1 Stochastic processes and filtrations

A stochastic process can be thought of as a random function.

Definition 1.1. A **stochastic process** is a collection $(X_t)_{t\in\mathscr{T}}$ indexed by a **"time parameter"** $t \in \mathscr{T}$.
For fixed $\omega \in \Omega$, the mapping $t \to X_t(\omega)$ is called a **sample path** or **trajectory**.

Example 1.2. Typically, we will consider one of the following three cases:

$$\begin{aligned} &\mathscr{T} = \{0, 1, 2, \dots\} \qquad \text{(discrete time)} \\ &\mathscr{T} = [0, T] \qquad \text{(continuous time, finite time horizon } T > 0) \\ &\mathscr{T} = [0, \infty) \qquad \text{(continuous time, infinite time horizon)} \end{aligned}$$

A filtration is a model of the available historical information as time passes.

Definition 1.3. A **filtration** is a collection of σ-algebras $\{\mathscr{F}_t\}_{t\in\mathscr{T}}$ with the property

$$\mathscr{F}_s \subset \mathscr{F}_t \qquad \text{if } s, t \in \mathscr{T} \text{ are such that } s \le t.$$

Similarly to the σ-algebra generated by a random variable, we can associate a filtration with each stochastic process. The σ-algebra $\mathscr{F}_t$ should be the historical information about the evolution of the process up until time t.

Definition 1.4. Let $\{X_t\}_{t\in\mathscr{T}}$ be a stochastic process. Then the family of σ-algebras

$$\mathscr{F}_t := \sigma(X_s \,:\, s \in \mathscr{T},\, s \le t)$$

is called the filtration generated by the process, which called the **natural filtration**.

Example 1.5. For $n = 0, 1, 2, \dots$, let X_n be the number of insurance claims an insurance company receives on day n. Then, at day n, the insurance company will be able to observe the information encoded by

$$\mathscr{F}_n = \sigma(X_0, X_1, \dots, X_n).$$

Example 1.6. For $t \in [0, T]$, let X_t denote the price at which a given stock is traded at an exchange. Then, at time t, a trader will be able to observe the price trajectory X_s, $0 \le s \le t$, and thus have access to the information encoded in

$$\mathscr{F}_t = \sigma(X_s \,:\, 0 \le s \le t).$$

We consider the situation in which a filtration $\{\mathscr{F}_t\}_{t\in\mathscr{T}}$ is given *a priori* and we want to define a stochastic process associated with this.

Definition 1.7. We say that a stochastic process $\{X_t\}_{t\in\mathscr{T}}$ is **adapted** to the filtration $\{\mathscr{F}_t\}_{t\in\mathscr{T}}$ if X_t is $\mathscr{F}_t$-measurable for each $t \in \mathscr{T}$.

Example 1.8. In Example 1.6, let

$$M_t := \max_{0\le s\le t} X_s$$

denote the maximum stock price within the interval $[0, t]$ (often called the **running maximum** of the stochastic process $\{X_t\}$). Then $\{M_t\}_{t\in[0,T]}$ is adapted with respect to the natural filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$ of X_t. The future maximum,

$$\widehat{M}_t := \max_{t\le u\le T} X_u,$$

however, is typically not adapted to $\{\mathscr{F}_t\}_{t\in[0,T]}$ of X_t.

1.1 Example Problems

Problem 1.1. Let Ω be the outcome of the flips of two coins. Let X_1 denote the number of heads showing on the first coin, and let X_2 denote the number of heads showing on the second coin. Let $\mathcal{F}_1 = \sigma(X_1)$ and $\mathcal{F}_2 = \sigma(X_1, X_2)$. Show that $\mathcal{F}_1$ and $\mathcal{F}_2$ form a filtration.

Solution 1.1. In this problem, we explicitly compute the natural filtration. We have

$$\Omega = \{HH, TT, HT, TH\}$$

To compute $\mathcal{F}_1 = \sigma(X_1)$, recall that $\mathcal{F}_1$ contains all the preimages of $\mathscr{B}(\mathbb{R})$ under X_1, i.e.

$$\mathcal{F}_1 = \{ \underbrace{\emptyset}_{X^{-1}(\emptyset)}, \underbrace{\Omega}_{X^{-1}(\mathbb{R})}, \underbrace{\{HH, HT\}}_{X^{-1}(1)}, \underbrace{\{TH, TT\}}_{X^{-1}(0)} \}.$$

To compute $\mathcal{F}_2 = \sigma(X_1, X_2)$ recall that $\mathcal{F}_2$ contains all the preimages of $\mathscr{B}(\mathbb{R}^2)$ under $X = (X_1, X_2)$, i.e.

$$\mathcal{F}_2 = \{ \underbrace{\emptyset}_{X^{-1}(\emptyset)}, \underbrace{\Omega}_{X^{-1}(\mathbb{R}^2)}, \underbrace{\{HH\}}_{X^{-1}(1,1)}, \underbrace{\{TT\}}_{X^{-1}(0,0)}, \underbrace{\{TH\}}_{X^{-1}(0,1)}, \underbrace{\{HT\}}_{X^{-1}(1,1)}, \underbrace{\{HH, HT\}}_{X^{-1}(1,\mathbb{R})}, \underbrace{\{TH, TT\}}_{X^{-1}(0,\mathbb{R})}, \dots \} = \mathcal{P}(\Omega).$$

where $\mathcal{P}(\Omega)$ is the power set of Ω, since the information of the number of heads on each coin encodes all possible subsets of Ω. One can easily see that the power set is the right set since $\mathcal{F}_2$ contains all the elements of Ω.

