Laplace's Equation in Polar Coordinates

Problem 1. Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition

$$u = 1 + 2\sin(\theta) \quad \text{on } r = a.$$

Solution 1. After converting to polar coordinates, our PDE can be written as the following problem on the circle

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < r < a, \quad -\pi \leq \theta \leq \pi \\ u(r,-\pi) = u(r,\pi) & 0 < r < a \\ u_\theta(r,-\pi) = u_\theta(r,\pi) & 0 < r < a \\ u(a,\theta) = 1 + 2\sin(\theta) & -\pi \leq \theta \leq \pi \\ \lim_{r\to 0} u(r,\theta) < \infty & -\pi \leq \theta \leq \pi \end{cases}$$

Step 1 — Separation of Variables: The PDE has periodic homogeneous angular boundary conditions, so we look for a solution of the form $u(r,\theta) = R(r)\Theta(\theta)$. For such a solution, the PDE implies

$$\Delta u = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \implies -\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda.$$

This results in the ODEs

$$r^2R''(r) + rR(r) - \lambda R'(r) = 0 \text{ and } \Theta''(\theta) + \lambda\Theta(\theta) = 0$$

with angular boundary conditions

$$R(r)\Theta(-\pi) = R(r)\Theta(\pi) = 0, R(r)\Theta'(-\pi) = R(r)\Theta'(\pi) = 0$$

and radial boundary conditions

$$R(a)\Theta(\theta) = 1 + 2\sin(\theta) \qquad \lim_{r\to 0} R(r)\Theta(\theta) < \infty.$$

For non-trivial solutions to the angle problem, we require $R(r) \not\equiv 0$, $\Theta(-\pi) = \Theta(\pi)$, $\Theta'(-\pi) = \Theta'(\pi)$.

Step 2 — Eigenvalue Problem: We now solve the periodic angular eigenvalue problem

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(\pi) - \Theta(-\pi) = \Theta'(\pi) - \Theta'(-\pi) = 0. \end{cases}$$

The eigenvalues and corresponding eigenfunctions are given by the full Fourier series

$$\lambda_0 = 0,\ \Theta_0(x) = 1, \quad \lambda_n = n^2,\ \Theta_n(x) = \cos(nx),\ \Phi_n(x) = \sin(nx), \quad n = 1, 2, \ldots.$$

Step 3 — Radial Problem: We now solve the radial problem for each eigenvalue. The ODE

$$r^2R'' + rR' - \lambda R = 0$$

is an Euler ODE with solutions

$$R_0(r) = C_0 \log r + D_0, \qquad R_n(r) = C_n r^{-n} + D_n r^n, \qquad n = 1, 2, \ldots$$

Since the solution should be regular at 0 ($\lim_{r\to 0} R(r) < \infty$), we need $C_n = 0$ for all $n \geq 0$, so our solution is of the form

$$R_0(r) = D_0, \qquad R_n(r) = D_n r^n, \qquad n = 1, 2, \ldots$$

for some arbitrary coefficients D_0 and D_n.

Step 4 — General Solution: Using the principle of superposition, and summing all the eigenfunctions gives us the general solution

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

where A_0, A_n and B_n are yet to be determined coefficients.

Step 5 — Particular Solution: To find constants A_0, A_n and B_n we need to use the boundary condition. Using the boundary condition we get

$$1 + 2\sin(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

Instead of solving for the Fourier series like usual, we can just equate coefficients to see that

$$A_0 = 1, \qquad a^1 A_1 = 2 \implies A_1 = \frac{2}{a},$$

and the rest of the coefficients are 0.

Step 6 — Final Answer: To summarize, the solution to the PDE is given by

$$u(r,\theta) = 1 + \frac{2}{a} r \sin(\theta).$$

Problem 2. Solve $u_{xx} + u_{yy} = 0$ in the wedge $r < a$, $0 < \theta < \beta$ with the BCs

$$u = 2\theta \text{ on r = a}, \quad u = 0 \text{ on } \theta = 0, \quad \text{and } u = \beta \text{ on } \theta = \beta.$$

Solution 2. After converting to polar coordinates, our PDE can be written as the following problem on the wedge

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < r < a, \quad 0 < \theta < \beta \\ u(r,0) = 0 & 0 < r < a \\ u(r,\beta) = \beta & 0 < r < a \\ u(a,\theta) = 2\theta & 0 < \theta < \beta. \end{cases}$$

