NAME AND NUMBER

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What is a Number?

A number is an exact noun, i.e. a hash to an event. In early childhood, we learn numbers by subitizing (finding a bijection to a small order set); in later childhood, we learn to count numbers.

This protoknole aims to characterize some common types of numbers. It is not about specific numbers. It is the author's opinion that the general public's working definition of number is in serious need of an update. The smarter we make the crowd when it comes to discussing data, the easier it is for experts to communcate complex findings.

Why start here?

Mathematics is an extension of natural language, and languages are devices for telling stories. You can't tell a story about the color blue to an audience who only knows of black, white, and red. Stories get better by slowly pushing the limits of what language can do. There is an order to things here: one starts with the lexicon, then moves on to grammar, then syntax, and finally semantics. Today, science fiction and romance writers invoke basic principles from physics such as relativity theory and wave-particle duality. When it comes to math, though, most writers are silent. Why? I think it's mainly because being wrong in math is a very loud thing.

If this is the case, then only a math-oriented individual can lay the groundwork so that stories can become more and more sophisticated.

Three types of number

Classical numbers, matrices, functions. Each one gets a page.

Sets of Numbers

- $\mathbf{Z}$ denotes the integers.
- $\mathbf{Q}$ denotes the rational numbers.
- $\overline{\mathbf{Q}}$ denotes the algebraic numbers ($\overline{X}$ denotes the closure of a set X).
- $\mathbf{C}$ denotes the analytic numbers (commonly called the "complex numbers").
- $\mathbf{T} = \overline{B_1(0)} \setminus B_1(0)$ denotes the circular numbers.[1]
- $\mathbf{R}$ denotes the continuous numbers (commonly called the "real numbers").
- $\mathbf{H}$ denotes the quaternions.
- $\mathcal{H}$ denotes the upper half of the complex plane.

How the Sets are Related

The chain of subsets

$$\mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C} \subset \mathbf{H}$$

describes the connections between about half of these sets; also, $\mathbf{T}$ and $\overline{\mathbf{Q}}$ and $\mathcal{H}$ are all subsets of $\mathbf{C}$. Indeed, some technically inclined readers might be wondering why I'm even bothering to include $\mathbf{H}$ at this level of exposition. Those readers are missing the point.

What the Sets are About

The integers are about **indication**: how many of a thing exists at any given event. The rationals are about **luminosity**; we arrive at the rationals from the integers via the gambit of Archimedes.[2] The continuous numbers are about **mass**; we arrive at them by obliterating equivalence classes (fractions suck at describing continuous numbers, which is why analysis of real-valued functions eventually turns into a blitzkrieg of integrals and summations). By relinquishing total order, we arrive at the analytic numbers, which are about **electromagnetism**. Finally, to arrive at the quaternions, we dismiss commutativity of multiplication, yielding the powerful rotation phenomenon known as **parallel transport**.

Orbification of the Sets

This is about introducting **torsion** (wraparoundness) to each number system. When you do this to $\mathbf{Z}$, you get the divisor loop $\mathbf{Z}/(n)$, which behaves like a clock. On $\mathbf{Q}$, this yields a cyclotomy lens (important in number theory). On $\mathbf{R}$, this yields $\mathbf{T}$. Doing this over $\mathbf{C}$ is so useful that the space has a special name: it is called the **Riemann sphere**, and it is denoted by $\overline{\mathbf{C}}$. I don't know of a name for $\overline{\mathbf{H}}$, but I think it is just as special (if not more) than what's happening in $\mathbf{C}$, so I've been calling it the **Dyson gasket**.

[1] This is also called the unit circle. $B_r(p)$ denotes the open ball of radius r centered at p.

[2] One thing to notice about the rationals is that one can take a midpoint given any two points. So, relaxing this a bit and then applying it recursively, within $[0, 1]$ we are guaranteed the existence of 0.1, and then within $[0, 0.1]$ we always have 0.000001, and then within $[0, 0.000001]$ we always have 0.0000000000000001, ad nauseam.

