# 1 Martingales

**Definition 1.1.** Let probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\{\mathcal{F}_t\}_{t\in\mathscr{T}}$. Then the quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathscr{T}}, \mathbb{P})$ is also called a **filtered probability space**.

In this course, $\mathscr{T}$ will typically be the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ or $\mathbb{R}^+ = [0, \infty)$ the non-negative numbers. A martingale is a stochastic process defined with respect to a filtered probability space. Loosely speaking, it represents the total payout of a fair game. That is, the expected value in the future is equal to its current value.

**Definition 1.2.** Let $X = \{X_t\}_{t\in\mathscr{T}}$ be a stochastic process satisfies the following two conditions.

- $X$ is **adapted** to $\{\mathcal{F}_t\}_{t\in\mathscr{T}}$, i.e., $X_t$ is $\mathcal{F}_t$ measurable for all $t \in \mathscr{T}$.
- $\mathbb{E}[|X_t|] < \infty$ for all $t \in \mathscr{T}$.

$X$ is called a **martingale** (with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathscr{T}}$) if

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s \qquad \text{for all } s, t \in \mathscr{T} \text{ with } s \le t. \tag{1}$$

If we say that $X = \{X_t\}_{t\in\mathscr{T}}$ is a martingale without specifying the filtration, we mean that $X = \{X_t\}_{t\in\mathscr{T}}$ is a martingale w.r.t. its natural filtration $\mathcal{F}_t^X = \sigma(X_s | s \in \mathscr{T},\ s \le t)$.

**Remark 1.3.** The condition (1) is equivalent to

$$\mathbb{E}[X_t - X_s | \mathcal{F}_s] = 0 \qquad \text{for all } s, t \in \mathscr{T} \text{ with } s \le t. \tag{2}$$

If we let $X_t$ denote the total payouts of a game at time $t$, then $X_t - X_s$ represents the gain (or loss) accumulated between times $t$ and $s$. Condition (2) implies that based on all the information available at time $s$, the expected value of this gain (or loss) is zero. In this sense, a martingale can be understood the mathematical formalization of a fair game.

**Remark 1.4.** In discrete time, $\mathscr{T} = \mathbb{N}$, the condition (1) is equivalent to

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n \qquad \text{for all } n \ge 0. \tag{3}$$

The proof is a direct application of the tower property.

**Example 1.5** (Simple Random Walk)**.** Let $Y_1, Y_2, \dots$ be i.i.d. Rademacher random variables, i.e. $\mathbb{P}(Y = \pm 1) = \frac{1}{2}$. Then $\{X_n\}_{n\ge 0}$ defined through

$$X_0 = 0 \qquad \text{and} \qquad X_n = \sum_{k=1}^{n} Y_k \tag{4}$$

is a martingale in discrete time with respect to the natural filtration $\mathcal{F}_n^X$. Indeed, since $Y_{n+1}$ is independent of $\mathcal{F}_n^X$,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n + Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n|\mathcal{F}_n] + \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = X_n + \mathbb{E}[Y_{n+1}] = X_n$$

which satisfies condition (3).

## 1.1 Properties

Naturally, the expected value of the earnings of a fair game is equal to zero.

**Proposition 1.6**

*If $\{X_t\}_{t\in\mathscr{T}}$ is a martingale, then*

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] \qquad \text{for all } t \in \mathscr{T}.$$

We have the following formula for the second moment of the earnings between time $s$ and $t$.

**Proposition 1.7**

*Let $\{X_t\}_{t\in\mathscr{T}}$ be a martingale with $\mathbb{E}[(X_t)^2] < \infty$ for all $t \in \mathscr{T}$. Then, for $s, t \in \mathscr{T}$ with $s \leq t$,*

$$\mathbb{E}[(X_t - X_s)^2|\mathcal{F}_s] = \mathbb{E}[X_t^2|\mathcal{F}_s] - X_s^2.$$

