COMP90051 **Statistical Machine Learning**

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6. PGM Representation

THE UNIVERSITY OF
MELBOURNE

Next Lectures

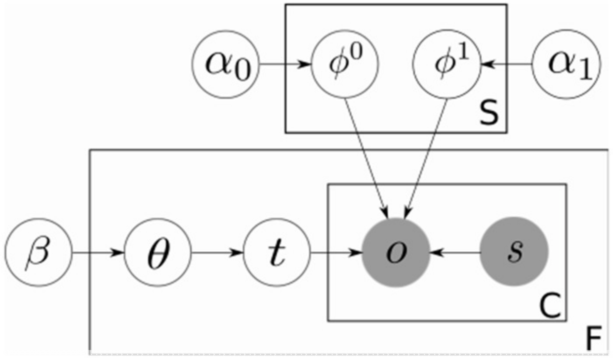
- Representation of joint distributions
- Examples
- Probabilistic inference
  - Computing other distributions from joint
- Statistical inference
  - Learn parameters from (missing) data

Probabilistic Graphical Models

Marriage of graph theory and probability theory.
Tool of choice for Bayesian statistical learning.

We'll stick with easier discrete case,
ideas generalise to continuous.

Motivation by practical importance



- **Many applications**
  - Phylogenetic trees
  - Pedigrees, Linkage analysis
  - Error-control codes
  - Speech recognition
  - Document topic models
  - Probabilistic parsing
  - Image segmentation
  - …
- **Unifies many previously-discovered algorithms**
  - HMMs
  - Kalman filters
  - Mixture models
  - LDA
  - MRFs
  - CRF
  - Logistic, linear regression
  - …

Motivation by way of comparison

Bayesian statistical learning

- Model joint distribution of X’s,Y and parameter r.v.’s
  - “Priors”: marginals on parameters
- Training: update prior to posterior using observed data
- Prediction: output posterior, or some function of it (MAP)

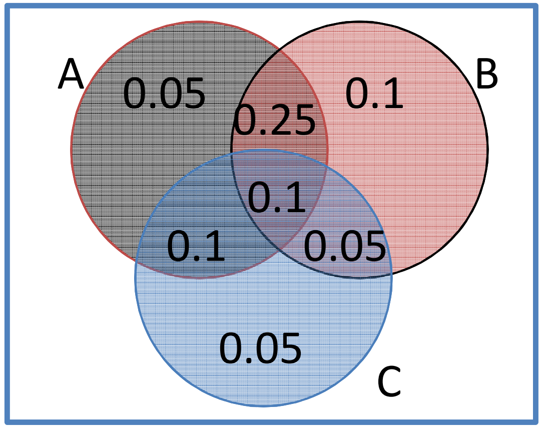
PGMs aka “Bayes Nets”

- Efficient joint representation
  - Independence made explicit
  - Trade-off between expressiveness and need for data, easy to make
  - Easy for practitioners to model
- Algorithms for fit parameters, compute marginals, posterior

Everything Starts at the Joint Distribution

- All joint distributions on discrete r.v.'s can be represented as tables
- #rows grows exponentially with #r.v.'s
- Example: Truth Tables
  - *M* Boolean r.v.'s require 2^M-1 rows
  - Table assigns probability per row

A	B	C	Prob
0	0	0	0.30
0	0	1	0.05
0	1	0	0.10
0	1	1	0.05
1	0	0	0.05
1	0	1	0.10
1	1	0	0.25
1	1	1	?



The Good: What we can do with the joint

- **Probabilistic inference** from joint on r.v.’s
  - Computing any other distributions involving our r.v.’s
- Pattern: want a distribution, have joint; use: Bayes rule + marginalisation
- Example: **naïve Bayes classifier**
  - Predict class y of instance $\boldsymbol{x}$ by maximising

$$\Pr(Y = y|\boldsymbol{X} = \boldsymbol{x}) = \frac{\Pr(Y=y,\boldsymbol{X}=\boldsymbol{x})}{\Pr(\boldsymbol{X}=\boldsymbol{x})} = \frac{\Pr(Y=y,\boldsymbol{X}=\boldsymbol{x})}{\sum_y \Pr(\boldsymbol{X}=\boldsymbol{x},Y=y)}$$

The Bad & Ugly: Tables *waaaaay* too large*!!*

- **The Bad**: Computational complexity
  - Tables have exponential number of rows in number of r.v.'s
  - Therefore → poor space & time to marginalise
- **The Ugly**: Model complexity
  - Way too flexible
  - Way too many parameters to fit
→ need lots of data OR will overfit
- Antidote: assume independence!

A	B	C	Prob
0	0	0	0.30
0	0	1	0.05
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1	0	1	0.10
1	1	0	0.25
1	1	1	?

Example: You're late!

