Graph Homomorphisms

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1 Cores

A *graph homomorphism from G to H* is a function h from the vertices of G to the vertices of H that preserves edges. That is, if $(x, y) \in E(G)$, then $(h(x), h(y)) \in E(H)$.

Retract (folding) An endomorphism h onto a subgraph H such that $x \in H$ implies $h(x) = x$.

Core The (unique) vertex-minimal graph in a homomorphism equivalence class.

Antichain A set of objects unrelated by homomorphisms.

1.1 Results from [1]

1. A graph homomorphism is uniquely represented as the composition of a retract relation and a subgraph relation.
2. A homomorphism equivalence class is uniquely represented as a core.
3. A core is uniquely represented as an antichain of connected cores.
4. A graph G is uniquely represented as the infinite sequence $|Hom(F_i, G)|$ for any enumeration of all finite graphs F_i.

1.2 Proofs

1. Given a homomorphism h from G to H, its retract is the image of G under H, which is a subgraph of H. Given a retract of G which is a subgraph of H, their composition will be a homomorphism because both preserve edges.
2. Every homomorphism equivalence class has a core. If an equivalence class has two cores, then there are homomorphism from each to the other, which define a bijection, which must preserve edges both ways. Thus the cores are isomorphic.

3. Every core is the disjoint union of some connected components. Each component must be a core, or else there would be a homomorphism from the whole graph to a smaller graph. Likewise, there can be no homomorphism between components, or else there would be an endomorphism from the whole graph to a proper subgraph of it. Thus the components of any core form an antichain.

[1]: Peter J. Cameron. Graph homomorphisms (class notes). September 2006. http://www.maths.qmul.ac.uk/ pjc/csgnotes/hom1.pdf&ei=mGLGTKbnBIS8lQfn08nhAQ&usg=AI
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