Relational Structure Homomorphisms

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1 Digraphs

A directed graph, or *digraph*, consists of a set of objects called *vertices* along with a set of ordered pairs of vertices called *arcs*.

A *homomorphism* between two digraphs is a function ϕ from the vertices of one to the vertices of the other which preserves arcs. More formally, a homomorphism ϕ from G_1 to G_2 is a function from the vertices of G_1 to the vertices of G_2 such that if (x, y) is an arc in G_1, then $(\phi(x), \phi(y))$ must be an arc in G_2. For convenience, let us overload homomorphisms ϕ by writing $\phi((x, y))$ to mean $(\phi(x), \phi(y))$.

2 Structures

A *signature* is a pair (S, A), where S is a set of objects called *relation symbols*, and A is a function from relation symbols to their *arity*, a natural number.

A relational structure, or just *structure*, over a given signature (S, A) is a set of objects called *vertices* and a set of *facts*. Each fact is a relation symbol R from S applied to an n-tuple of vertices, where n is $A(R)$. A fact with relation symbol R applied to vertices $x_1, ..., x_n$ is written $R(x_1, ..., x_n)$.

3 Homomorphisms

The homomorphism relation is *reflexive* because the identity function is always a homomorphism from a thing to itself. It is also *transitive*, because the composition of a homomorphism from X to Y and a homomorphism from Y to Z is itself a homomorphism from X to Z. Thus the homomorphism relation forms a *quasi-order*. But homomorphisms are not, in general, antisymmetric (that is, they do not neccessarily form a *partial order*). Take, for instance, the digraphs $(\{x, y, z\}, \{(x, y), (x, z)\})$ and $(\{a, b\}, \{(a, b)\})$ (a path of length 2 and a path of length 1). These digraphs are not isomorphic, yet there are homomorphisms from each to the other. When there are homomorphisms from each of two things to the other, we call them *homomorphically equivalent*.

Write $h(R(x_1, ..., x_n))$ for $R(h(x_1), ..., h(x_n))$.

4 Representing Structures as Colored Digraphs

We will define a function $\chi()$ which takes a structure with relations symbols S, whose maximum arity is k, and produces a colored digraph over the colors $S \cup \{1, ...k\}$. The vertices of $\chi(M)$ are the vertices and edges of M. And for every fact $f = R(x_1, ..., x_n)$ in M, $\chi(M)$ has an edge (f, f) of color R and edges (f, x_i) of color i for each i.

Claim:

$$M_1 \to M_2 \text{ iff } \chi(M_1) \to \chi(M_2)$$

Proof: We will show that both the forward and reverse implications hold. For the foward implication, assume that h is a homomorphism from M_1 to M_2. We will construct the homomorphism h' from $\chi(M_1)$ to $\chi(M_2)$ as follows: $h'(x) = h(x)$ for all vertices x of M_1, and $h'(R(x_1, ..., x_n)) = h(R(x_1, ..., x_n)) = R(h(x_1), ..., h(x_n))$ for each fact $R(x_1, ..., x_n)$ of M_1. Now we must show that h' preserves edges. First consider the edges of the form (f, f) and color R, where $f = R(x_1, ..., x_n)$ is a fact of M_1. Each such edge will be mapped to $(h(f), h(f))$ of color R. This is indeed an edge of $\chi(M_2)$, because $h(f)$ must be a fact in M_2. Now consider the other edges, which have form (f, x_i) of color i, where $f = R(x_1, ..., x_n)$ and $i = 1..n$. Each will be mapped to $(h(f), h(x_i))$ of color i. This is an edge of $\chi(M_2)$ because $h(f) = R(h(x_1), ..., h(x_i), ..., h(x_n))$ is a fact of M_2.

