1 Stochastic Differential Equations

Stochastic differential equations is the stochastic calculus counterpart of ordinary differential equations. In these types of problems we are given a differential equations involving stochastic processes, and we want to find the stochastic process that satisfies these differential equations. In the following subsections, we go over some applications of Itô's lemma to solve some classical SDEs.

1.1 Geometric Brownian Motion

We want to find a stochastic process that models the behavior of a stock price. Notice that W_t is not a realistic model of a stock price since it can take both positive and negative values. At the very least, a model for a stock price should always be non-negative, and its movement should be proportional to its current value.

Definition 1.1. Given a standard Brownian motion $\{W_t\}_{t\geq 0}$, a stochastic process $\{S_t\}_{t\geq 0}$ is called a **geometric Brownian motion** if it satisfies the stochastic differential equation

$$\mathrm{d}S_t = \sigma S_t \,\mathrm{d}W_t + \mu S_t \,\mathrm{d}t \tag{1}$$

for some constants $\sigma \geq 0$ and $\alpha \in \mathbb{R}$.

In (1), the term $\mu S_t \,\mathrm{d}t$ induces a proportionally constant rate of average growth. The term $\sigma S_t \,\mathrm{d}W_t$ induces proportionally constant random fluctuations. This is a reasonable model for a stock price for two reasons:

- The microscopic fluctuations of an asset price over a very short time interval $[t, t+\varepsilon]$ are approximately proportional to S_t,
- the microscopic fluctuations of the value of an investment of x units of cash made at time t will only depend on x and not on S_t.

This implies that for an x amount of cash, one can buy $\xi = x/S_t$ shares. The fluctuation of the value will thus be

$$\text{instantaneous change in value} = \xi \,\mathrm{d}S_t = x \cdot \frac{\mathrm{d}S_t}{S_t} = x \cdot \sigma \,\mathrm{d}W_t + x \cdot \mu \,\mathrm{d}t.$$

The SDE in (1) can be explicitly solved (Problem 1.1).

Proposition 1.2

The solution to (1) is

$$S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}.$$

Remark 1.3. This equation should remind us of the deterministic growth equation. Indeed, the ODE

$$df(t) = kf(t)\,dt \iff \frac{df}{dt} = kf(t)$$

is solved by separating variables, which gives us $f(t) = f(0)e^{kf(t)}$ which agrees with the solution of geometric Brownian motion up to a correction term.

We often call μ the drift and σ the volatility. By taking logarithms, we have

$$\log S_t = \log S_0 + \sigma W_t + \Big(\mu - \frac{\sigma^2}{2}\Big)t,$$

where the right-hand side is normally distributed with mean $\log S_0 + (\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$. Hence, S_t has a **log-normal distribution**. Furthermore, we expect to see fluctuations that are constant in time and a linear trend in log stock prices. We have shown several times that

$$Z_t = S_0 e^{\sigma W_t - \frac{\sigma^2}{2} t}$$

is a martingale. This means that the $S_0 e^{\sigma W_t - \frac{\sigma^2}{2} t}$ has no tendency to go up or down so

$$S_t = S_0 e^{\sigma W_t - \frac{\sigma^2}{2} t} e^{\mu t}$$

has a mean rate of return of the stock is μ. A generalization of geometric Brownian motion is given in Problem 1.1.

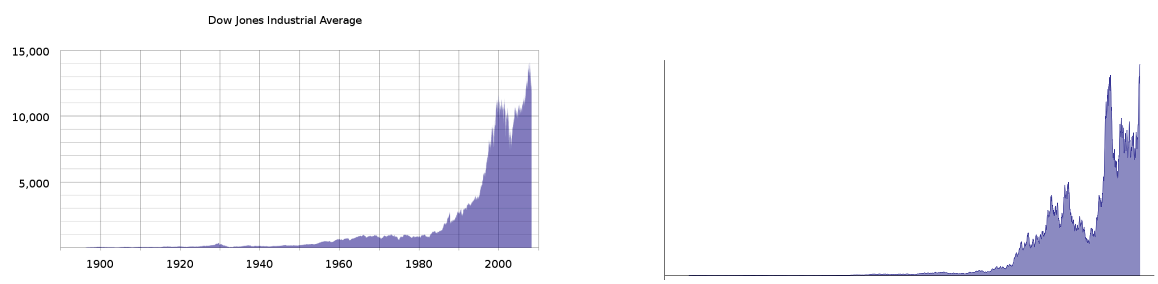

Figure 1: DJIA in absolute units vs. $S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$. For the DJIA, fluctuations are much stronger when the value of the index is high

