1 Martingales

Definition 1. Let probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\{\mathcal{F}_t\}_{t\in\mathscr{T}}$. Then the quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathscr{T}}, \mathbb{P})$ is also called a **filtered probability space**.

In this course, $\mathscr{T}$ will typically be the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ or $\mathbb{R}^+ = [0, \infty)$ the non-negative numbers. A martingale is a stochastic process defined with respect to a filtered probability space. Loosely speaking, it represents the total payout of a fair game. That is, the expected value in the future is equal to its current value.

Definition 2. Let $X = \{X_t\}_{t\in\mathscr{T}}$ be a stochastic process satisfies the following two conditions.

- X is **adapted** to $\{\mathcal{F}_t\}_{t\in\mathscr{T}}$, i.e., X_t is $\mathcal{F}_t$ measurable for all $t \in \mathscr{T}$.
- $\mathbb{E}[|X_t|] < \infty$ for all $t \in \mathscr{T}$.

X is called a **martingale** (with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathscr{T}}$) if

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \qquad \text{for all } s, t \in \mathscr{T} \text{ with } s \le t. \tag{1}$$

If we say that $X = \{X_t\}_{t\in\mathscr{T}}$ is a martingale without specifying the filtration, we mean that $X = \{X_t\}_{t\in\mathscr{T}}$ is a martingale w.r.t. its natural filtration $\mathcal{F}_t^X = \sigma(X_s | s \in \mathscr{T},\, s \le t)$.

Remark 1. The condition (1) is equivalent to

$$\mathbb{E}[X_t - X_s|\mathcal{F}_s] = 0 \qquad \text{for all } s, t \in \mathscr{T} \text{ with } s \le t. \tag{2}$$

If we let X_t denote the total payouts of a game at time t, then $X_t - X_s$ represents the gain (or loss) accumulated between times t and s. Condition (2) implies that based on all the information available at time s, the expected value of this gain (or loss) is zero. In this sense, a martingale can be understood the mathematical formalization of a fair game.

Remark 2. In discrete time, $\mathscr{T} = \{0, 1, \dots\}$, the condition (1) is equivalent to

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n \qquad \text{for all } n \ge 0. \tag{3}$$

The proof is a direct application of the tower property.

Example 1 (Simple Random Walk)**.** Let $Y_1, Y_2, \dots$ be i.i.d. Rademacher random variables, i.e. $\mathbb{P}(Y = \pm 1) = \frac{1}{2}$. Then $\{X_n\}_{n=0,1,2,\dots}$ defined through

$$X_0 = 0 \qquad \text{and} \qquad X_n = \sum_{k=1}^{n} Y_k \tag{4}$$

is a martingale in discrete time with respect to the natural filtration $\mathcal{F}_n^X$. Indeed, since Y_{n+1} is independent of $\mathcal{F}_n^X$,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n + Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n|\mathcal{F}_n] + \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = X_n + \mathbb{E}[Y_{n+1}] = X_n$$

which satisfies condition (3).

1.1 Properties

Naturally, the expected value of the earnings of a fair game is equal to zero.

Proposition 1

If $\{X_t\}_{t\in\mathscr{T}}$ is a martingale, then

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] \qquad \text{for all } t \in \mathscr{T}.$$

We have the following formula for the second moment of the earnings between time s and t.

Proposition 2

Let $\{X_t\}_{t\in\mathscr{T}}$ be a martingale with $\mathbb{E}[(X_t)^2] < \infty$ for all $t \in \mathscr{T}$. Then, for $s, t \in \mathscr{T}$ with $s \le t$,

$$\mathbb{E}[(X_t - X_s)^2 | \mathcal{F}_s] = \mathbb{E}[X_t^2 | \mathcal{F}_s] - X_s^2.$$

In particular,

$$\mathbb{E}[(X_t - X_s)^2] = \mathbb{E}[X_t^2] - \mathbb{E}[X_s^2].$$

Example 2. (The martingale betting strategy) Let X_t be a simple random walk. We now take an *adapted* stochastic process $\{\xi_n\}_{n=0,1,\dots}$ where $\xi_0 = 1$ and, for $n \ge 1$,

$$\xi_n = \begin{cases} 2^n, & \text{if } Y_1 = \cdots = Y_n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

This represents a betting strategy where we double our bet until we win. Then the gambler's total return at time n is

$$\begin{aligned} V_n &= \sum_{k=1}^{n} \xi_{k-1}(X_k - X_{k-1}) \\ &= \xi_0 Y_1 + \cdots + \xi_{n-1} Y_n \\ &= \begin{cases} -1 - 2 - \cdots - 2^{n-1} = -(2^n - 1), & \text{if } Y_1 = \cdots = Y_n = -1 \\ +1, & \text{otherwise.} \end{cases} \end{aligned}$$

One can show that with probability one there will eventually be some (random) integer n such that $Y_n = 1$, in which case the gambler will have won \$1.

Example 3 (General Betting Strategies)**.** In general, let $\{X_n\}_{n\ge 0}$ be a martingale denoting the outcomes of a fair game. We let the process $\{\xi_n\}_{n\ge 0}$ be an adapted process denoting a betting strategy. This means that the ξ_n bet is a function of the information up to the nth game.

Suppose we are at game k, if we bet ξ_k on the kth game, then we earn $\xi_k(X_{k+1} - X_k)$ on the kth game. Our earnings associated with this betting strategy is therefore

$$V_0 = 0, \qquad V_n = \sum_{k=0}^{n-1} \xi_k (X_{k+1} - X_k). \tag{5}$$

The question is if one can come up with a smart betting strategy such that $\mathbb{E}[V_n] > 0 = \mathbb{E}[V_0]$ for some n?

