A-Star 2016 Winter Math Camp

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1 Introduction

Welcome to A-Star Winter Math Camp 2016! This is my fourth A-Star camp.

- I've attended once as a student before.
- I've taught the AMC class twice before in the summer of 2015 and 2016.
- Number Theory and Geometry are my favourite subjects to teach :).

1.1 Schedule

Time	Subject
9-10:30 AM	Number Theory
10:45AM-12:15PM	Algebra
1:45-3:15PM	Geometry
3:30-5:00PM	Counting

Table 1: A-Star Teaching Schedule

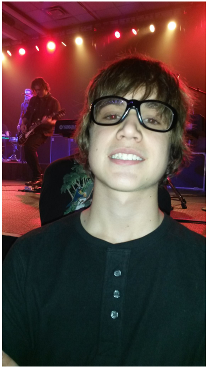
1.2 Icebreaker Activity

Three Truths and a Lie

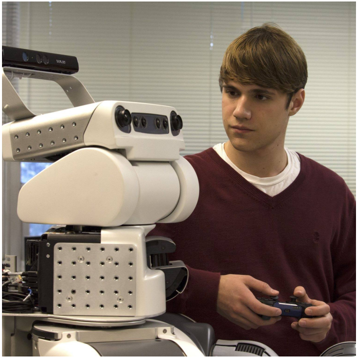
Write down three truths and one lie about yourself on your piece of paper. I'll guess which one is the lie! Good luck guessing which one is my lie.

- I've seen over 100 different bands live in concert.
- I've programmed a human sized robot.
- My family has 2 cats.
- I've competed in and won a crib race.

Concerts: Truth



Robot: Truth



Cats: (Deceptive) Lie!

We have 5...

Crib Race??: Truth

Celebration!

1.3 Math Time

The topic for today is divisibility and prime factorization.

2 Divisibility Rules

- 2 - Last digit is even.
- 3 - Sum of the digits is divisible by 3.
- 4 - Number formed by last two digits is divisible by 4.
- 5 - Last digit is either 0 or 5.
- 6 - Divisibility rules for both 2 and 3 hold.
- 7 - Take the last digit of the number and double it. Subtract this from the rest of the number. Repeat the process if necessary. Check to see if the final number obtained is divisible by 7. [2]

Lucky Seven

Choose **one** number below and determine if it is divisible by 7.

- 1729
- $2,718,281$
- $16,180,339$
- $31,415,926,535$

Taxicab Number

"It is a very interesting number; it is the smallest number expressible as the sum of two positive cubes in two different ways." - Srinivasa Ramanujan (1919)

$$\begin{aligned} 1729 &\rightarrow 172 - 2 \cdot 9 = 154 \\ 154 &\rightarrow 15 - 2 \cdot 4 = 7 \end{aligned}$$

Therefore, 1729 **is** divisible by 7.

Can you find the two ways Ramanujan referenced?

Euler's Number

$$\begin{aligned} 2718281 &\rightarrow 271828 - 2 \cdot 1 = 271826 \\ 271826 &\rightarrow 27182 - 2 \cdot 6 = 27170 \\ 27170 &\rightarrow 2717 - 2 \cdot 0 = 2717 \\ 2717 &\rightarrow 271 - 2 \cdot 7 = 257 \\ 257 &\rightarrow 25 - 2 \cdot 7 = 11 \end{aligned}$$

Therefore, 2718281 is **not** divisible by 7.

More on Euler's number (e) during Algebra lectures!

The Golden Ratio - $\phi = \frac{1+\sqrt{5}}{2} = 1.6180339\cdots$

$$\begin{aligned}
16180339 &\rightarrow 1618033 - 2\cdot 9 = 1618015 \\
1618015 &\rightarrow 161801 - 2\cdot 5 = 161791 \\
161791 &\rightarrow 16179 - 2\cdot 1 = 16177 \\
16177 &\rightarrow 1617 - 2\cdot 7 = 1603 \\
1603 &\rightarrow 160 - 2\cdot 3 = 154 \\
154 &\rightarrow 15 - 2\cdot 4 = 7
\end{aligned}$$

Hence, 16180339 **is** divisible by 7.

Pi

$31,415,926,535$ is too big of a number. Therefore, I wrote a computer program!

Seven.ipynb

It **is** divisible by 7.

