1 Solving the Wave Equation

Consider the wave equation on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = h(x) & x \in \mathbb{R}. \end{cases}$$

The solution to this PDE is given by D'Alembert's formula,

$$u(x,t) = \frac{g(x+ct)+g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds. \tag{1}$$

Problem 1.1. Solve the initial value problem

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = \tanh(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = \arctan(x) & x \in \mathbb{R}. \end{cases}$$

Solution 1.1. By D'Alembert's formula (1), the particular solution to this IVP is given by

$$u(x,t) = \frac{\tanh(x+2t)+\tanh(x-2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} \arctan(y)\,dy.$$

The integral term can be computed using integration by parts,

$$\begin{aligned} &\frac{1}{4}\int_{x-2t}^{x+2t} \arctan(y)\,dy \\ &= \frac{1}{4}\Big(y\arctan(y) - \frac{1}{2}\ln|1+y^2|\Big)\Big|_{y=x-2t}^{y=x+2t} \\ &= \frac{1}{4}\Big((x+2t)\arctan(x+2t) - (x-2t)\arctan(x-2t) - \frac{1}{2}\ln(1+(x+2t)^2) + \frac{1}{2}\ln(1+(x-2t)^2)\Big). \end{aligned}$$

Problem 1.2. Solve the following initial value problems

1. $$\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = h(x) & x \in \mathbb{R} \end{cases}$$

   with

   $$g(x) = \begin{cases} 0 & |x| \ge 1 \\ x^2 - x^4 & |x| < 1 \end{cases}, \qquad h(x) = 0.$$

2. $$\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = h(x) & x \in \mathbb{R} \end{cases}$$

   with

   $$g(x) = 0, \qquad h(x) = \begin{cases} 0 & |x| \ge 1 \\ x^2 - x^4 & |x| < 1 \end{cases}.$$

Solution 1.2.

(1) Since $h(x) = 0$, by D'Alembert's formula (1), the particular solution to this IVP is given by

$$u(x,t) = \frac{g(x+2t) + g(x-2t)}{2}.$$

Since $g(x)$ changes form based on the value of $|x|$, we can break our solution into 4 cases:

A. $|x+2t| \geq 1$, $|x-2t| \geq 1$: On this region, $g(x+2t) = 0$ and $g(x-2t) = 0$, so

$$u(x,t) = 0.$$

B. $|x+2t| < 1$, $|x-2t| \geq 1$: On this region, $g(x+2t) = (x+2t)^2 - (x+2t)^4$ and $g(x-2t) = 0$, so

$$u(x,t) = \frac{(x+2t)^2 - (x+2t)^4}{2}.$$

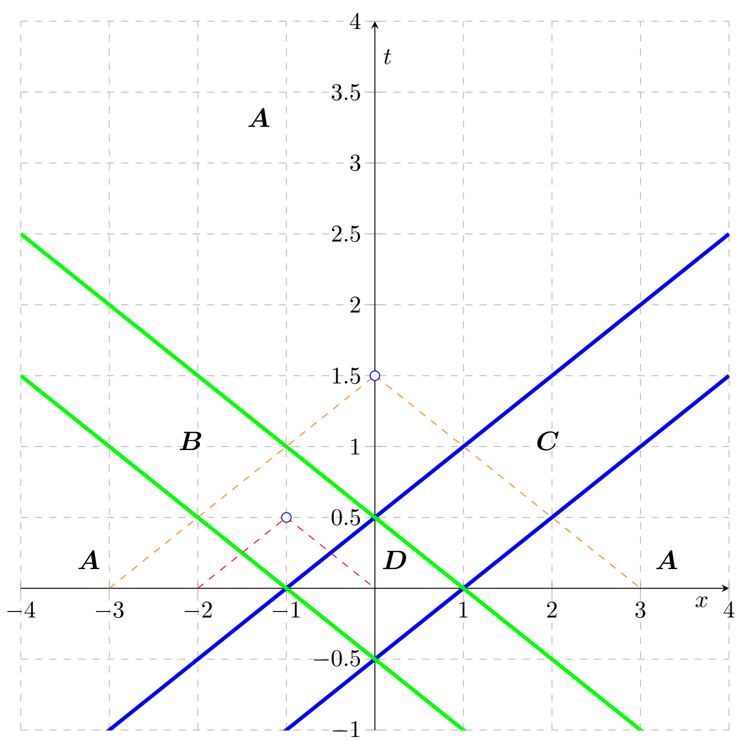
C. $|x+2t| \geq 1$, $|x-2t| < 1$: On this region, $g(x+2t) = 0$ and $g(x-2t) = (x-2t)^2 - (x-2t)^4$, so

$$u(x,t) = \frac{(x-2t)^2 - (x-2t)^4}{2}.$$

D. $|x+2t| < 1$, $|x-2t| < 1$: On this region, $g(x+2t) = (x+2t)^2 - (x+2t)^4$ and $g(x-2t) = (x-2t)^2 - (x-2t)^4$, so

$$u(x,t) = \frac{(x+2t)^2 - (x+2t)^4 + (x-2t)^2 - (x-2t)^4}{2}.$$

Characteristic Lines:



