Solving The Wave Equation

Consider the wave equation on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi(x) & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x) & x \in \mathbb{R}. \end{cases}$$

The particular solution to this PDE is given by

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds.$$

Problems

Wave Equation on $\mathbb{R}$

Problem 1. Solve the initial value problem

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \tanh(x) & x \in \mathbb{R}, \\ u_t(x,0) = \arctan(x) & x \in \mathbb{R}. \end{cases}$$

Solution 1. By D'Alembert's formula, the particular solution to this IVP is given by

$$u(x,t) = \frac{\tanh(x+2t) + \tanh(x-2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} \arctan(y)\,dy.$$

The integral term can be computed using integration by parts,

$$\begin{aligned} &\frac{1}{4}\int_{x-2t}^{x+2t} \arctan(y)\,dy \\ &= \frac{1}{4}\Big(y\arctan(y) - \frac{1}{2}\ln|1+y^2|\Big)\Big|_{y=x-2t}^{y=x+2t} \\ &= \frac{1}{4}\Big((x+2t)\arctan(x+2t) - (x-2t)\arctan(x-2t) - \frac{1}{2}\ln(1+(x+2t)^2) + \frac{1}{2}\ln(1+(x-2t)^2)\Big). \end{aligned}$$

Problem 2. Solve the following initial value problems

1. $$\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x) & x \in \mathbb{R}, \\ u_t(x,0) = h(x) & x \in \mathbb{R} \end{cases}$$

   with

   $$g(x) = \begin{cases} 0 & |x| \geq 1 \\ x^2 - x^4 & |x| < 1 \end{cases}, \qquad h(x) = 0.$$

2. $$\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x) & x \in \mathbb{R}, \\ u_t(x,0) = h(x) & x \in \mathbb{R} \end{cases}$$

   with

   $$g(x) = 0, \qquad h(x) = \begin{cases} 0 & |x| \geq 1 \\ x^2 - x^4 & |x| < 1 \end{cases}.$$

Solution 2.

(1) Since $h(x) = 0$, by D'Alembert's formula, the particular solution to this IVP is given by

$$u(x,t) = \frac{g(x+2t) + g(x-2t)}{2}.$$

Since $g(x)$ changes form based on the value of $|x|$, we can break our solution into 4 cases:

A. $|x+2t| \geq 1$, $|x-2t| \geq 1$: On this region, $g(x+2t) = 0$ and $g(x-2t) = 0$, so

$$u(x,t) = 0.$$

B. $|x+2t| < 1$, $|x-2t| \geq 1$: On this region, $g(x+2t) = (x+2t)^2 - (x+2t)^4$ and $g(x-2t) = 0$, so

$$u(x,t) = \frac{(x+2t)^2 - (x+2t)^4}{2}.$$

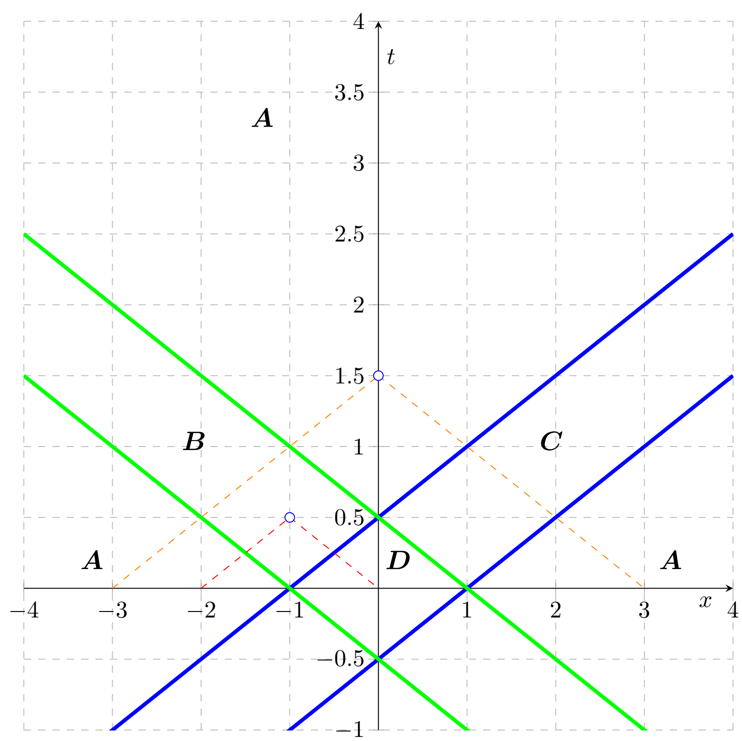
C. $|x+2t| \geq 1$, $|x-2t| < 1$: On this region, $g(x+2t) = 0$ and $g(x-2t) = (x-2t)^2 - (x-2t)^4$, so

