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Least squares degree reduction of Bézier curves

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In this paper we investigate the problem of reducing the degree of Bézier curves approximately from n to a prescribed target degree m whereby (parametric) continuity of any order $\leq \frac{m-1}{2}$ can be preserved at the two endpoints. The computations are carried out by minimizing the (constrained) L_2-norm between the two curves. In addition, a complete algorithm is given for performing the degree reduction within a prescribed error tolerance by help of subdivision. This work is an evident improvement on a previous paper (Eck, M *Comput.-Aided Geom. Des.* Vol 10 (1993) pp 237-251) about degree reduction in the sense that the algorithm presented is faster and much easier to implement, while still producing very good results.

Keywords: Bézier curve, degree reduction, constrained Legendre polynomials, constrained least squares approximation, endpoint constraints

In general, degree reduction of a Bézier curve

$$\mathbf{x}_n(t) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t),\ t \in [0, 1] \tag{1}$$

of degree n in $\mathscr{R}^s$ amounts to finding a Bézier curve

$$\bar{\mathbf{x}}_m(t) = \sum_{i=0}^{m} \bar{\mathbf{b}}_i B_i^m(t),\ t \in [0, 1] \tag{2}$$

of prescribed degree $m < n$ in $\mathscr{R}^s$ such that a suitable distance function $d(\mathbf{x}_n, \bar{\mathbf{x}}_m)$ between the two curves is minimized.

In the literature[1-8] one can find several schemes producing solutions for this approximation problem. These schemes mainly differ in the choice of the distance function and in requiring the solution to be either *best* or only *nearly best* relative to the distance function. For instance, one special type of degree reduction schemes works recursively by lowering the degree only by one in every step—a procedure commonly known as *economization.*

An example for such a stepwise method was recently given in Eck[9] where a very simple geometric construction of the new control points in each step is described. However, this general construction contains some scalar-valued degrees of freedom (in the following denoted as factors $\{\lambda_i\}$) which are then chosen in such a way that the maximal Euclidean distance

$$d_\infty(\mathbf{x}_n, \bar{\mathbf{x}}_{n-1}) = \max_{t \in [0,1]} \|\mathbf{x}_n(t) - \bar{\mathbf{x}}_{n-1}(t)\| \tag{3}$$

between $\mathbf{x}_n$ and $\bar{\mathbf{x}}_{n-1}$ with respect to the given parameterization is minimized.

Here, the derivation is mainly based on the so-called *constrained Chebyshev polynomials*[5,10]. These polynomials are equioscillating, which, in general, is characteristic for polynomial best approximations relative to the L_∞ norm. Moreover, the adjective *constrained* expresses the fact that preselected orders of (parametric) contact between $\mathbf{x}_n$ and $\bar{\mathbf{x}}_{n-1}$ can be achieved at the two curve endpoints during this minimization process.

Unfortunately the constrained Chebyshev polynomials are, in general, not known explicitly so their coefficients have to be determined numerically by a *Remes-type* algorithm, which itself needs a lot of implementation effort. Consequently, the factors $\{\lambda_i\}$ also cannot be given explicitly.

This major disadvantage is avoided in the current paper so that the algorithm presented is faster, more stable and much easier to implement. In more detail, we minimize the *least squares* distance function

$$d_2(\mathbf{x}_n, \bar{\mathbf{x}}_{n-1}) = \sqrt{\int_0^1 \|\mathbf{x}_n(t) - \bar{\mathbf{x}}_{n-1}(t)\|^2 \mathrm{d}t} \tag{4}$$

in every step using so-called *constrained Legendre polynomials.* These polynomials are explicitly known so that we are now able to derive explicit expressions (theorem 2) for the factors $\{\lambda_i\}$ of the general construction. Moreover, if this procedure is repeated to reduce the degree from n to m the best approximation property still holds. The same result is also valid for surfaces.

The paper is organized as follows. In the first section the general construction is briefly recalled together with some of its main properties. In the second section the remaining factors of this construction are specified by minimizing Equation 4, but the underlying derivations are shifted to the appendix in order to make the paper more readable. In the third section the methods presented here and in Eck[9] are compared for the case

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of degree reduction from n to $n-1$. We will see that even in this case the difference relative to the distance d_∞ is really small. The fourth section discusses the behaviour of the solution if the degree reduction algorithm is repeated stepwise in order to solve the original problem of degree reduction from n to m. Summarizing these ideas, we give a complete algorithm in section five, which carries out the degree reduction within a prescribed error tolerance. Finally, section six briefly discusses the obvious extension to tensor product Bézier surfaces.

