Week 2: First Order Semi-Linear PDEs

Introduction

We want to find a formal solution to the first order semilinear PDEs of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u).$$

Using a change of variables corresponding to characteristic lines, we can reduce the problem to a system of 3 ODEs. The solution follows by simply solving two ODEs in the resulting system.

Step 1: Formally, we want to solve the following system of PDEs

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{c(x,y,u)}.$$

Step 2: We first find the characteristic curve by solving the first pair,

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} \Leftrightarrow \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \Leftrightarrow \frac{dx}{dy} = \frac{a(x,y)}{b(x,y)}.$$

We will get a characteristic line of the form $C = f(x,y)$.

Step 3: We now can now find the general solution by solving either

$$\frac{dx}{a(x,y)} = \frac{du}{c(x,y,u)} \text{ or } \frac{dy}{b(x,y)} = \frac{du}{c(x,y,u)}.$$

We choose to solve the ODE that is easier to solve. We may need to use the characteristic curve and the implicit function theorem to write $f(x,y) = C$ as $y = y(C,x)$ or $x = x(C,y)$ to eliminate a variable to solve this ODE. When we solve the ODE, we must remember to write the constant of integration we get when we solve ODEs as a function $F(C) = F(f(x,y))$ instead.

Step 4: If we are given some initial conditions, then we can find the specific form of $F(C)$.

We will explain why this method works in the next section. We start by formally solving several examples using this technique.

Problems

Problem 1. Solve the initial value problem

$$u_t - 3u = 0, \quad u(0,x) = e^{-x^2}.$$

Solution 1. This is a linear first order PDE, so we can solve it using characteristic lines.

Step 1: We want to solve $u_t = 3u$. This gives us the system of equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{3u}.$$

Step 2: We begin by finding the characteristic curve. It suffices to solve

$$\frac{dt}{1} = \frac{dx}{0} \Rightarrow \frac{dx}{dt} = 0 \Rightarrow x = C.$$

Therefore, the characteristic curves are given by $C = x$.

Step 3: We now solve the first and third equation,

$$\frac{dt}{1} = \frac{du}{3u} \Rightarrow \frac{du}{dt} = 3u.$$

This is a separable ODE, which has solution

$$\log u = 3t + f(C) \Rightarrow u = e^{f(C)}e^{3t}.$$

Since $f(C)$ is an arbitrary function, we can redefine $e^{f(C)} =: g(C)$. Since $C = x$, we have our general solution is

$$u(t,x) = g(x)e^{3t}.$$

Step 4: We now use the initial value to solve for g. Since $u(0,x) = e^{-x^2}$ we have

$$e^{-x^2} = u(0,x) = g(x)e^{3\cdot 0} \Rightarrow g(x) = e^{-x^2},$$

so our particular solution is of the form

$$u(t,x) = e^{-x^2}e^{3t} = e^{3t-x^2}.$$

We can easily verify that these formal computations gives us a solution to the PDE.

Problem 2. Solve the initial value problem

$$u_t - 4u_x + u = 0, \quad u(0,x) = \frac{1}{1+x^2}.$$

Solution 2. This is a linear first order PDE, so we can solve it using characteristic lines.

Step 1: We want to solve $u_t - 4u_x = -u$. This gives us the system of equations

$$\frac{dt}{1} = \frac{dx}{-4} = \frac{du}{-u}.$$

Step 2: We begin by finding the characteristic curve. It suffices to solve

$$\frac{dt}{1} = \frac{dx}{-4} \Rightarrow \frac{dx}{dt} = -4 \Rightarrow x = -4t + C.$$

Therefore, the characteristic curves are given by $C = x + 4t$.

Step 3: We now solve the first and third equation,

$$\frac{dt}{1} = \frac{du}{-u} \Rightarrow \frac{du}{dt} = -u.$$

This is a separable ODE, which has solution

$$\log u = -t + f(C) \Rightarrow u = e^{f(C)}e^{-t}$$

Since $f(C)$ is an arbitrary function, we might can redefine $e^{f(C)} =: g(C)$. Since $C = x + 4t$, we have our general solution is

$$u(t,x) = g(x+4t)e^{-t}.$$

Step 4: We now use the initial value to solve for g. Since $u(0,x) = \frac{1}{1+x^2}$ we have

$$\frac{1}{1+x^2} = u(0,x) = g(x)e^{3\cdot 0} \Rightarrow g(x) = \frac{1}{1+x^2},$$

so our particular solution is of the form

$$u(t,x) = \frac{1}{1+(x+4t)^2} \cdot e^{-t}.$$

We can easily verify that these formal computations gives us a solution to the PDE.

Problem 3. Solve the initial value problem

$$2xyu_x + (x^2+y^2)u_y = 0$$

with $u(x,y) = \exp(x/(x-y))$ on $\{x+y=1\}$.

Solution 3. This is a linear first order PDE, so we can solve it using characteristic lines.

Step 1: We have the system of equations

$$\frac{dx}{2xy} = \frac{dy}{(x^2+y^2)} = \frac{du}{0}.$$

Step 2: We begin by finding the characteristic curve. It suffices to solve

$$\frac{dx}{2xy} = \frac{dy}{(x^2+y^2)} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\cdot\frac{x}{y} + \frac{1}{2}\cdot\frac{y}{x}.$$

This is a Homogenous ODE, which can be solved using the change of variables $w = \frac{y}{x}$. We have $\frac{dy}{dx} = x\frac{dw}{dx} + w$, so under this change of variables we have

$$x\frac{dw}{dx} + w = \frac{1}{2}\cdot w^{-1} + \frac{1}{2}\cdot w \Rightarrow x\frac{dw}{dx} = \frac{1}{2}\cdot w^{-1} - \frac{1}{2}\cdot w = \frac{1-w^2}{2w}.$$

This is a separable equation, so

$$\frac{2w}{1-w^2}dw = \frac{1}{x}dx \Rightarrow -\ln(1-w^2) = \ln x + D \Rightarrow e^{-D} = x(1-w^2) = \frac{x^2-y^2}{x}.$$

If we set $C = e^{-D}$, then $C = \frac{x^2-y^2}{x}$ is our characteristic curve.

Step 3: We now solve the first and third equation,

$$\frac{dx}{2xy} = \frac{du}{0} \Rightarrow \frac{du}{dx} = 0 \Rightarrow u = f(C).$$

Since $C = \frac{x^2-y^2}{x}$, we have our general solution is

$$u(x,y) = f\Big(\frac{x^2-y^2}{x}\Big).$$

Step 4: We now use the initial value to solve for f. Since $u(x,y) = \exp(x/(x-y))$ when $x+y=1$, we have

$$e^{\frac{x}{x-y}} = u(x,y)\big|_{x+y=1} = f\Big(\frac{x^2-y^2}{x}\Big)\Big|_{x+y=1} = f\Big(\frac{(x-y)(x+y)}{x}\Big)\Big|_{x+y=1} = f\Big(\frac{x-y}{x}\Big).$$

If we set $z = \frac{x-y}{x}$, then the above implies $e^{\frac{1}{z}} = f(z)$, so our particular solution is of the form

$$u(x,y) = e^{\frac{x}{x^2-y^2}}.$$

We can easily verify that these formal computations gives us a solution to the PDE.

Problem 4. Solve the initial value problem

$$xu_x + yu_y = xe^{-u}$$

with $u(x,y) = 0$ on $\{y = x^2\}$.

Solution 4. This is a semilinear first order PDE, so we can solve it using characteristic lines.

Step 1: We have the system of equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xe^{-u}}.$$

Step 2: We begin by finding the characteristic curve. It suffices to solve

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{dy}{dx} = \frac{y}{x}.$$

This is a separable ODE, which has solution

$$y = Cx$$

Therefore, our characteristic curve is $C = \frac{y}{x}$.

Step 3: We now solve the first and third equation,

$$\frac{dx}{x} = \frac{du}{xe^{-u}} \Rightarrow \frac{du}{dx} = e^{-u} \Rightarrow e^u = x + f(C).$$

Since $C = \frac{y}{x}$, we have our general solution is

$$u(x,y) = \ln\Big(x + f\Big(\frac{y}{x}\Big)\Big).$$

Step 4: We now use the initial value to solve for f. Since $u(x,y) = 0$ when $y = x^2$, we have

$$0 = u(x,y)\big|_{y=x^2} = \ln\Big(x + f\Big(\frac{y}{x}\Big)\Big)\Big|_{y=x^2} = \ln(x + f(x)) \Rightarrow f(x) = 1 - x.$$

Therefore, our particular solution is of the form

$$u(x,y) = \ln\Big(x + 1 - \frac{y}{x}\Big).$$

We can easily verify that these formal computations gives us a solution to the PDE.

Problem 5. Solve the initial value problem

$$xu_x + yu_y = 2x(x^2 - y^2)$$

when $u(1,y) = y^2$.

Solution 5. This is a linear first order PDE, so we can solve it using characteristic lines.

Step 1: We have the system of equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{2x(x^2-y^2)}.$$

Step 2: We begin by finding the characteristic curve. It suffices to solve

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{dy}{dx} = \frac{y}{x}.$$

This is a separable ODE, which has solution

$$y = Cx$$

Therefore, our characteristic curve is $C = \frac{y}{x}$.

Step 3: We now solve the first and third equation,

$$\frac{dx}{x} = \frac{du}{2x(x^2-y^2)} \Rightarrow \frac{du}{dx} = 2(x^2-y^2).$$

There is a y variable appearing in this ODE that we must eliminate it. Since $y = Cx$, we need to solve

$$\frac{du}{dx} = 2(x^2 - C^2x^2) \Rightarrow u = \frac{2(1-C^2)}{3}x^3 + f(C).$$

Since $C = \frac{y}{x}$, we have the general solution

$$u(x,y) = \frac{2}{3}\Big(1 - \Big(\frac{y}{x}\Big)^2\Big)x^3 + f\Big(\frac{y}{x}\Big).$$

Step 4: We now use the initial value to solve for f. Since $u(1,y) = y^2$ we have

$$y^2 = u(1,y) = \frac{2}{3}\Big(1-y^2\Big) + f(y) \Rightarrow f(y) = y^2 - \frac{2}{3}\Big(1-y^2\Big) = -\frac{2}{3} + \frac{5}{3}y^2.$$

Therefore, our particular solution is of the form

$$u(x,y) = \frac{2}{3}\Big(1 - \Big(\frac{y}{x}\Big)^2\Big)x^3 - \frac{2}{3} + \frac{5}{3}\Big(\frac{y}{x}\Big)^2.$$

We can easily verify that these formal computations gives us a solution to the PDE.

Problem 6. Find the general solution of

$$-2u_x + 4u_y = e^{x+3y} - 5u.$$

Solution 6. This is a linear first order PDE, so we can solve it using characteristic lines.

Step 1: We have the system of equations

$$\frac{dx}{-2} = \frac{dy}{4} = \frac{du}{e^{x+3y} - 5u}.$$

Step 2: We begin by finding the characteristic curve. It suffices to solve

$$\frac{dx}{-2} = \frac{dy}{4} \Rightarrow \frac{dy}{dx} = -2 \Rightarrow y = -2x + C.$$

Therefore, our characteristic curve is $C = y + 2x$.

Step 3: We now solve the first and third equation,

$$\frac{dx}{-2} = \frac{du}{e^{x+3y} - 5u} \Rightarrow \frac{du}{dx} = -\frac{1}{2}(e^{x+3y} - 5u) \Rightarrow \frac{du}{dx} - \frac{5}{2}u = -\frac{1}{2}e^{x+3y}.$$

There is a y variable appearing in this ODE that we must eliminate it. Since $y = C - 2x$, we need to solve

$$\frac{du}{dx} - \frac{5}{2}u = -\frac{1}{2}e^{-5x+3C}.$$

This is a linear ODE, which can be solved using an integrating factor of the form $\phi(x) = e^{-\frac{5}{2}x}$, which gives us

$$u = e^{\frac{5}{2}x}\left(-\frac{1}{2}\int e^{-5x+3C}e^{-\frac{5}{2}x}\,dx\right) = -\frac{1}{2}e^{\frac{5}{2}x}\left(\frac{2e^{-\frac{15}{2}x} + 3C}{-15} + f(C)\right) \Rightarrow u = \frac{1}{15}e^{-5x+3C} - \frac{1}{2}f(C)e^{\frac{5}{2}x}.$$

Since $C = y + 2x$, if we set $g(z) = -\frac{1}{2}f(z)$ then we get the general solution

$$u(x,y) = \frac{1}{15}e^{x+3y} + g(y+2x)e^{\frac{5}{2}x}.$$

We can easily verify that this solves our PDE.