$$u(x,t) = \frac{(x-2t)^2 - (x-2t)^4}{2}.$$

D. $|x+2t| < 1$, $|x-2t| < 1$: On this region, $g(x+2t) = (x+2t)^2 - (x+2t)^4$ and $g(x-2t) = (x-2t)^2 - (x-2t)^4$, so

$$u(x,t) = \frac{(x+2t)^2 - (x+2t)^4 + (x-2t)^2 - (x-2t)^4}{2}.$$

Characteristic Lines:



Description of Picture: The initial condition is supported on the interval $[-1,1]$. The wave propagates right along the lines $x-2t = C \in [-1,1]$ (between the blue characteristic lines) and left along the lines $x+2t = C \in [-1,1]$ (between the green characteristic lines). The behavior on each of the regions can be determined by drawing the domain of dependence at the point (x,t) and seeing if the corners lie in the interval $[-1,1]$. For example, at the point $(-1,0.5)$ the left corner does not lie in $[-1,1]$, while the right corner is in $[-1,1]$, which corresponds to case B above. Similarly, at the point $(0,1.5)$ both corners do not lie in $[-1,1]$, which corresponds to case A above.

(2) Since $g(x) = 0$, by D'Alembert's formula, the particular solution to this IVP is given by

$$u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy.$$

Since $h(x)$ changes form based on the value of $|x|$, we can break our solution into 5 cases:

A. $x-2t \le -1 \le 1 \le x+2t$: On this region, we can split our region of integration into

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy &= \frac{1}{4}\int_{x-2t}^{-1} h(y)\,dy + \frac{1}{4}\int_{-1}^{1} h(y)\,dy + \frac{1}{4}\int_{1}^{x+2t} h(y)\,dy \\ &= \frac{1}{4}\int_{-1}^{1} y^2 - y^4\,dy \\ &= \frac{1}{4}\Big(\frac{y^3}{3} - \frac{y^5}{5}\Big)\Big|_{y=-1}^{y=1} = \frac{1}{15}. \end{aligned}$$

B. $x-2t \le -1 \le x+2t \le 1$: On this region, we can split our region of integration into

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy &= \frac{1}{4}\int_{x-2t}^{-1} h(y)\,dy + \frac{1}{4}\int_{-1}^{x+2t} h(y)\,dy \\ &= \frac{1}{4}\int_{-1}^{x+2t} y^2 - y^4\,dy \\ &= \frac{1}{4}\Big(\frac{y^3}{3} - \frac{y^5}{5}\Big)\Big|_{y=-1}^{y=x+2t} = \frac{(x+2t)^3}{12} - \frac{(x+2t)^5}{20} + \frac{1}{30}. \end{aligned}$$

C. $-1 \le x-2t \le 1 \le x+2t$: On this region, we can split our region of integration into

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy &= \frac{1}{4}\int_{x-2t}^{1} h(y)\,dy + \frac{1}{4}\int_{1}^{x+2t} h(y)\,dy \\ &= \frac{1}{4}\int_{x-2t}^{1} y^2 - y^4\,dy \\ &= \frac{1}{4}\Big(\frac{y^3}{3} - \frac{y^5}{5}\Big)\Big|_{y=x-2t}^{y=1} = \frac{1}{30} - \frac{(x-2t)^3}{12} + \frac{(x-2t)^5}{20}. \end{aligned}$$

