Solving The Heat Equation

Consider the heat equation on the whole line

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \phi(x) & x \in \mathbb{R}. \end{cases}$$

The particular solution to this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} \phi(y)\, dy.$$

Problems

Heat Equation on $\mathbb{R}$

Problem 1. Solve the following PDE

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R},\ t > 0, \\ u(x,0) = e^{ax} & x \in \mathbb{R}, \end{cases}$$

where $a \in \mathbb{R}$.

Solution 1. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{ay}\, dy.$$

Since the initial data is an exponential, we can complete the square to simplify the integral. Completing the square in y, we have

$$\begin{aligned} -\frac{(y-x)^2}{4kt} + ay &= -\frac{y^2 - 2xy + x^2 - 4ktay}{4kt} \\ &= -\frac{y^2 - (2x + 4kta)y + x^2}{4kt} \\ &= -\frac{(y - (x + 2kta))^2 - (x + 2kta)^2 + x^2}{4kt} \\ &= -\frac{(y - (x + 2kta))^2}{4kt} + ax + a^2kt. \end{aligned}$$

Therefore, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+2kta))^2}{4kt} + ax + a^2kt}\, dy = e^{ax+a^2kt} \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+2kta))^2}{4kt}}\, dy}_{=1} = e^{ax+a^2kt}.$$

We used the fact that this Gaussian integral corresponds to the p.d.f. of a Gaussian random variable with mean $(x + 2kta)$ and variance $2kt$, so its integral over $\mathbb{R}$ is equal to 1.

Alternate Solution: The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{ay}\, dy.$$

If set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, e^{aX}$. In particular, using the formula for the moment generating function of the Gaussian random variable, we have

$$u(x,t) = \mathbb{E}\, e^{aX} = e^{a\mu + \frac{1}{2}a^2\sigma^2} = e^{ax + a^2kt}.$$

Heat Equation on the Half Line

Problem 2. Solve the following PDE

$$\begin{cases} u_t = k u_{xx} & 0 < x < \infty,\ t > 0, \\ u(x,0) = g(x) & 0 < x < \infty, \\ u(0,t) = h(t) & t > 0 \end{cases}$$

where

$$g(x) = \begin{cases} 1 & x < 1 \\ 0 & x \geq 1 \end{cases}, \qquad h(t) = 0.$$

Solution 2. We want to solve the heat equation on the half line with Dirichlet boundary conditions. We can use an odd reflection to extend the initial condition,

$$g_{odd}(x) = \begin{cases} -g(-x) & x < 0 \\ 0 & x = 0 \\ g(x) & x > 0 \end{cases} = \begin{cases} 0 & x \leq 1 \\ -1 & -1 < x < 0 \\ 0 & x = 0 \\ 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}.$$

The particular solution to the extended PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} g_{odd}(y)\, dy.$$

We can compute this by splitting the region of integration,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-1}^{0} e^{-\frac{(y-x)^2}{4kt}} (-1)\, dy + \frac{1}{\sqrt{4\pi kt}} \int_{0}^{1} e^{-\frac{(y-x)^2}{4kt}} 1\, dy.$$

Using the change of variables $z = \frac{y-x}{\sqrt{4kt}}$ to simplify the fist integral, we see

$$-\frac{1}{\sqrt{4\pi kt}} \int_{-1}^{0} e^{-\frac{(y-x)^2}{4kt}}\, dy = -\frac{1}{\sqrt{\pi}} \int_{\frac{-1-x}{\sqrt{4kt}}}^{\frac{-x}{\sqrt{4kt}}} e^{-z^2}\, dz = \frac{1}{2}\left(\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{1+x}{\sqrt{4kt}}\right)\right),$$

and using the change of variables $z = \frac{y-x}{\sqrt{4kt}}$ to simplify the second integral, we see

$$\frac{1}{\sqrt{4\pi kt}} \int_{0}^{1} e^{-\frac{(y-x)^2}{4kt}}\, dy = \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4kt}}}^{\frac{1-x}{\sqrt{4kt}}} e^{-z^2}\, dz = \frac{1}{2}\left(\operatorname{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right).$$

Therefore, for $x > 0$ and $t > 0$, the solution to the PDE is given by

$$u(x,t) = \frac{1}{2}\left(\operatorname{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) + 2 \cdot \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{1+x}{\sqrt{4kt}}\right)\right).$$

