Classifying PDEs

Problem 1. Consider first order equations and determine if they are linear homogeneous, linear inhomogeneous, or nonlinear; for nonlinear equations, indicate if they are also semilinear, or quasilinear:

$$u_t + xu_x - u = 0, \tag{1}$$
$$u_t + u_x - u^2 = 0, \tag{2}$$
$$u_t + uu_x + x = 0. \tag{3}$$

Solution 1.

(1) Since the coefficients in front of u_t, u_x, and u are functions of x and t only, the equation is linear. There is also no term that depends only on x or t, so it is homogeneous. To prove that the equation is linear, notice that

$$\begin{aligned} L[au + bv] &= (au + bv)_t + x(au + bv)_x - (au + bv) \\ &= a(u_t + xu_x - u) + b(v_t + xv_x - v) \\ &= aL[u] + bL[v]. \end{aligned}$$

(2) There is a u^2 term, so the function is nonlinear. However, the coefficients of the highest order terms are functions of x and t, so the function is semilinear. To prove that the operator is non-linear, we show that the scaling property fails,

$$L[2x] = (2x)_t + (2x)_x - (2x)^2 = 2 - 4x^2 \neq 2 - 2x^2 = 2L[x].$$

(3) There is a uu_x term, so the function is nonlinear. However, the coefficients of the highest order terms are functions of x, t and u, so the function is quasilinear.

Solving Basic PDEs

PDEs with only one term

Problem 2. Find the general solutions to the following equations:

$$u_{xxy} = 0, \tag{1}$$
$$u_{xyz} = \sin(x) + \sin(y)\sin(z). \tag{2}$$

Solution 2.

(1) We integrate out each of the partial derivatives and introduce an integration constant in each step,

$$\begin{aligned} & u_{xxy} = 0 \\ \Rightarrow\ & u_{xx} = f(x) \\ \Rightarrow\ & u_x = \tilde{f}(x) + g(y) && \tilde{f}_x = f \\ \Rightarrow\ & u = \tilde{\tilde{f}}(x) + xg(y) + h(y) && \tilde{\tilde{f}}_{xx} = f \end{aligned}$$

where $\tilde{\tilde{f}}$ is a twice differentiable function.

(2) We integrate out each of the partial derivatives and introduce an integration constant in each step,

$$\begin{aligned}
&u_{xyz} = \sin(x) + \sin(y)\sin(z) \\
\Rightarrow\ &u_{xy} = z\sin(x) - \sin(y)\cos(z) + f(x,y) \\
\Rightarrow\ &u_x = yz\sin(x) + \cos(y)\cos(z) + \tilde{f}(x,y) + g(x,z) && \tilde{f}_y = f \\
\Rightarrow\ &u = -yz\cos(x) + x\cos(y)\cos(z) + \tilde{\tilde{f}}(x,y) + \tilde{g}(x,z) + h(y,z) && \tilde{g}_x = g,\ \tilde{\tilde{f}}_{yx} = f,
\end{aligned}$$

where $\tilde{\tilde{f}}$ is differentiable in each of its coordinates, and $\tilde{g}$ is differentiable in its first coordinate.

Semilinear First Order PDEs

Problem 3. Find the general solutions to the following equations

$$u_t - 4u_x + u = 0, \tag{1}$$

$$-2u_x + 4u_y = e^{x+3y} - 5u. \tag{2}$$

Solution 3.

(1) We have the system of equations

$$\frac{dt}{1} = \frac{dx}{-4} = \frac{du}{-u}.$$

Characteristic Curve: We start by solving the equation involving the first and second term,

$$\frac{dt}{1} = \frac{dx}{-4} \Rightarrow \frac{dx}{dt} = -4 \Rightarrow C = x + 4t.$$

General Solution: We now solve the equation involving the first and third term,

$$\frac{dt}{1} = \frac{du}{-u} \Rightarrow \frac{du}{dt} = -u.$$

This is a separable ODE, which has solution

$$\log|u| = -t + f(C) \Rightarrow u = \pm e^{f(C)}e^{-t}$$

Since $f(C)$ is an arbitrary function, we might can redefine $\pm e^{f(C)} =: g(C)$. Since $C = x + 4t$, we have our general solution is

$$u(t,x) = g(x+4t)e^{-t}.$$

(2) We have the system of equations

$$\frac{dx}{-2} = \frac{dy}{4} = \frac{du}{e^{x+3y} - 5u}.$$

Characteristic Curve: We start by solving the equation involving the first and second term,

$$\frac{dx}{-2} = \frac{dy}{4} \Rightarrow \frac{dy}{dx} = -2 \Rightarrow C = y + 2x.$$