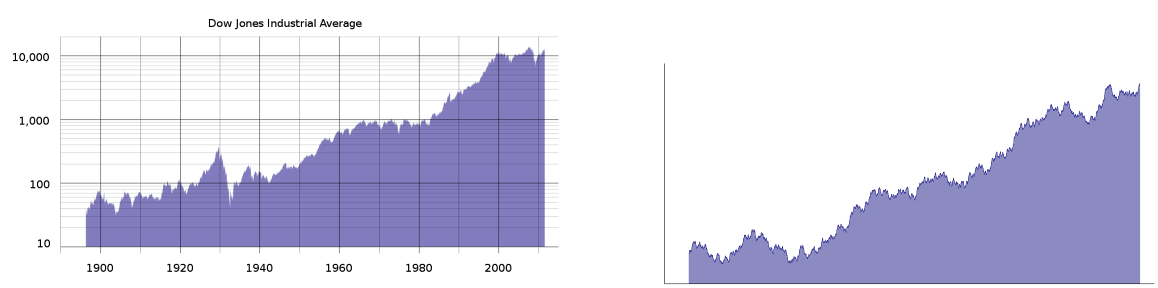

Figure 2: DJIA in logarithmic units vs. $Y_t = Y_0 + \sigma W_t + (\mu - \frac{\sigma^2}{2})t$. The historic development of the DJIA on a logarithmic scale has a linear trend.

1.2 Vasicek Interest Rate Model

We now introduce a stochastic process that models the behavior of interest rates.

Definition 1.4. Given a standard Brownian motion $\{W_t\}_{t\geq 0}$, a stochastic process $\{R_t\}_{t\geq 0}$ is called an **interest rate process** if it satisfies the stochastic differential equation

$$\mathrm{d}R_t = \sigma\,\mathrm{d}W_t + (\alpha - \beta R_t)\,\mathrm{d}t \tag{2}$$

for some constants $\sigma, \alpha, \beta > 0$.

This is a mean reverting model for the interest rate. In particular, if $R(t) > \frac{\alpha}{\beta}$ then the drift is negative so $R(t)$ is pushed down towards $\frac{\alpha}{\beta}$. The opposite happens if $R(t) < \frac{\alpha}{\beta}$. One downside of this model is that the interest rates can be negative. More complicated interest rate models exist, but they do not necessarily admit a closed form. The SDE in (2) can be solved explicitly (Problem 1.2)

Proposition 1.5

The solution to (2) is

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dW_s$$

Remark 1.6. This equation should remind us of the deterministic linear equation. Indeed, the ODE

$$df(t) = (\alpha - \beta f(t))dt \iff \frac{df}{dt} - \beta f = \alpha$$

is solved using the integrating factor $e^{-\beta t}$, which gives us $f(t) = e^{-\beta t}f(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t})$ which agrees with the solution of interest model up to a correction term.

1.3 Example Problems

1.3.1 Proofs of Main Results

Problem 1.1. Solve the SDE

$$\mathrm{d}S_t = \sigma S_t\,\mathrm{d}W_t + \mu S_t\,\mathrm{d}t.$$

Solution 1.1. Recall that the solution to the ODE

$$\frac{dy}{dt} = ky \tag{3}$$

can be solved by separating the vairables,

$$\frac{dy}{y} = kdt \implies d\ln(y) = kdt \implies \ln(y(t)) - \ln(y(0)) = kt \implies y(t) = y(0)e^{kt}.$$

The SDE looks very similar to (3). We can apply Itô's lemma (using the shorthand in differentials) to see that

$$\begin{aligned} d\log(S_t) &= \frac{1}{S_t}\,dS_t - \frac{1}{2}\frac{1}{S_t^2}dS_t dS_t \\ &= \sigma\,\mathrm{d}W_t + \mu\,\mathrm{d}t - \frac{1}{2}\sigma^2\,\mathrm{d}t \end{aligned}$$

because

$$\mathrm{d}S_t = \sigma S_t\,\mathrm{d}W_t + \mu S_t\,\mathrm{d}t$$

and

$$dS_t dS_t = \sigma^2 S_t^2\,\mathrm{d}W_t\mathrm{d}W_t + 2\mu\sigma S_t^2\,\mathrm{d}W_t\mathrm{d}t + \mu^2 S_t^2\,\mathrm{d}t\mathrm{d}t = \sigma^2 S_t^2\,\mathrm{d}t.$$