*In particular,*

$$\mathbb{E}[(X_t - X_s)^2] = \mathbb{E}[X_t^2] - \mathbb{E}[X_s^2].$$

**Example 1.8** (Martingale Betting Strategy)**.** Let $X_t$ be a simple random walk defined Example 1.5. We now take an *adapted* stochastic process $\{\xi_n\}_{n\geq 0}$ where $\xi_0 = 1$ and, for $n \geq 1$,

$$\xi_n = \begin{cases} 2^n, & \text{if } Y_1 = \cdots = Y_n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

This represents a betting strategy where we double our bet until we win. Then the gambler's total return at time $n \geq 1$ is

$$\begin{aligned} V_n &= \sum_{k=0}^{n-1} \xi_k (X_{k+1} - X_k) \\ &= \xi_0 Y_1 + \cdots + \xi_{n-1} Y_n \\ &= \begin{cases} -1 - 2 - \cdots - 2^{n-1} = -(2^n - 1), & \text{if } Y_1 = \cdots = Y_n = -1 \\ +1, & \text{otherwise.} \end{cases} \end{aligned}.$$

One can show that with probability one there will eventually be some (random) integer $n$ such that $Y_n = 1$, in which case the gambler will have won \$1.

**Example 1.9** (General Betting Strategies)**.** In general, let $\{X_n\}_{n\geq 0}$ be a martingale denoting the outcomes of a fair game. We let the process $\{\xi_n\}_{n\geq 0}$ be an adapted process denoting a betting strategy. This means that the $\xi_n$ bet is a function of the information up to the $n$th game.

Suppose we are at game $k$, if we bet $\xi_k$ on the $k$th game, then we earn $\xi_k(X_{k+1} - X_k)$ on the $k$th game. Our earnings associated with this betting strategy is therefore

$$V_0 = 0, \qquad V_n = \sum_{k=0}^{n-1} \xi_k (X_{k+1} - X_k). \tag{5}$$

A natural question is if one can come up with a smart betting strategy such that $\mathbb{E}[V_n] > 0 = \mathbb{E}[V_0]$ for some $n$? The answer to that question is no, and it is demonstrated in the following theorem. That is, no betting strategy that can turn a martingale into a favorable game.

**Theorem 1.10**

*Suppose $\{\xi_n\}_{n\geq 0}$ is an adapted process such that for every $n$ there exists a constant $C_n$ such that $|\xi_n(\omega)| \leq C_n$ for all $\omega \in \Omega$. If $\{X_n\}_{n\geq 0}$ is a martingale, then $\{V_n\}_{n\geq 0}$ defined in (5) is again a martingale. In particular, we have $\mathbb{E}[V_n] = 0$ for all $n$.*

## 1.2 Example Problems

### 1.2.1 Proofs of Results

**Problem 1.1.** Prove Proposition 1.6.

**Solution 1.1.** It follows from (1.2) that

$$\mathbb{E}[X_t] = \mathbb{E}\big[\mathbb{E}[X_t|\mathcal{F}_0]\big] = \mathbb{E}[X_0].$$

**Problem 1.2.** Prove Proposition 1.7.

**Solution 1.2.** We have

$$\begin{aligned}
\mathbb{E}[(X_t - X_s)^2|\mathcal{F}_s] &= \mathbb{E}[X_t^2 - 2X_tX_s + X_s^2|\mathcal{F}_s] \\
&= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2\mathbb{E}[X_tX_s|\mathcal{F}_s] + \mathbb{E}[X_s^2|\mathcal{F}_s] \\
&= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2X_s\mathbb{E}[X_t|\mathcal{F}_s] + X_s^2 \\
&= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2X_sX_s + X_s^2 \\
&= \mathbb{E}[X_t^2|\mathcal{F}_s] - X_s^2.
\end{aligned}$$

The second identity follows from the first by taking expectations.

**Problem 1.3.** Prove Theorem 1.10.

**Solution 1.3.** We check the properties of a martingale.

(i) Clearly, $\{V_n\}_{n\geq 0}$ is adapted.