The answer to that question is no, and it is demonstrated in the following theorem. That is, no betting strategy that can turn a martingale into a favorable game.

Theorem 1

Suppose $\{\xi_n\}_{n=0,1,\dots}$ is an adapted process such that for every n there exists a constant C_n such that $|\xi_n(\omega)| \le C_n$ for all $\omega \in \Omega$. If $\{X_n\}_{n=0,1,\dots}$ is a martingale, then $\{V_n\}_{n=0,1,\dots}$ defined in (5) is again a martingale. In particular, we have $\mathbb{E}[V_n] = 0$ for all n.

1.2 Example Problems

1.2.1 Proofs of Results

Problem 1.1. Prove Proposition 1.

Solution 1.1. It follows from (2) that

$$\mathbb{E}[X_t] = \mathbb{E}\big[\mathbb{E}[X_t|\mathcal{F}_0]\big] = \mathbb{E}[X_0].$$

Problem 1.2. Prove Proposition 2.

Solution 1.2. We have

$$\begin{aligned}\mathbb{E}[(X_t - X_s)^2|\mathcal{F}_s] &= \mathbb{E}[X_t^2 - 2X_tX_s + X_s^2|\mathcal{F}_s]\\ &= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2\mathbb{E}[X_tX_s|\mathcal{F}_s] + \mathbb{E}[X_s^2|\mathcal{F}_s]\\ &= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2X_s\mathbb{E}[X_t|\mathcal{F}_s] + X_s^2\\ &= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2X_sX_s + X_s^2\\ &= \mathbb{E}[X_t^2|\mathcal{F}_s] - X_s^2.\end{aligned}$$

The second identity follows from the first by taking expectations.

Problem 1.3. Prove Theorem 1.

Solution 1.3. We check the properties of a martingale.

(i) Clearly, $\{V_n\}_{n=0,1,\dots}$ is adapted.

(ii) The fact that $|\xi_k| \le C_k$ and letting $C := \max\{C_1, \dots, C_{n-1}\}$ gives that

$$\begin{aligned}\mathbb{E}[|V_n|] = \mathbb{E}\bigg[\bigg|\sum_{k=1}^{n}\xi_{k-1}(X_k - X_{k-1})\bigg|\bigg] &\le \sum_{k=1}^{n}\mathbb{E}[|\xi_{k-1}(X_k - X_{k-1})|]\\ &\le \sum_{k=1}^{n} C_{k-1}\mathbb{E}[|X_k - X_{k-1}|] \le C\sum_{k=1}^{n}\big(\mathbb{E}[|X_k|] + \mathbb{E}[|X_{k-1}|]\big) < \infty.\end{aligned}$$

(iii) Next, we have

$$\begin{aligned}\mathbb{E}[V_{n+1} - V_n|\,\mathcal{F}_n\,] &= \mathbb{E}[\,\xi_n(X_{n+1} - X_n)\,|\,\mathcal{F}_n\,]\\ &= \xi_n\mathbb{E}[\,X_{n+1} - X_n\,|\,\mathcal{F}_n\,] = 0\end{aligned}$$

Therefore $\{V_n\}_{n=0,1,\dots}$ is indeed a martingale. Finally, the martingale property Proposition 1 implies that

$$\mathbb{E}[V_n] = \mathbb{E}[V_0] = 0, \qquad \text{for all } n.$$

1.2.2 Definitions and Properties of Martingales

Problem 1.4. Let $Y_1, Y_2, \dots$ be independent (though not necessarily identically distributed) random variables with common expectation $\mathbb{E}[Y_k] = 0$ for all k. Show that $\{X_n\}_{n=0,1,2,\dots}$ defined by

$$X_0 = 0 \qquad \text{and} \qquad X_n = \sum_{k=1}^{n} Y_k$$

is a martingale in discrete time with respect to its natural filtration $\mathcal{F}_n^X$.

Solution 1.4. Since Y_{n+1} is independent of $\mathcal{F}_n^X$,

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n + Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n|\mathcal{F}_n] + \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = X_n + \mathbb{E}[Y_{n+1}] = X_n.$$

Problem 1.5. Let $Y_1, Y_2, \dots$ be independent and nonnegative (though not necessarily identically distributed) random variables with common expectation $\mathbb{E}[Y_k] = 1$ for all k.

1. Show that $\{X_n\}_{n\geq 0}$ defined through

$$X_0 = 1 \qquad \text{and} \qquad X_n = \prod_{k=1}^{n} Y_k$$

   is a martingale.

2. Let Y_k be of the form $Y_k = e^{Z_k - c_k}$ for independent random variables Z_k with distribution $N(0, \sigma_k^2)$ and certain constants c_k. That is, determine c_k such that $\{X_n\}_{n\geq 0}$ is a martingale.

Problem 1.6. Let X be a random variable such that $\mathbb{E}[|X|] < \infty$ and $\mathscr{T}$ either $\{0, 1, 2, \dots\}$ or $[0, \infty)$. Show that

$$X_t := \mathbb{E}[X|\mathcal{F}_t], \qquad t \in \mathscr{T},$$

is a martingale.

Problem 1.7. Consider an urn with balls of two colors, red and green. Assume that initially there is on ball of each color in the urn. At each time step, a ball is chosen at random from the urn. If a red ball is chosen, it is returned and in addition another red ball is added to the urn. Similarly, if a green ball is chosen, it is returned together with another green balls. Let X_n denote the number of red balls in the urn after n draws. Then $X_0 = 1$ and X_n is a time-inhomogeneous Markov Chain. Let $M_n = X_n/(n + 2)$ be the fraction of red balls after n draws. Then M_n is a martingale with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s | s \in \mathscr{T}, s \leq t)$.