Week 1: ODE Review

Introduction

We summarize the main ODEs we will see in this course

1. **Separable Equations:** An ODE of the form

$$\frac{dy}{dx} = f(x)(g(y))^{-1}$$

is separable. The ODE is solved by integrating

$$\int f(x)\,dx = \int g(y)\,dy.$$

1*. **Homogeneous Equations (Almost Separable):** An ODE of the form

$$\frac{dy}{dx} = f(x,y)$$

is homogeneous of order n if

$$f(tx,ty) = t^n f(x,y).$$

The ODE is solved by using the change of variable $u = y/x$ or $u = x/y$, which reduces the ODE to a separable equation.

2. **First Order Linear Equations:** An ODE of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is a first order linear ODE. The ODE is solved by multiplying both sides by an integrating factor $I(x) = e^{\int P(x)\,dx}$ and integrating both sides.

2*. **Bernoulli Equations (Almost Linear):** An ODE of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a Bernoulli equation. The ODE is solved by using the change of variable $z = y^{1-n}$, which reduces the ODE to a first order linear equation.

3. **Homogeneous Second Order Constant Coefficients:** An ODE of the form

$$ay'' + by' + cy = 0$$

is a homogeneous second order constant coefficient ODE. The ODE is solved by finding the roots r_1 and r_2 of the characteristic polynomial

$$C(r) = ar^2 + br + c = 0$$

and the general form of the solution is given by

$$y(x) = \begin{cases} C_1 e^{r_1 x} + C_2 e^{r_2 x} & r_1, r_2 \in \mathbb{R}, r_1 \neq r_2 \\ C_1 e^{rx} + C_2 x e^{rx} & r_1, r_2 \in \mathbb{R}, r_1 = r_2 = r \\ C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x) & r_1 = \alpha + i\beta, r_2 = \alpha - i\beta, \beta \neq 0 \end{cases}.$$

3*. **Euler Equations (Almost Constant Coefficient):** An ODE of the form

$$ax^2y'' + bxy' + cy = 0$$

are called Euler ODEs. The ODE is solved by finding the roots r_1 and r_2 of the characteristic polynomial

$$C(x) = ax(x-1) + bx + c = 0$$

and the general form of the solution is given by

$$y(x) = \begin{cases} C_1x^{r_1} + C_2x^{r_2} & r_1, r_2 \in \mathbb{R}, r_1 \neq r_2 \\ C_1x^r + C_2\log(x)x^r & r_1, r_2 \in \mathbb{R}, r_1 = r_2 = r \\ C_1x^\alpha\cos(\beta\log x) + C_2x^\alpha\sin(\beta\log x) & r_1 = \alpha + i\beta, r_2 = \alpha - i\beta, \beta \neq 0 \end{cases}.$$

Problems:

Problem 1. Find a general solution of

$$\frac{dy}{dx} = \frac{-\sqrt{1+y^2}}{x\log x}.$$

Solution 1. We first separate the variables

$$\frac{1}{\sqrt{1+y^2}}\,dy + \frac{1}{x\log x}\,dx = 0,\ \ x \neq 0, 1.$$

Integrating, we have

$$\log|y + \sqrt{y^2+1}| + \log|\log|x|| = C$$

or equivalently (we can remove the absolute value by absorbing the signs into C)

$$\log(|x|)\cdot(y + \sqrt{y^2+1}) = C.$$

Note: The integral with respect to y is a bit tricky. First using trigonometric substitution with $y = \tan\theta$,

$$\int \frac{1}{\sqrt{1+y^2}}\,dy = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \ln|\sec\theta + \tan\theta| = \log|y + \sqrt{y^2+1}|.$$

Problem 2. Solve the ODE

$$\frac{dy}{dx} = \frac{xe^{y/x} + y}{x}, \quad y(1) = 0.$$

Solution 2. We have

$$\frac{dy}{dx} = \frac{xe^{y/x} + y}{x}$$

is a homogeneous equation of order $n = 0$. Using the substitution $y = ux$, we have the following separable equation

$$x\frac{du}{dx} + u = e^u + u \Rightarrow \frac{du}{dx} = \frac{e^u}{x}.$$

Integrating, we see for $x \neq 0$

$$-e^{-u} = \log|x| + C \Rightarrow -e^{-y/x} = \log|x| + C.$$

Plugging in the initial data $y(1) = 0$, we have

$$-e^0 = 0 + C \Rightarrow C = -1.$$

The final particular solution is

$$e^{-y/x} + \log x = 1, \; e > x > 0.$$

The upper bound on x can be found by seeing $e^{-y/x} = 1 - \log x$ and the right hand side must be positive.

Problem 3. Find a general solution of

$$y' + 2y = \frac{3}{4}e^{-2x}.$$

Solution 3. This is a first order linear equation. We first find the integrating factor

$$I(x) = e^{\int 2\,dx} = e^{2x}.$$

Using the integrating factor, we see

$$e^{2x}y = \int \frac{3}{4}\,dx \Rightarrow e^{2x}y = \frac{3}{4}x + C \Rightarrow y = \frac{3}{4}xe^{-2x} + Ce^{-2x}.$$

Problem 4. Find a general solution of

$$xy' + y = y^2 \log x.$$

Solution 4. We have a Bernoulli equation. Rearranging terms, we see

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \log x.$$

Letting $z = \frac{1}{y}$, we have

$$\frac{dz}{dx} - \frac{1}{x}z = -\frac{\log x}{x}.$$

This is a first order linear equation, so first solving for the integrating factor

$$I(x) = e^{-\int \frac{1}{x}\,dx} = \frac{1}{x}.$$

Using the integrating factor, we see

$$\frac{1}{x}z = -\int \frac{\log x}{x^2}\,dx \Rightarrow \frac{1}{xy} = \frac{\log(x)}{x} + \frac{1}{x} + C \Rightarrow 1 = y\log x + y + Cxy.$$

Problem 5. Find a general solution of

$$r^2R''(r) + rR'(r) - \left(\frac{n\pi}{\beta}\right)^2 R(r) = 0.$$

where $n > 0$ and $\beta > 0$.

Solution 5. This is an Euler ODE with characteristic equation $C(r) = r(r-1)+r-(\frac{n\pi}{\beta})^2$ with roots $r = \pm\frac{n\pi}{\beta}$. The general solution is of the form

$$R(r) = Ar^{\frac{n\pi}{\beta}} + Br^{-\frac{n\pi}{\beta}}$$

for some coefficients A and B.

Problem 6. Find a general solution of

$$X''(x) - \beta^2 X(x) = 0$$

where $\beta > 0$.

Solution 6. This is an second order constant coefficient ODE with characteristic equation $C(x) = x^2 - \beta^2$ with roots $r = \pm\beta$, which corresponds to the solution

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x)$$

for some coefficients A and B.