Description of Picture: The initial condition is supported on the interval $[-1,1]$. The wave propagates right along the lines $x-2t=C\in[-1,1]$ (between the blue characteristic lines) and left along the lines $x+2t=C\in[-1,1]$ (between the green characteristic lines). The behavior on each of the regions can be determined by drawing the domain of dependence at the point (x,t) and seeing if the corners lie in the interval $[-1,1]$. For example, at the point $(-1,0.5)$ the left corner does not lie in $[-1,1]$, while the right corner is in $[-1,1]$, which corresponds to case B above. Similarly, at the point $(0,1.5)$ both corners do not lie in $[-1,1]$, which corresponds to case A above.

(2) Since $g(x)=0$, by D'Alembert's formula (1), the particular solution to this IVP is given by

$$u(x,t)=\frac{1}{4}\int_{x-2t}^{x+2t}h(y)\,dy.$$

Since $h(x)$ changes form based on the value of $|x|$, we can break our solution into 5 cases:

A. $x-2t\le -1\le 1\le x+2t$: On this region, we can split our region of integration into

$$\begin{aligned}u(x,t)=\frac{1}{4}\int_{x-2t}^{x+2t}h(y)\,dy&=\frac{1}{4}\int_{x-2t}^{-1}h(y)\,dy+\frac{1}{4}\int_{-1}^{1}h(y)\,dy+\frac{1}{4}\int_{1}^{x+2t}h(y)\,dy\\&=\frac{1}{4}\int_{-1}^{1}y^2-y^4\,dy\\&=\frac{1}{4}\Big(\frac{y^3}{3}-\frac{y^5}{5}\Big)\Big|_{y=-1}^{y=1}=\frac{1}{15}.\end{aligned}$$

B. $x-2t\le -1\le x+2t\le 1$: On this region, we can split our region of integration into

$$\begin{aligned}u(x,t)=\frac{1}{4}\int_{x-2t}^{x+2t}h(y)\,dy&=\frac{1}{4}\int_{x-2t}^{-1}h(y)\,dy+\frac{1}{4}\int_{-1}^{x+2t}h(y)\,dy\\&=\frac{1}{4}\int_{-1}^{x+2t}y^2-y^4\,dy\\&=\frac{1}{4}\Big(\frac{y^3}{3}-\frac{y^5}{5}\Big)\Big|_{y=-1}^{y=x+2t}=\frac{(x+2t)^3}{12}-\frac{(x+2t)^5}{20}+\frac{1}{30}.\end{aligned}$$

C. $-1\le x-2t\le 1\le x+2t$: On this region, we can split our region of integration into

$$\begin{aligned}u(x,t)=\frac{1}{4}\int_{x-2t}^{x+2t}h(y)\,dy&=\frac{1}{4}\int_{x-2t}^{1}h(y)\,dy+\frac{1}{4}\int_{1}^{x+2t}h(y)\,dy\\&=\frac{1}{4}\int_{x-2t}^{1}y^2-y^4\,dy\\&=\frac{1}{4}\Big(\frac{y^3}{3}-\frac{y^5}{5}\Big)\Big|_{y=x-2t}^{y=1}=\frac{1}{30}-\frac{(x-2t)^3}{12}+\frac{(x-2t)^5}{20}.\end{aligned}$$