For the reverse implication, assume that h is a homomorphism from $\chi(M_1)$ to $\chi(M_2)$. The homomorphism h' from M_1 to M_2 is just the restriction of h onto the vertices of M_1. We must show that h' preserves facts. Any fact $f = R(x_1, ..., x_n)$ of M_1 will be mapped to $R(h(x_1), ..., h(x_n))$. Since (f, f) of color R is an edge of $\chi(M_1)$, $(h(f), h(f))$ of color R must be an edge of $\chi(M_2)$. Likewise, there must be edges (f, x_i) of color i for each i. The only way these $i + 1$ edges could be present in $\chi(M_2)$ is if $R(h(x_1), ..., h(x_n))$ was a fact in M_2.

[Older notes from here on out; some falsehoods present]

5 Older Notes

Homomorphism A *homomorphism from G to H* is a function ϕ from the vertices of G to the vertices of H that preserves edges. That is, if e is an edge of G, then the edge formed by applying ϕ to each component of e is an edge of H.

Retract A *retract*, or *folding*, of G is an endomorphism ϕ onto a subgraph H of G such that $x \in H$ implies $\phi(x) = x$.

Core An object for which every endomorphism is also an automorphism.

Antichain A set of objects unrelated by homomorphisms.

5.1 Results from [1]

1. A homomorphism equivalence class has at most one core.

2. A core is uniquely represented as an antichain of connected cores.

3. A graph G is uniquely represented as the infinite sequence $|Hom(F_i, G)|$ for any enumeration of all finite graphs F_i.

5.2 Proofs

1. If an equivalence class has two cores, then there are homomorphisms from each to the other, ϕ and ϕ'. Consider the compositions $\phi \o \phi'$ and $\phi' \o \phi$. The first is an endomorphism from the first object to itself, and hence an automorphism, and the second is an endomorphism from second object to itself, hence an automorphism. Since both $\phi \o \phi'$ and $\phi' \o \phi$ are bijections, so are ϕ and ϕ'. Now we can show that ϕ is an isomorphism. We already know that it is a bijective homomorphism, so we need only show that it's inverse ϕ^{-1} is a homomorphism. ϕ^{-1} is equal to $(\phi' \o \phi)^{-1} \o \phi'$, which is the composition of an automorphism and a homomorphism, which is a homomorphism. Thus ϕ is an isomorphism and the equivalence class's cores are isomorphic.

2. Every core is the disjoint union of some connected components. Each component must be a core, or else it would have an endomorphism which is not an automorphism and so would the whole object. Likewise, there can be no homomorphism between components, since it could be used to construct an endomorphism which is not an automorphism by mapping one component to the other, and every other component to itself. Thus the components of any core (which are themselves connected cores) form an antichain.

In [2], Bauslaugh points out that cores ought be defined as graphs for which every endomorphism is an automorphism, and *not* as a vertex-minimal member of a graph homomorphism equivalence class as suggested in [1]. For finite graphs, these definitions are equivalent, but for infinite graphs only the latter results in cores being unique. Consider, for instance, the (countably) infinite graph with vertices $\{0, 1, 2, ...\}$ and edges $\{(x, y)|x < y\}$. Under the vertex-minimal core definition, this graph has an infinite number of cores, given by the subgraphs induced by $\{n, n+1, n+2, ...\}$ for any $n \geq 1$. These are in the same homomorphism equivalence class – a forward homomorphism maps x to $x + n$, and a reverse homomorphism maps x to x. And each core is indeed vertex minimal – they each have infinitely many vertices, and there is no homomorphism to any finite graph, since that graph would have to include a clique of every order.

A jointly universal set of relational structures may have more than one core. Take as an example the set consisting of a triangle and a Grotzsch Graph.

1. Determining whether G is H-colorable is NP complete for fixed H and varying G.

2. If there is a homomorphism from G to H, what can you say about the existence of cores of G and H?

3. Is the image of every endomorphism isomorphic to a retract?

[1]: Peter J. Cameron. Graph homomorphisms (class notes). September 2006. http://www.maths.qmul.ac.uk/ pjc/csgnotes/hom1.pdf

[2]: Bruce Lloyd Bauslaugh. Homomorphisms of infinite directed graphs. December 1994. Simon Fraser University.