Week 11

Problem 1. Find the Fourier Transform of

$$f(x) = e^{-a|x|} \qquad a > 0.$$

Solution 1. We split the region of integration,

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} e^{-ikx} e^{-a|x|}\, dx \\
&= \int_{-\infty}^{0} e^{-ikx+ax}\, dx + \int_{0}^{\infty} e^{-ikx-ax}\, dx \\
&= \frac{e^{-ikx+ax}}{a-ik}\Big|_{x=-\infty}^{x=0} + \frac{e^{-ikx-ax}}{-a-ik}\Big|_{x=0}^{x=\infty} \\
&= \frac{1}{a-ik} + \frac{1}{a+ik} \\
&= \frac{2a}{a^2+k^2}.
\end{aligned}$$

Problem 2. Using the properties of the Fourier transform, recover the general formula for the solution $u(x,t)$ of Laplace's equation

$$u_{xx} + u_{yy} = 0,\ x \in \mathbb{R},\ y \geq 0 \qquad u(x,0) = \phi(x).$$

Solution 2.

Step 1 — Transform the Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to Laplace's equation, and consider its Fourier transform

$$\hat{u}(k,y) = \int_{-\infty}^{\infty} e^{-ikx} u(x,y)\, dx.$$

Since $u_{xx} + u_{yy} = 0$, taking the Fourier transform of both sides implies that

$$-k^2 \hat{u}(k,y) + \hat{u}_{yy}(k,y) = 0 \qquad y > 0.$$

The solution to this ODE (in y) is given by

$$\hat{u}(k,y) = A(k)e^{-ky} + B(k)e^{ky},$$

where $A(k)$ and $B(k)$ are some yet to be determined functions of k.

Step 2 — Find the Particular Solution: Since our solution should be bounded for $y \geq 0$, we have $B(k) = 0$ for $k > 0$ and $A(k) = 0$ for $k < 0$. The general solution can be simplified as

$$\hat{u}(k,y) = C(k)e^{-|k|y}, \qquad C(k) = \begin{cases} A(k) & k > 0 \\ A(0) + B(0) & k = 0 \\ B(k) & k < 0 \end{cases}$$

We can find $C(k)$ by using our initial condition,

$$u(x,0) = \phi(x) \implies \hat{u}(k,0) = \hat{\phi}(k) \implies C(k) = \hat{\phi}(k).$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,y) = \hat{\phi}(k)e^{-|k|y}.$$

Step 3 — Recover the Solution: We take the inverse Fourier transform of both sides to recover our original function. Let $S(x,y)$ be the inverse Fourier transform of $e^{-|k|y}$,

$$\begin{aligned}
S(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-y|k|}\,dk &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{ikx+yk}\,dk + \frac{1}{2\pi}\int_{0}^{\infty} e^{ikx-yk}\,dk \\
&= \frac{1}{2\pi}\frac{e^{ikx+yk}}{ix+y}\Big|_{k=-\infty}^{k=0} + \frac{1}{2\pi}\frac{e^{ikx-yk}}{ix-y}\Big|_{k=0}^{k=\infty} \\
&= \frac{1}{2\pi}\Big(\frac{1}{ix+y} - \frac{1}{ix-y}\Big) \\
&= \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}.
\end{aligned}$$

Since $\hat{u}(k,y) = \hat{\phi}(k)e^{-|k|y}$, taking the inverse Fourier transform of both sides implies

$$u(x,y) = (\phi(\cdot) * S(\cdot,y))(x) = \int_{-\infty}^{\infty} S(x-\tau,y)\phi(\tau)\,d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\tau)^2+y^2}\phi(\tau)\,d\tau.$$

Problem 3. Using the properties of the Fourier transform, recover the general formula for the solution $u(x,t)$ of the heat equation

$$u_t - u_{xx} = 0,\ x\in\mathbb{R},\ t\ge 0 \qquad u(x,0) = \phi(x).$$

Solution 3.

Step 1 — Transform the Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to the heat equation, and consider its Fourier transform

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} e^{-ikx}u(x,t)\,dx.$$

Since $u_t - u_{xx} = 0$, taking the Fourier transform of both sides implies that

$$\hat{u}_t(k,t) + k^2\hat{u}(k,t) = 0 \qquad y > 0.$$

The solution to this ODE (in t) is given by

$$\hat{u}(k,t) = A(k)e^{-k^2t},$$

where $A(k)$ is some yet to be determined function of k.

Step 2 — Find the Particular Solution: We can find $A(k)$ by using our initial condition,

$$u(x,0) = \phi(x) \implies \hat{u}(k,0) = \hat{\phi}(k) \implies A(k) = \hat{\phi}(k).$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,t) = \hat{\phi}(k)e^{-k^2t}.$$

Step 3 — Recover the Solution: We take the inverse Fourier transform of both sides to recover our original function. Let $S(x,t)$ be the inverse Fourier transform of $e^{-k^2 t}$,

$$\begin{aligned} S(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-k^2 t}\,dk &= \frac{1}{2\pi}e^{-\frac{x^2}{4t}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\,dk \\ &= \frac{1}{2\pi}e^{-\frac{x^2}{4t}}\cdot\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-z^2}\,dz && \text{(See the Remark)} \\ &= \frac{1}{\sqrt{4\pi t}}\cdot e^{-\frac{x^2}{4t}}. && \int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}. \end{aligned}$$

Since $\hat{u}(k,y) = \hat{\phi}(k)e^{-k^2 t}$, taking the inverse Fourier transform of both sides implies

$$u(x,t) = (\phi(\cdot) * S(\cdot,t))(x) = \int_{-\infty}^{\infty} S(x-\tau,t)\phi(\tau)\,d\tau = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{4t}}\phi(\tau)\,d\tau.$$

Remark: The imaginary change of variables $z = \sqrt{t}k - i\frac{x}{2\sqrt{t}}$ can be justified using complex analysis.

$$\int_{\mathbb{R}} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\,dk = \frac{1}{\sqrt{t}}\int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz.$$

Consider the contour integral over the closed rectangular path oriented counter clockwise,

(1) $k - i\frac{x}{2\sqrt{t}}$ for k from $-M$ to M

(2) $M + iy$ for y from $-\frac{x}{2\sqrt{t}}$ to 0

(3) k for k from M to $-M$

(4) $M + iy$ for y from 0 to $-\frac{x}{2\sqrt{t}}$.

Since e^{-z^2} is analytic, the integral over this closed path is 0, so

$$\int_{(1)} e^{-z^2}\,dz + \int_{(2)} e^{-z^2}\,dz + \int_{(3)} e^{-z^2}\,dz + \int_{(4)} e^{-z^2}\,dz = 0.$$

Since e^{-z^2} is small when the $\mathrm{Re}(z) = \pm M$, if we take $M \to \infty$, the integrals over the regions (2) and (4) vanish leaving us with

$$\int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz + \int_{\infty}^{-\infty} e^{-z^2}\,dz = 0 \implies \int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz = \int_{-\infty}^{\infty} e^{-z^2}\,dz.$$

Therefore,

$$\int_{\mathbb{R}} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\,dk = \frac{1}{\sqrt{t}}\int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz = \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-z^2}\,dz.$$