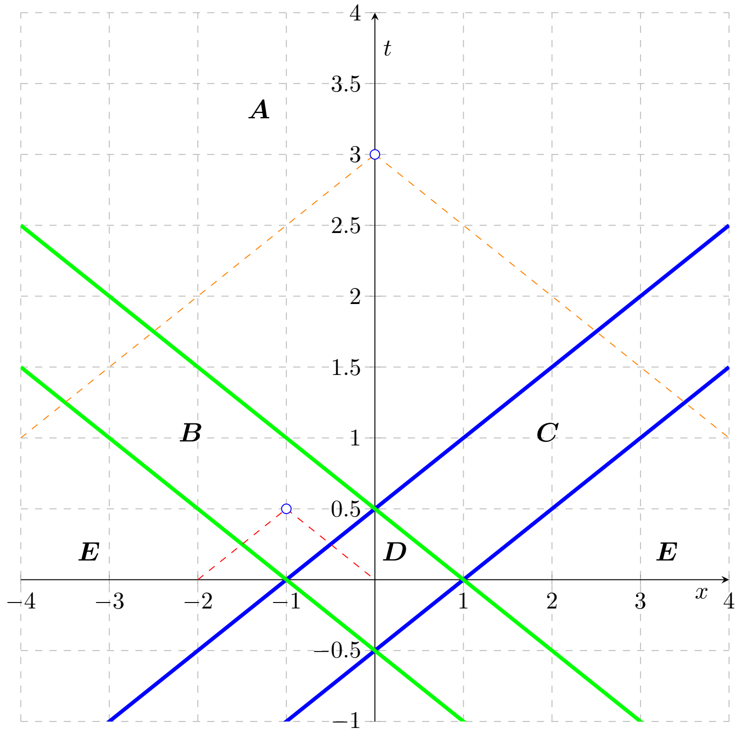
D. $-1 \le x-2t \le x+2t \le 1$: On this region, the integrand is always equal to $h(y) = y^2 - y^4$

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy &= \frac{1}{4}\int_{x-2t}^{x+2t} y^2 - y^4\,dy \\ &= \frac{1}{4}\Big(\frac{y^3}{3} - \frac{y^5}{5}\Big)\Big|_{y=x-2t}^{y=x+2t} \\ &= \frac{(x+2t)^3}{12} - \frac{(x+2t)^5}{20} - \frac{(x-2t)^3}{12} + \frac{(x-2t)^5}{20}. \end{aligned}$$

E. $x-2t \ge 1$, or $x+2t \le -1$: On this region, the integrand is always equal to $h(y) = 0$, so

$$u(x,t) = 0.$$

Characteristic Lines:



Description of Picture: The initial condition is supported on the interval $[-1,1]$. The behavior in each of the regions can be determined by drawing the domain of dependence at the point (x,t) and seeing how much of the interval $[-1,1]$ is contained in the base of the triangle. For example, at $(-1,0.5)$ the left corner of the base of the triangle is <-1 and the right corner of the base is in $[-1,1]$, which corresponds to case B above. Similarly, at $(0,3)$ the left corner of the base of the orange triangle is <-1 and the right corner of the base is in >1, which corresponds to case A above.