THE GENERAL CONSTRUCTION

Using the notation in Equations 1 and 2, the well-known procedure of elevating the degree of a Bézier curve from $m = n-1$ to n can be written in terms of the control points as

$$\mathbf{b}_i = \frac{1}{n}\left(i \cdot \bar{\mathbf{b}}_{i-1} + (n-i)\cdot \bar{\mathbf{b}}_i\right) \tag{5}$$

for $i = 0, \ldots, n$.

Now we aim to invert of this process. Therefore we solve the overdetermined system of Equation 5 for the unknowns $\{\bar{\mathbf{b}}_i\}_{i=0}^{n-1}$ twice: first by neglecting the last equation ($i = n$) and second by ignoring the first equation ($i = 0$) and obtain recursively defined (auxiliary) points

$$\bar{\mathbf{b}}_i^I = \frac{1}{n-i}\left(n \cdot \mathbf{b}_i - i \cdot \bar{\mathbf{b}}_{i-1}^I\right) \tag{6}$$

for $i = 0, \ldots, n-1$ and

$$\bar{\mathbf{b}}_{i-1}^{II} = \frac{1}{i}\left(n \cdot \mathbf{b}_i - (n-i)\cdot \bar{\mathbf{b}}_i^{II}\right) \tag{7}$$

for $i = n, \ldots, 1$ which are distinguished by upper indices I and II.

These two sets $\{\bar{\mathbf{b}}_i^I\}_{i=0}^{n-1}$ and $\{\bar{\mathbf{b}}_i^{II}\}_{i=0}^{n-1}$ represent the control polygons of two different Bézier curves of degree $n-1$, denoted by $\bar{\mathbf{x}}_{n-1}^I$ and $\bar{\mathbf{x}}_{n-1}^{II}$. Both curves are in a relationship to the given curve $\mathbf{x}_n$ of degree n which is stated below.

Lemma 1: The curves $\bar{\mathbf{x}}_{n-1}^I$ and $\bar{\mathbf{x}}_{n-1}^{II}$ enjoy the following properties:

- $\mathbf{x}_n$ and $\bar{\mathbf{x}}_{n-1}^I$ meet in $t = 0$
- $\mathbf{x}_n$ and $\bar{\mathbf{x}}_{n-1}^{II}$ meet in $t = 1$

with (parametric) contact of order $n-1$.

Based on this knowledge, the key idea of the degree reduction process is to write the unknown points $\{\bar{\mathbf{b}}_i\}_{i=0}^{n-1}$ of the final degree reduced curve $\bar{\mathbf{x}}_{n-1}$ as

$$\bar{\mathbf{b}}_i = (1-\lambda_i)\cdot \bar{\mathbf{b}}_i^I + \lambda_i \cdot \bar{\mathbf{b}}_i^{II} \tag{8}$$

for $i = 0, \ldots, n-1$ with help of the auxiliary points defined in Equations 6 and 7.

From an algebraic point of view the introduction of the unknown factors $\{\lambda_i \in \mathscr{R}\}$ in Equation 8 shifts the problem of computing the points $\{\bar{\mathbf{b}}_i\}$ to the problem of computing the factors $\{\lambda_i\}$. But, there are also geometric reasons for using this set-up since every coefficient $\bar{\mathbf{b}}_i$ splits the line segment from $\bar{\mathbf{b}}_i^I$ to $\bar{\mathbf{b}}_i^{II}$ in the ratio $\lambda_i/(1-\lambda_i)$.

Let us denote the nth *forward difference* $\Delta^n \mathbf{b}_0$ of the control points $\{\mathbf{b}_i\}_{i=0}^n$ as usual by

$$\Delta^n \mathbf{b}_0 = \sum_{j=0}^{n} (-1)^{n+j}\binom{n}{j}\mathbf{b}_j$$

Then the following theorem, proved in Eck[9], shows the importance of Equation 8 for degree reduction.

Theorem 1: Consider a curve $\mathbf{x}_n$ with $\Delta^n \mathbf{b}_0 \neq 0$. Further let arbitrary real values $\{h_i \in \mathscr{R}\}_{i=0}^n$ with $\gamma := \Delta^n h_0 \neq 0$ be given. Then the equality

$$\mathbf{x}_n(t) - \bar{\mathbf{x}}_{n-1}(t) \equiv \mathbf{a} \cdot \sum_{i=0}^{n} h_i B_i^n(t) \tag{9}$$

is identically fulfilled iff the control points of $\bar{\mathbf{x}}_{n-1}$ are given by Equation 8 using

$$\lambda_i = \gamma^{-1} \cdot \sum_{j=0}^{i} (-1)^{n+j}\binom{n}{j} h_j \tag{10}$$

$$\mathbf{a} = \gamma^{-1} \cdot \Delta^n \mathbf{b}_0 \tag{11}$$

Remark: The restriction $\Delta^n \mathbf{b}_0 \neq 0$ in theorem 1 is necessary since otherwise the curve $\mathbf{x}_n$ would be of degree $< n$ and the factors $\{\lambda_i\}$ could be chosen arbitrarily.