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with homogeneous angular boundary conditions. We set

$$u(r,\theta) = v(r,\theta) + w(r,\theta)$$

where $w(r,\theta)$ is chosen to satisfy the inhomogeneous boundary conditions. Like usual, we can take $w(r,\theta)$ to be a polynomial of the form

$$w(r,\theta) = (A\theta^2 + B\theta + C) \cdot \beta$$

for some constants A, B, C. Substituting $w(r,\theta)$ in the boundary conditions gives

$$C = 0 = w(r,0)$$
$$(A\beta^2 + B\beta + C)\beta = \beta = w(r,\beta).$$

By inspection it is clear that $A = 0$, $B = 1/\beta$, and $C = 0$ zero works. Therefore,

$$w(r, \theta) = \theta.$$

Step 2 — Separation of Variables: Since $v(r, \theta) = u(r, \theta) - w(r, \theta)$, our choice of $w(r, \theta)$ implies

$$\begin{cases} v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0 & 0 < r < a, \quad 0 < \theta < \beta \\ v(r, 0) = 0 & 0 < r < a \\ v(r, \beta) = 0 & 0 < r < a \\ v(a, \theta) = 2\theta - \theta = \theta & 0 < \theta < \beta. \end{cases} \tag{$*$}$$

This now has homogeneous angular boundary conditions, so we can use separation of variables and look for a solution of the form $v(r, \theta) = R(r)\Theta(\theta)$. For such a solution, the PDE implies

$$\Delta v = R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0 \implies -\frac{r^2 R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda.$$

This results in the ODEs

$$r^2 R''(r) + rR(r) - \lambda R'(r) = 0 \text{ and } \Theta''(\theta) + \lambda\Theta(\theta) = 0$$

with angular boundary conditions

$$R(r)\Theta(0) = R(r)\Theta(\beta) = 0.$$

and radial boundary conditions (and regularity condition)

$$R(a)\Theta(\theta) = \theta, \quad \lim_{r \to 0} R(r)\Theta(\theta) < \infty$$

For non-trivial solutions to the angle problem, we require $R(r) \not\equiv 0$, $\Theta(0) = \Theta(\beta)$.

Step 3 — Eigenvalue Problem: We now solve the angular eigenvalue problem

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 & 0 < \theta < \beta \\ \Theta(0) = \Theta(\beta) = 0. \end{cases}$$

This is a standard problem, and the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{\beta}\right)^2, \ \Theta_n(\theta) = \sin\left(\frac{n\pi}{\beta}\theta\right), \quad n = 1, 2, \ldots.$$

Step 4 — Radial Problem: For each eigenvalue, we solve the radial problem

$$r^2 R''(r) + rR'(r) - \left(\frac{n\pi}{\beta}\right)^2 R(r) = 0.$$

This is an Euler ODE with characteristic equation $C(r) = r(r-1) + r - (\frac{n\pi}{\beta})^2$ and roots $r = \pm\frac{n\pi}{\beta}$, which has general solution of the form

$$R_n(r) = A_n r^{\frac{n\pi}{\beta}} + B_n r^{-\frac{n\pi}{\beta}}$$

for some yet to be determined coefficients A_n and B_n. Since the solution should be regular at 0 ($\lim_{r \to 0} R(r) < \infty$), we need $B_n = 0$, so our solution is of the form

$$R_n(r) = A_n r^{\frac{n\pi}{\beta}}, \qquad n = 1, 2, \ldots$$

for some yet to be determined coefficient A_n. Using the principle of superposition, and taking a linear combination of the eigenfunctions gives the general solution

$$v(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\beta}} \sin\Big(\frac{n\pi}{\beta}\theta\Big).$$

Step 5 — Particular Solution: We now use the radial boundary condition to find A_n. Plugging the general solution into the boundary conditions, $R(a)\Theta(\theta) = \theta$ implies

$$\sum_{n=1}^{\infty} A_n a^{\frac{n\pi}{\beta}} \sin\Big(\frac{n\pi}{\beta}\theta\Big) = \theta.$$

