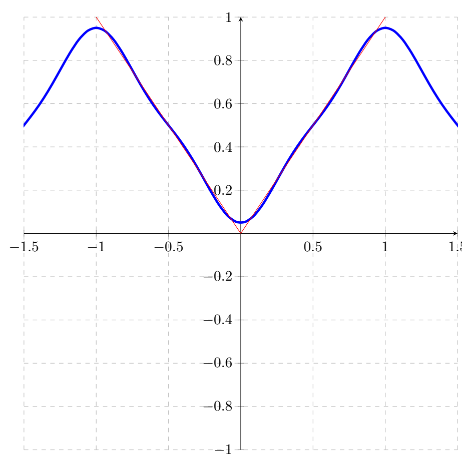
The corresponding Fourier series of $|x|$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{\pi^2 n^2} \cos(n\pi x).$$

Plots: The plots of the first 3 non-zero terms of the series are displayed below:



(a)



(b)

Remark. The series in part (a) and part (b) are also the respective Fourier sine and cosine series of $f(x) = x$ on $[0, 1]$.

Problem 1.2. Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < \pi, \ 0 < t < \infty \\ u(x,0) = 0 & 0 < x < \pi \\ u_t(x,0) = x & 0 < x < \pi \\ u_x(0,t) = u_x(\pi,t) = 0 & 0 < t < \infty. \end{cases}$$

Solution 1.2. This is a homogeneous PDE with vanishing Neumann boundary conditions.

Step 1 — Separation of Variables: We first find a solution to the homogeneous equation.

$$T''(t)X(x) - c^2 T(t)X''(x) = 0 \implies \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

with boundary conditions

$$T(t)X'(0) = T(t)X'(\pi) = 0 \implies X'(0) = X'(\pi) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < \pi \\ X'(0) = X'(\pi) = 0. \end{cases}$$

This is a standard eigenvalue problem with solution

Eigenvalues: $\lambda_n = n^2$ for $n = 0, 1, 2, \ldots$

Eigenfunctions: $X_n = \cos(nx)$ and $X_0 = 1$.

Step 3 — Time Problem: When $n = 0$, the time problem is

$$T_0''(t) = 0$$

which has solution

$$T_0(t) = A_0 + B_0 t.$$

The time problem related to the eigenvalues λ_n for $n \geq 1$ is

$$T_n''(t) + c^2 n^2 T_n(t) = 0 \text{ for } n = 1, 2, \ldots$$

which has solution

$$T_n(t) = A_n \cos(cnt) + B_n \sin(cnt).$$

Step 4 — General Solution: By the principle of superposition, the general form of our solution is

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = A_0 + B_0 t + \sum_{n=1}^{\infty} \Big(A_n \cos(cnt) + B_n \sin(cnt) \Big) \cos(nx).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution by solving for the constants A_n and B_n. The initial conditions imply

$$u(x,0) = \phi(x) \implies A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) = 0$$

and

$$u_t(x,0) = \psi(x) \implies B_0 + \sum_{n=1}^{\infty} B_n cn \cos(nx) = x.$$

Clearly the first initial condition implies that $A_n = 0$ for all $n \geq 0$. To find the B_n coefficients, we decompose x into its Fourier cosine series (or equivalently, decomposing $|x|$ into its full Fourier series on $[-\pi, \pi]$)

$$x = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{\pi n^2} \cos(nx)$$

and equate coefficients to conclude

$$B_0 = \frac{\pi}{2}, \qquad B_n cn = \frac{2((-1)^n - 1)}{\pi n^2} \implies B_n = \frac{2((-1)^n - 1)}{c\pi n^3}.$$

Therefore, our particular solution is

$$u(x,t) = B_0 t + \sum_{n=1}^{\infty} B_n \sin(cnt) \cos(nx)$$

where $B_0 = \frac{\pi}{2}$ and $B_n = \frac{2((-1)^n - 1)}{c\pi n^3}$.