Graph Homomorphisms

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1 Graphs

A *graph* is a set of *vertices* along with a set of unordered pairs of distinct vertices called *edges*. A *digraph* is like a graph, but it's edges are ordered pairs. A *relational structure*, or just *structure*, has a set of *vertices*, a set of *relations* with natural *arity*, and a set of n-tuples of vertices for each relation of arity n.

Unless otherwise mentioned, the following definitions and theorems should apply equally well to all three kinds of objects: graphs, digraphs, and structures.

2 Cores

Homomorphism A *homomorphism from G to H* is a function ϕ from the vertices of G to the vertices of H that preserves edges. That is, if e is an edge of G, then the edge formed by applying ϕ to each component of e is an edge of H.

Retract A *retract*, or *folding*, of G is an endomorphism ϕ onto a subgraph H of G such that $x \in H$ implies $\phi(x) = x$.

Core An object for which every endomorphism is also an automorphism.

Antichain A set of objects unrelated by homomorphisms.

2.1 Results from [1]

1. A homomorphism equivalence class has at most one core.
2. A core is uniquely represented as an antichain of connected cores.
3. A graph G is uniquely represented as the infinite sequence $|Hom(F_i, G)|$ for any enumeration of all finite graphs F_i.

2.2 Proofs

1. If an equivalence class has two cores, then there are homomorphisms from each to the other, ϕ and ϕ'. Consider the compositions $\phi \circ \phi'$ and $\phi' \circ \phi$. The first is an endomorphism from the first object to itself, and hence an automorphism, and the second is an endomorphism from second object to itself, hence an automorphism. Since both $\phi \circ \phi'$ and $\phi' \circ \phi$ are bijections, so are ϕ and ϕ'. Now we can show that ϕ is an isomorphism. We already know that it is a bijective homomorphism, so we need only show that it's inverse ϕ^{-1} is a homomorphism. ϕ^{-1} is equal to $(\phi' \circ \phi)^{-1} \circ \phi'$, which is the composition of an automorphism and a homomorphism, which is a homomorphism. Thus ϕ is an isomorphism and the equivalence class's cores are isomorphic.

2. Every core is the disjoint union of some connected components. Each component must be a core, or else it would have an endomorphism which is not an automorphism and so would the whole object. Likewise, there can be no homomorphism between components, since it could be used to construct an endomorphism which is not an automorphism by mapping one component to the other, and every other component to itself. Thus the components of any core (which are themselves connected cores) form an antichain.

In [2], Bauslaugh points out that cores ought be defined as graphs for which every endomorphism is an automorphism, and *not* as a vertex-minimal member of a graph homomorphism equivalence class as suggested in [1]. For finite graphs, these definitions are equivalent, but for infinite graphs only the latter results in cores being unique. Consider, for instance, the (countably) infinite graph with vertices $0, 1, 2, ...$ and edges $(x, y)|x < y$. Under the vertex-minimal core definition, this graph has an infinite number of cores, given by the subgraphs induced by $n, n+1, n+2, ...$ for any $n \geq 1$. These are in the same homomorphism equivalence class – a forward homomorphism maps x to $x + n$, and a reverse homomorphism maps x to x. And each core is indeed vertex minimal – they each have infinitely many vertices, and there is no homomorphism to any finite graph, since that graph would have to include a clique of every order.

A jointly universal set of relational structures may have more than one core. Take as an example the set consisting of a triangle and a Grotzsch Graph.

[1]: Peter J. Cameron. Graph homomorphisms (class notes). September 2006. http://www.maths.qmul.ac.uk/ pjc/csgnotes/hom1.pdf

[2]: Bruce Lloyd Bauslaugh. Homomorphisms of infinite directed graphs. December 1994. Simon Fraser University.