Remark 1.9. Informally, $\mathcal{F}_1$ contains the information knowing how many heads on the first flip, and $\mathcal{F}_2$ contains the information knowing how many heads on the first and second flip. Clearly $\mathcal{F}_2$ contains more information so $\mathcal{F}_1 \subset \mathcal{F}_2$ which satisfies the criteria of a filtration.

Problem 1.2. Let S_t denote the stock price at times $t = 0, 1, 2$. Suppose that $S_0 = 100$. Suppose that at each time step, the stock will go up \$1 with probability p and down \$1 with probability $1 - p$. Furthermore, the changes of each price change is independent.

1. What is the underlying sample space Ω for this stochastic process?
2. Find $S_1, S_2, \sigma(S_1)$ and $\sigma(S_2)$.
3. What is the natural filtration for $S_{t t\in\{0,1,2\}}$?
4. Find $\mathbb{E}[S_1 \mid \mathcal{F}_0]$ and $\mathbb{E}[S_2 \mid \mathcal{F}_1]$.

Solution 1.2.

Part 1: Ω contains all possible movements of the stock. We have

$$\Omega = \{uu, dd, ud, du\}$$

where u denotes the stock going up and d denotes the stock going down.

Part 2: We have that S_1 is the function

$$S_1(\omega) = \begin{cases} 101 & \omega \in \{uu, ud\} \\ 99 & \omega \in \{dd, du\} \end{cases}$$

and

$$\sigma(S_1) = \{\emptyset, \Omega, \{uu, ud\}, \{dd, du\}\}.$$

Likewise, we have

$$S_2(\omega) = \begin{cases} 102 & \omega \in \{uu\} \\ 100 & \omega \in \{ud, du\} \\ 98 & \omega \in \{dd\} \end{cases}$$

and

$$\sigma(S_2) = \{\emptyset, \Omega, \{uu\}, \{dd\}, \{ud, du\}, \{uu, dd\}, \{uu, ud, du\}, \{dd, ud, du\}\}.$$

Notice that the elements of $\sigma(S_2)$ can never contain ud without du because it is impossible to know if the stock moved up and down or down and up knowing only its stock price at time 2.

Part 3: The natural filtration is

$$\mathcal{F}_0 = \sigma(S_0) = \{\emptyset, \Omega\}$$
$$\mathcal{F}_1 = \sigma(S_0, S_1) = \{\emptyset, \Omega, \{uu, ud\}, \{dd, du\}\}$$
$$\mathcal{F}_2 = \sigma(S_0, S_1, S_2) = \mathcal{P}(\Omega).$$

We can easily see that $\mathcal{F}_2$ is equal to $\mathcal{P}(\Omega)$ because it is generated by all the elements of Ω. In particular, the events $S_1 = 101, S_2 = 100$ and $S_1 = 99, S_2 = 100$ implies we can differentiate between ud and du, which was not possible only knowing S_2.

Part 4: Since $\mathcal{F}_0$ is the trivial σ-algebra (and therefore independent of S_1) we have

$$\mathbb{E}[S_1 \mid \mathcal{F}_0] = \mathbb{E}[S_1] = 101p + 99(1-p).$$

Since the σ-algebra correspond to the σ algebra generated by a random variable, we have

$$\mathbb{E}[S_2 \mid \mathcal{F}_1] = \mathbb{E}[S_2 \mid S_1].$$

First of all, we have

$$p_{S_2|S_1}(s_2 \mid 101) = \begin{cases} p & s_2 = 102 \\ 1-p & s_2 = 100 \\ 0 & s_2 = 98 \end{cases} \qquad p_{S_2|S_1}(s_2 \mid 99) = \begin{cases} 0 & s_2 = 102 \\ p & s_2 = 100 \\ 1-p & s_2 = 98 \end{cases}$$

so integrating with respect to the conditional probability implies that

$$\mathbb{E}[S_2 \mid S_1 = 101] = 102p + 100(1-p) \qquad \mathbb{E}[S_2 \mid S_1 = 99] = 100p + 98(1-p)$$

so

$$\mathbb{E}[S_2 \mid S_1](\omega) = \begin{cases} 102p + 100(1-p) & \omega = \{uu, ud\} \\ 100p + 98(1-p) & \omega = \{dd, du\} \end{cases}.$$

Remark 1.10. An alternative approach will be to recognize that S_1 takes only two values so a measurable function with respect to $\sigma(S_1)$ can only take two values

$$\mathbb{E}[S_2 \mid S_1] = \begin{cases} a_1 & \omega \in \{uu, ud\} \\ a_2 & \omega \in \{du, dd\}. \end{cases}$$

By the definition of conditional expectation, we must have

$$a_1 p^2 + a_1 p(1-p) = \mathbb{E}[\mathbb{E}[S_2 \mid S_1] \mathbb{1}_{uu,ud}] = \mathbb{E}[S_2 \mathbb{1}_{uu,ud}] = 102p^2 + 100p(1-p) \Rightarrow a_1 = 100 + 2p.$$

and

$$a_2(1-p)^2 + a_2 p(1-p) = \mathbb{E}[\mathbb{E}[S_2 \mid S_1] \mathbb{1}_{dd,du}] = \mathbb{E}[S_2 \mathbb{1}_{dd,du}] = 98(1-p)^2 + 100p(1-p) \Rightarrow a_2 = 98 + 2p.$$

which coincides with above.