Problem 3. Solve the initial value problem

$$\begin{cases} u_{tt} - 4u_{xx} = f(x,t) & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x) & x \in \mathbb{R}, \\ u_t(x,0) = h(x) & x \in \mathbb{R} \end{cases}$$

with

$$f(x,t) = \begin{cases} \sin(x) & 0 < t < \pi \\ 0 & t \geq \pi \end{cases}, \qquad g(x) = 0, \qquad h(x) = 0.$$

Solution 3. Since $g(x) = 0$ and $h(x) = 0$, by D'Alembert's formula with the source term the particular solution to this IVP is given by

$$\begin{aligned} u(x,t) = \frac{1}{4} \iint_\Delta f(y,s)\, dyds &= \frac{1}{4} \int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \sin(y) \mathbb{1}_{[0,\pi]}(s)\, dyds \\ &= \frac{1}{4} \int_0^{\min(t,\pi)} \int_{x-2(t-s)}^{x+2(t-s)} \sin(y)\, dyds. \end{aligned}$$

If you draw the region of integration, we are basically chopping off Δ above the line $t = \pi$ and integrating the remaining trapezoid (or triangle if t is small enough). We have two cases,

A. $t < \pi$: On this region, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \sin(y)\,dyds \\
&= \frac{1}{4}\int_0^t \Big(-\cos(y)\Big|_{y=x-2(t-s)}^{y=x+2(t-s)}\Big) ds \\
&= \frac{1}{4}\int_0^t -\cos(x+2(t-s)) + \cos(x-2(t-s)) ds. \\
&= \frac{1}{8}\Big(\sin(x+2(t-s)) + \sin(x-2(t-s))\Big)\Big|_{s=0}^{s=t} \\
&= \frac{1}{4}\sin(x) - \frac{1}{8}\sin(x+2t) - \frac{1}{8}\sin(x-2t).
\end{aligned}$$

B. $t \geq \pi$: On this region, we can split our region of integration into

$$\begin{aligned}
u(x,t) &= \frac{1}{4}\int_0^\pi \int_{x-2(t-s)}^{x+2(t-s)} \sin(y)\,dyds \\
&= \frac{1}{4}\int_0^\pi \Big(-\cos(y)\Big|_{y=x-2(t-s)}^{y=x+2(t-s)}\Big) ds \\
&= \frac{1}{4}\int_0^\pi -\cos(x+2(t-s)) + \cos(x-2(t-s)) ds. \\
&= \frac{1}{8}\Big(\sin(x+2(t-s)) + \sin(x-2(t-s))\Big)\Big|_{s=0}^{s=\pi} \\
&= 0.
\end{aligned}$$

Wave Equation on the Half Line

Problem 4. Solve the following PDE

$$\begin{cases} u_{tt} = 4u_{xx} & 0 < x < \infty, \quad t > 0 \\ u(x,0) = 1 & 0 < x < \infty, \\ u_t(x,0) = 0 & 0 < x < \infty \\ u(0,t) = 0 & t > 0 \end{cases}.$$

Solution 4. We want to solve the wave equation on the half line with Dirichlet boundary conditions. We can use an odd reflection to extend the initial condition,

$$g_{odd}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}, \quad h_{odd}(x) = 0.$$

The particular solution to the extended PDE is

$$u(x,t) = \frac{g_{odd}(x+2t) + g_{odd}(x-2t)}{2}.$$

We now examine the cases depending on the sign of $x - 2t$:

1. For $x - 2t > 0$, we have $g_{odd}(x \pm 2t) = 1$ so the solution is given by

$$u(x,t) = \frac{1+1}{2} = 1.$$

2. For $x - 2t < 0$, we have $g_{odd}(x-2t) = -1$ while $g_{odd}(x+2t) = 1$, so
$$u(x,t) = \frac{1-1}{2} = 0.$$

3. When $x - 2t = 0$, we have $g_{odd}(x-2t) = 0$ so
$$u(x,t) = \frac{1}{2}.$$

In summary, for $x > 0$ and $t > 0$, the solution is given by
$$u(x,t) = \begin{cases} 1 & x > 2t \\ \frac{1}{2} & x = 2t \\ 0 & x < 2t \end{cases}.$$

From here, it is clear that there is a singularity at the line $x = 2t$. This is because the odd extension of g is not continuous at $x = 0$.