In [8]:
```python
#Author:  Justin Stevens
#A-Star Winter Math Camp, 2016
#Determines if a number is divisible by 7

def divis_sev(x):
    """Inputs an integer x and prints out a list of numbers generated by following the -2 last digit rule
        Returns True or False based on whether the integer is divisible by 7."""
    cur_num=x
    while cur_num>7:
        print(cur_num)
        trunc_num=cur_num//10  #Removes last digit from the number
        last_dig=cur_num%10   #Stores the last digit in last_dig
        cur_num=trunc_num-2*last_dig  #Applies the divisibility rule for 7
    if cur_num>0:
        print(cur_num)
    if cur_num%7==0:
        return True
    else:
        return False
```

In [13]:
```python
divis_sev(31415926535)
```

```
31415926535
3141592643
314159258
31415909
3141572
314153
31409
3122
308
14
```

Out[13]: True

2.1 Explanation of the Magic

Let the number that we want to determine its divisibility by 7 be N. Let the last digit of N be x. Then, we can represent N as

$$N = 10a + x.$$

Note that we want to prove that 7 divides N implies that 7 also divides $a - 2x$.

To do so, we will multiply N by some integer.

Magic Continued

The magic integer is 5. The reason is because 5 and -2 leave the same remainder when dividing by 7.

If 7 divides N, then 7 should also divide $5N$. From the expression above for N, we have

$$5N = 50a + 5x.$$

Now, the question is, how do we get $a - 2x$ out of this?

Moving Around

We think to take the difference between $5N$ and $a - 2x$. Since we know that $5N$ is divisible by 7 if the difference is divisible by 7, then $a - 2x$ must also be divisible by 7.

Using the expression for $5N$ we found on the previous slide,

$$\begin{aligned} 5N - (a - 2x) &= 50a + 5x - (a - 2x) \\ &= 49a + 7x. \end{aligned}$$

This is clearly a multiple of 7, therefore, our proof is complete!

2.2 More Divisibility Rules

- 8 - The numbers formed by the last three digits are divisible by 8.
- 9 - The sum of the digits is divisible by 9.
- 10 - The number ends in 0.
- 11 - Let E be the sum of the digits in an even place. Let O be the sum of the digits in an odd place. 11 must divide the difference $E - O$ for the number to be divisible by 11.
- 12 - Combination of divisibility rules for 3 and 4.
- 13 - Same as the divisibility rule for 7, except replace $-2x$ with $+4x$.

3 Factorials

One of my favourite problems in number theory has to do with factorials. The factorial of a positive integer n is defined as the product of all the natural numbers less than or equal to n. In other words,

$$n! = n \times (n-1) \times (n-2) \times \cdots 1.$$

For instance, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

3.1 Zeros at the end of a Factorial

Note that $6! = 720$ ends in one factorial. The number

$$25! = 15511210043330985984000000$$

ends in 6 zeros.

Problem

How many zeros does $100!$ end in?

How Does Zero Work?

Zeros at the end of a number come from powers of 10. For instance, we can rewrite

$$25! = 15511210043330985984 \times 10^6.$$

Therefore, the problem is equivalent to finding the largest power of 10 that divides $100!$. One way to mathematically write this is $v_{10}(100!)$.

Since $10 = 2 \cdot 5$, the largest power of 10 that divides $100!$ is the **minimum** of $v_2(100!)$ and $v_5(100!)$.

V for Vendetta

We begin by calculating $v_2(100!)$. We write out

$$100! = 100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot 1.$$

Consider all the numbers in the product above.

How many of them are multiples of 2? Multiples of 4? Multiples of 8? Multiples of 16? Multiples of 32? Multiples of 64?

Floor Function

The number of multiples of 2 in $100!$ is simply the number of even numbers in the product. Half of the numbers are even, therefore, there are $\frac{100}{2} = 50$ multiples of 2.

For other powers of 2 that do not evenly divide into 100, we must introduce the floor function.

Definition

The floor function of a real number x is defined as the largest integer less than or equal to x. In other words, it is the result of truncating x. For instance, $\lfloor 3.14159 \rfloor = 3$ and $\lfloor -16.3 \rfloor = -17$.

Using our new friend, the floor function, we answer the question about multiples.

- There are $\lfloor \frac{100}{2} \rfloor = 50$ multiples of 2.
- There are $\lfloor \frac{100}{4} \rfloor = 25$ multiples of 4.
- There are $\lfloor \frac{100}{8} \rfloor = 12$ multiples of 8.
- There are $\lfloor \frac{100}{16} \rfloor = 6$ multiples of 16.
- There are $\lfloor \frac{100}{32} \rfloor = 3$ multiples of 32.
- There are $\lfloor \frac{100}{64} \rfloor = 1$ multiple of 64.