General Solution: We now solve the equation involving the first and second term,

$$\frac{dx}{-2} = \frac{du}{e^{x+3y} - 5u} \Rightarrow \frac{du}{dx} = -\frac{1}{2}(e^{x+3y} - 5u) \Rightarrow \frac{du}{dx} - \frac{5}{2}u = -\frac{1}{2}e^{x+3y}.$$

There is a y variable appearing in this ODE that we must eliminate it. Since $y = C - 2x$, we need to solve

$$\frac{du}{dx} - \frac{5}{2}u = -\frac{1}{2}e^{-5x+3C}.$$

This is a linear ODE, which can be solved using an integrating factor of the form $\phi(x) = e^{-\frac{5}{2}x}$, which gives us

$$u = e^{\frac{5}{2}x}\bigg(-\frac{1}{2}\int e^{-5x+3C}e^{-\frac{5}{2}x}\,dx\bigg) = -\frac{1}{2}e^{\frac{5}{2}x}\bigg(\frac{2e^{-\frac{15}{2}x+3C}}{-15} + f(C)\bigg) \Rightarrow u = \frac{1}{15}e^{-5x+3C} - \frac{1}{2}f(C)e^{\frac{5}{2}x}.$$

Since $C = y + 2x$, if we set $g(z) = -\frac{1}{2}f(z)$ then we get the general solution

$$u(x,y) = \frac{1}{15}e^{x+3y} + g(y+2x)e^{\frac{5}{2}x}.$$

Problem 4. Solve the initial value problem

$$2xyu_x + (x^2+y^2)u_y = 0$$

with $u(x,y) = \exp(x/(x-y))$ on $\{x+y=1\}$.

Solution 4. We have the system of equations

$$\frac{dx}{2xy} = \frac{dy}{(x^2+y^2)} = \frac{du}{0}.$$

Characteristic Curves: We start by solving the equation involving the first and second term,

$$\frac{dx}{2xy} = \frac{dy}{(x^2+y^2)} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\cdot\frac{x}{y} + \frac{1}{2}\cdot\frac{y}{x}.$$

This is a Homogenous ODE, which can be solved using the change of variables $w = \frac{y}{x}$. We have $\frac{dy}{dx} = x\frac{dw}{dx} + w$, so under this change of variables we have

$$x\frac{dw}{dx} + w = \frac{1}{2}\cdot w^{-1} + \frac{1}{2}\cdot w \Rightarrow x\frac{dw}{dx} = \frac{1}{2}\cdot w^{-1} - \frac{1}{2}\cdot w = \frac{1-w^2}{2w}.$$

This is a separable equation, so

$$\frac{2w}{1-w^2}dw = \frac{1}{x}dx \Rightarrow -\ln(1-w^2) = \ln x + D \Rightarrow e^{-D} = x(1-w^2) = \frac{x^2-y^2}{x}.$$

If we set $C = e^{-D}$, then $C = \frac{x^2-y^2}{x}$ is our characteristic curve.

General Solution: We now solve the equation involving the first and third term,

$$\frac{dx}{2xy} = \frac{du}{0} \Rightarrow \frac{du}{dx} = 0 \Rightarrow u = f(C).$$

Since $C = \frac{x^2-y^2}{x}$, we have our general solution is

$$u(x,y) = f\Big(\frac{x^2-y^2}{x}\Big).$$

Particular Solution: We now use the initial value to solve for f. Since $u(x,y) = \exp(x/(x-y))$ when $x+y=1$, we have

$$e^{\frac{x}{x-y}} = u(x,y)\big|_{x+y=1} = f\Big(\frac{x^2-y^2}{x}\Big)\Big|_{x+y=1} = f\Big(\frac{(x-y)(x+y)}{x}\Big)\Big|_{x+y=1} = f\Big(\frac{x-y}{x}\Big).$$