By the Fourier sine series, we have

$$a^{\frac{n\pi}{\beta}} A_n = \frac{2}{\beta}\int_0^\beta \theta \sin\Big(\frac{n\pi}{\beta}\theta\Big)\, d\theta = \theta \implies A_n = a^{-\frac{n\pi}{\beta}} \frac{2}{\beta}\int_0^\beta \theta \sin\Big(\frac{n\pi}{\beta}\theta\Big)\, d\theta.$$

Step 6 — Final Answer: To summarize, since $u(r,\theta) = v(r,\theta) + w(r,\theta)$ we have

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\beta}} \sin\Big(\frac{n\pi}{\beta}\theta\Big) + \theta.$$

where the coefficients A_n are given by

$$A_n = a^{-\frac{n\pi}{\beta}} \frac{2}{\beta}\int_0^\beta \theta \sin\Big(\frac{n\pi}{\beta}\theta\Big)\, d\theta.$$

Problem 3. Solve $u_{xx} + u_{yy} = 0$ in the region $\{\alpha < \theta < \beta, a < r < b\}$ with the boundary conditions $u = 0$ on the two sides $\theta = \alpha$ and $\theta = \beta$, $u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.

Solution 3. After converting to polar coordinates, our PDE can be written as the following problem on the wedge of an annuli

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < a < r < b, \quad \alpha < \theta < \beta \\ u(r,\alpha) = 0 & 0 < a < r < b \\ u(r,\beta) = 0 & 0 < a < r < b \\ u(a,\theta) = g(\theta) & \alpha < \theta < \beta \\ u(b,\theta) = h(\theta) & \alpha < \theta < \beta \end{cases}$$

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with symmetric homogeneous angular boundary conditions. We use rotation invariance, and set

$$v(r,\theta) = u(r,\theta + \alpha).$$

By rotational invariance, it is easy to see that $v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ and the domain of $v(r,\theta)$ is the centered wedge of the annuli $\{0 < \theta < \beta - \alpha, a < r < b\}$.

Step 2 — Separation of Variables: Since $v(r,\theta) = u(r,\theta+\alpha)$, our choice of $w(r,\theta)$ implies

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 & 0 < a < r < b, \quad 0 < \theta < \beta - \alpha \\ v(r,0) = 0 & 0 < a < r < b \\ v(r,\beta-\alpha) = 0 & 0 < a < r < b \\ v(a,\theta) = g(\theta+\alpha) & 0 < \theta < \beta - \alpha \\ v(b,\theta) = h(\theta+\alpha) & 0 < \theta < \beta - \alpha. \end{cases}$$

This PDE now has symmetric homogeneous angular boundary conditions, so we look for a solution of the form $v(r,\theta) = R(r)\Theta(\theta)$. For such a solution, the PDE implies

$$\Delta v = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \implies -\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda.$$

This results in the ODEs

$$r^2R''(r) + rR(r) - \lambda R'(r) = 0 \text{ and } \Theta''(\theta) + \lambda\Theta(\theta) = 0$$

with angular boundary conditions

$$R(r)\Theta(0) = R(r)\Theta(\beta - \alpha) = 0,$$

and radial boundary conditions

$$R(a)\Theta(\theta) = g(\theta + \alpha), \quad R(b)\Theta(\theta) = h(\theta + \alpha).$$

For non-trivial solutions to the angle problem, we require $R(r) \not\equiv 0$, $\Theta(0) = \Theta(\beta - \alpha)$.

Step 3 — Eigenvalue Problem: We now solve the angular eigenvalue problem

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 & 0 < \theta < \beta \\ \Theta(0) = \Theta(\beta - \alpha) = 0. \end{cases}$$

This is a standard problem, and the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \Big(\frac{n\pi}{\beta - \alpha}\Big)^2, \ \Theta_n(\theta) = \sin\Big(\frac{n\pi}{\beta - \alpha}\theta\Big), \quad n = 1, 2, \dots.$$

Step 4 — Radial Problem: For each eigenvalue, we solve the radial problem

$$r^2R''(r) + rR'(r) - \Big(\frac{n\pi}{\beta - \alpha}\Big)^2 R(r) = 0.$$

This is an Euler ODE with characteristic equation $C(r) = r(r-1) + r - (\frac{n\pi}{\beta-\alpha})^2$ and roots $r = \pm\frac{n\pi}{\beta-\alpha}$, which has general solution of the form

$$R_n(r) = A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}$$

for some yet to be determined coefficients A_n and B_n. Using the principle of superposition, and taking a linear combination of the eigenfunctions gives the general solution

$$v(r,\theta) = \sum_{n=1}^{\infty} \big(A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}\big) \sin\Big(\frac{n\pi}{\beta - \alpha}\theta\Big).$$