How Much Power Does 2 Have?

I claim that the number of powers of 2 in $100!$ is the sum of all the numbers above:

$$50 + 25 + 12 + 6 + 3 + 1 = 97.$$

For the numbers in the product $100!$ that have a highest power of 2^1, we have counted them once in the number 50.

For those that have a highest power of 2^2, they contribute a total of 2 to the product $100!$. We have counted them *once already* in the number 50 since they are also multiples of 2. Since they should contribute a total of 2 to the product, we add them one time more in the number 25.

Similarly, for the numbers that have a highest power of 2^3, they should contribute a total of 3 to the product $100!$. They have been counted once in the number 50 and once in the number 25, therefore, we should add them one time more in the number 12.

This logic extends to the powers $2^4, 2^5$, and 2^6.

Hence, $v_2(100!) = 97$. Are we done now?

Forgot About Magic 5

Nope! We also must compute $v_5(100!)$. We use the same method as above to determine that:

- There are $\lfloor \frac{100}{5} \rfloor = 20$ multiples of 5^1.
- There are $\lfloor \frac{100}{25} \rfloor = 4$ multiples of 5^2.

Therefore, $v_5(100!) = 20 + 4 = 24$.

Finishing the Problem

Therefore, $5^{24} \parallel 100!$ and $2^{97} \parallel 100!$. Hence, the largest power of 10 that divides $100!$ is 24 and the number of zeros at the end of $100!$ is $\boxed{24}$.

We indeed verify through the use of Mathematica that

$$100! = 106992388562667004907159682643816214685929638952175999\\ 932299156089414639761565182862536979208272237582511852109168 6\\ 4000000000000000000000000.$$

3.2 Legendre's Formula

Adrien-Marie Legendre (1752-1833) generalized this problem.

Theorem

The number of powers of a prime p that divide into $n!$ is

$$v_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor.$$

Summation Symbol

The $\sum$ symbol represents a summation. The k at the bottom is the variable that is being summed over. The 1 and ∞ are the ranges for the sum. For instance,

$$\sum_{k=1}^{4}(k^2) = 1^2 + 2^2 + 3^2 + 4^2.$$

In the case of the sum above,

$$\sum_{k=1}^{\infty}\lfloor\frac{n}{p^k}\rfloor = \lfloor\frac{n}{p^1}\rfloor + \lfloor\frac{n}{p^2}\rfloor + \lfloor\frac{n}{p^3}\rfloor + \lfloor\frac{n}{p^4}\rfloor + \cdots.$$

Define $s_p(n)$ to be the sum of the digits when the number n is expressed in base p. Then, an alternative way of writing Legendre's Formula is

$$v_p(n!) = \frac{n - s_p(n)}{p - 1}.$$

For instance, 100 in base 2 is $100 = 1100100_2$. The sum of the digits is $s_2(100) = 3$. Therefore,

$$v_2(100!) = \frac{100 - 3}{1} = 97.$$

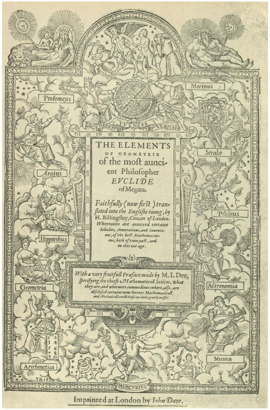
Furthermore, $100 = 400_5$. The sum of the digits is $s_5(100) = 4$. Therefore,

$$v_5(100!) = \frac{100 - 4}{4} = 24.$$

4 Euclid's Elements

Around the time of 300 BC, a great Greek mathematician rose from Alexandria by the name of Euclid. He wrote a series of 13 books known as *Elements*. Elements is thought by many to be the most successful and influential textbook ever written. It has been published the second most of any book, next to the Bible. [3]

The book covers both Euclidean geometry and elementary number theory. This chapter will focus solely on **Book VII, Proposition 1.**


Ptolomeus
Marinus
VIRESCIT VVLNERE VERITAS
THE ELEMENTS
OF GEOMETRIE
of the most aunci-
ent Philosopher
EVCLIDE
of Megara.

Faithfully (now first) tran-
slated into the Englishe toung, by
H. Billingsley, Citizen of London.
Whereunto are annexed certaine
Scholies, Annotations, and Inventi-
ons, of the best Mathematici-
ens, both of time past, and
in this our age.