SPECIFYING THE FACTORS

Next, we will specify the still unknown factors $\{\lambda_i\}$ of the general construction so that the best approximant $\bar{\mathbf{x}}_{n-1}$ is found which minimizes the distance function $d_2(\mathbf{x}_n, \bar{\mathbf{x}}_{n-1})$ in Equation 4 with respect to some endpoint constraints.

In the Appendix it is shown in full detail that this best approximant is determined by Equation A15 with the help of *constrained Legendre polynomials*. So, comparing Equations A15 and 9 we immediately deduce that the real numbers $\{h_i\}_{i=0}^n$ in theorem 1 must be

$$h_i = (-1)^{n+i}\binom{n}{i}^{-1}\binom{n}{i-a}\binom{n}{i+a}$$

Then using the well known binomial identity[11]

$$\sum_{i=\alpha}^{n-\alpha}\binom{n}{i-\alpha}\binom{n}{i+\alpha} = \binom{2n}{n+2\alpha}$$

we come directly to theorem 2.

Theorem 2: Consider a curve $\mathbf{x}_n$ with $\Delta^n \mathbf{b}_0 \neq 0$. Iff the factors $\{\lambda_i\}_{i=0}^{n-1}$ are given by

$$\lambda_i = \binom{2n}{n+2\alpha}^{-1} \cdot \sum_{j=0}^{i}\binom{n}{j-\alpha}\binom{n}{j+\alpha} \tag{12}$$

then the control points $\{\bar{\mathbf{b}}_i\}_{i=0}^{n-1}$ from Equation 8 determine the curve $\bar{\mathbf{x}}_{n-1}$ that minimizes $d_2(\mathbf{x}_n, \bar{\mathbf{x}}_{n-1})$ with the additional constraints for $t_0 = 0$ resp. $t_0 = 1$

and $2\alpha \le n$

$$\frac{d^r}{dt^r}\mathbf{x}_n(t)\bigg|_{t=t_0} = \frac{d^r}{dt^r}\bar{\mathbf{x}}_{n-1}(t)\bigg|_{t=t_0},\ 0 \le r \le \alpha - 1$$

Corollary 1: For a curve $\bar{\mathbf{x}}_{n-1}$, as in theorem 2,

$$d_\infty(\mathbf{x}_n, \bar{\mathbf{x}}_{n-1}) = s_{n,\alpha}\cdot\binom{2n}{n}^{-1}\cdot\|\Delta^n\mathbf{b}_0\| \tag{13}$$

where the *disturbance factor* $s_{n,\alpha} = \binom{2n}{n+2\alpha}^{-1}\cdot\binom{2n}{n}\cdot M_{n,\alpha}$ is used.

Remarks: (1) Here, the disturbance factor $s_{n,\alpha}$ is introduced because it represents the quantity of how much better or worse (relative to the maximal deviation) the $C^{\alpha-1}$-solution is in relation to unconstrained approximation ($s_{n,0} = 1$).

(2) The real value $M_{n,\alpha}$, used in corollary 1 and defined in Equation A12, represents the maximal value of the respective constrained Legendre polynomial. Unfortunately, this value is, in general, not explicitly known. Therefore, in Equation A14 for the most relevant values of n and α a numerical upper bound $\tilde{M}_{n,\alpha}$ is provided which can be used to define the upper bound $\tilde{s}_{n,\alpha} = \binom{2n}{n+2\alpha}^{-1}\cdot\binom{2n}{n}\cdot\tilde{M}_{n,\alpha}$ of $s_{n,\alpha}$.

(3) The factors $\{\lambda_i\}$ specified in theorem 2 depend only on α and the degree n; they do not depend on the control points $\{\mathbf{b}_i\}$. Moreover, these factors share the following two properties:

- $\lambda_i = 1 - \lambda_{n-i-1}$
- $0 = \lambda_0 = \ldots = \lambda_{\alpha-1} < \lambda_\alpha \ldots < \lambda_{n-\alpha-1} < \lambda_{n-\alpha} = \ldots = \lambda_{n-1} = 1$

Thus, the new points $\bar{\mathbf{b}}_i$ always lie between $\bar{\mathbf{b}}_i^I$ and $\bar{\mathbf{b}}_i^{II}$.

(4) The case $n = 2\alpha$ simply represents the well understood case of Hermite interpolation.