(ii) Since $|\xi_k| \leq C_k$, we define $C := \max\{C_1, \ldots, C_{n-1}\}$ so that

$$\begin{aligned}
\mathbb{E}[|V_n|] = \mathbb{E}\left[\left|\sum_{k=1}^n \xi_{k-1}(X_k - X_{k-1})\right|\right] &\leq \sum_{k=1}^n \mathbb{E}[|\xi_{k-1}(X_k - X_{k-1})|] \\
&\leq \sum_{k=1}^n C_{k-1}\mathbb{E}[|X_k - X_{k-1}|] \leq C\sum_{k=1}^n \big(\mathbb{E}[|X_k|] + \mathbb{E}[|X_{k-1}|]\big) < \infty.
\end{aligned}$$

(iii) Next, we have

$$\begin{aligned}
\mathbb{E}[V_{n+1} - V_n|\,\mathcal{F}_n\,] &= \mathbb{E}[\,\xi_n(X_{n+1} - X_n)\,|\,\mathcal{F}_n\,] \\
&= \xi_n\mathbb{E}[\,X_{n+1} - X_n\,|\,\mathcal{F}_n\,] = 0
\end{aligned}$$

so $\{V_n\}_{n\geq 0}$ is a martingale. Finally, the martingale property Proposition 1.6 implies that

$$\mathbb{E}[V_n] = \mathbb{E}[V_0] = 0, \qquad \text{for all } n.$$

### 1.2.2 Definitions and Properties of Martingales

**Problem 1.4.** Let $Y_1, Y_2, \dots$ be independent (though not necessarily identically distributed) random variables with common expectation $\mathbb{E}[Y_k] = 0$ for all $k$. Show that $\{X_n\}_{n=0,1,2,\dots}$ defined by

$$X_0 = 0 \qquad \text{and} \qquad X_n = \sum_{k=1}^{n} Y_k$$

is a martingale in discrete time with respect to its natural filtration $\mathcal{F}_n^X$.

**Solution 1.4.** Since $Y_{n+1}$ is independent of $\mathcal{F}_n^X$,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n + Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n|\mathcal{F}_n] + \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = X_n + \mathbb{E}[Y_{n+1}] = X_n.$$

**Problem 1.5.** Let $Y_1, Y_2, \dots$ be independent and nonnegative (though not necessarily identically distributed) random variables with common expectation $\mathbb{E}[Y_k] = 1$ for all $k$.

1. Show that $\{X_n\}_{n\geq 0}$ defined through

$$X_0 = 1 \qquad \text{and} \qquad X_n = \prod_{k=1}^{n} Y_k$$

   is a martingale.

2. Let $Y_k$ be of the form $Y_k = e^{Z_k - c_k}$ for independent random variables $Z_k$ with distribution $N(0, \sigma_k^2)$ and certain constants $c_k$. That is, determine $c_k$ such that $\{X_n\}_{n\geq 0}$ is a martingale.

**Solution 1.5.**

**Part 1:** The fact that $X$ is adapted and integrable is clear. To show (3), notice that by independence,

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[Y_{n+1} \prod_{k=1}^{n} Y_k \,\middle|\, \mathcal{F}_n\right] = \prod_{k=1}^{n} Y_k \,\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \prod_{k=1}^{n} Y_k \,\mathbb{E}[Y_{n+1}] = \prod_{k=1}^{n} Y_k = X_n.$$

**Part 2:** From part 1, it suffices to find a constant so that $\mathbb{E}[Y_k] = 1$. We have by the moment generating function formula for the Gaussian,

$$\mathbb{E}[Y_k] = e^{-c_k}\,\mathbb{E}[e^{Z_k}] = e^{-c_k} e^{\frac{\sigma^2}{2}} = 1 \iff c_k = \frac{\sigma^2}{2}.$$

**Problem 1.6.** Let $X$ be a random variable such that $\mathbb{E}[|X|] < \infty$ and $\mathscr{T}$ either $\{0, 1, 2, \dots\}$ or $[0, \infty)$. Show that

$$X_t := \mathbb{E}[X|\mathcal{F}_t], \qquad t \in \mathscr{T},$$

is a martingale.

