MATH 104A

Homework 1

General Instructions: You have to integrate all the problems that require coding and/or numerical computation in a single *jupyter* notebook. Make sure all your codes have a preamble which describes purpose of the code, all the input variables, the expected output, your name, and the date of the last time you modified it. Write your own code, individually. Do not copy codes! The solutions to the problems that do not require coding must be uploaded as a single pdf or as part of the *jupyter* notebook.

1. Review and state the following theorems of Calculus:
   (a) The Intermediate Value Theorem
   (b) The Mean Value Theorem
   (c) Rolle's Theorem
   (d) The Mean Value Theorem for Integrals
   (e) The Weighted Mean Value Theorem for Integrals

2. Write a computer code to implement the Composite Trapezoidal Rule quadrature

$$T_h[f] = h\Big[\frac{1}{2}f(x_0) + f(x_1) + ... + f(x_{N-1}) + \frac{1}{2}f(x_N)\Big] \tag{1}$$

   to approximate the definite integral

$$I[f] = \int_a^b f(x)dx \tag{2}$$

   using the equally spaced points $x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots x_N = x_0 + Nh = b$, where $h = \frac{b-a}{N}$.

   (a) Test your code with $f(x) = \frac{1}{(1+x)^2}$ in $[0, 2]$ by computing the error $|I[f] - T_h[f]|$ for $h = \frac{2}{20}, \frac{2}{40}, \frac{2}{80}$, and verify that T_h has a convergent trend at the expected quadratic rate.
   (b) Let $f(x) = \sqrt{x}$ in $[0, 1]$. Compute $T_{1/N}$ for $N = 16, 32, 64, 128$. Do you see a second order convergence to the exact value of the integral? Explain.

3. Consider the definite integral

$$I[\cos x^2] = \int_0^{\sqrt{\pi/2}} \cos x^2 dx \tag{3}$$

We cannot calculate its exact value but we can compute accurate approximations to it using $T_h[\cos x^2]$. Let

$$q(h) = \frac{T_{h/2}[\cos x^2] - T_h[\cos x^2]}{T_{h/4}[\cos x^2] - T_{h/2}[\cos x^2]}. \tag{4}$$

   (a) Using your code, find a value of h for which $q(h)$ is approximately equal to 4.

   (b) Get an approximation of the error, $I[\cos x^2] - T_h[\cos x^2]$, for that particular value of h.

   (c) Use this error approximation to obtain the *extrapolated*, improved, approximation

$$S_h[\cos x^2] = T_h[\cos x^2] + \frac{4}{3}(T_{h/2}[\cos x^2] - T_h[\cos x^2]) \tag{5}$$

   (d) Explain why $S_h[\cos x^2]$ is more accurate and converges faster to $I[\cos x^2]$ than $T_h[\cos x^2]$.

4. Let V be a vector space. Prove that a norm $||\cdot||$ on V that defines a continuous function $||\cdot|| : V \to [0, \infty)$.

5. Let $V = \mathbb{R}^2$. Sketch the unit ball for the norms $||\cdot||_1$, $||\cdot||_2$, and $||\cdot||_\infty$.

6. We say that a sequence of functions $\{f_n\}$ defined on $[a, b]$ converges uniformly to a function f if for each $\epsilon > 0$, there is N, which depends only on ϵ and $[a, b]$ but is independent of x, such that

$$|f_n(x) - f(x)| < \epsilon, \text{ if } n > N, \text{ for all } x \in [a, b]$$

(6)

   Define the sequence of numbers $M_n = ||f_n - f||_\infty$. Prove that $\{f_n\}$ converges uniformly to f in $[a, b]$ if and only if M_n converges to zero as $n \to \infty$.

7. (a) Prove that the sequence of functions given by

$$f_n(x) = \left(\frac{n-1}{n}\right)x^2 + \frac{1}{n}x, \; 0 \le x \le 1 \tag{7}$$

   converges uniformly to $f(x) = x^2$ in $[0, 1]$.

   (b) Does the sequence $f_n(x) = x^n$ defined on $[0, 1]$ converge uniformly?