1 Eigenvalue Problems

Problem 1.1. Find the positive eigenvalues and eigenfunctions of

$$\begin{cases} X^{(4)} = \lambda X & 0 < x < L \\ X(0) = X(L) = X''(0) = X''(L) = 0 \end{cases}$$

Solution 1.1. We want to find non-trivial solutions to the eigenvalue problem i.e. $X(x) \neq 0$.

Step 1: We are interested in positive eigenvalues, so we can set $\lambda = \beta^4 > 0$, where $\beta > 0$. We first find the general solution to the ODE

$$X^{(4)} = \beta^4 X \qquad 0 < x < L.$$

This is a fourth order constant coefficient ODE with roots $r = \pm\beta, \pm\beta i$, which corresponds to the solution

$$X(x) = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x).$$

Step 2: We now solve for the values of β that satisfy the boundary conditions. Plugging the solution into the initial conditions gives

$$\begin{aligned} A + C &= 0 \\ A\cos(\beta L) + B\sin(\beta L) + C\cosh(\beta L) + D\sinh(\beta L) &= 0 \\ -\beta^2 A + \beta^2 C &= 0 \\ -A\beta^2\cos(\beta L) - B\beta^2\sin(\beta L) + C\beta^2\cosh(\beta L) + D\beta^2\sinh(\beta L) &= 0. \end{aligned}$$

Since $\beta > 0$, the first and third equation implies $A + C = 0$ and $-A + C = 0$ which can only happen when $A = C = 0$. We now have to solve the system

$$\begin{aligned} B\sin(\beta L) + D\sinh(\beta L) &= 0 \\ -B\beta^2\sin(\beta L) + D\beta^2\sinh(\beta L) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2\sin(\beta L) & \beta^2\sinh(\beta L) \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has a non-trivial solution when

$$\det\left(\begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2\sin(\beta L) & \beta^2\sinh(\beta L) \end{bmatrix} \right) = 0 \implies 2\beta^2\sinh(\beta L)\sin(\beta L) = 0.$$

Since $\beta > 0$ and $\sinh(\beta L) > 0$, this simplifies to $\sin(\beta L) = 0$, which occurs precisely when

$$\beta_n = \frac{n\pi}{L}, \quad n \geq 1.$$

Therefore, the corresponding eigenvalues are $\lambda_n = \frac{n^4\pi^4}{L^4}$. Furthermore, notice that from the equation

$$B\sin(\beta_n L) + D\sinh(\beta_n L) = 0,$$

we must have $D = 0$ since $\sinh(\beta_n L) > 0$ and $\sin(\beta_n L) = 0$. Finally, for $n \geq 1$, the corresponding eigenfunction (taking $B = 1$) for the eigenvalue $\lambda_n = (\frac{n\pi}{L})^4$ is

$$X_n(x) = \sin\Big(\frac{n\pi x}{L}\Big).$$

2 Separation of Variables

Problem 2.1. Consider wave equation with the Neumann boundary condition on the left and weird b.c. on the right:

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0 && 0 < x < l, \\ &u_x(0,t) = 0, \\ &(u_x + i\alpha u_t)(l,t) = 0 \end{aligned}$$

with $\alpha \in \mathbb{R}$.

1. Separate variables
2. Find the "weird" eigenvalue problem for the ODE;
3. Solve the problem;
4. Find the simple solution $u(x,t) = X(x)T(t)$.

Hint: You may assume that all the eigenvalues are real (which is the case).

Solution 2.1. We will use the method of separation of variables to find a complex solution to this PDE.

Step 1 — Separation of Variables: We look for a separated solution $u(x,t) = T(t)X(x)$ to our PDE. Plugging this into our PDE gives

$$T''(t)X(x) - c^2 T(t)X''(x) = 0 \implies \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda.$$

This gives the following ODEs

$$X''(x) - \lambda X(x) = 0 \quad \text{and} \quad T''(t) - c^2 \lambda T(t) = 0,$$

with boundary conditions

$$X'(0)T(t) = 0 \quad \text{and} \quad X'(l)T(t) + i\alpha X(l)T'(t) = 0.$$

Instead of trying to find an eigenvalue problem for $X(x)$ like in the usual cases, we first solve the second order ODEs, which gives

$$X_\lambda(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}, \qquad T_\lambda(t) = Ce^{c\sqrt{\lambda}t} + De^{-c\sqrt{\lambda}t} \qquad \text{when } \lambda \neq 0.$$

and

$$X_0(x) = A + Bx, \qquad T_0(t) = C + Dt \qquad \text{when } \lambda = 0.$$

Step 2 — Solve for λ: We use the boundary conditions to solve for $\lambda \in \mathbb{R}$ (by the hint), and its corresponding eigenfunctions X_λ and T_λ.

