The Fourier Series & its Applications

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MTH-450 Final Project Presentation

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Introduction

In 1822, Mathematician Joseph Fourier boldly stated "there is no function f, or part of a function, which cannot be expressed by a trigonometric series." [3]

What is a Fourier Series

- An Infinite Sum of Trigonometric Functions
- Used to Approximate any Function f

About Joseph Fourier

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- Tragically died at the age of 62 from falling down the steps in his own home

Bernoulli & d'Alembret's Wave Equation

- Over 50 years before Fourier published his ideas, d'Alembret sought to model a vibrating string
- Based on some criteria, he arrived at the solution

$$u(x,t) = \sum_{n=1}^{N} b_n \sin(nx)\cos(nt).$$

- Daniel Bernoulli proposed any string's starting position could be modeled by an infinite trigonometric series
- In "The Analytical Theory of Heat"(1822), Fourier extended this idea to any function f, no matter how complex or discontinuous.

Fourier Series Representation

Theorem

Let $f : [-\pi, \pi] \to \mathbb{R}$ be an arbitrary function. If f can be expressed as an infinite trigonometric series in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

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then the coefficients a_0, a_n, and b_n are given by

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- $b_n = \int_{-\pi}^{\pi} f(x) \sin(\pi nx)dx.$

Even & Odd Functions

These properties help eliminate some computation

Lemma

If f is an even function $(f(x) = f(-x))$*, then* $b_n = 0$.

Lemma

If f is an odd function $(f(x) = -f(-x))$*, then* $a_0 = a_n = 0$.

A Simple Example

Let $f(x) = x^2$. Then, using Theorem 1, f can be represented by the Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Note that x^2 is an even function, so $b_n = 0$.

A Simple Example, $f(x) = x^2$

Deriving the coefficients, we have,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$$

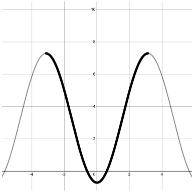
and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx) dx = \frac{4(-1)^n}{n^2}.$$

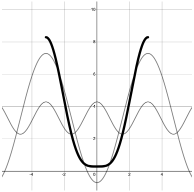
Thus substituting these coefficients in the general Fourier series formula, we have

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx).$$

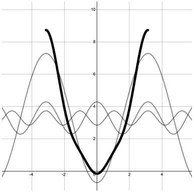
Fourier Series of $f(x) = x^2$



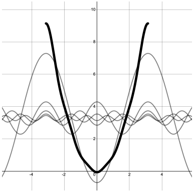
$n = 1$



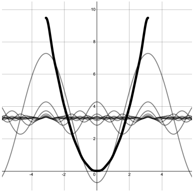
$n = 2$



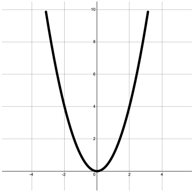
$n = 3$



$n = 5$



$n = 10$



$n \to \infty$

Generalized Fourier Series

Lemma

Let $f : [a,b] \to \mathbb{R}$ be an arbitrary function. If f can be expressed as an infinite trigonometric series in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{b-a}\right) + b_n \sin\left(\frac{2\pi nx}{b-a}\right),$$

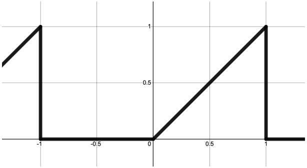
then the coefficients a_0, a_n, and b_n are given by

- $a_0 = \frac{1}{b-a}\int_a^b f(x)dx,$
- $a_n = \int_a^b f(x)\cos\left(\frac{2\pi nx}{b-a}\right)dx,$
- $b_n = \int_a^b f(x)\sin\left(\frac{2\pi nx}{b-a}\right)dx.$

A More Complex Example

Let g be the function defined by

$$g(x) = \begin{cases} 0, & -1 \leq x < 0 \\ x, & 0 \leq x < 1. \end{cases}$$



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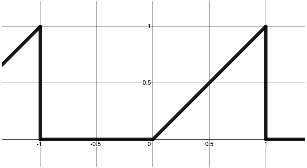


To provide a sample calculation,

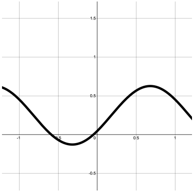
$$a_n = \int_a^b g(x) \cos\left(\frac{2\pi nx}{b-a}\right) dx = \frac{(-1)^n - 1}{\pi^2 n^2}.$$

A More Complex Example

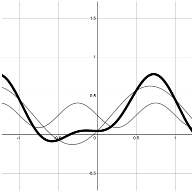
Then, the Fourier Series of g takes the form

$$\begin{aligned} g(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{b-a}\right) + b_n \sin\left(\frac{2\pi nx}{b-a}\right) \\ &= \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi^2 n^2} \cos(\pi nx) - \frac{(-1)^n}{\pi n} \sin(\pi nx). \end{aligned}$$

Fourier Series of g



$n = 1$



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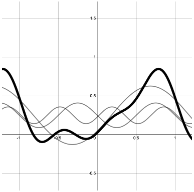

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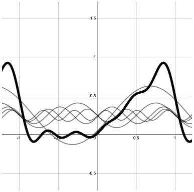

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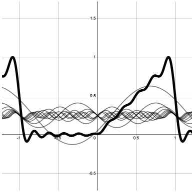

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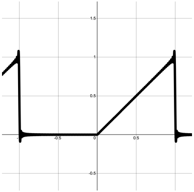


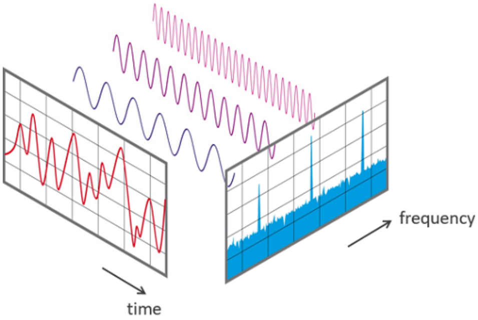
$n \to \infty$

Continuity & Differentiability

- Using the Weierstrass–M Test, we prove that the Fourier Series is everywhere continuous and infinitely differentiable.
- Term-by-Term Differentiability proves that the Fourier Series is infinitely differentiable

Applications of the Fourier Transform

- Discrete Fourier Transform
- Fast Fourier Transform
- Media Compression
- Fun pictures!



Questions?

References

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