Therefore, we can integrate to see that

$$\log(S_t) - \log(S_0) = \sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t \implies S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}.$$

Problem 1.2. Solve the SDE

$$\mathrm{d}R_t = \sigma\,\mathrm{d}W_t + (\alpha - \beta R_t)\,\mathrm{d}t$$

Solution 1.2. Recall that the solution to the linear ODE

$$\frac{dy}{dt} + \beta y = \alpha \tag{4}$$

can be solved using the integrating factor $e^{\beta t}$

$$\frac{dy}{dt}e^{\beta t} + \beta y e^{\beta t} = \alpha e^{\beta t} \implies d(ye^{\beta t}) = \alpha e^{\beta t} \implies ye^{\beta t} - y(0) = \frac{\alpha}{\beta}(e^{\beta t} - 1)$$

so

$$y(t) = e^{-\beta t}y(0) + \frac{\alpha}{\beta}(1 - e^{\beta t})$$

The SDE looks very similar to (4). We can apply Itô's lemma (using the shorthand in differentials) to $f(x,t) = xe^{\beta t}$

$$\begin{aligned} d(R_t e^{\beta t}) &= \beta R_t e^{\beta t}\,dt + e^{\beta t}\,dR_t + \frac{1}{2}\cdot 0\,dR_t dR_t \\ &= \beta R_t e^{\beta t}\,dt + \sigma e^{\beta t}\,dW_t + (\alpha - \beta R_t)e^{\beta t}\,dt \\ &= \sigma e^{\beta t}\,dW_t + \alpha e^{\beta t}\,dt \end{aligned}$$

because

$$\mathrm{d}R_t = \sigma\,\mathrm{d}W_t + (\alpha - \beta R_t)\,\mathrm{d}t$$

Therefore, we can integrate to see that

$$R_t e^{\beta t} - R_0 = \sigma \int_0^t e^{\beta s}\,dW_s + \frac{\alpha}{\beta}(e^{\beta t} - 1)$$

which can be rearranged to give

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dW_s.$$

2 Black–Scholes Model

We now derive one of the fundamental results in option pricing. The **Black–Scholes model** gives the price of a European call option.

Let S_t be a stock price. Suppose that it can be modeled by a Geometric–Brownian motion, that is, S_t satisfies the SDE (1)

$$dS_t = \sigma S_t\, dW_t + \mu S_t\, dt$$

which has explicit solution by Proposition 1.2

$$S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}.$$

Suppose that the risk free interest rate is given by r. An option is a financial contract that gives the holder a choice of what to do at a specific date called the maturity. A European call option gives the holder a choice to buy a particular stock S at price K at maturity. That is at maturity, the holder will make

$$\begin{cases} S - K & S > K \\ 0 & S < K \end{cases}$$

since if $S > K$ the holder can resell the stock on the market to make $S - K$, and if $S \leq K$, then the holder will not exercise the option .

Mathematically, we can think of the call option as a function of the stock price with fixed terminal value.

Definition 2.1. Let $T > 0$. A **European call option** with **strike price** K and **maturity** T is a function of a stochastic process with **payoff**

$$c(T, S_T) = (S_T - K)_+ = \max(S_t - K, 0).$$

This option lowers the risk of investing in a stock since it guarantees the maximum price you can buy a stock at. Of course, such a contract has some value, so our goal is to determine what the fair value of buying this contract.

Let $c(t, x)$ denote the value of the call option at time t when the value of the stock satisfies $S_t = x$. We do not have access to S_t in the future, but by knowing $c(t, x)$, we will get a formula for the call option as a function of the future stock price. The Black–Scholes formula gives the price of the call option at all times as a function of the underlying stocks drift μ, volatility σ^2, the risk free interest rate r, and the time to maturity $T - t$.

