Week 12

Problem 1. (Strauss 6.4.9) Solve $u_{xx} + u_{yy} = 0$ in the wedge $r < a$, $0 < \theta < \beta$ with the BCs

$$u = \theta \text{ on r = a}, \quad u = 0 \text{ on } \theta = 0, \quad \text{and } u = \beta \text{ on } \theta = \beta.$$

Solution 1. After converting to polar coordinates, our PDE can be written as the following problem on the wedge

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < r < a, \quad 0 < \theta < \beta \\ u(r,0) = 0 & 0 < r < a \\ u(r,\beta) = \beta & 0 < r < a \\ u(a,\theta) = \theta & 0 < \theta < \beta. \end{cases}$$

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with homogeneous angular boundary conditions. We set

$$w(r,\theta) = v(r,\theta) + w(r,\theta)$$

where $w(r,\theta)$ is chosen to satisfy the inhomogeneous boundary conditions. Like usual, we can take $w(r,\theta)$ to be a polynomial of the form

$$w(r,\theta) = (A\theta^2 + B\theta + C) \cdot \beta$$

for some constants $A, B, \dots, C$. Substituting $w(r,\theta)$ in the boundary conditions gives

$$C = 0 = w(r,0)$$
$$A\beta^2 + B\beta + C = \beta = w(r,\beta).$$

By inspection it is clear that $C = 0$, and $B = 1$ with the rest of the coefficients zero works. Therefore,

$$w(r,\theta) = \theta.$$

Step 2 — Separation of Variables: Since $v(r,\theta) = u(r,\theta) - w(r,\theta)$, our choice of $w(r,\theta)$ implies

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 & 0 < r < a, \quad 0 < \theta < \beta \\ v(r,0) = 0 & 0 < r < a \\ v(r,\beta) = 0 & 0 < r < a \\ v(a,\theta) = \theta - \theta = 0 & 0 < \theta < \beta. \end{cases} \qquad (*)$$

This now has homogeneous angular boundary conditions, so we can use separation of variables and look for a solution of the form $v(r,\theta) = R(r)\Theta(\theta)$. For such a solution, the PDE implies

$$\Delta v = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \implies -\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda.$$

This results in the ODEs

$$r^2R''(r) + rR(r) - \lambda R'(r) = 0 \text{ and } \Theta''(\theta) + \lambda\Theta(\theta) = 0$$

with angular boundary conditions

$$R(r)\Theta(0) = R(r)\Theta(\beta) = 0.$$

and radial boundary conditions (and regularity condition)

$$R(a)\Theta(\theta) = 0, \quad \lim_{r\to 0} R(r)\Theta(\theta) < \infty$$

For non-trivial solutions to the angle problem, we require $R(r) \not\equiv 0$, $\Theta(0) = \Theta(\beta)$.

Step 3 — Eigenvalue Problem: We now solve the angular eigenvalue problem

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 & 0 < \theta < \beta \\ \Theta(0) = \Theta(\beta) = 0. \end{cases}$$

This is a standard problem, and the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \Big(\frac{n\pi}{\beta}\Big)^2,\ \Theta_n(\theta) = \sin\Big(\frac{n\pi}{\beta}\theta\Big), \quad n = 1, 2, \dots.$$

Step 4 — Radial Problem: For each eigenvalue, we solve the radial problem

$$r^2 R''(r) + rR'(r) - \Big(\frac{n\pi}{\beta}\Big)^2 R(r) = 0.$$

This is an Euler ODE with characteristic equation $C(r) = r(r-1) + r - (\frac{n\pi}{\beta})^2$ and roots $r = \pm\frac{n\pi}{\beta}$, which has general solution of the form

$$R_n(r) = A_n r^{\frac{n\pi}{\beta}} + B_n r^{-\frac{n\pi}{\beta}}$$

for some yet to be determined coefficients A_n and B_n. Since the solution should be regular at 0 ($\lim_{r\to 0} R(r) < \infty$), we need $B_n = 0$, so our solution is of the form

$$R_n(r) = A_n r^{\frac{n\pi}{\beta}}, \qquad n = 1, 2, \dots$$

for some yet to be determined coefficient A_n. Using the principle of superposition, and taking a linear combination of the eigenfunctions gives the general solution

$$v(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\beta}} \sin\Big(\frac{n\pi}{\beta}\theta\Big).$$

