1 Generating Random Variables

1.1 Quantiles

Suppose that we are given X and a value $p \in (0,1)$ and we are interested in computing the value of t such that

$$F_X(t) = \mathbb{P}(X \le t) = p.$$

If F_X is invertible, then $t = F_X^{-1}(p)$. However, not all CDFs are invertible so how does one define such a t in general. This generalized notation of an inverse is called a quantile function.

Definition 1 (p-quantile)**.** Let $p \in [0,1]$. The p-quantile (or $100 \times p$th percentile) of the distribution of X with CDF F_X is the smallest number c_p that satisfies $F_X(c_p) \ge p$. In other words,

$$c_p = \inf\{x \in \mathbb{R} : F_X(x) \ge p\}.$$

Definition 2 (Median)**.** The *median* of a distribution is its 0.5 quantile.

The quantile function takes a probability p and returns its p-quantile.

Definition 3 (Quantile Function)**.** The *quantile function* $F_X{-}1[0,1] \mapsto \mathbb{R}$ is the function given by

$$F_X^{-1}(p) := c_p = \inf\{x \in \mathbb{R} : F_X(x) \ge p\}.$$

The quantile function is also called *generalized inverse function*, because it is a well defined function even if F_X is not strictly increasing like in the case of discrete random variables. This is why we use the same notation F_X^{-1}, even though it is not the inverse in the traditional sense. In fact, the quantile behaves exactly like an inverse function, where the equalities are often replaced by inequalities.

Proposition 1 (*Properties of the Generalized Inverse*)

The quantile function satisfies F_X^{-1} for F_X satisfies

1. *For all $x \in \mathbb{R}$, $F_X^{-1}(F_X(x)) \le x$*
2. *For all $p \in [0,1]$, $F_X(F_X^{-1}(p)) \ge p$*
3. *$F_X^{-1}(p) \le x \Leftrightarrow p \le F_X(x)$*
4. *$F_X^{-1}(p)$ is non-decreasing and left-continuous (except for the endpoints $p = 0$ or $p = 1$)*

We can compute the quantile in the following way

- If the distribution function F_X is continuous and strictly increasing, it has an inverse F_X^{-1} so

$$c_p = F_X^{-1}(p).$$

- If F_X has jumps or flat regions, then $F_X(x) = p$ may not have any solution or it might have infinitely many. In this case, the function $F_X^{-1}(p)$ is the left continuous step function that interpolates between the points (p, x) where x is the location of the jumps of F_X.

1.2 Inverse Transform Sampling

It is "easy" to sample from the continuous uniform distribution $\mathrm{Unif}(0,1)$ on a computer. These uniform random variables can be used to generate samples from any distribution.

Theorem 1 (***Inverse Transform Sampling***)

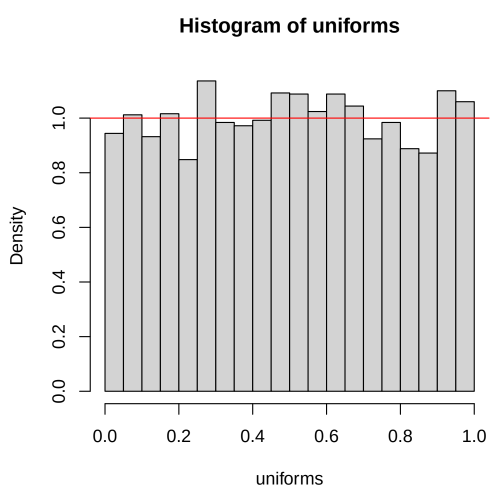
Let F_X be any cumulative distribution function of some random variable X and $U \sim U(0,1)$. Then the random variable $Y = F_X^{-1}(U)$ has CDF F_X.

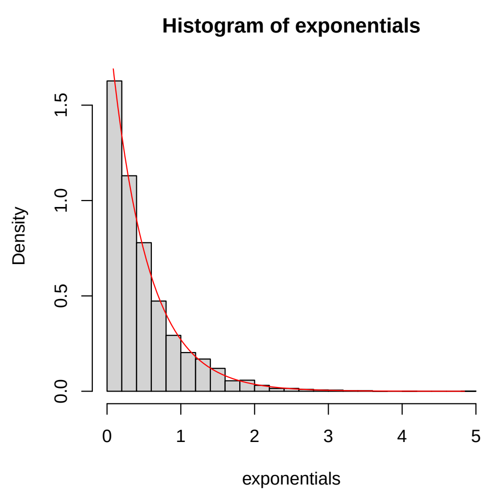
Remark 1. This is a generalization of a simple concept. For instance, if we want to generate a flip of a coin (a Ber(0.5) random variable), then we can sample a number uniformly u from $[0,1]$ and define $x = X(u) = 0$ if $u_1 \in [0, 0.5]$ and $x = X(u) = 1$ if $u_1 \in [0.5, 1]$. One can check that this coincides with the inverse transform sampling method (see Problem 1.4.)