Theorem 2.2 (*Price of a European Call*)

The price of the European call option $c(t, x)$ at time $t < T$ with stock price $S_t = x$ is,

$$c(t, x) = x\Phi(d_+(T - t, x)) - Ke^{-r(T-t)}\Phi(d_-(T - t, x))$$

where Φ is the cdf of a standard normal distribution and

$$d_\pm(t, x) = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{x}{K} + (r \pm \frac{\sigma^2}{2})T\right].$$

As a simple sanity check, we have that

$$\lim_{t\to T} c(t, S_t) = \begin{cases} S_T\Phi(+\infty) - K\Phi(+\infty) & S_T > K \\ S_T\Phi(-\infty) - K\Phi(-\infty) & S_T < K \end{cases} = \begin{cases} S_T - K & S_T > K \\ 0 & S_T < K \end{cases}$$

since $\lim_{t\to T} d_\pm(t, S_t) = +\infty$ if $S_T > K$ and $\lim_{t\to T} d_\pm(t, S_t) = -\infty$ if $S_t < K$.

2.1 Derivation of the Black–Scholes Model

Our strategy to determine the price is to design an financial instrument called a **replicating portfolio** made up of stocks and risk free bonds. If there are **no arbitrage** opportunities, then the price of the replica portfolio and the European call option should have the same value.

2.1.1 The Evolution of the Replicating Portfolio

We want to make a replicating portfolio that is self financing, i.e. after the initial purchase of the replicating portfolio, we are unable to add or withdraw any money until maturity.

Suppose that we can buy a risk free bond B_t with interest rate r. The price of the bond satisfies the ODE

$$dB_t = rB_t\,dt$$

which has solution $B_t = B_0 e^{rt}$. Let $X(t)$ denote the value of a self-financing portfolio. At time t, we hold $\Delta(t)$ shares of the stock at time t, so the value of the portfolio in stocks is $\Delta(t)S_t$. The remaining balance $B_t = X_t - \Delta(t)S_t$ is invested in bonds. In summary,

$$X(t) = \Delta(t)S_t + B_t = \Delta(t)S_t + (X(t) - \Delta(t)S_t).$$

The differential satisfied by the replicating portfolio is

$$\begin{aligned} dX(t) = \Delta(t)\,dS_t + \,dB_t &= \Delta(t)\,dS_t + r(X(t) - \Delta(t)S_t)\,dt \\ &= \Delta(t)(\sigma S_t\,dW_t + \mu S_t\,dt) + r(X(t) - \Delta(t)S_t)\,dt \\ &= rX(t)\,dt + \Delta(t)(\mu - r)S_t\,dt + \Delta(t)\sigma S_t\,dW_t \end{aligned} \tag{5}$$

Remark 2.3. The first term can be interpreted appearing above is the underlying rate of return on the portfolio, the second term is the risk premium for investing in a stock, and the last term is the volatility.

2.1.2 The Evolution of the Call Option

Recall that $c(t, S_t)$ is the value of the call option at time t and $c(t, x)$ is a non-random function of the stock price S_t. By Itô formula applied to $c(t, x)$ and (1) we have

$$\begin{aligned} \mathrm{d}c(t,S_t) &= c_t(t,S_t)\mathrm{d}t + c_x(t,S_t)\mathrm{d}S_t + \frac{1}{2}c_{xx}(t,S_t)\mathrm{d}S_t\mathrm{d}S_t \\ &= \left(c_t(t,S_t) + c_x(t,S_t)\mu S_t + \frac{\sigma^2}{2}c_{xx}(t,S_t)S_t^2\right)\mathrm{d}t + c_x(t,S_t)\sigma S_t\,\mathrm{d}W_t. \end{aligned} \tag{6}$$

2.1.3 Equating the Evolutions

We will require that the present value of the replicating portfolio to be equal to the value of the option at all t, since if it were not the case then an arbitrage opportunity exists (we can buy the lower value one and sell the higher value one to make risk free profit). That is, we need that for all $t < T$,

$$e^{-rt}X(t) = e^{-rt}c(t, S_t).$$

This happens if its initial value $X(0) = c(0, S_0)$ and its discounted evolution satisfy

$$d(e^{-rt}X(t)) = d(e^{-rt}c(t, S_t)).$$

Applying Itô's lemma (or the product rule), this implies that

$$-re^{-rt}X(t)\,dt + e^{-rt}\,dX(t) = -re^{-rt}c(t, S_t)\,dt + e^{-rt}\,dc(t, S_t).$$