- Modeling a tardy lecturer. Boolean r.v.'s
  - B: Ben teaches the class
  - S: It is sunny (o.w. bad weather)
  - L: The lecturer arrives slightly late (o.w. on time)
- Assume: Ben sometimes delayed by bad weather, Ben more likely late than co-lecturer
  - $\Pr(S|B) = \Pr(S)$, $\Pr(S) = 0.3$, $\Pr(B) = 0.6$
- Lateness not independent on weather, lecturer
  - Need $\Pr(L|B = b, S = s)$ for all combinations
- Need just 6 parameters

$\Pr(L = true\|\dots)$		B	
		False	True
S	False	0.2	0.1
	True	0.1	0.05

Independence: not a dirty word

Lazy Lecturer Model	Model details	# params
Our model with S, B independence	$\Pr(S, B)$ factors to $\Pr(S)\Pr(B)$	2
	$\Pr(L \mid B, S)$ modelled in full	4
Assumption-free model	$\Pr(L, B, S)$ modelled in full	7

- Independence assumptions
  - Can be reasonable in light of domain expertise
  - Allow us to factor → Key to tractable models

Factoring Joint Distributions

- Chain Rule: for any ordering of r.v.'s can always factor:

$$\Pr(X_1, X_2, \ldots, X_k) = \prod_{i=1}^{k} \Pr(X_i | X_{i+1}, \ldots, X_k)$$

- Model's independence assumptions correspond to
  - Dropping conditioning r.v.'s in the factors!
  - Example unconditional indep.: $\Pr(X_1 | X_2) = \Pr(X_1)$
  - Example conditional indep.: $\Pr(X_1 | X_2, X_3) = \Pr(X_1 | X_2)$
- Example: independent r.v.'s $\Pr(X_1, \ldots, X_k) = \prod_{i=1}^{k} \Pr(X_i)$
- Simpler factors speed inference and avoid overfitting

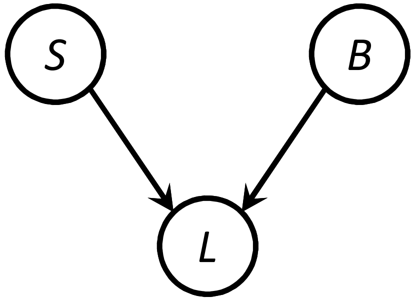
Directed PGM

- Nodes
- Edges (acyclic)

- Random variables
- Conditional dependence
  - Node table: $\Pr(child|parents)$
  - Child directly depends on parents
- Joint factorisation

$$\Pr(X_1, X_2, \dots, X_k) = \prod_{i=1}^{k} \Pr\left(X_i | X_j \in parents(X_i)\right)$$

Tardy Lecturer Example

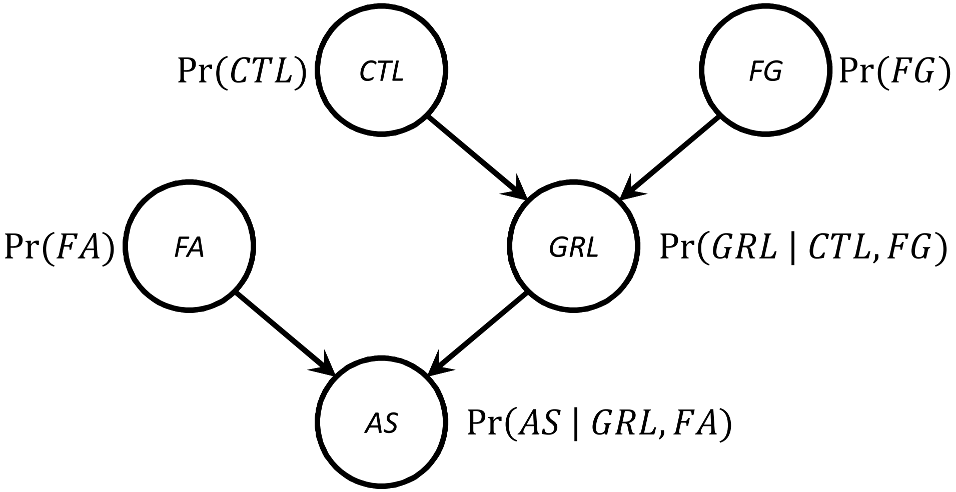


$\Pr(S)$ $\Pr(B)$

$\Pr(L|S,B)$

Example: Nuclear power plant

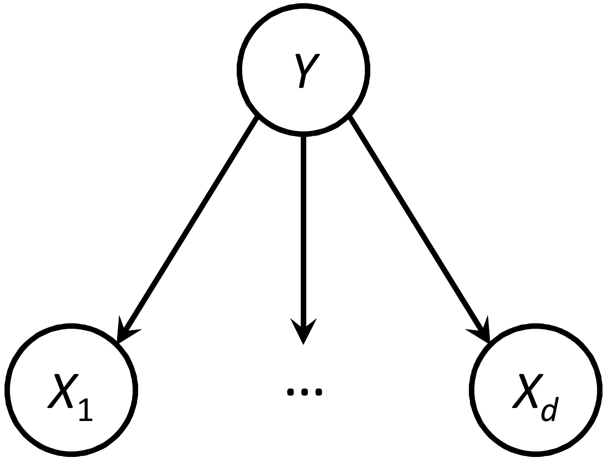
- Core temperature → Temperature Gauge → Alarm
- Model uncertainty in monitoring failure
  - GRL: gauge reads low
  - CTL: core temperature low
  - FG: faulty gauge
  - FA: faulty alarm
  - AS: alarm sounds
- PGMs to the rescue!