Tables 1 and *2* show some explicitly calculated factors λ_i and some of the disturbance factors $s_{n,\alpha}$ together with the corresponding upper bounds $\tilde{s}_{n,\alpha}$, each for $\alpha > 0$. Particularly, *Table 2* illustrates that the constrained approximations are sometimes even better (relative to the maximal deviation) than the corresponding unconstrained approximations. And although such effects can always arise if one uses a criterion A to produce approximations and then a (different) criterion B to compare them, the result is of interest for the present application. However, for $\alpha \le 2$ all entries in *Table 2* are of moderate size so that we can recommend this kind of boundary constraint for practical purposes.

Finally, in *Figures 1* and *2* two computed examples are shown. Here, the same Bézier curve of degree 8 has been degree reduced requiring C^0-continuity ($\alpha = 1$) and C^1-continuity ($\alpha = 2$) at the endpoints.

A COMPARISON

For the purpose of comparison, it might be of some interest to consider how much worse the maximal deviation d_∞ of the constrained best L_2 approximation $\bar{\mathbf{x}}_{n-1}$ is in relation to the solution in Eck[9] where this norm is minimized directly.

Therefore, we briefly recall that this constrained best L_∞ approximation $\hat{\mathbf{x}}_{n-1}$, computed numerically by a Remes-type algorithm, satisfies

$$d_\infty(\mathbf{x}_n, \hat{\mathbf{x}}_{n-1}) = \hat{s}_{n,\alpha}\cdot 2^{1-2n}\cdot\|\Delta^n\mathbf{b}_0\| \tag{14}$$

where $\hat{s}_{n,\alpha} \ge 1$ again is a disturbance factor of the unconstrained case $\alpha = 0$. However, this factor has to be computed numerically, too.

Now from Equations 13 and 14 we see that the following *comparison coefficient* $q_{n,\alpha} \ge 1$ is the appropriate measure:

$$q_{n,\alpha} = \frac{d_\infty(\mathbf{x}_n, \bar{\mathbf{x}}_{n-1})}{d_\infty(\mathbf{x}_n, \hat{\mathbf{x}}_{n-1})} = \frac{2^{2n-1}}{\binom{2n}{n}}\cdot\frac{s_{n,\alpha}}{\hat{s}_{n,\alpha}} \tag{15}$$

Moreover, notice that these coefficients do not depend on the curve to be approximated but only on n and α.

In *Table 3* the coefficients $q_{n,\alpha}$ are listed for small values of n and $\alpha = 0, 1, 2, 3$. Although we can make no statement which is valid for general n and α, we observe that all the entries in this table are of moderate size so that minimization of the least squares norm instead of the maximum norm represents a very powerful alternative. This behaviour is established further in the following section.

STEPWISE DEGREE REDUCTION

Next, we investigate the natural extension of the degree reduction method considered up to now, that is the reduction of the polynomial degree from n to any $m < n$. To do so, we apply the above described least squares method iteratively, reducing the degree by one in each step, until the target degree is reached.

A first consequence of performing this process with a fixed value $\alpha \le \frac{m-1}{2}$ in every iteration is that the

Table 1 The factors λ_i for constrained L_2-approximation

	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
$\alpha = 1$							
$n = 3$	0	$\frac{1}{2}$	1				
$n = 4$	0	$\frac{3}{14}$	$\frac{11}{14}$	1			
$n = 5$	0	$\frac{1}{12}$	$\frac{1}{2}$	$\frac{11}{12}$	1		
$\alpha = 2$							
$n = 5$	0	0	$\frac{1}{2}$	1	1		
$n = 6$	0	0	$\frac{5}{22}$	$\frac{17}{22}$	1	1	
$n = 7$	0	0	$\frac{5}{52}$	$\frac{1}{2}$	$\frac{47}{52}$	1	1

Table 2 The exact disturbance factors $s_{n,\alpha}$ and their bounds $\tilde{s}_{n,\alpha}$ (rounded to four digits)

n	$s_{n,1}$	$s_{n,2}$	$s_{n,3}$	$\tilde{s}_{n,1}$	$\tilde{s}_{n,2}$	$\tilde{s}_{n,3}$
3	0.9623			0.9771		
4	0.8036			0.8056		
5	0.7250	2.2540		0.7250	2.2903	
6	0.6778	1.6070		0.6789	1.6177	
7	0.6463	1.2903	6.3819	0.6491	1.2927	6.9479
8	0.6237	1.1032	4.0236	0.6282	1.1034	4.3412
9	0.6067	0.9801	2.9250	0.6126	0.9803	3.1023