With a very fruitfull Præface made by M. I. Dee,
specifying the chiefe Mathematicall Scieces, what
they are, and wherunto commodious: where, also, are
disclosed certaine new Secrets Mathematicall
and Mechanicall, vntill these our daies, greatly missed.

Aratus
Strabo
Polibius
Hipparchus
Geometria
Astronomia
Arithmetica
Musica
MERCVRIVS
Imprinted at London by Iohn Daye.

Figure 1: “Frontispiece of Sir Henry Billingsley’s first English version of Elements in 1570" - Source: Wikipedia [3]

4.1 Division Algorithm

The way division is commonly introduced in primary school is seen in the picture below:

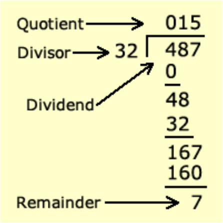


Figure 2: Source: CalculatorSoup

The division algorithm rigorizes this process. In the integers, $\mathbb{Z}$, the statement of the division algorithm is below:

Theorem

For every integer pair a, b, there exists distinct integer quotients and remainders, q and r, that satisfy

$$a = bq + r \quad | \quad 0 \leq r < |b|.$$

The proof of this comes from either the well-ordering principle or induction.

4.2 Proof of Division Algorithm

We consider the case when b is positive for simplicity. Consider the set

$$S = \{a - bq \quad | \quad q \in \mathbb{Z}^+, \, a - bq > 0\}.$$

In other words, this set consists of the positive integer values of $a - bq$ for q also being a positive integer. In order to continue with the proof, we must cite a famous Lemma from set theory.

Lemma (Well-ordering principle)
Every non-empty subset of positive integers has a least element.

Therefore, the set S has a *minimum element*, say when $q = q_1$ and $r = r_1$. I will prove that $0 \leq r_1 < b$.

Assume for the sake of contradiction otherwise and that

$$a - bq_1 = r_1 \geq b. \tag{1}$$

However, then I claim that $a - b(q_1 + 1)$ is a smaller member of set S.

Indeed, since $q_1 + 1 \in \mathbb{Z}^+$, the first condition is satisfied.

Furthermore, using 1, $a - b(q_1 + 1) = a - bq_1 - b \geq 0$. Therefore, both conditions are satisfied, and we have found a smaller member of set S. This contradicts the minimality of q_1 and r_1. Hence, $0 \leq r_1 < b$.

5 $\star$ Division in Other Domains

While the statement of the division algorithm may now seem like a mere formality, it is actually very vital to our number system. Without the division algorithm, we would not have unique prime factorization amongst the integers.

Furthermore, it is applicable when considering domains other than the integers, such as $\mathbb{Z}[i]$ (Gaussian integers) and $\mathbb{Z}[\omega]$ (Eisenstein integers).

With Respect to Gauss

The Gaussian integers are lattice points in the complex plane. [4]

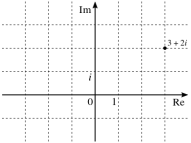


Figure 3: Source: Wikipedia

Rigorously, they are defined as the set

$$S = \{a + bi \quad | \quad a, b \in \mathbb{Z}\}.$$

5.1 Gaussian Division Problem [1]

Problem

Find the quotient and remainder when we divide $z = -1+4i$ by $w = 1+2i$ in $\mathbb{Z}[i]$.

To begin with, before I give a rigorous definition of division in $\mathbb{Z}[i]$, I want you to explore possible quotients and remainders. That is, with no restrictions other than sticking to the Gaussian integers, find a pair q, r such that

$$z = -1 + 4i = (1+2i)q + r = wq + r.$$

We'll discuss our findings in a few minutes!

Here are some examples of possible pairs (q, r):

- $-1 + 4i = (1 + 2i)(1) + (-2 + 2i)$, therefore, $(q, r) = (1, -2 + 2i)$.
- $-1 + 4i = (1 + 2i)(2) + (-3)$, therefore, $(q, r) = (2, -3)$.
- $-1 + 4i = (1 + 2i)(i) + (1 + 3i)$, therefore, $(q, r) = (i, 1 + 3i)$.
- $-1+4i = (1+2i)(-i)+(-3+5i)$, therefore, $(q, r) = (-i, -3+5i)$.
- $-1 + 4i = (1 + 2i)(1 + i) + i$, therefore, $(q, r) = (1 + i, i)$.