If we set $z = \frac{x-y}{x}$, then the above implies $e^{\frac{1}{z}} = f(z)$, so our particular solution is of the form

$$u(x,y) = e^{\frac{x}{x^2-y^2}}.$$

Semilinear First Order PDEs in Higher Dimensions

Problem 5. Find the general solution to the equation

$$u_x + 3u_y - 2u_z = u.$$

Solution 5. We have the system of equations,

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{-2} = \frac{du}{u}.$$

Characteristic Curves Part 1: We start by solving the equation involving the first and second term,

$$\frac{dx}{1} = \frac{dy}{3} \implies C = 3x - y.$$

Characteristic Curves Part 2: We now solve the equation involving the first and third term,

$$\frac{dx}{1} = \frac{dz}{-2} \implies D = 2x + z.$$

General Solution: We now solve the equation involving the first and fourth term,

$$\frac{dx}{1} = \frac{du}{u} \implies x = \log|u| + f(C,D) \implies u(x,y,z) = f(C,D)e^x = f(3x-y, 2x+z)e^x,$$

for some function f of two variables.

Problem 6. Find the general solution to the equation

$$u_t + yu_x + xu_y = 0.$$

Find the particular solution when $u(0,x,y) = f(x,y)$.

Solution 6. We have the system of equations,

$$\frac{dt}{1} = \frac{dx}{y} = \frac{dy}{x} = \frac{du}{0}.$$

Characteristic Curves Part 1: We start by solving the equation involving the second and third term,

$$\frac{dx}{y} = \frac{dy}{x} \implies x^2 = y^2 + C \implies C = x^2 - y^2.$$

Characteristic Curves Part 2: We now solve the equation involving the first and second term using the fact $y = \sqrt{x^2 - C}$,

$$\frac{dt}{1} = \frac{dx}{y} = \frac{dx}{\sqrt{x^2 - C}} \implies t = \log|\sqrt{x^2 - C} + x| + D = \log|y + x| + D \implies D = \frac{(x+y)}{e^t}.$$

General Solution: We now solve the equation involving the first and fourth term,

$$\frac{dt}{1} = \frac{du}{0} \implies u(t,x,y) = g(C,D) = g\Big(x^2 - y^2, \frac{(x+y)}{e^t}\Big).$$

Particular Solution: Plugging in our initial conditions, we have

$$u(0,x,y) = g(x^2 - y^2, x + y) = f(x,y).$$

We set $u = x^2 - y^2$ and $v = x + y$. Our goal is to write x and y as some functions of u and v. We see that

$$u = x^2 - y^2 = (x-y)(x+y) = (x-y)v \implies x - y = \frac{u}{v}.$$

Since $x + y = v$ and $x - y = \frac{u}{v}$, we can add and subtract our answers to conclude

$$x = \frac{1}{2}\Big(v + \frac{u}{v}\Big) \qquad y = \frac{1}{2}\Big(v - \frac{u}{v}\Big),$$

so

$$u(0,x,y) = g(u,v) = f\Big(\frac{1}{2}\Big(v + \frac{u}{v}\Big), \frac{1}{2}\Big(v - \frac{u}{v}\Big)\Big).$$

Therefore, our particular solution is given by

$$\begin{aligned} u(t,x,y) &= g\Big(x^2 - y^2, \frac{(x+y)}{e^t}\Big) \\ &= f\Big(\frac{1}{2}\Big(\frac{x+y}{e^t} + \frac{x^2-y^2}{\frac{(x+y)}{e^t}}\Big), \frac{1}{2}\Big(\frac{x+y}{e^t} - \frac{x^2-y^2}{\frac{(x+y)}{e^t}}\Big)\Big) \\ &= f\Big(\frac{1}{2}\Big(\frac{x+y}{e^t} + \frac{x-y}{e^{-t}}\Big), \frac{1}{2}\Big(\frac{x+y}{e^t} - \frac{x-y}{e^{-t}}\Big)\Big). \end{aligned}$$

Remark: We were a bit sloppy with the constants and domains of our functions above. The constants C and D changed each line and writing $x^2 - y^2 = C$ implicitly in terms of y depends on the value of C. We should check our general solution to ensure that it is a solution to our PDE by differentiating.