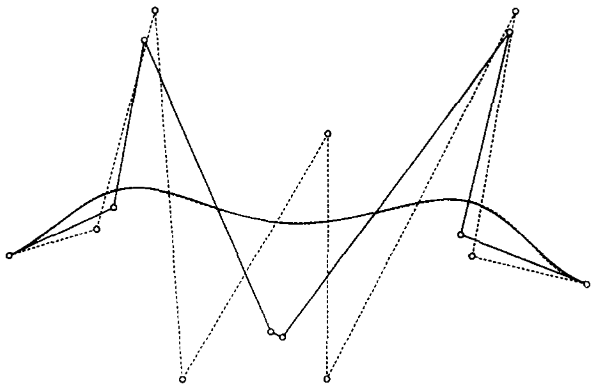

Figure 1 Reduction from $n = 8$ (dashed) to $m = 7$ (solid) with C^0-continuity

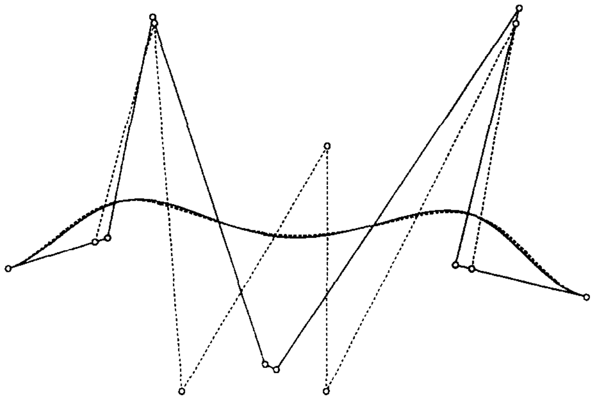

Figure 2 Reduction from $n = 8$ (dashed) to $m = 7$ (solid) with C^1-continuity

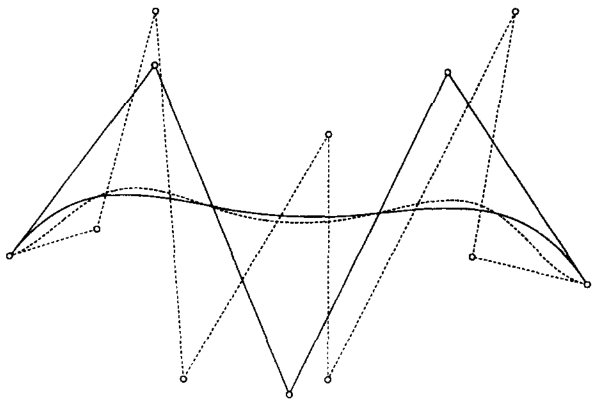

Figure 3 Reduction from $n = 8$ (dashed) to $m = 4$ (solid) with C^0-continuity

original curve $\mathbf{x}_n$ and the final approximation $\bar{\mathbf{x}}_m$ still agree up to order $\alpha - 1$ at the two endpoints. In *Figure 3* the degree of the Bézier curve used previously is reduced from 8 to 4 with C^0-continuity at the boundaries.

A second and very important consequence is that the best approximation property with respect to the L_2-norm still holds. This nice behaviour is not valid in case of minimizing the deviation function d_∞ in each step.

However, a more general comparison as carried out in the previous section is not possible here since the respective error formulas for arbitrary m depend in a non-trivial manner on the control points. Nevertheless, to get an idea of the quality of the present method, in *Figure 4* the result from Eck[9] is shown using the same input as in *Figure 3*. Obviously, the results are very similar.

Another remaining problem is to find a reasonable upper bound for $d_\infty(\mathbf{x}_n, \bar{\mathbf{x}}_m)$. A simple bound can be obtained by adding up all the maximal errors appearing in each step. Here, practical investigations have shown that this bound is good enough. Thus, in the following algorithm this idea is used.

DEGREE REDUCTION ALGORITHM

In this section we summarize the investigations of this paper by stating an algorithm for degree reduction from a more practical point of view. Here, in addition, a prescribed error tolerance ε on the maximal pointwise deviation must hold as follows:

$$d_\infty(\mathbf{x}_n, \bar{\mathbf{x}}_m) \leq \varepsilon$$

Now, to satisfy this restriction during the stepwise algorithm two aspects have to be considered.

First of all, we need to determine how large the partial error $\tilde{\varepsilon}$ in each of the separate $n - m$ steps is allowed to be. A very simple solution would be $\tilde{\varepsilon} = \varepsilon/(n - m)$. However, in the following algorithm we are more careful and proceed by subtracting the error of every degree reduction step subsequently from ε (the remaining error is denoted by δ). The new partial error bound $\tilde{\varepsilon}$ is then computed as the quotient of δ and the number of remaining degree reduction steps.