Why include Matrices?

In a first linear algebra course, one makes a distinction between vector and scalar quantities. Gilbert Strang notes in his book *Introduction to Linear Algebra* that vectors arose from a need to add apples and oranges. So if vectors are numbers, and vectors are basically the scalars of the matrix world (you need tensor theory actually make this precise), then I reckon matrices, at least the ones with numbers in them, ought to be considered numbers in their own right.

Groups of Matrices

Below, R is some ring (an algebraic structure that vaguely resembles the integers).

- $\mathrm{GL}(R)$ denotes the general linear group, whose matrices are always invertible.[3]
- $\mathrm{SL}(R)$ denotes the special linear group, whose matrices always have determinant 1.
- $\mathrm{SO}(R)$ denotes the special orthogonal group,[4] which consists of matrices M where $M^{-1} = M^t$.

The point is that most of the matrices you see on a day to day basis in the technical world are so tame that we can classify them into little boxes based on how they behave. Being invertible is almost always a good thing. Having determinant ± 1 is tantamount to preserving volume (most matrices considered here are encoding some sort of change of basis), while having positive determinant is tantamount to preserving orientation. Orthogonality implies isometry, that is, any orthogonal matrix corresponds to a tranformation that preserves distances.

Matrix Multiplication

Order almost always matters when multiplying matrices.[5] This means that matrices can be used to describe the multiplication within almost any mathematical context. For example, analytic numbers can be described using matrices of continuous numbers as follows:

$$x + iy \mapsto \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \quad \text{so that} \quad \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ax - by \\ ay + bx \end{bmatrix}$$

Analogously, quaternions can be represented in terms of matrices of analytic numbers:

$$A + Ri + Gj + Bk \mapsto \begin{bmatrix} A + Ri & G + Bi \\ -G + Bi & A - Ri \end{bmatrix}$$

This calculation is incredibly messy without matrices. Are matrices numbers? Well, they simplify calculations into easily crunchable procedures. Sounds like a number to me.

[3] That is, their determinant is always nonzero.

[4] M^t denotes the transpose, which is just the matrix you get after reflecting about the NW-SE diagonal.

[5] You can convince yourself this is true by thinking about the similarities between multiplying matrices and composing functions.

Oh, come on

Yeah, I consider functions to be a special type of number, too. There are two main classification hierarchies for most students of mathematics: growth classification identifies how fast an algorithm (tied to some function, in theory) can solve a problem; smoothness classification identifies how amenable a function is to approximation by polynomials.[6]

For growth classes, I merely give examples. For smoothness, I describe things a bit.

Classes of Functions, by Growth

- $O(1)$: Lookup in a constant-size hash table.
- $O(\log n)$: Sorted (binary) search.
- $O(n)$: Unsorted (linear) search.
- $O(n \log n)$: Fast Fourier transform.
- $O(n^2)$: Insertion sort.
- $O(2^n)$: Traversing a full binary tree of depth n.
- $O(n!)$: Most stupid stuff.

Classes of Functions, by Smoothness

- $\mathcal{C}^0$. These functions are called **continuous**, and (surprise!) most functions seen in real analysis courses are of this type.
- $\mathcal{C}^1$. The "1" here refers to the property of having a single continuous derivative (sometimes more). For example, the absolute value function $|x|$ is $\mathcal{C}^0$ but not C^1 since its derivative jumps at the origin.
- $\mathcal{C}^n$. Same deal as above, but with n derivatives.
- $\mathcal{C}^\infty$. These functions are called **smooth** and each one has a Taylor series expansion somewhere. All polynomials of any degree are part of this class.
- $\mathcal{C}^\omega$. These functions are called **analytic** and are even better than smooth functions (and thus harder to come by). The standard example of something smooth but not analytic are the bump functions.

[6] A polynomial is a string of exponents in terms of some arbitrary variable whose coefficients come from a ring R. For instance, $4a^7 + 20a + 1337 \in \mathbf{Z}[a]$.