Problem 7. Find the general solution of

$$xu_x - yu_y + y^2u = y^2, \text{ where } y \neq 0$$

Solution 7. This is a linear first order PDE, so we can solve it using characteristic lines.

Step 1: We want to solve $xu_x - yu_y = y^2 - y^2u$. We have the system of equations

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{du}{y^2 - y^2u}.$$

Step 2: We begin by finding the characteristic curve. It suffices to solve

$$\frac{dx}{x} = \frac{dy}{-y} \Rightarrow \frac{dy}{dx} = -\frac{y}{x}.$$

This is a separable ODE, with solution

$$\ln y = -\ln x + D \Rightarrow e^D = xy$$

If we set $C = e^D$ then our characteristic curve is $C = xy$.

Step 3: We now solve the second and third equation,

$$\frac{dy}{-y} = \frac{du}{y^2 - y^2u} \Rightarrow \frac{du}{dy} = -y + yu \Rightarrow \frac{du}{dx} - yu = -y.$$

This is a linear ODE, which can be solved using an integrating factor of the form $\phi(y) = e^{-y^2/2}$, which gives us

$$u = e^{y^2/2}\left(\int -ye^{-y^2/2}\,dy\right) = e^{y^2/2}\left(e^{-y^2/2} + f(C)\right) = 1 + f(C)e^{y^2/2}.$$

Since $C = xy$, we get the general solution

$$u(x,y) = 1 + f(xy)e^{y^2/2}.$$

We can easily verify that this solves our PDE.

Problem 8. Solve the initial value problem

$$u_y + yu_x = 0, \quad u(0,y) = \sin(y^2).$$

In which region of the xy plane is the solution uniquely determined by the initial condition.

Solution 8. This is a linear first order PDE, so we can solve it using characteristic lines.

Step 1: We have the system of equations

$$\frac{dx}{y} = \frac{dy}{1} = \frac{du}{0}.$$

Step 2: We begin by finding the characteristic curve. It suffices to solve

$$\frac{dx}{y} = \frac{dy}{1} \Rightarrow \frac{dy}{dx} = \frac{1}{y}.$$

This is a separable ODE, with solution

$$\frac{y^2}{2} = x + C \Rightarrow C = \frac{y^2}{2} - x.$$

Step 3: We now solve the second and third equation,

$$\frac{dy}{1} = \frac{du}{0} \Rightarrow \frac{du}{dy} = 0 \Rightarrow u = f(C).$$

Since $C = \frac{y^2}{2} - x$, we get the general solution

$$u(x,y) = f(\frac{y^2}{2} - x).$$

Step 4: We now use the initial value to solve for f. Since $u(0,y) = \sin(y^2)$ we have

$$\sin(y^2) = u(0,y) = f(\frac{y^2}{2}).$$

If we set $z = \frac{y^2}{2}$, then we have

$$f(z) = \sin(2z)$$

Therefore, our particular solution is of the form

$$u(x,y) = \sin(y^2 - 2x).$$

We can easily verify that these formal computations gives us a solution to the PDE. However, since $z = \frac{y^2}{2} \geq 0$, this initial condition only specified the values of $f(z)$ for $z \geq 0$. Therefore, our solution is only uniquely determined for $\frac{y^2}{2} - x \geq 0 \Rightarrow x \leq \frac{y^2}{2}$.

Proof of the Coordinate Method

We now explain why the above method works for certain semilinear PDEs that can be reduced to solvable ODEs. We essentially do a change of variables to reduce the PDE into an ODE. Suppose we have a PDE of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u). \tag{$*$}$$

Step 1: We want to solve the equation

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} \Rightarrow \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}.$$

Suppose we can find a family of solutions $f(x,y) = C$, called the characteristic curves. Since this is a solution to the ODE, by implicitly differentiating, we must have

$$0 = \frac{d}{dx} f(x,y) = f_x(x,y) + f_y(x,y)\frac{dy}{dx} = f_x(x,y) + f_y(x,y)\frac{b(x,y)}{a(x,y)}$$

and therefore, our the family of solutions must satisfy the condition

$$0 = a(x,y)f_x(x,y) + b(x,y)f_y(x,y). \tag{1}$$

We will see that using this $f(x,y)$, we can reduce our PDE into either an ODE with respect to x (Choice 1) or an ODE with respect to y (Choice 2).

