1 Classifying PDEs

The *order* of the PDE is the order of the largest derivative in the PDE. Let F be a nonlinear function. There are 4 major classifications of kth order PDE:

1. Linear: A PDE is *linear* if the coefficients in front of the partial derivative terms are all functions of the space variable $\vec{x} \in \mathbb{R}^n$,
$$\sum_{|\alpha|\le k} a_\alpha(\vec{x}) D^\alpha u = f(\vec{x}).$$
A linear PDE is *homogeneous* if there is no term that depends only on the space variables, i.e. $f(\vec{x}) \equiv 0$. Likewise, a linear PDE is *inhomogeneous* if $f(\vec{x}) \neq 0$.

2. Semilinear: A PDE is *semilinear* if the coefficients in front of the highest order partial derivative terms are all functions of the space variable $\vec{x} \in \mathbb{R}^n$,
$$\sum_{|\alpha|=k} a_\alpha(\vec{x}) D^\alpha u + F(D^{k-1}u, \dots, Du, u, \vec{x}) = 0.$$

3. Quasilinear: A PDE is *quasilinear* if the coefficients in front of the highest order partial derivative terms are all functions of the space variable $\vec{x} \in \mathbb{R}^n$ or lower derivative terms,
$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, Du, u, \vec{x}) D^\alpha u + F(D^{k-1}u, \dots, Du, u, \vec{x}) = 0.$$

4. Fully Nonlinear: A PDE is *fully nonlinear* if it is not of the above 3 forms. That is, the PDE is fully nonlinear if it depends nonlinearly on the highest order partial derivative terms,
$$F(D^k u, \dots, Du, u, \vec{x}) = 0.$$

Problem 1.1. Consider first order equations and determine if they are linear homogeneous, linear inhomogeneous, or nonlinear; for nonlinear equations, indicate if they are also semilinear, or quasilinear:

$$u_y + x u_x - u = 0, \tag{1}$$
$$u_y + u_x - u^2 = 0, \tag{2}$$
$$u_y + u u_x + x = 0. \tag{3}$$

Solution 1.1.

(1) Since the coefficients in front of u_y, u_x, and u are functions of x and y only, the equation is linear. There is also no term that depends only on x or y, so it is homogeneous. To prove that the equation is linear, notice that

$$\begin{aligned} L[au+bv] &= (au+bv)_y + x(au+bv)_x - (au+bv) \\ &= a(u_y + xu_x - u) + b(v_y + xv_x - v) \\ &= aL[u] + bL[v]. \end{aligned}$$

(2) There is a u^2 term, so the function is nonlinear. However, the coefficients of the highest order terms are functions of x and y, so the function is semilinear. To prove that the operator is nonlinear, we show that the scaling property fails for a nice function such as $u(x,y) = x$,

$$L[2x] = (2x)_y + (2x)_x - (2x)^2 = 2 - 4x^2 \neq 2 - 2x^2 = 2((x)_y + (x)_x - x^2) = 2L[x].$$

(3) There is a uu_x term, so the function is nonlinear. However, the coefficients of the highest order terms are functions of x, y and u, so the function is quasilinear.

2 Solving Basic PDEs

We review some techniques from ODEs. The only difference in these multivariable examples is the integration constant is now a function of the other variables.

Problem 2.1. Find the general solutions to the following equations:

$$u_{xxy} = 0, \tag{1}$$
$$u_{xyz} = \sin(x) + \sin(y)\sin(z). \tag{2}$$

Solution 2.1.

(1) We integrate out each of the partial derivatives by treating the remaining variables as constants and introduce an integration constant in each step,

$$\begin{aligned} u_{xxy} &= 0 \\ \Rightarrow\ u_{xx} &= f_{xx}(x) \\ \Rightarrow\ u_x &= f_x(x) + g(y) \\ \Rightarrow\ u &= f(x) + xg(y) + h(y) \end{aligned}$$

where $f(x)$ is a twice differentiable function.

(2) We integrate out each of the partial derivatives by treating the remaining variables as constants and introduce an integration constant in each step,

$$\begin{aligned} u_{xyz} &= \sin(x) + \sin(y)\sin(z) \\ \Rightarrow\ u_{xy} &= z\sin(x) - \sin(y)\cos(z) + f_{xy}(x,y) \\ \Rightarrow\ u_x &= yz\sin(x) + \cos(y)\cos(z) + f_x(x,y) + g_x(x,z) \\ \Rightarrow\ u &= -yz\cos(x) + x\cos(y)\cos(z) + f(x,y) + g(x,z) + h(y,z) \end{aligned}$$

where $f(x,y)$ is differentiable in x and y, and $g(x,z)$ is differentiable in x.

Problem 2.2. Find the general solution to

$$u_{xy} = 2u_x + e^{x+y}.$$

Solution 2.2. To simplify notation, we define $v(x,y) = u_x(x,y)$. Treating x as a constant, we first solve the ODE

$$v_y = 2v + e^{x+y} \implies v_y - 2v = e^{x+y}.$$

This is a linear inhomogeneous ODE in y, so it can be solved using the integrating factor

$$I(y) = e^{\int -2\,dy} = e^{-2y}.$$

We multiply both sides by e^{-2y} and integrate to solve for v,

$$e^{-2y}v_y - 2e^{-2y}v = e^{x-y} \Rightarrow (e^{-2y}v)_y = e^{x-y} \Rightarrow e^{-2y}v = -e^{x-y} + f(x) \Rightarrow v = -e^{x+y} + e^{2y}f_x(x).$$

Since $v = u_x$, we can now integrate in x to recover u,

$$u_x = -e^{x+y} + e^{2y}f_x(x) \implies u = -e^{x+y} + f(x)e^{2y} + g(y),$$

where $f(x)$ is a differentiable function.