A second problem occurs if the error, calculated by Equation 13, in a certain degree reduction step, let say from j to $j - 1$, is larger than the prescribed partial bound $\tilde{\varepsilon}$. In this case it is necessary to subdivide the curve segment of degree j. For instance, if we subdivide at k equidistant parameter values $t_i = i/(k + 1)$, for $i = 1, \ldots, k$, then the error on each of the resulting $k + 1$ segments is decreased by a factor of $1/(k + 1)^j$ (here $k = 0$ means no subdivision). Using this fact, we can always determine a number k so that the partial error bound $\tilde{\varepsilon}$ is satisfied:

$$k = \left\lfloor \left(\frac{s_{j,\alpha} \cdot \|\Delta^j \mathbf{b}_0\|}{\tilde{\varepsilon} \cdot \binom{2j}{j}} \right)^{1/j} \right\rfloor$$

For these reasons, the output of the following degree reduction algorithm, in general, will be a spline curve consisting of several Bézier segments. Thus, it is not a surprising generalization if we allow a Bézier spline

Table 3 The comparison coefficients $q_{n,\alpha}$ (rounded to two digits)

n	$q_{n,0}$	$q_{n,1}$	$q_{n,2}$	$q_{n,3}$
3	1.60	1.00		
4	1.83	1.07		
5	2.03	1.15	1.00	
6	2.22	1.22	1.05	
7	2.39	1.29	1.11	1.00
8	2.55	1.36	1.16	1.04

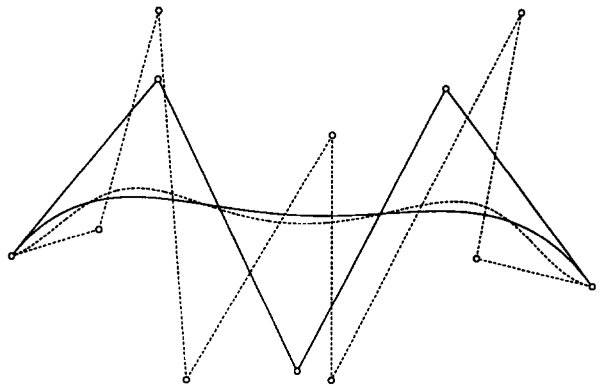

Figure 4 Reduction from $n=8$ (dashed) to $m=4$ (solid) with C^0-continuity using the method in Eck[9]

curve as input, where the degree of every segment is assumed to be n.

Input

- *seg*: number of spline segments of the input Bézier spline
- *n*: degree of the input Bézier spline
- **b**: vector-valued control polygon of input Bézier spline, where **b**[j][i] denotes the ith control point of the jth spline segment
- *m*: target degree of the output Bézier spline
- *alpha*: endpoint continuity enforced during the degree reduction
- *eps*: prescribed error tolerance

The following functions and subroutines are called by the algorithm:

- **Error**(j, *deg*, **b**, *alpha*): returns the error value for the jth segment of the Bézier spline **b** of degree *deg* calculated using Equation 13 together with Equation A14
- **Floor**(x): returns the greatest integer less than or equal to x
- **Subdivide**(j, *deg*, **b**, *split*, *tseg*, **tb**): subdivides the jth segment of the Bézier spline **b** of degree *deg* into *split* + 1 equidistant segments and stores the resulting segments as *tseg*th, …, (*tseg* + *split*)th segment of the Bézier spline **tb** of degree *deg*
- **Reduce**(k, *deg*, **tb**, *alpha*): overwrites the kth segment of the Bézier spline **tb** of degree *deg* by the degree reduced Bézier segment using Equations 8 together with 6, 7 and 12

Algorithm

```
for (j = 1 to seg by 1) delta[j] = eps;
for (deg = n to m + 1 by - 1)
  tseg = 1;
  for (j = 1 to seg by 1)
      terr = Error(j, deg, b, alpha);
      split = Floor((terr * (deg - m)/delta[j])^(1/deg));
      Subdivide(j, deg, b, split, tseg, tb);
      for (k = tseg to tseg + split by 1)
          tdelta[k] = delta[j] - terr/(split + 1)^ deg;
          Reduce(k, deg, tb, alpha);
      endfor
      tseg = tseg + split + 1;
  endfor
  seg = tseg;
  for (j = 1 to tseg by 1)
    delta[j] = tdelta[j];
    for (k = 0 to deg - 1 by 1) b[j][k] = tb[j][k];
  endfor
endfor
for (j = 1 to seg by 1) error[j] = eps - delta[j];
```

Output

- *seg*: number of spline segments of the output Bézier spline
- **b**: vector-valued control polygon of output Bézier spline
- *error*: error bound for each output segment

Now, applying this algorithm to the example of *Figure 3* where the maximal error satisfies the bound $d_\infty(\mathbf{x}_8, \bar{\mathbf{x}}_4) \leq 0.0668$ we can get, e.g. the following two results:

- For $\varepsilon = 0.015$ and $\alpha = 1$ the output spline consists of three quartic segments with break-points $\frac{1}{4}$ and $\frac{1}{2}$. The overall error bound is $d_\infty(\mathbf{x}_8, \bar{\mathbf{x}}_4) \leq 0.0144$.
- For $\varepsilon = 0.0001$ and $\alpha = 1$ the output spline consists of seven quartic segments with break-points $\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{6}$ and $\frac{11}{12}$. The overall error bound is $d_\infty(\mathbf{x}_8, \bar{\mathbf{x}}_4) \leq 0.00007$.