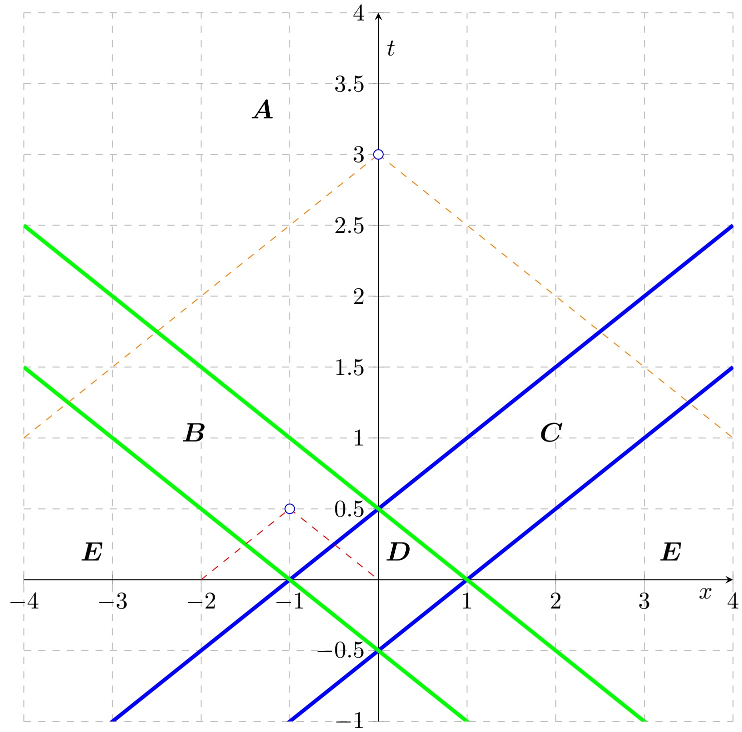
D. $-1\le x-2t\le x+2t\le 1$: On this region, the integrand is always equal to $h(y)=y^2-y^4$

$$\begin{aligned}u(x,t)=\frac{1}{4}\int_{x-2t}^{x+2t}h(y)\,dy&=\frac{1}{4}\int_{x-2t}^{x+2t}y^2-y^4\,dy\\&=\frac{1}{4}\Big(\frac{y^3}{3}-\frac{y^5}{5}\Big)\Big|_{y=x-2t}^{y=x+2t}\\&=\frac{(x+2t)^3}{12}-\frac{(x+2t)^5}{20}-\frac{(x-2t)^3}{12}+\frac{(x-2t)^5}{20}.\end{aligned}$$

E. $x-2t\ge 1$, or $x+2t\le -1$: On this region, the integrand is always equal to $h(y)=0$, so

$$u(x,t)=0.$$

Characteristic Lines:



Description of Picture: The initial condition is supported on the interval $[-1, 1]$. The behavior in each of the regions can be determined by drawing the domain of dependence at the point (x, t) and seeing how much of the interval $[-1, 1]$ is contained in the base of the triangle. For example, at $(-1, 0.5)$ the left corner of the base of the triangle is < -1 and the right corner of the base is in $[-1, 1]$, which corresponds to case B above. Similarly, at $(0, 3)$ the left corner of the base of the orange triangle is < -1 and the right corner of the base is in > 1, which corresponds to case A above.

Problem 1.3. Solve the initial value problem

$$\begin{cases} u_{tt} - 4u_{xx} = f(x,t) & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = h(x) & x \in \mathbb{R} \end{cases}$$

with

$$f(x,t) = \begin{cases} \sin(x) & 0 < t < \pi \\ 0 & t \geq \pi \end{cases}, \qquad g(x) = 0, \qquad h(x) = 0.$$

Solution 1.3. Since $g(x) = 0$ and $h(x) = 0$, by D'Alembert's formula (1) the particular solution to this IVP is given by

$$\begin{aligned} u(x,t) = \frac{1}{4} \iint_{\Delta} f(y,s)\, dyds &= \frac{1}{4} \int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \sin(y) \mathbb{1}_{[0,\pi]}(s)\, dyds \\ &= \frac{1}{4} \int_0^{\min(t,\pi)} \int_{x-2(t-s)}^{x+2(t-s)} \sin(y)\, dyds. \end{aligned}$$

If you draw the region of integration, we are basically chopping off Δ above the line $t = \pi$ and integrating the remaining trapezoid (or triangle if t is small enough). We have two cases,

A. $t < \pi$: On this region, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \sin(y)\,dyds \\
&= \frac{1}{4}\int_0^t \left(-\cos(y)\Big|_{y=x-2(t-s)}^{y=x+2(t-s)} \right) ds \\
&= \frac{1}{4}\int_0^t -\cos(x+2(t-s)) + \cos(x-2(t-s)) ds. \\
&= \frac{1}{8}\Big(\sin(x+2(t-s)) + \sin(x-2(t-s)) \Big)\Big|_{s=0}^{s=t} \\
&= \frac{1}{4}\sin(x) - \frac{1}{8}\sin(x+2t) - \frac{1}{8}\sin(x-2t).
\end{aligned}$$

B. $t \geq \pi$: On this region, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{4}\int_0^\pi \int_{x-2(t-s)}^{x+2(t-s)} \sin(y)\,dyds \\
&= \frac{1}{4}\int_0^\pi \left(-\cos(y)\Big|_{y=x-2(t-s)}^{y=x+2(t-s)} \right) ds \\
&= \frac{1}{4}\int_0^\pi -\cos(x+2(t-s)) + \cos(x-2(t-s)) ds. \\
&= \frac{1}{8}\Big(\sin(x+2(t-s)) + \sin(x-2(t-s)) \Big)\Big|_{s=0}^{s=\pi} \\
&= 0.
\end{aligned}$$

Problem 1.4. Find the general solution of the *Goursat* problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x > c|t|; \\ u|_{x=-ct} = g(t), & t < 0; \\ u|_{x=ct} = h(t), & t > 0, \end{cases}$$

where g and h satisfy the compatibility condition $g(0) = h(0)$.

Solution 1.4. Recall that the general solution of $u_{tt} - c^2 u_{xx} = 0$ is given by

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \text{ for } x > c|t|,$$

for some yet to be determined functions ϕ and ψ. Using the initial conditions, we can recover the specific form of ϕ and ψ. For $t < 0$, the first boundary condition implies,

$$u|_{x=-ct} = g(t) \implies \phi(0) + \psi(-2ct) = g(t) \overset{s=-2ct}{\implies} \psi(s) = g\Big(-\frac{s}{2c}\Big) - \phi(0) \text{ for } s > 0$$

and for $t > 0$, the second boundary condition implies

$$u|_{x=ct} = h(t) \implies \phi(2ct) + \psi(0) = h(t) \overset{s=2ct}{\implies} \phi(s) = h\Big(\frac{s}{2c}\Big) - \psi(0) \text{ for } s > 0.$$

If we take limits as $s \to 0$ from right, the condition $g(0) = h(0)$ implies that

$$\psi(0) = g(0) - \phi(0) \quad \text{and} \quad \phi(0) = h(0) - \psi(0) \implies \psi(0) + \phi(0) = g(0) = h(0) = \frac{g(0)+h(0)}{2}.$$

Therefore, our particular solution is given by

$$u(x,t) = h\Big(\frac{x+ct}{2c}\Big) + g\Big(\frac{ct-x}{2c}\Big) - (\phi(0)+\psi(0)) = h\Big(\frac{x+ct}{2c}\Big) + g\Big(\frac{ct-x}{2c}\Big) - \frac{g(0)+h(0)}{2}, \tag{2}$$

since $x+ct>0$ and $ct-x<0$ for $x>c|t|$, the solution is uniquely defined on this region.

Problem 1.5. Find the general solution of the *Goursat* problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & x > c|t|; \\ u|_{x=-ct} = g(t), & t<0; \\ u|_{x=ct} = h(t), & t>0, \end{cases}$$

where g and h satisfy the compatibility condition $g(0)=h(0)$.

Solution 1.5. This is the inhomogeneous variant of Problem 1.4.

Inhomogeneous solution: We computed the homogeneous solution in the previous exercise. It suffices to find the solution to the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & x > c|t|; \\ u|_{x=-ct} = 0, & t<0; \\ u|_{x=ct} = 0, & t>0. \end{cases}$$

Since we want to parametrize by the characteristic coordinates $\xi = x+ct$ and $\eta = x-ct$, we use the change of variables

$$x = \frac{\xi+\eta}{2} \quad \text{and} \quad t = \frac{\xi-\eta}{2c}.$$

Under this change of variables, we have

$$\frac{\partial}{\partial\xi} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial\eta} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2c}\frac{\partial}{\partial t},$$

so

$$\frac{\partial^2}{\partial\xi\partial\eta} = \Big(\frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial t}\Big)\Big(\frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2c}\frac{\partial}{\partial t}\Big) = -\frac{1}{4c^2}\Big(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\Big).$$

Therefore,

$$u_{tt} - c^2 u_{xx} = f(x,t) \implies u_{\xi\eta} = -\frac{1}{4c^2} f\Big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\Big).$$

Integrating with respect to η then ξ, we get

$$u(\xi,\eta) = -\frac{1}{4c^2}\int_{\xi_0}^{\xi}\int_{\eta_0}^{\eta} f\Big(\frac{\xi'+\eta'}{2}, \frac{\xi'-\eta'}{2c}\Big) d\eta' d\xi' + \phi(\xi) + \psi(\eta),$$

where ϕ and ψ are differentiable functions. We can choose the lower limit to be anything we wish, so we choose $\xi_0 = 0$ and $\eta_0 = 0$ (this particular choice will become apparent later on in the computation),

$$u(\xi,\eta) = -\frac{1}{4c^2}\int_{0}^{\xi}\int_{0}^{\eta} f\Big(\frac{\xi'+\eta'}{2}, \frac{\xi'-\eta'}{2c}\Big) d\eta' d\xi' + \phi(\xi) + \psi(\eta).$$