Summarizing in a Table

Quotient	Remainder
1	$-2+2i$
2	-3
i	$1+3i$
$-i$	$-3+5i$
$1+i$	i

Table 2: Division Algorithm applied to $z=-1+4i$ divided by $w=1+2i$.

Magnitude

Now the question lies on which remainder is best. When we worked with integers, we simply had the condition $0 \leq r < |b|$. However, how do we compare the values of two imaginary numbers such as $-2+2i$ and -3?

In order to do this, we recall the magnitude of a complex number $z = a+bi$. By definition,

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2},$$

where $\bar{z}$ is the complex conjugate.

The magnitude was also equivalent to the Euclidean distance between a point in the complex plane and the origin.

5.2 Norms

Since with Euclidean Domains, we want to work with integers, we define the **norm** of a complex number $z = a + bi$ to be

$$N(a + bi) = z\bar{z} = a^2 + b^2.$$

The norm function is used in comparing lengths of Gaussian Integers when using the division algorithm.

Note that the norm function over $\mathbb{Z}$ was $N(b) = |b|$.

Making the Table Normal

Quotient	Remainder	Norm
1	$-2+2i$	8
2	-3	9
i	$1+3i$	10
$-i$	$-3+5i$	34
$1+i$	i	$\boxed{1}$

Table 3: Extension of Table 2 with Norms

We therefore see that the best way to divide $z = -1 + 4i$ by $w = 1 + 2i$ of the quotients attempted is

$$z = -1 + 4i = (1 + 2i)(1 + i) + i = wq + r.$$

Note that

$$N(r) = 1 < N(w) = 5.$$

This property is unique to the quotient and remainder pair we've found.

In general, the statement of the division algorithm over $\mathbb{Z}[i]$ ensures the existance and uniqueness of a pair (q, r) for which

$$z = wq + r \quad | \quad N(r) < N(w).$$

5.3 Visualizing Division in Gaussian Integers

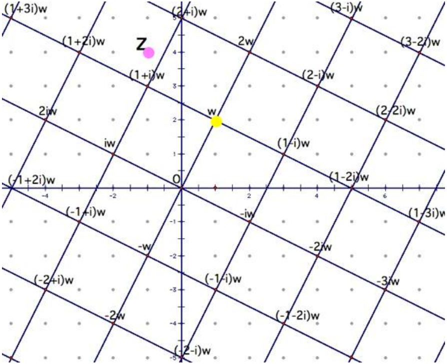


Figure 4: Source: Clay Kitchings [1]

6 Base Numbers

Base numbers are the heart of computers with both binary and hexadecimal. Binary can refer to the 2 states of a switch - on or off. Hexadecimal can be used to describe locations in computer memory or colours with HTML.

Definition

When we write numbers using the first b whole numbers (i.e. $0, 1, 2, \cdots, b-1$), this is a base b system. [5]

We can think of base conversions as different ways of *grouping numbers*.

6.1 Binary

The most common and applicable base is binary. In binary, the only two usable digits are 0 and 1. Therefore, we have to write every number as a sum of powers of 2.

For instance, to write 19 in binary, we would write

$$19 = 16 + 2 + 1 = 2^4 + 2^1 + 2^0 = 10011_2.$$

Similarly, to convert from binary to decimal, we find the power of 2 that each 1 corresponds with:

$$101010_2 = 2^5 + 2^3 + 2^1 = 32 + 8 + 2 = 42.$$

2016 Problems

When working with bases that are not binary, things get slightly more complicated. If we want to convert the positive number n into base b, we use a similar algorithm to binary.

We attempt to find the highest power of the base b that goes into the number n. We then subtract this from the number n and repeat until we get to the units digit.

Problem

Convert 2016 into base 8.

Magic 8 Ball

We begin by listing powers of 8:

$$8, 64, 512, 4096, \cdots.$$

The largest power of 8 that is less than 2016 is 512. We wish to divide 2016 by 512 to see what the quotient and remainder are. To do this, we introduce the division algorithm.

References

[1] Clay Kitchings. *Gaussian Integers & Division Algorithm Project*.

[2] A-Star. *A-Star Winter Math Camp AMC 10/12 Handout*. Star League, 2015.

[3] Wikipedia. Euclid's elements.

[4] Iurie Boreico. *A-Star Summer Math Camp USAMO Number Theory*. Star League, 2013.

[5] Matthew Crawford. *Introduction to Number Theory*. Art of Problem Solving, 2nd edition.

[6] Justin Stevens. *Olympiad Number Theory Through Challenging Problems*. AoPS Featured Articles, 3rd edition, 2016.