Zero Eigenvalue: When $\lambda = 0$, then the first initial condition implies that

$$B = 0,$$

so $X_0(x) = A$. The second initial condition, then implies that

$$i\alpha DA = 0 \implies D = 0,$$

otherwise we are left with a trivial solution. Therefore, we must have $T_0(t) = C$, which means that the constant function

$$u(x,t) = X_0(x)T_0(t) \equiv E$$

where $E \in \mathbb{C}$ is the solution corresponding to the zero eigenvalue.

Non-Zero Eigenvalues: We now find $\lambda \neq 0$ that satisfies the boundary conditions

$$T_\lambda(t)X_\lambda'(0) = 0, \tag{1}$$

$$X_\lambda'(l)T_\lambda(t) + i\alpha X_\lambda(l)T_\lambda'(t) = 0. \tag{2}$$

1. First Boundary Condition: For all t, we must have

$$(A\sqrt{\lambda} - B\sqrt{\lambda})(Ce^{c\sqrt{\lambda}t} + De^{-c\sqrt{\lambda}t}) = 0.$$

   Since $\sqrt{\lambda} \neq 0$ and the equation must hold for all t, we need $A = B \neq 0$. In other words, we need $X_\lambda(x)$ to be an even function.

2. Second Boundary Condition: For all t, we must have

$$A(\sqrt{\lambda}e^{\sqrt{\lambda}l} - \sqrt{\lambda}e^{-\sqrt{\lambda}l})(Ce^{c\sqrt{\lambda}t} + De^{-c\sqrt{\lambda}t}) + i\alpha cA(e^{\sqrt{\lambda}l} + e^{-\sqrt{\lambda}l})(C\sqrt{\lambda}e^{c\sqrt{\lambda}t} - D\sqrt{\lambda}e^{-c\sqrt{\lambda}t}) = 0$$

   and rearranging and dividing out by $A\sqrt{\lambda} \neq 0$ gives

$$(e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l})(Ce^{c\sqrt{\lambda}t} + De^{-c\sqrt{\lambda}t}) + i\alpha c(e^{\sqrt{\lambda}l} + e^{-\sqrt{\lambda}l})(Ce^{c\sqrt{\lambda}t} - De^{-c\sqrt{\lambda}t}) = 0$$

   We can now use the fact $e^{c\sqrt{\lambda}t}$ and $e^{-c\sqrt{\lambda}t}$ are linearly independent to equate coefficients to solve for λ such that the above equation holds for all t. If $D = 0$ and $C \neq 0$, we can divide by $e^{c\sqrt{\lambda}t}$ to conclude that

$$(e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l}) + (e^{\sqrt{\lambda}l} + e^{-\sqrt{\lambda}l})i\alpha c = 0 \implies \tanh(\sqrt{\lambda}l) = -i\alpha c \implies \sqrt{\lambda} = \frac{i(\pi n - \tan^{-1}(\alpha c))}{l}$$

   and if $C = 0$ and $D \neq 0$ we can divide by $e^{-c\sqrt{\lambda}t}$ to conclude that

$$(e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l}) - (e^{\sqrt{\lambda}l} + e^{-\sqrt{\lambda}l})i\alpha c = 0 \implies \tanh(\sqrt{\lambda}l) = i\alpha c \implies \sqrt{\lambda} = \frac{i(\pi n + \tan^{-1}(\alpha c))}{l}.$$

   We used the identity $\tanh(x) = -i\tan(ix)$ in our simplification above.

We get two families of solutions depending on the eigenvalue,

$$X_{\lambda_n}(x)T_{\lambda_n}(t) = (e^{\sqrt{\lambda_n}x} + e^{-\sqrt{\lambda_n}x})e^{c\sqrt{\lambda_n}t} \quad \text{where} \quad \sqrt{\lambda_n} = \frac{i(\pi n - \tan^{-1}(\alpha c))}{l}$$

and

$$X_{\gamma_n}(x)T_{\gamma_n}(t) = (e^{\sqrt{\gamma_n}x} + e^{-\sqrt{\gamma_n}x})e^{-c\sqrt{\gamma_n}t} \quad \text{where} \quad \sqrt{\gamma_n} = \frac{i(\pi n + \tan^{-1}(\alpha c))}{l}.$$

Step 3 — General Simple Solution: Therefore, taking the linear combination of our solutions, we have

$$u(x,t) = E + \sum_{i=-\infty}^{\infty} C_n(e^{\sqrt{\lambda_n}x} + e^{-\sqrt{\lambda_n}x})e^{c\sqrt{\lambda_n}t} + \sum_{i=-\infty}^{\infty} D_n(e^{\sqrt{\gamma_n}x} + e^{-\sqrt{\gamma_n}x})e^{-c\sqrt{\gamma_n}t}$$

where $C_n, D_n, E \in \mathbb{C}$ and

$$\sqrt{\lambda_n} = \frac{i(\pi n - \tan^{-1}(\alpha c))}{l} \quad \text{and} \quad \sqrt{\gamma_n} = \frac{i(\pi n + \tan^{-1}(\alpha c))}{l}.$$