Problem 3. Solve the following PDE

$$\begin{cases} u_t = k u_{xx} & 0 < x < \infty,\ t > 0, \\ u(x,0) = g(x) & 0 < x < \infty, \\ u_x(0,t) = h(t) & t > 0 \end{cases}$$

where

$$g(x) = xe^{-x}, \qquad h(t) = 0.$$

Solution 3. We want to solve the heat equation on the half line with Neumann boundary conditions. We can use an even reflection to extend the initial condition,

$$g_{even}(x) = \begin{cases} g(-x) & x < 0 \\ g(x) & x \geq 0 \end{cases} = \begin{cases} -xe^{x} & x < 0 \\ xe^{-x} & x \geq 0 \end{cases}.$$

The particular solution to the extended PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} g_{even}(y)\, dy.$$

We can compute this by splitting the region of integration,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\frac{(y-x)^2}{4kt}} (-ye^{y})\, dy + \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(y-x)^2}{4kt}} ye^{-y}\, dy.$$

Completing the square and integrating by parts, we see that

$$\begin{aligned}
&- \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\frac{(y-x)^2}{4kt}} ye^{y}\, dy \\
&= -\frac{e^{x+kt}}{\sqrt{4\pi kt}} \int_{-\infty}^{0} ye^{-\frac{(y-(x+2kt))^2}{4kt}}\, dy \\
&= -\frac{e^{x+kt}}{\sqrt{4\pi kt}} \int_{-\infty}^{0} (y-(x+2kt))e^{-\frac{(y-(x+2kt))^2}{4kt}}\, dy - (x+2kt)\cdot \frac{e^{x+kt}}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\frac{(y-(x+2kt))^2}{4kt}}\, dy \\
&= \frac{e^{x+kt}}{\sqrt{\pi}} \cdot \left(\frac{e^{-\frac{(y-(x+2kt))^2}{4kt}}}{2} \right)\Bigg|_{y=-\infty}^{y=0} - (x+2kt)\cdot \frac{e^{x+kt}}{\sqrt{\pi}} \cdot \int_{-\infty}^{\frac{-(x+2kt)}{\sqrt{4kt}}} e^{-z^2}\, dz \\
&= \frac{e^{x+kt}}{\sqrt{\pi}} \cdot \frac{e^{-\frac{(x+2kt)^2}{4kt}}}{2} - (x+2kt)\cdot e^{x+kt} \cdot \frac{1}{2}\left(-\operatorname{erf}\left(\frac{x+2kt}{\sqrt{4kt}} \right) + 1 \right),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(y-x)^2}{4kt}} ye^{-y}\, dy \\
&= \frac{e^{-x+kt}}{\sqrt{4\pi kt}} \int_{0}^{\infty} ye^{-\frac{(y-(x-2kt))^2}{4kt}}\, dy \\
&= \frac{e^{-x+kt}}{\sqrt{4\pi kt}} \int_{0}^{\infty} (y-(x-2kt))e^{-\frac{(y-(x-2kt))^2}{4kt}}\, dy + (x-2kt)\cdot \frac{e^{-x+kt}}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(y-(x-2kt))^2}{4kt}}\, dy \\
&= -\frac{e^{-x+kt}}{\sqrt{\pi}} \cdot \left(\frac{e^{-\frac{(y-(x-2kt))^2}{4kt}}}{2} \right)\Bigg|_{y=0}^{y=\infty} + (x-2kt)\cdot \frac{e^{-x+kt}}{\sqrt{\pi}} \cdot \int_{\frac{-(x-2kt)}{\sqrt{4kt}}}^{\infty} e^{-z^2}\, dz \\
&= \frac{e^{-x+kt}}{\sqrt{\pi}} \cdot \frac{e^{-\frac{(x-2kt)^2}{4kt}}}{2} + (x-2kt)\cdot e^{-x+kt} \cdot \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x-2kt}{\sqrt{4kt}} \right) \right).
\end{aligned}$$

Therefore, for $x > 0$ and $t > 0$, the solution to the PDE is given by

$$\begin{aligned}
u(x,t) &= \frac{e^{-x+kt}}{\sqrt{\pi}} \cdot \frac{e^{-\frac{(x-2kt)^2}{4kt}}}{2} + \frac{e^{x+kt}}{\sqrt{\pi}} \cdot \frac{e^{-\frac{(x+2kt)^2}{4kt}}}{2} \\
&\quad + (x-2kt)\cdot e^{-x+kt} \cdot \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x-2kt}{\sqrt{4kt}} \right) \right) - (x+2kt)\cdot e^{x+kt} \cdot \frac{1}{2}\left(-\operatorname{erf}\left(\frac{x+2kt}{\sqrt{4kt}} \right) + 1 \right).
\end{aligned}$$