Step 5 — Particular Solution: We now use the radial boundary condition to find A_n. Plugging the general solution into the boundary conditions, $R(a)\Theta(\theta) = 0$ implies

$$\sum_{n=1}^{\infty} A_n a^{\frac{n\pi}{\beta}} \sin\Big(\frac{n\pi}{\beta}\theta\Big) = 0.$$

By the Fourier sine series, we have

$$a^{\frac{n\pi}{\beta}} A_n = \frac{2}{\beta}\int_0^{\beta} 0 \sin\Big(\frac{n\pi}{\beta}\theta\Big)\, d\theta = 0 \implies A_n = 0.$$

Step 6 — Final Answer: To summarize, since $u(r,\theta) = v(r,\theta) + w(r,\theta)$ we have

$$u(r,\theta) = \theta.$$

Remark: We went through every step of the problem, to show how one would solve this problem in the scenario that the radial boundary conditions did not vanish. However, one can easily deduce from Step 2, that the unique solution to PDE ($*$) is $v \equiv 0$. From here, we have

$$u(r,\theta) = v(r,\theta) + w(r,\theta) = \theta.$$

Problem 2. (Strauss 6.1.13) Solve $u_{xx} + u_{yy} = 0$ in the region $\{\alpha < \theta < \beta, a < r < b\}$ with the boundary conditions $u = 0$ on the two sides $\theta = \alpha$ and $\theta = \beta$, $u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.

Solution 2. After converting to polar coordinates, our PDE can be written as the following problem on the wedge of an annuli

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < a < r < b, \quad \alpha < \theta < \beta \\ u(r,\alpha) = 0 & 0 < a < r < b \\ u(r,\beta) = 0 & 0 < a < r < b \\ u(a,\theta) = g(\theta) & \alpha < \theta < \beta \\ u(b,\theta) = h(\theta) & \alpha < \theta < \beta \end{cases}$$

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with symmetric homogeneous angular boundary conditions. We use rotation invariance, and set

$$v(r,\theta) = u(r,\theta + \alpha).$$

By rotational invariance, it is easy to see that $v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ and the domain of $v(r,\theta)$ is the centered wedge of the annuli $\{0 < \theta < \beta - \alpha, a < r < b\}$.

Step 2 — Separation of Variables: Since $v(r,\theta) = u(r,\theta+\alpha)$, our choice of $w(r,\theta)$ implies

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 & 0 < a < r < b, \quad 0 < \theta < \beta - \alpha \\ v(r,0) = 0 & 0 < a < r < b \\ v(r,\beta-\alpha) = 0 & 0 < a < r < b \\ v(a,\theta) = g(\theta+\alpha) & 0 < \theta < \beta - \alpha \\ v(b,\theta) = h(\theta+\alpha) & 0 < \theta < \beta - \alpha. \end{cases}$$

This PDE now has symmetric homogeneous angular boundary conditions, so we look for a solution of the form $v(r,\theta) = R(r)\Theta(\theta)$. For such a solution, the PDE implies

$$\Delta v = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \implies -\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda.$$

This results in the ODEs

$$r^2R''(r) + rR(r) - \lambda R'(r) = 0 \text{ and } \Theta''(\theta) + \lambda\Theta(\theta) = 0$$

with angular boundary conditions

$$R(r)\Theta(0) = R(r)\Theta(\beta-\alpha) = 0,$$

and radial boundary conditions

$$R(a)\Theta(\theta) = g(\theta+\alpha), \quad R(b)\Theta(\theta) = h(\theta+\alpha).$$

For non-trivial solutions to the angle problem, we require $R(r) \not\equiv 0$, $\Theta(0) = \Theta(\beta-\alpha)$.