**Solution 1.6.** Clearly $X_t$ is $\mathcal{F}_t$ measurable because the conditional expected value. Furthermore, by Jensen's inequality and the law of total expectation

$$\mathbb{E}[|X_t|] = \mathbb{E}[|\,\mathbb{E}[X|\mathcal{F}_t]|] \leq \mathbb{E}[\mathbb{E}[|X||\mathcal{F}_t]] = \mathbb{E}[|X|] < \infty.$$

Next, so show property (1) we have by the tower property that

$$\mathbb{E}\left[X_t|\mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_t]|\mathcal{F}_s\right] = \mathbb{E}[X|\mathcal{F}_s] = X_s.$$

# 2 Stopping time

For this section, we focus on discrete time martingales, but similar statements can be made in continuous time. A stopping time is a random variable whose probability it occurred before $n$ depends on the historical information up to time $n$. We can think of a stopping time as a rule that tells us when to stop playing a game, which naturally can only depends on past information.

**Definition 2.1.** A random time $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a **stopping time** if $\{\tau \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$.

Stopping times include random variables such as stopping after 5 losses in a row, or stopping after the $X_t$ exceeds a certain value.

**Example 2.2.** Let $\{X_n\}_{n\geq 0}$ be any adapted process and define

$$\tau = \min\{n : X_n \geq c\}.$$

Then $\tau$ is a stopping time, which is sometimes called the **first passage time** of the level $c$.

**Definition 2.3.** Given a stopping time, we can define $\sigma$-algebra

$$\mathscr{F}_\tau = \{A \in \mathcal{F} : \{\tau \leq n\} \cap A \in \mathcal{F}_n \text{ for all n}\}$$

which consists of the events that depend on the information up to a random stopping time $\tau$. If $X_n$ is $\mathcal{F}_n$ measurable, then the random variable $X_\tau$ is $\mathcal{F}_\tau$ measurable.

A stopping time can be interpreted as a strategy to stop a game based only on current and historical information. The next theorem states that we cannot come up with a clever stopping strategy that can turn a martingale into a favorable game.

**Theorem 2.4 (*Optional stopping theorem*)**

*Let $\{X_n\}_{n\geq 0}$ be a martingale and $\tau$ be a stopping time. Suppose that*

$$\mathbb{E}\,|X_\tau| < \infty, \quad \lim_{n\to\infty} \mathbb{E}\,|X_n| \mathbb{1}(n \leq \tau) = 0$$

*then*

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0].$$

**Remark 2.5.** Notice that the conditions of Theorem 2.4 are satisfied if

1. there exists a constant $C$ such that $\tau \leq C$ almost surely ; or
2. $|X_{n\wedge\tau}| \leq C$ for all $n$ almost surely.

Both of these are reasonable assumptions on a stopping time for betting, since the first says that we have to eventually stop playing, and the second prevents us from betting an arbitrarily amount of money (we can't bet more money than we have).

The integrability conditions are essential as demonstrated by the following example.

**Example 2.6.** Consider the situation of Example 1.8, where $\{X_n\}_{n\geq 0}$ is a simple random walk and $\{\xi_n\}_{n\geq 0}$ is the martingale betting strategy. We have seen that $\{V_n\}_{n\geq 0}$ is a martingale with $V_0 = 0$. We let

$$\tau(\omega) = \min\{n : Y_n(\omega) = +1\},$$

where $Y_n = X_n - X_{n-1}$, denote the first time we win a game. Then $\tau$ is a stopping time with $\mathbb{P}(\tau < \infty) = 1$ and $V_\tau = 1$. Therefore,

$$\mathbb{E}[V_\tau] = 1 \neq 0 = \mathbb{E}[V_0].$$

This does not contradict Theorem 2.4 because

$$\mathbb{P}(\tau = n) = 2^{-n} \qquad \mathbb{P}(\tau > n) = \sum_{k \geq n+1} 2^{-k} = 2^{-n}$$

so

$$\mathbb{E}\,|V_n| \mathbb{1}(n \leq \tau) = \mathbb{P}(\tau = n) + (2^n - 1)\,\mathbb{P}(\tau > n) = 1$$

which does not go to zero.

## 2.1 Example Problems

### 2.1.1 Proofs of Results

**Problem 2.1.** Prove the Optional Stopping Theorem (Theorem 2.4) under the assumption that $\tau$ is a bounded stopping time.