1.3 Sampling Algorithm

1. No matter what CDF F_X (discrete or continuous), we can sample observations as follows:
   (a) Sample $u \sim U(0,1)$ (eg via `runif()`)
   (b) Return $x = F_X^{-1}(u)$.
2. Repeating this n times independently gives n realizations of X.

Example 1. We sample `uniforms <- runif(5000)` and then `exponentials <- -log(1-uniforms)/2`.





1.4 Example Problems

Problem 1.1. Consider the random variable X with

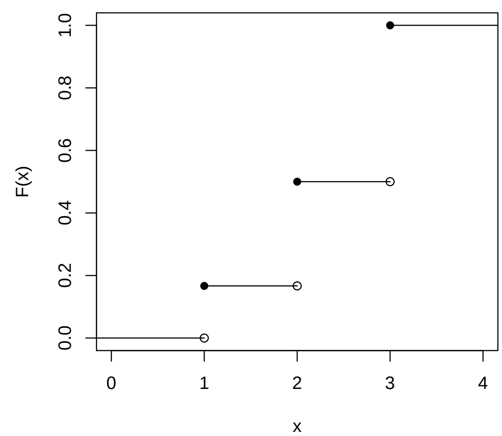
$$\mathbb{P}(X = 1) = 1/6, \qquad \mathbb{P}(X = 2) = 2/6 \qquad \mathbb{P}(X = 3) = 3/6.$$

Sketch the CDF of X and compute $F_X^{-1}(p)$ for $p \in (0,1)$.

Solution 1.1.

1. The cumulative distribution function is

$$F_X(x) = \begin{cases} 0, & x < 1, \\ \frac{1}{6}, & 1 \le x < 2 \\ \frac{1}{2}, & 2 \le x < 3, \\ 1 & 3 \le x \end{cases}$$

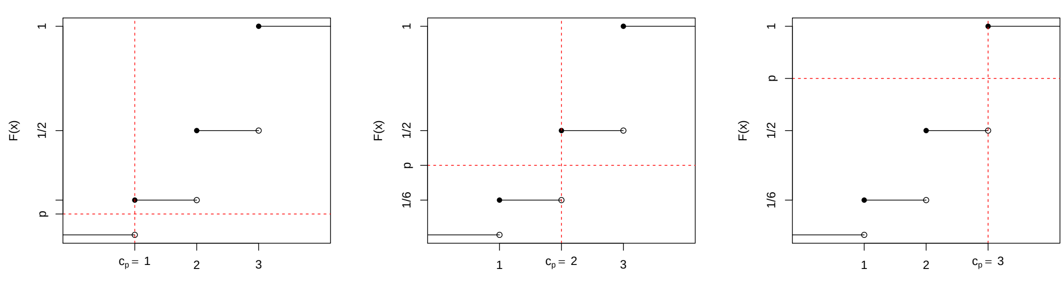


2. To compute the quantile function, we notice that the discontinuities of the CDF occur at $(1, \frac{1}{6}), (2, \frac{1}{2}), (3, 1)$. Therefore, the discontinuities for the quantile function occur at $(\frac{1}{6}, 1), (\frac{1}{2}, 2), (1, 3)$. Extending this to make the function left continuous implies

$$F_X^{-1}(p) = c_p = \inf\{x \in \mathbb{R} : F_X(x) \ge p\} = \begin{cases} 1, & 0 < p \le \frac{1}{6} \\ 2, & \frac{1}{6} < p \le \frac{1}{2} \\ 3, & \frac{1}{2} < p \le 1. \end{cases}$$

Remark 2. The end points of the intervals in the quantile function are the same as the p values of the CDF at the jumps. Furthermore, the $<$ inequality is always on the left of the x and the $\le$ inequality is always to the right of the x. This implies the quantile function is left continuous. Furthermore, the value of the quantile function on each interval is equal to the value of the quantile function at the right endpoint .

