Introduction to Analysis

Lecture Notes

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Contents

Notation

- $\mathbb{N} = \{0, 1, 2, ...\}$
- $\mathbb{Z}^+ = \{1, 2, 3, ...\}$

1. Introduction to Analysis

Discussion 1.1: Take $x \in \mathbb{R}$ which satisfies $x \geq 0$. Let $a_0 = [x]$. Since $x \geq 0$, we know $a_o \in \mathbb{N}$. By definition, we have $a_0 \leq x < a_0 + 1$ or equivalency

$$0 \leq 10(x - a_0) < 10 \tag{1.1}$$

Next take $a_1 = [10x - a_0]$. Eq.(1.1) implies $0 \leq a_1 \leq 9$. By definition, we have

$$a_1 \leq 10(x - a_0) < a_1 + 1$$

,or equivalency

$$a_0 + \frac{a_1}{10} \leq x < a_0 + \frac{a_1}{10} + \frac{1}{10}$$

Inductively, we suppose that we havve already found a finite list

$$a_0 \in \mathbb{N}, \quad a_1, ..., a_n \in \{0, 1, 2, ..., 9\}$$

such that for $1 \leq k \leq n$ we have

$$\sum_{\alpha=0}^{k} \frac{a_\alpha}{10^\alpha} \leq x < \sum_{\alpha=0}^{k} \frac{a_\alpha}{10^\alpha} + \frac{1}{10^k} \tag{1.2}$$

Eq.(1.2) implies

$$
\begin{aligned}
\sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha \leq 10^{n+1} x &< \sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha + 10 \\
\implies 0 \leq 10^{n+1} \left(x - \sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha \right) &< 1
\end{aligned} \tag{1.3}
$$

Take $a_{n+1} = \left[10^{n+1}x - \sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha\right]$. Eq.(1.3) implies $a_{n+1} \in \{0, 1, ..., 9\}$. By definition, $a_{n+1} \leq 10^{n+1-\alpha} - \sum_{\alpha=0}^{n} 10^{n+1-\alpha} a_\alpha < a_{n+1} + 1$ which is equivalent to

$$
\sum_{\alpha=0}^{n+1} \frac{a_\alpha}{10^\alpha} \leq x < \sum_{\alpha=0}^{n+1} \frac{a_\alpha}{10^\alpha} + \frac{1}{10^{n+1}}
$$

Discussion 1.1 leads to the following lemma

Lemma 1.1: Let $x \in \mathbb{R}$ with $x \geq 0$. Then it follow s that we have a unique squence $\{a_n\}_{n=0}^{\infty}$, $(a_i \in \mathbb{N},\ \forall i)$ such that $a_k \in \{0, 1, ..., 9\}$ for all $k \geq 1$ and that

$$
\sum_{\alpha=0}^{n} \frac{a_\alpha}{10^\alpha} \leq x < \sum_{\alpha=0}^{n} \frac{a_\alpha}{10^\alpha} + \frac{1}{10^n} \tag{1.4}
$$

holds for all $n \geq 0$.

Remark 1.1: Lemma 1.1 implies that

$$
\lim_{n \to \infty} \sum_{\alpha=0}^{n} \frac{a_\alpha}{10^\alpha} = x
$$

Lemma 1.1 leads to

Corollary 1.1: $\mathbb{Q}$ is dense in $\mathbb{R}$.

We cast this rsult in a boarder framewark.

Definition 1.1: Let X be non-empty set. A **metric** on X is a function $d : X \times X \to [0, \infty)$ such that

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We say that (X, d) is a **metric space**.

Definition 1.2: Let (X, d) be a metric space. Take E be a subset of X, We say that E is **dense** in X if for each $x \in X$ and any $\varepsilon > 0$, we can find some $a \in E$ such that $0 \leq d(x, a) < \varepsilon$.

