MATH 104A

Homework 2

General Instructions: You have to integrate all the problems that require coding and/or numerical computation in a single *jupyter* notebook. Make sure all your codes have a preamble which describes purpose of the code, all the input variables, the expected output, your name, and the date of the last time you modified it. Write your own code, individually. Do not copy codes! The solutions to the problems that do not require coding must be uploaded as a single pdf or as part of the *jupyter* notebook.

1. Let
$$f(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ 1 - x & \frac{1}{2} \le x \le 1 \end{cases}$$
Write a code to construct the corresponding Bernstein polynomials and use it to estimate the rate of convergence of $B_n(f)$ to f.

2. Generate you own version of the integral sign
$$\int$$
by using a composite, quadratic Bézier curve (you may write a short code for it). Make sure the curve is C^1.

3. Let V be a linear space with norm $||\cdot||$, W a subspace of V, and $f \in V$. Prove that the set of best approximations to f by elements in W is convex.

4. (a) Find the best uniform approximation to $f(x) = \sin(2x)$ on $[0, 2\pi]$ by polynomials of degree at most 2.
   (b) Let $f \in C[a, b]$. Find the best uniform approximation to f by a constant.

5. Let $V = \mathbb{R}^3$ with $||\cdot||_\infty$, $W = \text{span}\{(0, 1, 0), (0, 0, 1)\}$, and $f = (3, 6, 4)$. Prove that a best approximation to f is not unique.

6. Prove that every $p \in \mathbb{P}_n$ has a unique representation of the form:
$$p(x) = a_0 + a_1T_1(x) + \cdots + a_nT_n(x), \tag{1}$$
where T_j, for $j = 1, \ldots, n$ are the Chebyshev polynomials of degree j.

7. Plot T_1, T_2, T_3, T_6.