Step 5 — Particular Solution: We now use the radial boundary condition to find A_n. Plugging the general solution into the radial boundary conditions implies,

$$\sum_{n=1}^{\infty} \big(A_n a^{\frac{n\pi}{\beta-\alpha}} + B_n a^{-\frac{n\pi}{\beta-\alpha}}\big) \sin\Big(\frac{n\pi}{\beta - \alpha}\theta\Big) = g(\theta + \alpha)$$

and

$$\sum_{n=1}^{\infty} \big(A_n b^{\frac{n\pi}{\beta-\alpha}} + B_n b^{-\frac{n\pi}{\beta-\alpha}}\big) \sin\Big(\frac{n\pi}{\beta - \alpha}\theta\Big) = h(\theta + \alpha).$$

By the Fourier sine series, we have

$$A_n a^{\frac{n\pi}{\beta-\alpha}} + B_n a^{-\frac{n\pi}{\beta-\alpha}} = \frac{2}{\beta-\alpha}\int_0^{\beta-\alpha} g(\theta+\alpha)\sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big)\,d\theta = \frac{2}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta,$$

and

$$A_n b^{\frac{n\pi}{\beta-\alpha}} + B_n b^{-\frac{n\pi}{\beta-\alpha}} = \frac{2}{\beta-\alpha}\int_0^{\beta-\alpha} h(\theta+\alpha)\sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big)\,d\theta = \frac{2}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta.$$

This system can be written as a 2×2 matrix with linearly independent columns (since $a \neq b$), so we may solve for A_n and B_n if we wish. I leave it in this form, because the simplification does not produce a nicer answer.

Step 6 — Final Answer: To summarize, since $v(r,\theta) = u(r,\theta+\alpha)$ we have $u(r,\theta) = v(r,\theta-\alpha)$, we have

$$u(r,\theta) = \big(A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}\big)\sin\Big(\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\Big),$$

where A_n and B_n are solutions to the linear system

$$\begin{aligned} a^{\frac{n\pi}{\beta-\alpha}} A_n + a^{-\frac{n\pi}{\beta-\alpha}} B_n &= \tfrac{2}{\beta-\alpha}\textstyle\int_\alpha^\beta g(\theta)\sin\Big(\tfrac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta \\ b^{\frac{n\pi}{\beta-\alpha}} A_n + b^{-\frac{n\pi}{\beta-\alpha}} B_n &= \tfrac{2}{\beta-\alpha}\textstyle\int_\alpha^\beta h(\theta)\sin\Big(\tfrac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta \end{aligned}.$$

Remark: To solve this system, we can use the formula for the inverse of a 2×2 matrix. If we define $I_n = \frac{2}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta$ and $J_n = \frac{2}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta$, this gives

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \frac{1}{a^{\frac{n\pi}{\beta-\alpha}} b^{-\frac{n\pi}{\beta-\alpha}} - a^{-\frac{n\pi}{\beta-\alpha}} b^{\frac{n\pi}{\beta-\alpha}}} \begin{bmatrix} b^{-\frac{n\pi}{\beta-\alpha}} & -a^{-\frac{n\pi}{\beta-\alpha}} \\ -b^{\frac{n\pi}{\beta-\alpha}} & a^{\frac{n\pi}{\beta-\alpha}} \end{bmatrix} \times \begin{bmatrix} I_n \\ J_n \end{bmatrix}.$$

That is, for $C_n = a^{\frac{n\pi}{\beta-\alpha}} b^{-\frac{n\pi}{\beta-\alpha}} - a^{-\frac{n\pi}{\beta-\alpha}} b^{\frac{n\pi}{\beta-\alpha}}$ we have

$$A_n = \frac{1}{C_n}\cdot\Bigg(\frac{2b^{-\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta - \frac{2a^{-\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta\Bigg)$$

and

$$B_n = \frac{1}{C_n}\cdot\Bigg(-\frac{2b^{\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta + \frac{2a^{\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta\Bigg).$$