We now use the initial conditions to solve for ϕ and ψ. When $\xi = 0$, we must have $x=-ct$. On this line the initial condition $u|_{x=-ct} = 0$ implies that $u(0,\eta)$ must be 0 for all η, so

$$0 = u(0,\eta) = \phi(0) + \psi(\eta) \implies \psi(\eta) = -\phi(0).$$

Similarly, when $\eta = 0$, we must have $x = ct$. On this line the initial condition $u|_{x=ct} = 0$, so

$$0 = u(\xi, 0) = \phi(\xi) + \psi(0) \implies \phi(\xi) = -\psi(0).$$

Therefore, both $\phi(\xi)$ and $\psi(\eta)$ are constant functions, so adding these two conditions implies that

$$\phi(\xi) + \psi(\eta) = -\phi(0) - \psi(0) = -(\phi(\xi) + \psi(\eta)) \implies \phi(\xi) + \psi(\eta) = 0.$$

Since the $\phi(\xi) + \psi(\eta)$ term vanishes, changing back into the x and t coordinates (the Jacobian of this linear transformation is $2c$), we see that

$$u(\xi,\eta) = -\frac{1}{4c^2}\int_0^{\xi}\int_0^{\eta} f\left(\frac{\xi'+\eta'}{2}, \frac{\xi'-\eta'}{2c}\right) d\eta' d\xi' \iff u(x,t) = -\frac{1}{2c}\iint_{R(x,t)} f(x',t') dx'dt' \qquad (3)$$

where $R(x,t)$ is the rectangle in ξ and η,

$$R(x,t) = \{(\xi',\eta') : 0 \le \xi' \le \xi, 0 \le \eta' \le \eta\} = \{(x',t') : 0 \le x' + ct' \le x + ct, 0 \le x' - ct' \le x - ct\}.$$

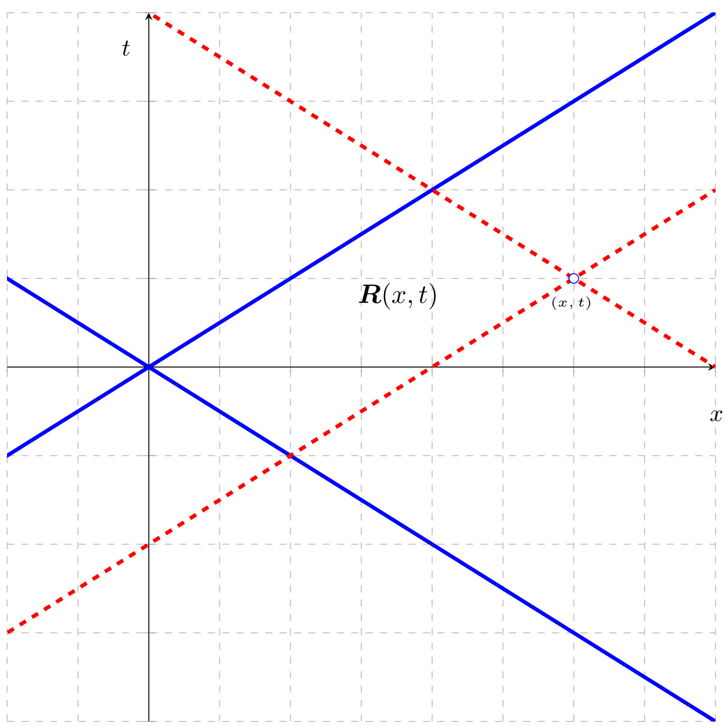
Full Solution: By linearity, the full solution of the inhomogeneous Goursat problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & x > c|t|; \\ u|_{x=-ct} = g(t), & t < 0; \\ u|_{x=ct} = h(t), & t > 0, \end{cases}$$

is given by the sum of the homogeneous (2) and inhomogeneous (3) solutions of the Goursat problem,

$$u(x,t) = h\Big(\frac{x+ct}{2c}\Big) + g\Big(\frac{ct-x}{2c}\Big) - \frac{g(0)+h(0)}{2} - \frac{1}{2c}\iint_{R(x,t)} f(x',t')dx'dt'.$$

Characteristic Lines:



Description of Picture: The region of integration $R(x,t)$ is given by the region bounded by the boundary $x = ct$ and $x = -ct$ and the characteristic lines passing through the point (x,t). In the picture above, the region of integration corresponding to the point (x,t) indicated by the blue hollow dot is the region bounded by the blue and dashed red lines.