Week 5: The Heat Equation on $\mathbb{R}$

Introduction

Consider the heat equation on the whole line

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi(x) & x \in \mathbb{R}. \end{cases}$$

The particular solution to this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} \phi(y)\, dy.$$

It is usually impossible to compute $u(x,t)$ except for some very specific initial data. We will go over a few of these cases in the following examples.

Problems

Problem 1. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = e^{ax} & x \in \mathbb{R}, \end{cases}$$

where $a \in \mathbb{R}$.

Solution 1. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{ay}\, dy.$$

Since the initial data is an exponential, we can complete the square to simplify the integral. Completing the square in y, we have

$$\begin{aligned} -\frac{(y-x)^2}{4kt} + ay &= -\frac{y^2 - 2xy + x^2 - 4ktay}{4kt} \\ &= -\frac{y^2 - (2x + 4kta)y + x^2}{4kt} \\ &= -\frac{(y - (x+2kta))^2 - (x+2kta)^2 + x^2}{4kt} \\ &= -\frac{(y-(x+2kta))^2}{4kt} + ax + a^2kt. \end{aligned}$$

Therefore, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+2kta))^2}{4kt} + ax + a^2kt}\, dy = e^{ax+a^2kt} \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+2kta))^2}{4kt}}\, dy}_{=1} = e^{ax+a^2kt}.$$

We used the fact that this Gaussian integral corresponds to the p.d.f. of a Gaussian random variable with mean $(x + 2kta)$ and variance $2kt$, so its integral over $\mathbb{R}$ is equal to 1.

Alternate Solution: The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{ay}\, dy.$$

If set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, e^{aX}$. In particular, using the formula for the moment generating function of the Gaussian random variable, we have

$$u(x,t) = \mathbb{E}\, e^{aX} = e^{a\mu + \frac{1}{2}a^2\sigma^2} = e^{ax + a^2 kt}.$$

Problem 2. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = \mathbb{1}_{[a,\infty)}(x) & x \in \mathbb{R}, \end{cases}$$

where $a \in \mathbb{R}$ and

$$\mathbb{1}_{[a,\infty)}(x) = \begin{cases} 1 & x \geq a \\ 0 & x < a. \end{cases}$$

Write your answer in terms of the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp.$$

Solution 2. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_a^\infty e^{-\frac{(y-x)^2}{4kt}}\, dy.$$

We can use a change of variables to write this function in terms of error functions. Using the change of variables

$$p = \frac{y-x}{\sqrt{4kt}}, \qquad \sqrt{4kt}\, dp = dy, \qquad y = a \implies p = \frac{a-x}{\sqrt{4kt}} \qquad y = \infty \implies p = \infty$$

we get

$$\begin{aligned} u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{a-x}{\sqrt{4kt}}}^\infty e^{-p^2}\, dp &= \frac{1}{\sqrt{\pi}} \int_{\frac{a-x}{\sqrt{4kt}}}^0 e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-p^2}\, dp \\ &= -\frac{1}{\sqrt{\pi}} \int_0^{\frac{a-x}{\sqrt{4kt}}} e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-p^2}\, dp \\ &= -\frac{1}{2}\text{erf}\left(\frac{a-x}{\sqrt{4kt}}\right) + \frac{1}{2}. \end{aligned}$$

Problem 3. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = \mathbb{1}_{[-1,1]}(x) & x \in \mathbb{R}, \end{cases}$$

where

$$\mathbb{1}_{[-1,1]}(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & x > 1. \end{cases}$$

Write your answer in terms of the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp.$$

Solution 3. We can use a change of variables to write this function in terms of error functions like in the previous problem. Instead, we can use linearity and our result in the last problem to simplify this problem. It is easy to check that

$$\mathbb{1}_{[-1,1]}(x) = \mathbb{1}_{[-1,\infty)}(x) - \mathbb{1}_{(1,\infty)}(x).$$

In particular, if we let u_1 be a solution to the heat equation with initial condition $\mathbb{1}_{[-1,\infty)}(x)$ and u_2 be a solution to the heat equation with initial condition $-\mathbb{1}_{(1,\infty)}(x)$, then it is easy to check that $u = u_1 + u_2$ is a solution to the PDE given in this problem. Therefore, by linearity and our result from the last problem, we have

$$\begin{aligned} u(x,t) = u_1(x,t) + u_2(x,t) &= -\frac{1}{2}\mathrm{erf}\left(\frac{-1-x}{\sqrt{4kt}}\right) + \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) - \frac{1}{2} \\ &= -\frac{1}{2}\mathrm{erf}\left(\frac{-1-x}{\sqrt{4kt}}\right) + \frac{1}{2}\mathrm{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) \\ &= \frac{1}{2}\mathrm{erf}\left(\frac{1+x}{\sqrt{4kt}}\right) - \frac{1}{2}\mathrm{erf}\left(\frac{x-1}{\sqrt{4kt}}\right) \end{aligned}$$

since the error function is odd.

Problem 4. Solve the following PDE

$$\begin{cases} u_t = k u_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = x & x \in \mathbb{R}. \end{cases}$$

Solution 4. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} y \, dy.$$

If we set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, X$. In particular, since the mean of this normal distribution is x, we have

$$u(x,t) = \mathbb{E}\, X = x.$$

Problem 5. Solve the following PDE

$$\begin{cases} u_t = k u_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = x^2 & x \in \mathbb{R}. \end{cases}$$

Solution 5. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} y^2 \, dy.$$

If we set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, X^2$. In particular, since the variance of this normal distribution is $2kt$ and the mean is x, we have

$$u(x,t) = \mathbb{E}\, X^2 = \mathrm{Var}(X) + (\mathbb{E}\, X)^2 = 2kt + x^2,$$

where we used the fact $\mathrm{Var}(X) = \mathbb{E}\, X^2 - (\mathbb{E}\, X)^2$.

Problem 6. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u(x,0) = \delta(x-a) & x \in \mathbb{R}. \end{cases}$$

Solution 6. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} \delta(y-a)\, dy.$$

By definition of the delta function, $\int_{\mathbb{R}} \Psi(y)\delta(y-a)\,dy = \Psi(a)$, we have

$$u(x,t) = \frac{e^{-\frac{(a-x)^2}{4kt}}}{\sqrt{4\pi kt}}.$$