where we used the fact that $dtdX(t) = 0$ and $dtdc(t, S_t) = 0$. Substituting the differentials computed in (5) and (6) implies that

$$\begin{aligned}
&- rX(t)\,dt + \left(rX(t)\,dt + \Delta(t)(\mu - r)S_t\,dt + \Delta(t)\sigma S_t\,dW_t\right)\\
&= -rc(t, S_t)\,dt + \left(c_t(t, S_t) + c_x(t, S_t)\mu S_t + \frac{\sigma^2}{2}c_{xx}(t, S_t)S_t^2\right)\mathrm{d}t + c_x(t, S_t)\sigma S_t\,\mathrm{d}W_t.
\end{aligned}$$

In order for the coefficients of the dW_t terms to cancel, we must have

$$\Delta(t) = c_x(t, S_t)$$

which is called the *delta-hedging rule*. Substituting this value for $\Delta(t)$ simplifies the coefficients of the dt term to give

$$-rc(t, S_t) + c_t(t, S_t) + rS_t c_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t) = 0.$$

which is called the **Black–Scholes PDE**. Solving this PDE subject to the terminal condition that $c(T, S_T) = (S_T - K)_+$ will give us the formula for $c(t, x)$,

Remark 2.4. Investing $X(0) = c(0, S_0)$ at the start and buying $c_x(t, S_t)$ units of the stock in the portfolio while investing the remainder into a risk free bond will replicate the call option.

2.1.4 Solving the Black–Scholes PDE

Since the Black–Scholes PDE must be valid for all values of S_t, it suffices to solve

$$\begin{cases} -rc(t,x) + c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = 0 & 0 < t < T, x > 0 \\ c(T,x) = (x-K)_+ & x > 0 \end{cases} \tag{7}$$

To find a unique solution, we also have to add some boundary conditions. At $x = 0$, (7) implies that

$$-rc(t, 0) = c_t(t, 0) \qquad 0 < t < T$$

This is a standard growth ODE, so it has the solution $c(t, 0) = c(0, 0)e^{-rt}$. Since $c(T, 0) = (0-K)_+ = 0$, we must have

$$0 = c(T, 0) = c(0, 0)e^{-rT} \implies c(0, 0) = 0,$$

which means that we require $c(t, x) = 0$ to be 0 for all $t \in [0, T]$. To figure out the boundary condition at $x = \infty$, notice that when x is very large, then the payout at time T is very likely to be $x - K$. To replicate a portfolio that pays $S_T - K$ at maturity at time t, you have to buy one stock at price S_t stock and sell the present value $e^{-r(T-t)}K$ of risk free bond. In particular, we must have

$$\lim_{x\to\infty}[c(t, x) - (x - e^{-r(T-t)}K)] = 0.$$

Therefore, we need to solve the PDE with boundary conditions

$$\begin{cases} -rc(t,x) + c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = 0 & 0 < t < T, x > 0 \\ c(T,x) = (x-K)_+ & x > 0 \\ c(t,0) = 0 & 0 \le t \le T \\ \lim_{x\to\infty}[c(t,x) - (x - e^{-r(T-t)}K)] = 0 & 0 \le t \le T. \end{cases} \tag{8}$$

From here, one can check that the function in Theorem 2.2 solves this PDE by differentiating. We provide a probabilistic argument for how to compute the present value $c(0, x)$ in Problem 2.1.

2.2 Example Problems

Problem 2.1. Find the solution to (8) at $t = 0$. In particular, find $c(0, S_0)$.

Solution 2.1. We first do some change of variables to simplify this problem. If we define

$$u(t,x) = e^{r(T-t)}c(t, e^{-r(T-t)}x), \tag{9}$$

then one has that by (8)

$$\begin{aligned}
&u_t(t,x) + \frac{\sigma^2}{2}x^2 u_{xx}(t,x) \\
&= -re^{r(T-t)}c(t, e^{-r(T-t)}x) + e^{r(T-t)}c_t(t, e^{-r(T-t)}x) + e^{r(T-t)}(e^{-r(T-t)}x)c_x(t, e^{-r(T-t)}x) \\
&\quad + \frac{\sigma^2}{2}e^{r(T-t)}(xe^{-r(T-t)})^2 c_{xx}(t, e^{-r(T-t)}x) \\
&= e^{r(T-t)}(-rc(t, e^{-r(T-t)}x) + c_t(t,x) + r(e^{-r(T-t)}x)c_x(t, e^{-r(T-t)}x) + \frac{1}{2}\sigma^2(e^{-r(T-t)}x)^2 c_{xx}(t, e^{-r(T-t)}x)) \\
&= 0
\end{aligned}$$