Step 3 — Eigenvalue Problem: We now solve the angular eigenvalue problem

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 & 0 < \theta < \beta \\ \Theta(0) = \Theta(\beta-\alpha) = 0. \end{cases}$$

This is a standard problem, and the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \Big(\frac{n\pi}{\beta-\alpha}\Big)^2,\ \Theta_n(\theta) = \sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big), \quad n = 1, 2, \ldots.$$

Step 4 — Radial Problem: For each eigenvalue, we solve the radial problem

$$r^2R''(r) + rR'(r) - \Big(\frac{n\pi}{\beta-\alpha}\Big)^2 R(r) = 0.$$

This is an Euler ODE with characteristic equation $C(r) = r(r-1) + r - (\frac{n\pi}{\beta-\alpha})^2$ and roots $r = \pm\frac{n\pi}{\beta-\alpha}$, which has general solution of the form

$$R_n(r) = A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}$$

for some yet to be determined coefficients A_n and B_n. Using the principle of superposition, and taking a linear combination of the eigenfunctions gives the general solution

$$v(r,\theta) = \sum_{n=1}^{\infty} \big(A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}\big) \sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big).$$

Step 5 — Particular Solution: We now use the radial boundary condition to find A_n. Plugging the general solution into the radial boundary conditions implies,

$$\sum_{n=1}^{\infty} \big(A_n a^{\frac{n\pi}{\beta-\alpha}} + B_n a^{-\frac{n\pi}{\beta-\alpha}}\big) \sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big) = g(\theta+\alpha)$$

and

$$\sum_{n=1}^{\infty} \big(A_n b^{\frac{n\pi}{\beta-\alpha}} + B_n b^{-\frac{n\pi}{\beta-\alpha}}\big) \sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big) = h(\theta+\alpha).$$

By the Fourier sine series, we have

$$A_n a^{\frac{n\pi}{\beta-\alpha}} + B_n a^{-\frac{n\pi}{\beta-\alpha}} = \frac{2}{\beta-\alpha}\int_0^{\beta-\alpha} g(\theta+\alpha)\sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big)\,d\theta = \frac{2}{\beta-\alpha}\int_\alpha^{\beta} g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta,$$

and

$$A_n b^{\frac{n\pi}{\beta-\alpha}} + B_n b^{-\frac{n\pi}{\beta-\alpha}} = \frac{2}{\beta-\alpha}\int_0^{\beta-\alpha} h(\theta+\alpha)\sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big)\,d\theta = \frac{2}{\beta-\alpha}\int_\alpha^{\beta} h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta.$$

This system can be written as a 2×2 matrix with linearly independent columns (since $a \neq b$), so we may solve for A_n and B_n if we wish. I leave it in this form, because the simplification does not produce a nicer answer.

Step 6 — Final Answer: To summarize, since $v(r,\theta) = u(r,\theta+\alpha)$ we have $u(r,\theta) = v(r,\theta-\alpha)$, we have

$$u(r,\theta) = \big(A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}\big) \sin\Big(\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\Big),$$

where A_n and B_n are solutions to the linear system

$$\begin{array}{l} a^{\frac{n\pi}{\beta-\alpha}} A_n + a^{-\frac{n\pi}{\beta-\alpha}} B_n = \frac{2}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta \\ b^{\frac{n\pi}{\beta-\alpha}} A_n + b^{-\frac{n\pi}{\beta-\alpha}} B_n = \frac{2}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta \end{array}.$$

Remark: To solve this system, we can use the formula for the inverse of a 2×2 matrix. If we define $I_n = \frac{2}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta$ and $J_n = \frac{2}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta$, this gives

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \frac{1}{a^{\frac{n\pi}{\beta-\alpha}}b^{-\frac{n\pi}{\beta-\alpha}} - a^{-\frac{n\pi}{\beta-\alpha}}b^{\frac{n\pi}{\beta-\alpha}}}\begin{bmatrix} b^{-\frac{n\pi}{\beta-\alpha}} & -a^{-\frac{n\pi}{\beta-\alpha}} \\ -b^{\frac{n\pi}{\beta-\alpha}} & a^{\frac{n\pi}{\beta-\alpha}} \end{bmatrix} \times \begin{bmatrix} I_n \\ J_n \end{bmatrix}.$$