Discussion 1.2: Take $x \in \mathbb{R}$, $N \in \mathbb{Z}^+$. Consider the list $kx - [kx]$, where $k = 0, 1, 2, ..., N$. Observe

$$kx - [kx] \in [0, 1) \text{ for } k = 0, 1, 2, ..., N \tag{1.5}$$

But

$$[0, 1) = \bigcup_{\alpha=0}^{N-1} \left[\frac{\alpha}{N}, \frac{\alpha+1}{N}\right) = \left[0, \frac{1}{N}\right) \cup \left[\frac{1}{N}, \frac{2}{N}\right) \cup ... \cup \left[\frac{N-1}{N}, 1\right) \tag{1.6}$$

By Pigeonhole principle, we define from Eq.(1.5) to Eq.(1.6) that there exists integers $0 \leq k_1 < k_2 \leq N$, together with some integer $0 \leq \alpha \leq N - 1$ such that

$$\begin{cases} k_1 x - [k_1 x] \in \left[\frac{\alpha}{N}, \frac{\alpha+1}{N}\right) \\ k_2 x - [k_2 x] \in \left[\frac{\alpha}{N}, \frac{\alpha+1}{N}\right) \end{cases} \tag{1.7}$$

Eq.(1.7) implies

$$|(k_2 - k_1)x - ([k_2 x] - [k_1 x])| < \frac{1}{N}$$

By construction, we know $1 \leq k_2 - k_1 \leq N$, and $[k_2 x] - [k_1 x] \in \mathbb{Z}$.

Discussion 1.2 leads to the following result

Lemma 1.2: Let $x \in \mathbb{R}$, $N \in \mathbb{Z}^+$. we can find some $k \in \{1, 2, ..., N\}$ and some $l \in \mathbb{Z}$ such that $|kx - l| < \frac{1}{N}$ or

$$\left|x - \frac{l}{k}\right| < \frac{1}{Nk} \leq \frac{1}{k^2}$$

Base on Lemma 1.2, we now prove

Lemma 1.3: Let $x \in \mathbb{R} \setminus \mathbb{Q}$, the following set is not finite.

$$E = \left\{\frac{l}{k} : l \in \mathbb{Z}, k \in \mathbb{Z}^+, \text{and } \left|x - \frac{l}{k}\right| < \frac{1}{k^2}\right\}$$

Proof: Assume towards contradiction that E is finite. So, that $E = \left\{\frac{1_1}{k_1}, \frac{1_2}{k_2}, ..., \frac{1_m}{k_m}\right\}$ such that for any $1 \leq \alpha \leq m$ we have

$$l_\alpha \in \mathbb{Z},\ k_\alpha \geq 1 \text{ and } 0 < \left| x - \frac{l_\alpha}{k_\alpha} \right| < \frac{1}{k^2} \tag{1.8}$$

Take

$$\begin{cases} \delta = \min\left\{ \left| x - \frac{l_\alpha}{k_\alpha} \right| : 1 \leq \alpha \leq m \right\} > 0 \\ N = \left[\frac{1}{\delta}\right] + 1 > \frac{1}{\delta} \end{cases} \tag{1.9}$$

However, in accordance to Lemma 1.2, there exists $1 \leq \tilde{k} \leq N$, $\tilde{l} \in \mathbb{Z}$ sucht that $\left|\tilde{k}x - \tilde{l}\right| < \frac{1}{N}$ or

$$\left| x - \frac{\tilde{l}}{\tilde{k}} \right| < \frac{1}{N\tilde{k}} \tag{1.10}$$

Eq.(1.10) implies

$$\left| x - \frac{\tilde{l}}{\tilde{k}} \right| < \frac{1}{N\tilde{k}} \leq \frac{1}{\tilde{k}^2} \tag{1.11}$$

Eq.(1.10) and Eq.(1.9) implies for all $1 \leq \alpha \leq m$

$$\left| x - \frac{\tilde{l}}{\tilde{k}} \right| < \frac{1}{N\tilde{k}} \leq \frac{1}{N} < \delta \leq \left| x - \frac{l_\alpha}{k_\alpha} \right| \tag{1.12}$$

Eq.(1.12) leads

$$\frac{\tilde{l}}{\tilde{k}} \notin E$$

A controdiction has been arrived.

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