and

$$u(T,x) = e^{r(T-T)}c(t, e^{r(T-T)}x) = c(T,x) = (x-K)_+$$

Therefore, we can conclude that u satisfies the PDE

$$\begin{cases} u_t(t,x) + \frac{\sigma^2}{2}x^2 u_{xx}(t,x) = 0 & 0 < t < T, x > 0 \\ u(T,x) = (x-K)_+. \end{cases} \tag{10}$$

We will solve u using tools from probability theory. Notice that using (1) that $u(t, S_t)$ satisfies the differential

$$\begin{aligned}
d(u(t,S_t)) &= u_t(t,S_t)\,dt + u_x(t,S_t)\,dS_t + \frac{1}{2}u_{xx}(t,S_t)dS_t dS_t \\
&= u_t(t,S_t)\,dt + u_x(t,S_t)(\sigma S_t\,dW_t + \mu S_t\,dt) + \frac{1}{2}\sigma^2 S_t^2 u_{xx}(t,S_t)\,dt \\
&= u_x(t,S_t)(\sigma S_t\,dW_t + \mu S_t\,dt)
\end{aligned}$$

since by (10)

$$u_t(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 u_{xx}(t,S_t) = 0.$$

Furthermore, we must also have that (10) holds for all μ, so it must hold for $\mu = 0$. In this simplified case, we have that

$$du(t,S_t) = u_x(t,S_t)\sigma S_t\,dW_t \implies u(t,S_t) - u(0,S_0) = \int_0^t u_x(s,S_s)\sigma S_s\,dW_s$$

so in particular, $u(t, S_t)$ is a martingale since $u(0, S_0)$ is not random. Therefore, we must have that for all $t \in [0, T]$,

$$\mathbb{E}[u(t,S_t)] = u(0,S_0)$$

Taking $t = T$ implies that by Proposition 1.2

$$u(0,S_0) = \mathbb{E}[(S_T - K)_+] = \mathbb{E}[(S_0 e^{\sigma W_T - \frac{\sigma^2}{2}T} - K)_+]$$

Since $W_t \sim N(0,t)$, this expected value can be computed by explicitly by completing the square and a change of variables to give

$$\begin{aligned}
u(0,S_0) &= \mathbb{E}[(S_0 e^{\sigma W_T - \frac{\sigma^2}{2}T} - K)_+] \\
&= \int_{-\infty}^{\infty} \left(S_0 e^{\sigma x - \frac{\sigma^2}{2}T} - K\right)_+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}\,dx \\
&= \int_{-\tilde{d}_-(T,S_0)}^{\infty} \left(S_0 e^{\sigma\sqrt{T}x - \frac{\sigma^2}{2}T} - K\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\
&= S_0 e^{-\frac{\sigma^2}{2}T} e^{\frac{\sigma^2}{2}T} \int_{-\tilde{d}_-(T,S_0)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sqrt{T}\sigma)^2}{2}}\,dx - \int_{-\tilde{d}_-(T,S_0)}^{\infty} K \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\
&= S_0 \int_{-\tilde{d}_-(T,S_0)-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx - \int_{-\tilde{d}_-(T,S_0)}^{\infty} K \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\
&= S_0 \Phi(\tilde{d}_+(T,S_0)) - K\Phi(\tilde{d}_-(T,S_0))
\end{aligned}$$

where Φ is the cdf of a standard normal distribution and

$$\tilde{d}_\pm(T,x) = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{K} \pm \frac{\sigma^2}{2}T\right].$$

Using the fact that $c(t,x) = e^{-r(T-t)}u(t, e^{r(T-t)}x)$ by (9) implies that

$$c(0,S_0) = e^{-rT}u(0, e^{rT}S_0) = S_0\Phi(d_+(T,x)) - Ke^{-rT}\Phi(d_-(T,S_0))$$

where Φ is the cdf of a standard normal distribution and

$$d_\pm(t,S_0) = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{K} + (r \pm \frac{\sigma^2}{2})T\right].$$