Here, we observe that the final segmentation is, in general, not equidistant, obviously a consequence of our careful error and subdivision strategy.

However, it should be mentioned that the resulting spline is neither optimal with respect to the choice of the break-points nor optimal with respect to the minimization of the least squares norm. In order to ensure the best approximation property here one cannot proceed subsequently; rather one has to compute a global least squares solution to determine all the segments simultaneously.

Nevertheless, this algorithm produces a reasonable solution to the degree reduction problem with very little computational effort.

TENSOR PRODUCT SURFACES

Eck[9,12] outlines how one can also solve in a very simple and straightforward manner the problem of degree reduction for tensor product Bézier surfaces once one has a suitable degree reduction method for Bézier curves. Here, the idea is to apply the curve algorithm first to every row of the original control net and afterwards to every column of the resulting new (auxiliary) control net.

The final control net is obviously one degree lower than the original surface in each parameter direction. Moreover, certain kinds of boundary conditions can be achieved if the curve algorithm is applied with $\alpha \geq 1$ each.

However, the same behaviour as already mentioned in the curve case can be observed here namely, whereas the methods in Eck[9,12] only produce *nearly best* or *very good* approximations to the chosen norm in the present case the resulting degree reduced surface really is the *best* L_2 approximation. This fact again follows easily from the orthogonality of the constrained Legendre polynomials.

Therefore, we can again expect very good results (even relative to the maximal deviation d_∞), although we have not yet implemented this algorithm.

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APPENDIX

In this Appendix we derive the characterizing Equation A15 of the best approximation, which is necessary for proving theorem 2.

It is well known from classical approximation theory[13,14] that the solution $\bar{\mathbf{x}}_{n-1}$ of the minimization problem (Equation 4) without boundary constraints is simply given by

$$\bar{\mathbf{x}}_{n-1}(t) = \mathbf{x}_n(t) - \mathbf{a} \cdot P_n(2t-1),\ t \in [0, 1] \tag{A1}$$

where

$$P_n(x) = \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} B_i^n\left(\frac{x+1}{2}\right) \tag{A2}$$

is the *Legendre polynomial* of degree n (here in Bernstein–Bézier representation). The validity of Equation A1 is based on the orthogonality property of these polynomials over the interval $[-1, 1]$ relative to the weight function $w(x) = 1$. Moreover, the coefficient $\mathbf{a}$ has to be chosen so that the right-hand side of Equation A1 is a polynomial of degree $n-1$:

$$\mathbf{a} = \binom{2n}{n}^{-1} \cdot \frac{1}{n!} \cdot \frac{\mathrm{d}^n \mathbf{x}_n(t)}{\mathrm{d}t^n} = \binom{2n}{n}^{-1} \cdot \Delta^n \mathbf{b}_0 \tag{A3}$$

However, the solution $\bar{\mathbf{x}}_{n-1}$ in Equation A1 does not agree with $\mathbf{x}_n$ for $t = 0$ and $t = 1$. Therefore, we extend this concept in such a way that, for any preselected integer $\alpha \geq 0$, the first $\alpha - 1$ derivatives of $\bar{\mathbf{x}}_{n-1}$ at the two segment endpoints agree with those of $\mathbf{x}_n$. To do so, we have to construct appropriate *constrained Legendre polynomials* playing the same role as the Legendre polynomials for unconstrained approximation.

These constrained Legendre polynomials $P_{n,\alpha}(x)$ of degree n ($n \geq 2\alpha$) should have α-fold zeroes at $x = -1$ resp. $x = 1$. Thus, the set-up

$$P_{n,\alpha}(x) = (1+x)^\alpha (1-x)^\alpha H_{n-2\alpha}(x) \tag{A4}$$

with an unknown polynomial $H_{n-2\alpha}(x)$ of degree $n - 2\alpha$ is appropriate.

