MATH 104A

Homework 6

General Instructions: You have to integrate all the problems that require coding and/or numerical computation in a single *jupyter* notebook. Make sure all your codes have a preamble which describes purpose of the code, all the input variables, the expected output, your name, and the date of the last time you modified it. Write your own code, individually. Do not copy codes! The solutions to the problems that do not require coding must be uploaded as a single pdf or as part of the *jupyter* notebook.

1. (Optional - Not Graded) Prove that any polynomial $P_n(x)$ of degree at most n can be written as

$$P_n(x) = \sum_{j=0}^{n} a_j \phi_j(x),$$

where $\phi_j(x)$ is a polynomial of degree exactly j, for $j = 0, 1, \dots, n$.

2. Suppose $\{\phi_0(x), \phi_1(x), \dots, \phi_n(x)\}$ is an orthogonal set of functions with respect to the L^2 inner product, i.e.

$$\langle \phi_j, \phi_k \rangle = \int_a^b \phi_j(x)\phi_k(x)dx = 0, \quad \text{if } j \neq k.$$

Prove the Pythagorean theorem

$$\|\phi_0 + \phi_1 + \dots \phi_n\|^2 = \|\phi_0\|^2 + \|\phi_1\|^2 + \dots \|\phi_n\|^2,$$

where $\|f\|^2 = \langle f, f \rangle$.

3. (Optional - Not Graded) The solution $P_n(x)$ to the Least Squares Approximation problem of f by a polynomial of degree at most n is given explicitly in terms of orthogonal polynomials $\psi_0(x), \psi_1(x), \dots, \psi_n(x)$, where ψ_j is a polynomial of degree j, by

$$P_n(x) = \sum_{j=0}^{n} a_j \psi_j(x), \quad a_j = \frac{\langle f, \psi_j \rangle}{\langle \psi_j, \psi_j \rangle}.$$

(a) Let $\mathcal{P}_n$ be the space of polynomials of degree at most n. Prove that the error $f - P_n$ is orthogonal to this space, i.e. $\langle f - P_n, q \rangle = 0$ for each $q \in \mathcal{P}_n$. (b) Using the analogy of vectors interpret this result geometrically (recall the concept of orthogonal projection).

4. (a) Obtain the first 4 Legendre polynomials in $[-1, 1]$. b) Find the least squares polynomial approximations of degrees 1, 2, and 3 for the function $f(x) = e^x$ on $[-1, 1]$. c) What is the polynomial least squares approximation of degree 4 for $f(x) = x^3$ on $[-1, 1]$? Explain.

5. Plot the monic Chebyshev polynomials $\tilde{T}_0(x)$, $\tilde{T}_1(x)$, $\tilde{T}_2(x)$, $\tilde{T}_3(x)$, and $\tilde{T}_4(x)$.

6. (Optional - Not Graded) Prove that

$$\frac{2}{\pi}\int_{-1}^{1} \frac{[T_n(x)]^2}{\sqrt{1-x^2}}dx = 1. \tag{1}$$

7. The concentration c of a radioactive material decays according to the law $c(t) = be^{-at}$ where t represents time in seconds, $a = 0.1$ sec^{-1}, and b is the initial concentration. a)Using the Least Squares method and the data table (Table 1) below find b. b) Find the error in the least squares approximation.

t_i (sec)	C_i
1	0.91
2	0.80
3	0.76
4	0.65

Table 1:

8. (Optional - Not Graded) Given a collection of data points $\{(x_i, y_i)\}_{i=1}^m$ find the best least squares approximation of the form $y = ax^2 + bx^3$.

9. (Optional - Not Graded) (a) Given a collection of data points $\{(x_i, y_i)\}_{i=1}^m$ find the best least squares approximation of the form $y = ax + bx^2$.
(b) Use this approximation to fit the data in Table 2.
(c) Find the error in the least squares approximation.

x_i	y_i
1	3.1
2	9.8
3	21.2
4	36.1

Table 2: