Week 8

For reference, we summarize the main steps involved in computing and classifying convergence of Fourier series. In general, the (full) Fourier series of $f \in \mathrm{PWS}[-L, L]$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big),$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

The coefficients of the Fourier series also obey several properties.

Theorem (Parseval's Identity). If $f \in L^2([-L, L])$ then

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f^2(x)\,dx.$$

Given a function f, we denote the Nth partial sum of the Fourier series on $[-L, L]$ with

$$f_N(x) = a_0 + \sum_{n=1}^{N} \Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big),$$

where a_0, (a_n) and b_n are the corresponding Fourier coefficients of f. Convergence can be summarized by the following theorems:

Theorem (Pointwise Convergence Theorem). If $f(x) \in \mathrm{PWS}[-L, L]$, then

$$\lim_{N\to\infty} f_N(x) = \frac{f_{ext}(x^+) + f_{ext}(x^-)}{2} \text{ on } (-L, L),$$

where f_{ext} is the appropriate periodic extension of f.

Theorem (Uniform Convergence Theorem). If f is continuous on $[-L, L]$, f' is piecewise continuous on $[-L, L]$ and $f(-L) = f(L)$, then f_N converges uniformly to f on $[-L, L]$.

We now work through several examples:

Problem 1. Obtain the Fourier (sine and cosine) series of $f(x) = x(\pi - x)$ on $0 \le x \le \pi$. Then show that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

Moreover, use Fourier sine series and Parseval's identity to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Solution 1.

Cosine Series: The cosine series is the even extension of $f(x)$ to $-\pi \le x \le \pi$. We compute the coefficients

a_0**:** Since $f_{ext}(x)$ is even, the coefficient is given by

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\int_0^\pi x(\pi - x)\,dx = \frac{\pi^2}{6}.$$

a_n**:** Since $f_{ext}(x)\cos(nx)$ is even, the coefficient is given by

$$\begin{aligned} a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos(nx)\,dx &= -2\cdot\frac{\pi n - 2\sin(\pi n) + \pi n\cos(\pi n)}{\pi n^3} \\ &= -2\cdot\frac{1+(-1)^n}{n^2} \\ &= \begin{cases} 0 & n \text{ is odd} \\ -\frac{4}{n^2} & n \text{ is even} \end{cases}. \end{aligned}$$

b_n**:** Since $f_{ext}(x)\sin(nx)$ is odd, the coefficient is given by

$$b_n = 0.$$

The corresponding Fourier cosine series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\Big(a_n\cos(nx) + b_n\sin(nx)\Big) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\cos(2nx).$$

Sine Series: The sine series is the odd extension of $f(x)$ to $-\pi \le x \le \pi$. We compute the coefficients

a_0**:** Since $f_{ext}(x)$ is odd, the coefficient is given by

$$a_0 = 0.$$

a_n**:** Since $f_{ext}(x)\cos(nx)$ is odd, the coefficient is given by

$$a_n = 0.$$

b_n**:** Since $f_{ext}(x)\sin(nx)$ is even, the coefficient is given by

$$\begin{aligned} b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)\,dx &= 2\cdot\frac{2 - \pi n\sin(\pi n) - 2\cos(\pi n)}{\pi n^3} \\ &= 4\cdot\frac{1-(-1)^n}{\pi n^3} \\ &= \begin{cases} \frac{8}{\pi n^3} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}. \end{aligned}$$

The corresponding Fourier sine series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\Big(a_n\cos(nx) + b_n\sin(nx)\Big) = \sum_{n=1}^{\infty}\frac{8}{\pi(2n-1)^3}\cdot\sin((2n-1)x).$$

Evaluating infinite series: We use the fact $\sin(\frac{(2k-1)\pi}{2}) = (-1)^{k+1}$ to compute the series. $f(x)$ is continuous at $\frac{\pi}{2}$, so the pointwise convergence theorem implies

$$f(\pi/2) = \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{8}{\pi(2n-1)^3}\cdot\sin((2n-1)x)\Big|_{x=\frac{\pi}{2}} = \sum_{n=1}^{\infty}\frac{8}{\pi(2n-1)^3}(-1)^{n+1}.$$

Rearranging terms implies

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

To compute the other series, we use Parseval's identity. Notice that for the odd extension of $x(\pi - x)$ to $[-\pi, \pi]$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f_{ext}(x)^2\,dx = \frac{2}{\pi}\int_0^{\pi} (x(\pi - x))^2\,dx = \frac{\pi^4}{15}.$$

Therefore, by Parseval's identity on the sine series, we have

$$\frac{\pi^4}{15} = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty} \frac{64}{\pi^2(2n-1)^6}$$

which implies

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

To recover the series in the question, we split the sum into its odd and even components,

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \sum_{n=1}^{\infty} \frac{1}{(2n)^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \frac{1}{64}\sum_{n=1}^{\infty} \frac{1}{n^6}.$$

Rearranging terms and using our formula for the sum of odd terms implies

$$\Big(1 - \frac{1}{64}\Big)\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960} \implies \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{960} \cdot \frac{64}{63} = \frac{\pi^6}{945}.$$

Problem 2. First, find Fourier series of $f(x) = \begin{cases} 0 & -2 \le x < 0 \\ 2 - x & 0 \le x \le 2 \end{cases}$ and plot extension of this function to whole line. Does it converge pointwise to $f(x)$ in (-2,2)? what about uniformly? Use the Pointwise Convergence at $x = 0$ and compute

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} =?$$

Solution 2. The Fourier series is given by

a_0**:** The coefficient is given by

$$a_0 = \frac{1}{4}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{4}\int_0^2 2 - x\,dx = \frac{1}{2}.$$

a_n**:** Using integration by parts, the coefficient is given by

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{1}{2}\int_0^2 (2 - x)\cos\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{2 - 2\cos(\pi n)}{\pi^2 n^2} = \frac{2 - 2(-1)^n}{\pi^2 n^2}.$$

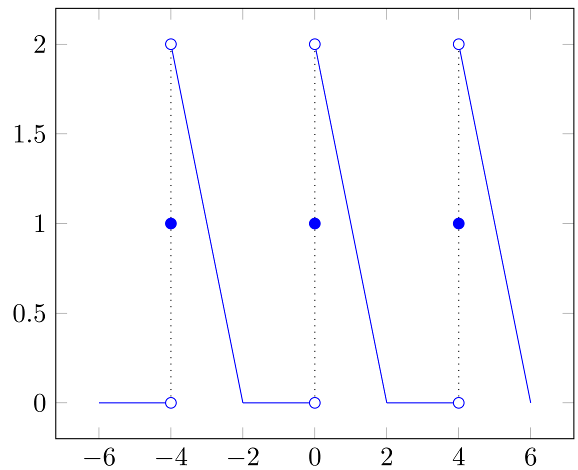
b_n**:** Using integration by parts, the coefficient is given by

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{1}{2}\int_0^2 (2 - x)\sin\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{2\pi n - 2\sin(\pi n)}{\pi^2 n^2} = \frac{2}{\pi n}.$$

The corresponding Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big) = \frac{1}{2} + \sum_{n=1}^{\infty}\Big(\frac{2-2(-1)^n}{\pi^2 n^2}\cdot\cos\Big(\frac{n\pi x}{2}\Big) + \frac{2}{\pi n}\cdot\sin\Big(\frac{n\pi x}{2}\Big)\Big).$$

By the pointwise convergence theorem, the series converges pointwise to the average of the left and right endpoints of the periodic extension of $f(x)$ to $\mathbb{R}$. The plot of the Fourier series is given below:



We now examine convergence.

1. Pointwise Convergence: By the pointwise convergence theorem, we have

$$\lim_{N\to\infty} f_N(0) = \frac{f_{ext}(0^+) + f_{ext}(0^-)}{2} = \frac{2+0}{2} = 1.$$

   In particular, since $f(0) = 2$ we have

$$\lim_{N\to\infty} |f_N(0) - f(0)| = \Big|\lim_{N\to\infty} f_N(0) - f(2)\Big| = 1 \neq 0,$$

   so the Fourier series does not converge pointwise on $(-2, 2)$.

2. Uniform Convergence: Uniform convergence is a stronger condition than pointwise convergence, that is, if $f_N(x)$ converges uniformly on $[-2, 2]$ then $f_N(x)$ converges pointwise on $(-2, 2)$. In particular, since $f_N(x)$ does not converge pointwise at $x = 1$, we have the Fourier series does not converge uniformly on $[-2, 2]$.

Lastly, we use the pointwise convergence theorem to compute the required series. At $x = 0$, we have

$$\lim_{N\to\infty} f_N(0) = \frac{f_{ext}(0^+) + f_{ext}(0^-)}{2} = \frac{2+0}{2} = 1.$$

Now evaluating the series at $x = 0$ gives

$$\begin{aligned}\frac{1}{2} + \sum_{n=1}^{\infty}\Big(\frac{2-2(-1)^n}{\pi^2 n^2}\cdot\cos\Big(\frac{n\pi x}{2}\Big) + \frac{2}{\pi n}\cdot\sin\Big(\frac{n\pi x}{2}\Big)\Big)\Big|_{x=0} &= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2-2(-1)^n}{\pi^2 n^2} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{4}{\pi^2(2n-1)^2}.\end{aligned}$$

Therefore, by the pointwise convergence theorem at $x = 0$, we have

$$\frac{1}{2} + \sum_{n=1}^{\infty}\frac{4}{\pi^2(2n-1)^2} = 1 \implies \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Problem 3. Without computing Fourier series of the following functions on $[-\pi, \pi]$, determine whether the Fourier series converges pointwise to $f(x)$ on $(-\pi, \pi)$ or uniformly to $f(x)$ on $[-\pi, \pi]$.

$$(1) \quad f(x) = \begin{cases} 2 & -\pi \leq x < 0 \\ \frac{3}{2} & x = 0 \\ \cos x & 0 < x \leq \pi \end{cases} \qquad (2) \quad f(x) = x^2 + x$$

Solution 3.

Part (1)

1. Pointwise Convergence: The Fourier series clearly converges pointwise whenever $f(x)$ is continuous by the pointwise convergence theorem. We now examine the behavior at the point of discontinuity, $x = 0$. By the pointwise convergence theorem, we have

$$\lim_{N \to \infty} f_N(0) = \frac{f_{ext}(0^+) + f_{ext}(0^-)}{2} = \frac{2+1}{2} = \frac{3}{2}.$$

In particular, since $f(0) = 1$ we have

$$\lim_{N \to \infty} |f_N(0) - f(0)| = \left| \lim_{N \to \infty} f_N(0) - f(0) \right| = 0,$$

so the Fourier series converges pointwise on $(-\pi, \pi)$.

2. Uniform Convergence: f is continuous, f' is piecewise continuous, but f does not satisfy the periodic boundary conditions $f(-\pi) = 2 \neq -1 = f(\pi)$. This is not enough to show that the series does not uniformly converge because the statement of the uniform convergence theorem only goes in one direction.

   To see the series does not converge uniformly we first check if the series converges at the endpoints. By the pointwise convergence theorem applied at the point π,

$$f_N(\pi) = \frac{f_{ext}(\pi^+) + f_{ext}(\pi^-)}{2} = \frac{f(-\pi) + f(\pi)}{2} = \frac{2-1}{2} = \frac{1}{2}$$

and therefore,

$$\lim_{N \to \infty} |f_N(\pi) - f(\pi)| \geq \left| \lim_{N \to \infty} f(\pi) - f_N(\pi) \right| = \frac{1}{2} \geq 0.$$

Since the series does not converge at one of the endpoints, we have

$$\lim_{N \to \infty} \|f_N(x) - f(x)\|_\infty := \lim_{N \to \infty} \sup_{x \in [-\pi, \pi]} |f_N(x) - f(x)| \geq \lim_{N \to \infty} |f_N(\pi) - f(\pi)| = \frac{3}{2} \neq 0,$$

so f_N does not converge uniformly on $[-\pi, \pi]$.