That is, for $C_n = a^{\frac{n\pi}{\beta-\alpha}}b^{-\frac{n\pi}{\beta-\alpha}} - a^{-\frac{n\pi}{\beta-\alpha}}b^{\frac{n\pi}{\beta-\alpha}}$ we have

$$A_n = \frac{1}{C_n}\cdot\bigg(\frac{2b^{-\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta - \frac{2a^{-\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta\bigg)$$

and

$$B_n = \frac{1}{C_n}\cdot\bigg(-\frac{2b^{\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta + \frac{2a^{\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta\bigg).$$

Problem 3. (Strauss 6.4.4) Derive Poisson's formula

$$u(r,\theta) = (r^2 - a^2)\int_{-\pi}^{\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\,\frac{d\phi}{2\pi}$$

for the exterior of a circle.

Solution 3. We continue off from the derivation on page 175 in Strauss. We know the particular solution is given by

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n\cos(n\theta) + B_n\sin(n\theta))$$

where

$$A_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)\,d\theta$$

and

$$B_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin(n\theta)\,d\theta.$$

We now sum the series explicitly. Using the fact

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,d\phi + \sum_{n=1}^{\infty}\Big(\frac{a}{r}\Big)^n\frac{1}{\pi}\int_{-\pi}^{\pi} h(\phi)\Big(\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta)\Big)\,d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,d\phi + \sum_{n=1}^{\infty}\Big(\frac{a}{r}\Big)^n\frac{1}{\pi}\int_{-\pi}^{\pi} h(\phi)\cos(n(\theta-\phi))\,d\phi \\ &= \int_{-\pi}^{\pi} h(\phi)\bigg(1 + 2\sum_{n=1}^{\infty}\Big(\frac{a}{r}\Big)^n\cos(n(\theta-\phi))\bigg)\frac{d\phi}{2\pi}. \end{aligned}$$

The sum on the inside can be summed using the geometric series, giving us

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty}\Big(\frac{a}{r}\Big)^n\cos(n(\theta-\phi)) &= 1 + \sum_{n=1}^{\infty}\Big(\frac{a}{r}\Big)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\Big(\frac{a}{r}\Big)^n e^{-in(\theta-\phi)} \\ &= 1 + \frac{ae^{i(\theta-\phi)}}{r - ae^{i(\theta-\phi)}} + \frac{ae^{-i(\theta-\phi)}}{r - ae^{-i(\theta-\phi)}} \\ &= \frac{r^2 - a^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}. \end{aligned}$$

That is,

$$u(r,\theta) = (r^2 - a^2)\int_{-\pi}^{\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\,\frac{d\phi}{2\pi}.$$

Uniqueness Proofs

We now do several uniqueness proofs using the energy method. We first explain the general methodology. Given a function $w \in C^2(D)$, where $D \subset \mathbb{R}^n$ is an open domain, we define energy of w as

$$E[w] = \int_D |\nabla w|^2 \, dx.$$

Our goal is to show that $E[w] \leq 0$ which will imply $E[w] = 0$. This will prove that $\nabla w = 0$ in D, which proves that w is constant on the interior. If we can show that $w = 0$ on ∂D, then we can prove uniqueness of continuous solutions.

Our main tool to prove the upper bound of the energy is Green's First identity.

Green's First Identity: Let D be a bounded open subset of $\mathbb{R}^n$. For $u, v \in C^2(\bar{D})$, we have

$$\int_D \nabla u \cdot \nabla v \, dx = \int_{\partial D} u \frac{\partial v}{\partial n} \, dS - \int_D u \Delta v \, dx. \tag{1}$$

Problem 4. (Strauss 6.4.11) Using the energy method, prove the uniqueness of the Robin problem

$$\Delta u = f \text{ in } D, \quad \frac{\partial u}{\partial n} + au = h \quad \text{on } \partial D$$

where D is any domain in three dimensions and where a is a positive constant.