Remark 3. To find individual points of the quantile at p, we find the smallest point where the graph $F_X(x)$ lies on or above the horizontal line p. This is demonstrated for $p \in (0, 1/6]$ (left), $p \in (1/6, 1/2]$ (middle) and $p \in (1/2, 1]$ (right).



Problem 1.2. Let $U \sim \text{Unif}(0, 1)$. We want to sample from the $\text{Exp}(2^{-1})$ distribution with density

$$f_X(x) = 2e^{-2x}, \quad x > 0$$

and 0 otherwise. Write Y as a function of U such that Y is equal in distribution to X.

Solution 1.2. The CDF on the support of X

$$F_X(x) = \int_0^x 2e^{-2t} dt = 1 - e^{-2x},$$

which is strictly increasing for on its support $x \geq 0$. Solving for $F_X(y) = x$ to recover the inverse gives $y = F_X^{-1}(x) = -\frac{1}{2}\log(1-x)$, so

$$F_X^{-1}(x) = -\frac{1}{2}\log(1-x)$$

for $x \in (0,1)$. Therefore, by Theorem 1

$$Y = -\frac{1}{2}\log(1-U)$$

has the same distribution $Y \sim \text{Exp}(2^{-1})$.

Problem 1.3. Suppose that we wish to generate a random observation, x, from a distribution with PDF given by

$$f_X(x) = \frac{1}{8\sqrt{x}}, \quad 0 < x < 16$$

and 0 otherwise. We generate an observation, u, from a continuous $\text{Unif}(0,1)$ distribution (using software) and get 0.1348. Determine the value $x = x(u)$, that this value u will produce.

Solution 1.3. We first compute the CDF on the support of X

$$F_X(x) = \int_0^x \frac{1}{8\sqrt{t}}\, dt = \frac{1}{4}\sqrt{x}, \quad 0 < x < 16.$$

which is strictly increasing on its support $0 < x < 16$. Solving for $F_X(y) = x$ to recover the inverse gives $y = F^{-1}(x) = (4x)^2$, so

$$F_X^{-1}(x) = (4x)^2$$

for $x \in (0,1)$. By the sampling algorithm, if $u = 0.1348$ the corresponding observation of x is

$$x = F_X^{-1}(u) = (4 \cdot 0.1348)^2 = 0.2907.$$

Problem 1.4. Explain how you would sample a biased flip of a coin with probability of heads p using a uniform random variable.

Solution 1.4. If X is the outcome of a biased flip of a coin with probability of heads p, then $X \sim \text{Bern}(p)$. This means that $f_X(1) = p$ and $f_X(0) = 1-p$. The CDF and quantile function is therefore,

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases} \qquad F_X^{-1}(x) = \begin{cases} 0 & 0 < x \leq 1-p \\ 1 & 1-p < x \leq 1 \end{cases}.$$

If $U \sim \text{Unif}(0,1)$, we know that X has the same distribution as $F_X^{-1}(U)$. Therefore, to generate a biased coin flip, we sample $u \sim \text{Unif}(0,1)$ and define $x(u) = 0$ if $u < 1-p$ and $x(u) = 1$ if $u > 1-p$.

Problem 1.5.

1. 75th percentile of the standard normal distribution
2. 58th percentile of the $N(5,9)$ distribution
3. Let $Z \sim N(0,1)$. Find c such that

$$\mathbb{P}(-c \le Z \le c) = 0.95$$

Solution 1.5.

1. We find
$$\Phi^{-1}(0.75) = 0.6745$$
2. We need to find the 0.58 quantile of X where $\mu = 5$ and $\sigma = \sqrt{9} = 3$,
$$F_X^{-1}(0.58) = 5 + 3F_Z^{-1}(0.58) = 5 + 3\Phi^{-1}(0.58) = 5 + 3 \cdot 0.2019 = 5.6057$$
3. We solve for c using the quantile function,
$$\begin{aligned} 0.95 = \mathbb{P}(-c \le Z \le c) &= \Phi(c) - \Phi(-c) \\ &\Leftrightarrow 0.95 = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1 \\ &\Leftrightarrow 0.975 = \Phi(c) \\ &\Leftrightarrow c = \Phi^{-1}(0.975) = 1.96 \end{aligned}$$

2 Proofs of Key Results

Problem 2.1. Prove Theorem 1 in the simpler case when F_X is invertible.