The polynomials $P_{n,\alpha}(x)$ should also be orthogonal over the interval $[-1, 1]$ with respect to the weight function $w(x) = 1$. Therefore, we can immediately deduce that the polynomials $H_{n-2\alpha}(x)$ in Equation A4 have to be orthogonal with respect to the weight function

$$w(x) = (1+x)^{2\alpha}(1-x)^{2\alpha} \tag{A5}$$

The orthogonal polynomials relative to this weight function are contained in the large variety of *Jacobi polynomials* $P_n^{(r,s)}(x)$ with $r, s > -1$. These polynomials of degree n have been thoroughly investigated in the past (see Szego[15]). We recall only two of their main properties here:

- Jacobi polynomials are orthogonal over $[-1, 1]$ with respect to the weight function

$$w(x) = (1+x)^r (1-x)^s \tag{A6}$$

- Jacobi polynomials are given explicitly in Bernstein–Bézier form by

$$P_n^{(r,s)}(x) = \sum_{i=0}^{n} \bar{c}_i B_i^n\left(\frac{x+1}{2}\right) \tag{A7}$$

with

$$\bar{c}_i = (-1)^{n+i} \frac{\binom{n+r}{i}\binom{n+s}{n-i}}{\binom{n}{i}} \tag{A8}$$

Now, comparing Equations A5 and A6, we directly obtain $H_{n-2\alpha}(x) = \gamma \cdot P_{n-2\alpha}^{(2\alpha,2\alpha)}(x)$, where γ is an arbitrary normalization factor. Here, the setting $\gamma = 4^{-\alpha}$ is advantageous because it simplifies the following representation formula (Equation A10) for the constrained Legendre polynomials. Altogether, we have

$$P_{n,\alpha}(x) = 4^{-\alpha}(1+x)^{\alpha}(1-x)^{\alpha} P_{n-2\alpha}^{(2\alpha,2\alpha)}(x) \tag{A9}$$

and substituting Equation A7 into Equation A9 the Bernstein–Bézier representation for the constrained Legendre polynomials is found after some calculation to be

$$P_{n,\alpha}(x) = \sum_{i=\alpha}^{n-\alpha} c_i B_i^n\left(\frac{x+1}{2}\right) \tag{A10}$$

with

$$c_i = (-1)^{n+i} \frac{\binom{n}{i-\alpha}\binom{n}{i+\alpha}}{\binom{n}{i}} \tag{A11}$$

Comparing Equations A2 with A10 and A11, we immediately find $P_{n,0}(x) = P_n(x)$, as expected. Hence, we know that the maximal absolute value

$$M_{n,\alpha} = \max_{x \in [-1,1]} |P_{n,\alpha}(x)| \tag{A12}$$

for $\alpha = 0$ and arbitrary n is always equal to 1. However, for $\alpha > 0$ these values are not explicitly known.

Thus, the need for upper bounds of $M_{n,\alpha}$ is manifest. The following bound, valid for arbitrary n and α, is due to Gawronski (personal communication):

$$M_{n,\alpha} \le 1 \tag{A13}$$

However, this general upper bound for $M_{n,\alpha}$ gives no reasonable values for small n and α. But these cases are especially important for practical applications. Therefore, we tried a somewhat different approach. At first, we computed numerically the maximal values for every combination of $n \le 30$ and $\alpha \le 3$. Then we determined the following three approximate upper bounds $\tilde{M}_{n,\alpha} \ge M_{n,\alpha}$ with the help of best fitting routines from the software package MATHEMATICA

$$\begin{aligned}
\tilde{M}_{n,1} &= -0.21800 - 0.02212 \cdot n \\
&\quad + 0.00025 \cdot n^2 + 0.32104 \cdot \log(3+n) \\
\tilde{M}_{n,2} &= -1.25815 - 0.02483 \cdot n \\
&\quad + 0.00020 \cdot n^2 + 0.59084 \cdot \log(7+n) \\
\tilde{M}_{n,3} &= -0.25695 + 0.00910 \cdot n \\
&\quad - 0.00013 \cdot n^2 + 0.08222 \cdot \log(9+n)
\end{aligned} \tag{A14}$$

which are valid for $2\alpha \le n \le 30$. Here, $\log(\cdot)$ denotes the natural logarithm.

These particular bounds are not based on any theoretical investigations. Instead, it has turned out that the maximal relative error is nearly two percent for the first two bounds and less than nine percent for the last bound, good enough for practical purposes.

Now coming back to our original problem, it is obvious that the best constrained approximant $\bar{\mathbf{x}}_{n-1}$ is defined using constrained Legendre polynomials (shifted to the interval [0, 1]) as

$$\bar{\mathbf{x}}_{n-1}(t) = \mathbf{x}_n(t) - \mathbf{a}_{\alpha} \cdot P_{n,\alpha}(2t-1),\ t \in [0,1] \tag{A15}$$

where the coefficient $\mathbf{a}_{\alpha}$ again must be chosen so that $\bar{\mathbf{x}}_{n-1}(t)$ is a polynomial of degree $n-1$.

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