Solution 4. Consider two solutions u and v of the Robin problem. Define $w = u - v$. We have w solves

$$\begin{cases} \Delta w = 0 & in \quad D \\ \frac{\partial w}{\partial n} + aw = 0 & on \quad \partial D. \end{cases}$$

Computing the energy of w, we have

$$\begin{aligned} E[w] &= \iiint_D |\nabla w|^2 \, dV \\ &= \iint_{\partial D} w \frac{\partial w}{\partial n} \, dS - \iiint_D w \cdot \Delta w \, dV && \text{(Green's First Identity)} \\ &= \iint_{\partial D} w \frac{\partial w}{\partial n} \, dS && (\Delta w = 0) \\ &= \iint_{\partial D} w(-aw) \, dS && \text{(Boundary Conditions)} \\ &= -a \iint_{\partial D} w^2 \, dS \leq 0. \end{aligned}$$

Since $E[w] \geq 0$, the inequality above implies $0 \leq E[w] \leq 0 \implies E[w] = 0$, so by continuity

$$\nabla w \equiv 0 \text{ in } D \quad \Longrightarrow \quad w = \text{constant in } \bar{D}.$$

However, we also know that $E[w] = -a \int_{\partial D} w^2 \, ds = 0$, which also means that $w = 0$ on ∂D. Therefore, we must have $w \equiv 0$ which implies $u = v$ on $\bar{D}$ if we require our solution to be continuous.

Problem 5. Using the energy method, prove the uniqueness of the Neumann problem on $D \subset \mathbb{R}^2$

$$\begin{cases} \Delta u = f & in \quad D \\ \frac{\partial u}{\partial n} = g & on \quad \partial D \end{cases}$$

up to a constant.

Solution 5. Consider two solutions u and v of the Neumann problem. Define $w = u - v$. We have w solves

$$\begin{cases} \Delta w = 0 & in \quad D \\ \frac{\partial w}{\partial n} = 0 & on \quad \partial D. \end{cases}$$

Computing the energy of w, we have

$$\begin{aligned} E[w] &= \iint_D |\nabla w|^2 \, dA \\ &= \int_{\partial D} w \frac{\partial w}{\partial n} \, ds - \iint_D w \cdot \Delta w \, dA && \text{(Green's First Identity)} \\ &= \int_{\partial D} w \frac{\partial w}{\partial n} \, ds && (\Delta w = 0) \\ &= \int_{\partial D} 0 \, ds = 0 && \text{(Boundary Conditions).} \end{aligned}$$

Since $E[w] \geq 0$ the inequality above implies $0 \leq E[w] = 0 \implies E[w] = 0$, so by continuity

$$\nabla w \equiv 0 \text{ in } D \quad \implies \quad w = \text{constant in } \bar{D}.$$

We do not get any information from boundary condition $\frac{\partial w}{\partial n} = 0$ to determine the explicit value of this constant. Therefore, $u = v + C$ for some constant $C \in \mathbb{R}$.

Problem 6. Using the energy method, prove the uniqueness of the Dirichlet problem on $D \subset \mathbb{R}^2$

$$\begin{cases} \Delta u = f & in \quad D \\ u = g & on \quad \partial D. \end{cases}$$

Solution 6. Consider two solutions u and v of the Dirichlet problem. Define $w = u - v$. We have w solves

$$\begin{cases} \Delta w = 0 & in \quad D \\ w = 0 & on \quad \partial D. \end{cases}$$

Computing the energy of w, we have

$$\begin{aligned} E[w] &= \iint_D |\nabla w|^2 \, dA \\ &= \int_{\partial D} w \frac{\partial w}{\partial n} \, ds - \iint_D w \cdot \Delta w \, dA && \text{(Green's First Identity)} \\ &= \int_{\partial D} w \frac{\partial w}{\partial n} \, ds && (\Delta w = 0) \\ &= \int_{\partial D} 0 \, ds = 0 && \text{(Boundary Conditions).} \end{aligned}$$

Since $E[w] \geq 0$ the inequality above implies $0 \leq E[w] = 0 \implies E[w] = 0$, so by continuity,

$$\nabla w \equiv 0 \text{ in } D \quad \implies \quad w = \text{constant in } \bar{D}.$$

Since $w = 0$ on ∂D, we must have $w \equiv 0$ which implies $u = v$ on $\bar{D}$ if we require our solution to be continuous.