Introduction to Analysis

Lecture Notes

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Contents

Notation

- $\mathbb{N} = \{0, 1, 2, ...\}$
- $\mathbb{Z}^+ = \{1, 2, 3, ...\}$

1. Introduction to Analysis

Discussion 1.1: Take $x \in \mathbb{R}$ which satisfies $x \geq 0$. Let $a_0 = [x]$. Since $x \geq 0$, we know $a_o \in \mathbb{N}$. By definition, we have $a_0 \leq x < a_0 + 1$ or equivalency

$$0 \leq 10(x - a_0) < 10 \tag{1.1}$$

Next take $a_1 = [10x - a_0]$. Equation (1.1) implies $0 \leq a_1 \leq 9$. By definition, we have

$$a_1 \leq 10(x - a_0) < a_1 + 1$$

,or equivalency

$$a_0 + \frac{a_1}{10} \leq x < a_0 + \frac{a_1}{10} + \frac{1}{10}$$

Inductively, we suppose that we havve already found a finite list

$$a_0 \in \mathbb{N}, \quad a_1, ..., a_n \in \{0, 1, 2, ..., 9\}$$

such that for $1 \leq k \leq n$ we have

$$\sum_{\alpha=0}^{k} \frac{a_\alpha}{10^\alpha} \leq x < \sum_{\alpha=0}^{k} \frac{a_\alpha}{10^\alpha} + \frac{1}{10^k} \tag{1.2}$$

Equation (1.2) implies

$$\sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha \leq 10^{n+1} x < \sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha + 10$$

$$\implies 0 \leq 10^{n+1} \left(x - \sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha \right) < 1 \tag{1.3}$$

Take $a_{n+1} = \left[10^{n+1} x - \sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha\right]$. Equation (1.3) implies $a_{n+1} \in \{0, 1, ..., 9\}$. By definition, $a_{n+1} \leq 10^{n+1-\alpha} - \sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha < a_{n+1} + 1$ which is equivalent to

$$\sum_{\alpha=0}^{n+1} \frac{a_\alpha}{10^\alpha} \leq x < \sum_{\alpha=0}^{n+1} \frac{a_\alpha}{10^\alpha} + \frac{1}{10^{n+1}}$$

Discussion 1.1 leads to the following lemma

Lemma 1.1: Let $x \in \mathbb{R}$ with $x \geq 0$. Then it follow s that we have a unique squence $\{a_n\}_{n=0}^{\infty}$, $(a_i \in \mathbb{N},\ \forall i)$ such that $a_k \in \{0, 1, ..., 9\}$ for all $k \geq 1$ and that

$$\sum_{\alpha=0}^{n} \frac{a_\alpha}{10^\alpha} \leq x < \sum_{\alpha=0}^{n} \frac{a_\alpha}{10^\alpha} + \frac{1}{10^n}$$

holds for all $n \geq 0$.

Remark 1.1: Lemma 1.1 imples that

$$\lim_{n\to\infty} \sum_{\alpha=0}^{n} \frac{a_\alpha}{10^\alpha} = x$$

Lemma 1.1 leads to

Corollary 1.1: $\mathbb{Q}$ is dense in $\mathbb{R}$.

We cast this rsult in a boarder framewark.

Definition 1.1: Let X be non-empty set. A **metric** on X is a function $d : X \times X \to [0, \infty)$ such that

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We say that (X, d) is a **metric space**.

2. test

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