Step 2 (Choice 1): We now explain why it suffices to solve the equations

$$\frac{dx}{a(x,y)} = \frac{du}{c(x,y,u)} \Rightarrow \frac{du}{dx} = \frac{c(x,y,u)}{a(x,y)}.$$

To reduce ($*$) to this ODE, we do a change of variables

$$\xi(x,y) = x, \quad \eta(x,y) = f(x,y)$$

where $f(x,y) = C$ is the function we found in Step 1. Notice that

$$u_x = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = u_\xi + u_\eta f_x(x,y)$$

and

$$u_y = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = u_\eta f_y(x,y)$$

therefore we have

$$\begin{aligned} a(x,y)u_x + b(x,y)u_y &= a(x,y)(u_\xi f_x(x,y)u_\eta) + b(x,y)f_y(x,y)u_\eta \\ &= a(x,y)u_\xi + (a(x,y)f_x(x,y) + b(x,y)f_y(x,y))u_\eta \\ &= a(x,y)u_\xi(\xi,\eta) \end{aligned}$$

since $f(x,y)$ satisfies (1). Since our PDE satisfies ($*$), we have shown that

$$u_\xi(\xi,\eta) = \frac{c(x(\xi,\eta),y(\xi,\eta),u)}{a(x(\xi,\eta),y(\xi,\eta))}$$

We can write this back in our original coordinates $\xi = x$, $\eta = f(x,y) = C$. If the characteristic curve $f(x,y) = C$ can be solved implictly for $y = y(x,C)$, then we can eliminate the y variable, which gives us

$$u_x(x,C) = \frac{c(x,y(x,C),u)}{a(x,y(x,C))}$$

This is precisely the ODE we are solving if we choose the first ODE in Step 3. This also explains why the integration constant is of the form $F(C)$ instead of just a constant.

Step 2 (Choice 2): It might be easier to instead solve the system

$$\frac{dy}{b(x,y)} = \frac{du}{c(x,y,u)} \Rightarrow \frac{du}{dy} = \frac{c(x,y,u)}{b(x,y)}.$$

To reduce $(*)$ to this ODE, we instead use the change of variables,

$$\xi(x,y) = y, \quad \eta(x,y) = f(x,y),$$

where $f(x,y) = C$ is the function we found in Step 1. Notice that

$$u_x = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = u_\eta f_x(x,y)$$

and

$$u_y = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = u_\xi + u_\eta f_y(x,y)$$

so we have

$$\begin{aligned} a(x,y)u_x + b(x,y)u_y &= a(x,y)(f_x(x,y)u_\eta) + b(x,y)(u_\xi + f_y(x,y)u_\eta) \\ &= b(x,y)u_\xi(\xi,\eta) + (a(x,y)f_x(x,y) + b(x,y)f_y(x,y))u_\eta \\ &= b(x,y)u_\xi(\xi,\eta) \end{aligned}$$

since $f(x,y)$ satisfies (1). Since our PDE satisfies $(*)$, we have shown that

$$u_\xi(\xi,\eta) = \frac{c(x(\xi,\eta), y(\xi,\eta), u)}{a(x(\xi,\eta), y(\xi,\eta))}$$

We can write this back in our original coordinates $\xi = y$, $\eta = f(x,y) = C$. If the characteristic curve $f(x,y) = C$ can be solved implictly for $x = x(y,C)$, then we can eliminate the x variable, which gives us

$$u_y(C,y) = \frac{c(x(y,C), y, u)}{a(x(y,C), y)}$$

This is precisely the ODE we are solving if we choose the second ODE in Step 3. This also explains why the integration constant is of the form $F(C)$ instead of just a constant.

Remark: Notice that the above procedure works even if one of the coefficients vanish (i.e. $a \equiv 0$, $b \equiv 0$ or $c \equiv 0$). We just lose some choice as to the equations we solve, since if $a \equiv 0$, then we must use Choice 2 because Choice 1 does not make sense. The derivation assumed that the coefficients were sufficiently nice such that we can find solutions to the ODEs at least locally. We should verify that our formal solutions are true solutions by checking our solutions after deriving them.