**Solution 2.1.** We first show that $X_{n\wedge\tau}$ is a martingale. We only consider the case $\mathscr{T} = \{0, 1, 2, \dots\}$. We have

$$X_{n\wedge\tau} - X_{(n-1)\wedge\tau} = \mathbb{1}\{\tau > n-1\}(X_n - X_{n-1}).$$

Thus, stopping the process is the same as using the betting strategy $\xi_n = \mathbb{1}\{\tau > n\}$, which is adapted since $\tau$ is a stopping time. More precisely, by writing $X_{n\wedge\tau}$ as a telescoping sum we have

$$X_{n\wedge\tau} = X_0 + \sum_{k=1}^{n}(X_{k\wedge\tau} - X_{(k-1)\wedge\tau}) = X_0 + \sum_{k=1}^{n} \mathbb{1}\{\tau > k-1\}(X_k - X_{k-1})$$

Therefore Theorem 1.10 implies that $X_{n\wedge\tau}$ is a martingale.

If $\tau$ is almost surely bounded by some constant $C$, then $X_{N\wedge\tau} = X_\tau$ for all $N > C$. Hence, by Proposition 1.6,

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_{N\wedge\tau}] = \mathbb{E}[X_{0\wedge\tau}] = \mathbb{E}[X_0].$$

**Remark 2.7.** The general proof of Theorem 2.4 uses the dominated convergence theorem to interchange the limit and expected value.

**Problem 2.2.** Show that for a random time $\tau : \Omega \to \{0, 1, \dots\} \cup \{\infty\}$, the following conditions are equivalent:

(a) $\tau$ is a stopping time.

(b) For every $n \geq 0$, we have $\{\tau \leq n\} \in \mathcal{F}_n$.

(c) For every $n \geq 0$, we have $\{\tau > n\} \in \mathcal{F}_n$.

(d) For every $n \geq 0$, we have $\{\tau = n\} \in \mathcal{F}_n$.

**Solution 2.2.** The equivalence of (a) and (b) is immediate by the definition. To see that (b) and (c) are equivalent, recall that

$$A \in \mathcal{F}_n \iff A^c \in \mathcal{F}_n.$$

Since $\{\tau \leq n\}^c = \{\tau > n\}$, it follows that $\{\tau \leq n\} \in \mathcal{F}_n \iff \{\tau > n\} \in \mathcal{F}_n$ so (b) and (c) are equivalent. To see that (b) and (c) are equivalent, notice that

$$\{\tau = n\} = \{\tau \leq n\} \setminus \{\tau \leq n-1\}$$

so $\{\tau = n\}$ if (b) holds since the $\sigma$-algebra is closed under unions and complements.

#### 2.1.2 Applications

**Problem 2.3.** Let $\{X_n\}_{n\geq 0}$ is a simple random walk, $a, b \in \mathbb{N}$, and

$$\tau = \min\{n \,|\, X_n = -a \text{ or } X_n = b\}.$$

Find

$$\mathbb{P}[X_\tau = b].$$

**Solution 2.3.** Recall that $\{X_n\}_{n\geq 0}$ is a martingale and $\tau$ is a stopping time. We can interpret $X_{n\wedge\tau}$ as the balance in a fair coin-tossing game between two players with respective capital $a$ and $b$. We are interested in the probability that the player with capital $b$ goes bankrupt before the other player, i.e., $\mathbb{P}(X_\tau = b)$.

We have that the stopping time $\tau$ satisfies $\mathbb{P}(\tau < \infty) = 1$ and $|X_{n\wedge\tau}| \leq a \vee b$ for all $n$ almost surely. Therefore, using Theorem 2.4 with uniformly bounded stopped martingales implies that

$$0 = \mathbb{E}[X_0] = \mathbb{E}[X_\tau] = b\mathbb{P}(X_\tau = b) - a\mathbb{P}(X_\tau = -a) = b\mathbb{P}(X_\tau = b) - a(1 - \mathbb{P}(X_\tau = b))$$

which gives

$$\mathbb{P}[X_\tau = b] = \frac{a}{a+b}.$$