Solution 2.1. Let F_Y denote the CDF of the random variable $Y = F_X^{-1}(U)$. Then,

$$F_Y(x) = \mathbb{P}(F_X^{-1}(U) \le x) = \mathbb{P}(F_X(F_X^{-1}(U)) \le F_X(x)) = \mathbb{P}(U \le F(x)).$$

Furthermore, if $U \sim U(0,1)$ then

$$F_Y(x) = \mathbb{P}(U \le F_X(x)) = \int_0^{F_X(x)} t \, dt = F_X(x).$$

The random variable $Y = F_X^{-1}(U)$ has the CDF F_X, as desired.

Problem 2.2. If F_X is a CDF, then its quantile function F_X^{-1} satisfies

$$F_X^{-1}(p) \le x \Leftrightarrow p \le F_X(x)$$

Solution 2.2. The proof relies on the fact that $F_X^{-1}(p)$ is the infimum of all $\{t : F_X(t) \ge p\}$, and therefore smaller than (or equal to) any $x \in \{t : F_X(t) \ge p\}$.

($\Longrightarrow$) Suppose that $F_X^{-1}(p) \le x$. This implies that $x \in \{t : F_X(t) \ge p\}$ so $p \le F_X(x)$.

($\Longleftarrow$) Suppose that $p \le F_X(x)$. This implies that $x \in \{t : F_X(t) \ge p\}$ so $F_X^{-1}(p) \le x$.

Problem 2.3. Prove Theorem 1.

Solution 2.3. Let F_Y denote the CDF of the random variable $Y = F_X^{-1}(U)$. Using the properties of the quantile function (Problem 2.2) that

$$F_X^{-1}(p) \le x \Leftrightarrow p \le F_X(x).$$

So we can conclude that

$$\{F_X^{-1}(U) \le x\} = \{U \le F_X(x)\}$$

Therefore, the CDF of Y is

$$F_Y(x) = \mathbb{P}(F_X^{-1}(U) \le x) = \mathbb{P}(U \le F_X(x)) = F_X(x).$$

Problem 2.4. Prove the following properties for the quantile function

1. For all $x \in \mathbb{R}$, $F_X^{-1}(F_X(x)) \le x$
2. For all $p \in [0,1]$, $F_X(F_X^{-1}(p)) \ge p$
3. $F_X^{-1}(p) \le x \Leftrightarrow p \le F_X(x)$
4. $F_X^{-1}(p)$ is non-decreasing and left-continuous (except for the endpoints $p = 0$ or $p = 1$)

Solution 2.4.

1. We have
$$F_X^{-1}(F_X(x)) = \inf_{t \in \mathbb{R}}\{F_X(t) \ge F_X(x)\} \le x$$
since $x \in \{t \in \mathbb{R} : F_X(t) \ge F_X(x)\}$.

2. Since F_X is right continuous and increasing we have $\{F_X(x) \ge p\}$ is a closed set, so it attains its infimum. Therefore, $c_p \in \{F_X(x) \ge p\}$ so
$$F_X(F_X^{-1}(p)) = F_X(c_p) \ge p.$$

3. This was shown in Problem 2.2.

4. Suppose that $p_1 \le p_2$. Then
$$F_X^{-1}(p_1) = \inf_{x \in \mathbb{R}}\{F_X(x) \ge p_1\} \le \inf_{x \in \mathbb{R}}\{F_X(x) \ge p_2\} = F_X^{-1}(p_2)$$
since $\{F_X(x) \ge p_1\} \subseteq \{F_X(x) \ge p_2\}$, so F_X^{-1} is non-decreasing.
To see left continuity, notice that monotone functions can only have jump discontinuities, so it suffices to show that $\sup_{q<p} F_X^{-1}(q) = F_X^{-1}(p)$. For each $q < p$ and $\epsilon > 0$, we have by definition of the supremum
$$\sup_{q<p} F_X^{-1}(q) + \epsilon \ge F_X^{-1}(q) \overset{(3)}{\Longrightarrow} F_X(\sup_{q<p} F_X^{-1}(q) + \epsilon) \ge q.$$
So taking $\epsilon \to 0$ by right continuity of F_X implies that $F_X(\sup_{q<p} F_X^{-1}(q)) \ge q$ for all $q < p$ so $F_X(\sup_{q<p} F_X^{-1}(q)) \ge p$. Property 3 above implies that
$$\sup_{q<p} F_X^{-1}(q) \ge F_X^{-1}(p).$$
This combined with monotonicity $\sup_{q<p} F_X^{-1}(q) \le F_X^{-1}(p)$ implies that $\sup_{q<p} F_X^{-1}(q) = F_X^{-1}(p)$ as required.