Week 11

Problem 1. (Strauss 6.1.2) Find the solutions that depend only on r of the equation $u_{xx}+u_{yy}+u_{zz} = k^2u$, where k is a positive constant. (*Hint:* Substitute $u = v/r$.)

Solution 1. Recall that in $\mathbb{R}^3$, if we do a change of variables to spherical coordinates,

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right).$$

If we are looking for solutions that only depend on r, that is $u(r,\phi,\psi) = u(r)$ then we can safely ignore the terms on the right, so $u_{xx} + u_{yy} + u_{zz} = k^2u$ can be expressed in spherical coordinates as

$$u_{rr} + \frac{2}{r}u_r = k^2u.$$

This is a second order ODE, which we can solve using the substitution $u = v/r$. Notice

$$u_r = \frac{v_r}{r} - \frac{v}{r^2}, \quad u_{rr} = \frac{v_{rr}}{r} - 2\frac{v_r}{r^2} + 2\frac{v}{r^3}$$

so under this change of variables, we have

$$u_{rr} + \frac{2}{r}u_r = k^2u \implies \frac{v_{rr}}{r} = k^2\frac{v}{r} \implies v_{rr} - k^2v = 0.$$

This is a second order constant coefficient ODE with roots $r = \pm k$, so

$$v = Ae^{kr} + Be^{-kr} \implies u = A\frac{e^{kr}}{r} + B\frac{e^{-kr}}{r},$$

is the general solution.

Problem 2. (Strauss 6.1.5) Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x,y)$ vanishing on $r = a$.

Solution 2. Since we are on the disk, and neither our source or initial conditions depend on the angle θ we can use rotational invariance to solve this problem. Recall that in $\mathbb{R}^2$, if we do a change of variables to polar form,

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

If we are looking for solutions that only depend on r, that is $u(r,\theta) = u(r)$, then we can safely ignore the terms on the right, so $u_{xx} + u_{yy} = 1$ can be expressed in spherical coordinates as

$$u_{rr} + \frac{1}{r}u_r = 1 \implies ru_{rr} + u_r = r \implies (u_r r)' = r.$$

This ODE can be solved by directly integrating, which implies

$$u_r r = \frac{r^2}{2} + C_1 \implies u_r = r + \frac{C_1}{r} \implies u(r) = \frac{r^2}{4} + C_1\log r + C_2.$$

We now use the boundary conditions to solve for the coefficients. We have the implicit condition $\lim_{r\to 0} u(r) < \infty$ and the boundary condition $u(a) = 0$. Therefore, we must have

$$\lim_{r\to 0} u(r) = \lim_{r\to 0} \frac{r^2}{4} + C_1\log r + C_2 < \infty \text{ and } 0 = u(a) = \frac{a^2}{4} + C_1\log a + C_2.$$

The first condition implies that $C_1 = 0$ and the second condition implies $C_2 = -\frac{a^2}{4}$. Therefore,

$$u(r) = \frac{r^2}{4} - \frac{a^2}{4},$$

is the particular solution.

Problem 3. (Strauss 6.1.6) Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing on both parts of the boundary $r = a$ and $r = b$.

Solution 3. Since we are on the annulus, and neither our source or initial conditions depend on the angle θ we can use rotational invariance to solve this problem. Following the steps in problem 2, we have the general solution to $u_{xx} + u_{yy} = 1$ in polar coordinates is given by

$$u(r) = \frac{r^2}{4} + C_1 \log r + C_2.$$

We now use the boundary conditions to solve for the coefficients. We have the conditions $u(a) = 0$ and $u(b) = 0$, which implies

$$u(a) = \frac{a^2}{4} + C_1 \log a + C_2 = 0 \text{ and } u(b) = \frac{b^2}{4} + C_1 \log b + C_2 = 0.$$

This can be easily solved to give $C_1 = -\frac{b^2-a^2}{4(\log(b)-\log(a))}$ and $C_2 = -\frac{a^2}{4} + \frac{b^2-a^2}{4(\log(b)-\log(a))} \log(a)$, giving us the particular solution

$$u(r) = \frac{r^2 - a^2}{4} - \frac{(b^2 - a^2)(\log(r) - \log(a))}{4(\log(b) - \log(a))}.$$

Problem 4. (Strauss 6.1.10) Prove the uniqueness of the Dirichlet problem $\Delta u = f$ in D, $u = g$ on the boundary of D by the energy method. That is, after subtracting two solution $w = u - v$, multiply the Laplace equation for w by w itself and use the divergence theorem.

Solution 4. Assume that u and v are both solutions to the $\Delta u = f$ in D and $u = g$ on ∂D. If we define $w = u - v$ then $\Delta w = 0$ in D and $w = 0$ on ∂D. Therefore, by integration by parts

$$0 = -\int_D w \Delta w \, dx = \int_D |\nabla w|^2 \, dx - \int_{\partial D} w \frac{\partial w}{\partial \nu} \, dS = \int_D |\nabla w|^2 \, dx$$

which implies that $\nabla w \equiv 0$ in D (in other words, all partials of w are 0 on D). Since $w = 0$ on ∂D we must have $w \equiv 0$ which implies $u = v$ on $\bar{D}$.

Problem 5. (Strauss 6.1.12) Check the validity of the maximum principle for the harmonic function $(1 - x^2 - y^2)/(1 - 2x + x^2 + y^2)$ in the disk $\bar{D} = \{x^2 + y^2 \leq 1\}$. Explain.

Solution 5. One can easily check that

$$\frac{\partial^2}{\partial x^2} \frac{(1 - x^2 - y^2)}{(1 - 2x + x^2 + y^2)} = \frac{4(x - 1)(x^2 - 2x - 3y^2 + 1)}{(x^2 - 2x + y^2 + 1)^3} = \frac{\partial^2}{\partial y^2} \frac{(1 - x^2 - y^2)}{(1 - 2x + x^2 + y^2)}$$

so $u(x, y) = \frac{(1-x^2-y^2)}{(1-2x+x^2+y^2)}$ is a solution to $u_{xx} + u_{yy} = 0$. If we factor our solution, notice

$$u(x, y) = \frac{(1 - x^2 - y^2)}{(1 - 2x + x^2 + y^2)} = \frac{1 - (x^2 + y^2)}{(x - 1)^2 + y^2}.$$

Notice that on the interior $D = \{x^2 + y^2 < 1\}$, the numerator is positive so

$$\max_{(x,y) \in D} u(x, y) > 0$$

while on the boundary $\partial D = \{x^2 + y^2 = 1\}$ the numerator is 0, so

$$\max_{(x,y)\in\partial D\setminus(1,0)} u(x,y) = 0,$$

(our function is not defined at $(1,0)$ so we ignore this point). In particular, for this example we have

$$\max_{(x,y)\in\partial D\setminus(1,0)} u(x,y) < \max_{(x,y)\in D} u(x,y),$$

which appears to contradict the maximum principle. However, this is not a counterexample because the maximum principle does not apply to this case, because $u(x,y)$ is not continuous on $\bar{D} = \{x^2+y^2 \leq 1\}$ since there is a discontinuity at the point $(1,0)$.