# SPACE AND POINT

Justin T. Chun, December 2021

*This protoknole assumes the content of "Name and Number."*

## Sets, elements, equivalence, inclusion

The central notion in mathematical foundations is that of **set** and **element**. Usually, elements are themselves defined in terms of sets (where the base case is $\varnothing$, the set with zero elements), but I don't think this is worth worrying about until one hits a true roadblock. Set and element yield the binary relations **equivalence** and **inclusion**. For two sets $X$ and $Y$,

$$X = Y$$

means that $X$ and $Y$ have exactly the same elements, whereas

$$X \in Y$$

means that $X$ is an element of $Y$. Even at this scale, we can frame things in terms of **space** and **point**, i.e.

*set is to space as element is to point.*

This theme of connecting things together so that they align in terms of spaces and points, comprises the modus operandi of this protoknole.

## Subsets, containment, power sets, pointed sets

Sets of elements all belonging to some larger set $T$ are called **subsets** of $T$. We write $X \subseteq Y$ when $X$ is a subset of $Y$ and say that $Y$ **contains** $X$. The set of all subsets of a set $T$ is called the **power set** of $T$. Certain operations on the power set of $T$ always make sense; namely, the **union** $X \cup Y$, the **intersection** $X \cap Y$, and the **relative complement** $X \setminus Y$. The power set is partially ordered by containment; thus, power sets are in a way the initial **posets** (short for "partially ordered sets"). Once again,

*power set is to space as subset is to point.*

A **pointed set** is a set where exactly one element is recognized as special. For example,

$$(\{-1, 0, 1, 2, 3\},\ 0)$$

works.

## Equivalence relations, partitions, multisets

An **equivalence relation** on a set is essentially a subset-specifying rule that **partitions** the set into disjoint chunks. This is the archetypal **multiset**: simply pair each subset with however many elements that subset contains. Here's an example of this:

$$\{\ (\{1, 3, 5\},\ 3), \quad (\{2, 8, 7\},\ 3), \quad (\{0, 4, 9, 6\},\ 4)\ \}.$$

## Rings, unit groups, modules, vectorspaces, tensor products

Before examining geometric and analytic generalizations of posets and multisets, it may be instructive to look at how space and point generalize in an algebraic setting. A first instance of this phenomenon would be

*ring is to space as units group is to point,*

where a **ring** is any sort of set that somewhat resembles the integers, and the **units group** is all the elements of a ring that are invertible under multiplication. Also noteworthy are

*module is to space as base ring is to point,*

and the very closely related

*vectorspace is to space as ground field is to point.*

I should probably add that this is an incredibly bizarre way to think about vectorspaces. The more orthodox way to look at them is to describe the space relative to an **origin**, making the entire space interpretable as a pointed set. You can even grade the space, framing it as a **flag**! As an example, look at $\mathbf{C}^6$:

$$0 \in \mathbf{C} \in \mathbf{C}^2 \in \mathbf{C}^3 \in \mathbf{C}^4 \in \mathbf{C}^5 \in \mathbf{C}^6.$$

Anyway, modules over a ring are the simplext context in which one can define a **tensor product**.

## Continuous maps, preimage, open sets, topologies

Just as the tensor product is a swiss-army tool for algebraic computations, there exists a similarly powerful object in the geometric setting: the **continuous map**. It is defined in terms of the **preimage** of select **open sets**, i.e. $f : X \to Y$ is continuous when

$$\forall\, V \subseteq Y \text{ open}, \quad f^{-1}(V) \text{ is also open.}$$

For any topological game, the pokedex of opens is called the **topology**. This is why continuous maps are such a razor-sharp tool when doing analysis: since topologies work only to preserve the notion of "closeness," every neato downstream aspect of a space can be gleaned by simply paying attention to what continuous maps do on that space.

## Topological spaces, Hausdorff spaces, metric spaces

Continuous functions preserve **compactness**, **connectedness**, path-connectedness, Lindelofness (weak compactness), and separability. We call a set equipped with a topology a **topological space**. Briefly, topspaces beget **Hausdorff spaces** beget **metric spaces** beget **normed vector spaces** beget **inner product spaces**. When the norm on a vectorspace is **complete** (fun fact: all Hausdorff spaces have unique limits), that's a **Banach space**; a Banach space with an inner product is called a **Hilbert space**.

Okay – that should be enough incentive to present the actual definition. Let $2^X$ denote the power set of some set $X$. The subset $\tau \subseteq 2^X$ is a topology if

1. $\varnothing \in \tau$ and $X \in \tau$,
2. $\tau$ is closed under union, and
3. $\tau$ is closed under finite intersections.

The set of topologies on $X$ forms a **complete lattice** ("toset"?), wherein every subset-pair has both an **infimum** (meet) and a **supremum** (join).

## $\sigma$-algebras, measures, integrals

Time to switch gears. A subset $\Sigma \subseteq 2^X$ is a **$\sigma$-algebra** if

1. $\Sigma$ is closed under relative complement,
2. $\varnothing \in \Sigma$, and
3. $\Sigma$ is closed under countable union.

The initial $\sigma$-algebra here is called the **Borel $\sigma$-algebra**. Elements of the Borel $\sigma$-algebra are called **Borel sets**. That is, Borel sets are any set in a topspace formable from open sets through countable union, countable intersection, and relative complement. This $\sigma$-algebra then forms a foundation upon which all of **measure** and **integration** theory is built.

From Borel sets we get measures, and from measures we get integrals (the **Lebesgue integral** in particular). All of this analytic equipment is usually communicated by specifying a 3-tuple $(X, \Sigma, \mu)$ called a **measure space**. Here, $\mu$ is the measure, a function from $\Sigma$ to $[-\infty, \infty]$ such that

1. $\mu(E) \geqslant 0 \quad \forall E \in \Sigma$,
2. $\mu(\varnothing) = 0$, and
3. $\forall$ countable collections $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in $\Sigma$, we have

$$\mu\left(\bigsqcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

## Recap: spaces, points, index sets

With both topspaces and measure spaces, there seems to be some sort of intermediate level between space and point. With topspaces this is the topology, and with measure spaces this is the $\sigma$-algebra. In both cases, there is a mediating **index set** that takes care of keeping track of all the details floating around. Usually this index set is denoted by $I$ when we care enough to reference it directly.

*I was going to keep plugging and explain Fourier transforms, expected value, and Lie algebras, but I'll save it for an upcoming protoknole. This should be enough for now.*