Part (2)

1. Pointwise Convergence: $f(x) = x^2 + x$ is continuous, so $f(x^+) = f(x^-)$ for all $x \in (-\pi, \pi)$. By the pointwise convergence theorem, for all $x \in (-\pi, \pi)$ we have

$$\lim_{N \to \infty} f_N(x) = \frac{f_{ext}(x^+) + f_{ext}(x^-)}{2} = f(x).$$

Therefore, f_N converges pointwise on $(-\pi, \pi)$.

2. Uniform Convergence: f is continuous, f' is piecewise continuous, but f does not satisfy the periodic boundary conditions $f(-\pi) = \pi^2 - \pi \neq \pi^2 + \pi = f(\pi)$. This is not enough to show that the series does not uniformly converge because the statement of the uniform convergence theorem only goes in one direction.

   To see the series does not converge uniformly we first check if the series converges at the endpoints. By the pointwise convergence theorem applied at the point π,
$$f_N(\pi) = \frac{f_{ext}(\pi^+) + f_{ext}(\pi^-)}{2} = \frac{f(-\pi) + f(\pi)}{2} = \frac{\pi^2 + \pi + \pi^2 - \pi}{2} = \pi^2$$
   and therefore,
$$\lim_{N\to\infty} |f_N(\pi) - f(\pi)| \geq \lim_{N\to\infty} f(\pi) - f_N(\pi) = \pi^2 + \pi - \pi^2 = \pi.$$
   Since the series does not converge at one of the endpoints, we have
$$\lim_{N\to\infty} \|f_N(x) - f(x)\|_\infty := \lim_{N\to\infty} \sup_{x\in[-\pi,\pi]} |f_N(x) - f(x)| \geq \lim_{N\to\infty} |f_N(\pi) - f(\pi)| \geq \pi \neq 0,$$
   so f_N does not converge uniformly on $[-\pi, \pi]$.

Problem 4. (Strauss 5.4.12) Find the Fourier sine series of $f(x) = x$ on the interval $(0, L)$. Apply Parseval's identity to show $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution 4. We take $L = \pi$. The sine series is the odd extension of $f(x) = x$ to $-\pi \leq x \leq \pi$. We just compute the coefficients

a_0: Since $f_{ext}(x) = x$ is odd, the coefficient is given by
$$a_0 = 0.$$
a_n: Since $f_{ext}(x)\cos(nx) = x\cos(nx)$ is odd, the coefficient is given by
$$a_n = 0.$$
b_n: Since $f_{ext}(x)\sin(nx) = x\sin(nx)$ is even, the coefficient is given by
$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx = 2\cdot\frac{\sin(\pi n) - \pi n\cos(\pi n)}{\pi n^2} = \frac{2(-1)^{n+1}}{n}.$$
The corresponding Fourier sine series is given by
$$f(x) = a_0 + \sum_{n=1}^\infty \Big(a_n\cos(nx) + b_n\sin(nx)\Big) = \sum_{n=1}^\infty \frac{2(-1)^{n+1}}{n}\cdot\sin((2n-1)x).$$
We now apply Parseval's identity. Notice that for the odd extension of x to $[-\pi, \pi]$,
$$\frac{1}{\pi}\int_{-\pi}^\pi f_{ext}(x)^2\,dx = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2\pi^2}{3}.$$
Therefore, by Parseval's identity on the sine series, we have
$$\frac{2\pi^2}{3} = 2a_0^2 + \sum_{n=1}^\infty (a_n^2 + b_n^2) = \sum_{n=1}^\infty \frac{4}{n^2}$$
which implies
$$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.$$