Joint $\Pr(CTL, FG, FA, GRL, AS)$ given by

$$\Pr(AS|FA, GRL)\Pr(FA)\Pr(GRL|CTL, FG)\Pr(CTL)\Pr(FG)$$

Naïve Bayes



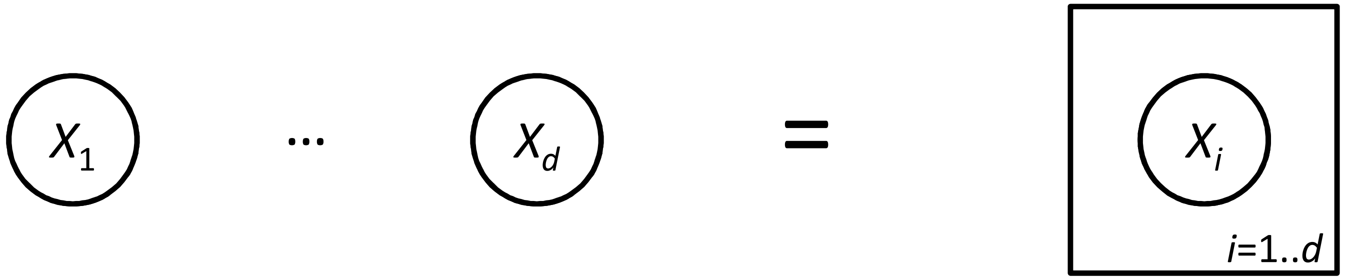
$Y \sim \mathrm{Bernoulli}(\theta)$

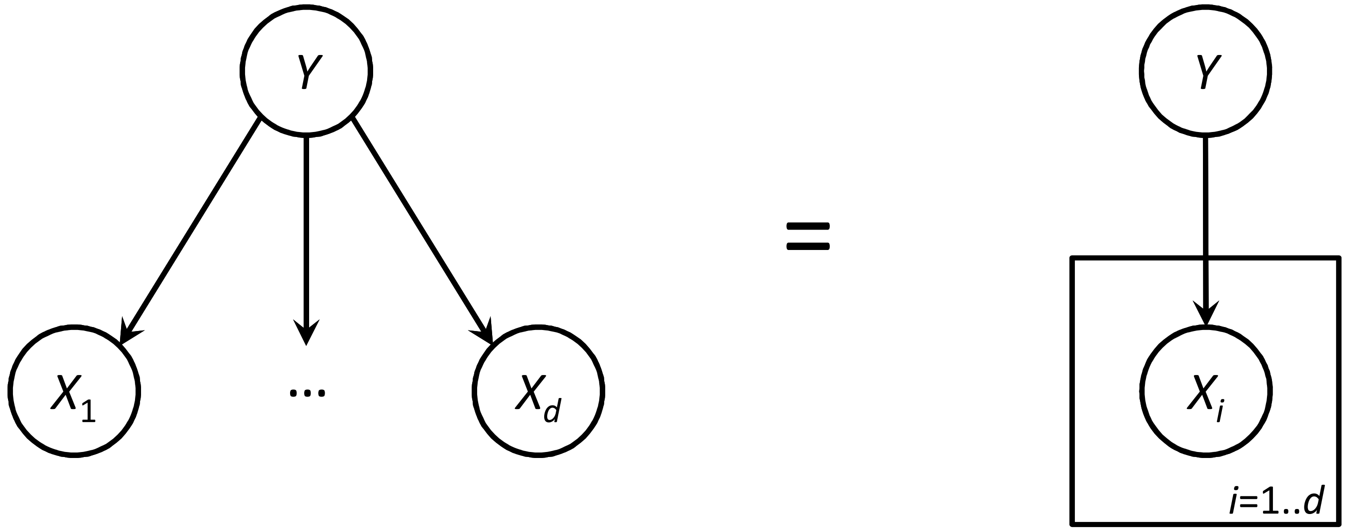
$X_j | Y \sim \mathrm{Bernoulli}\left(\theta_{j,Y}\right)$

$$\begin{aligned}
&\Pr(Y, X_1, \dots, X_d) \\
&\quad = \Pr(X_1, \dots, X_d, Y) \\
&\quad = \Pr(X_1 | X_2, \dots, X_d, Y) \Pr(X_2 | X_3, \dots, X_d, Y) \dots \Pr(X_{d+1} | X_d, Y) \Pr(X_d | Y) \Pr(Y) \\
&\quad = \Pr(X_1 | Y) \Pr(X_2 | Y) \dots \Pr(X_d | Y) \Pr(Y)
\end{aligned}$$

Prediction: predict label maximising $\Pr(Y | X_1, \dots, X_d)$

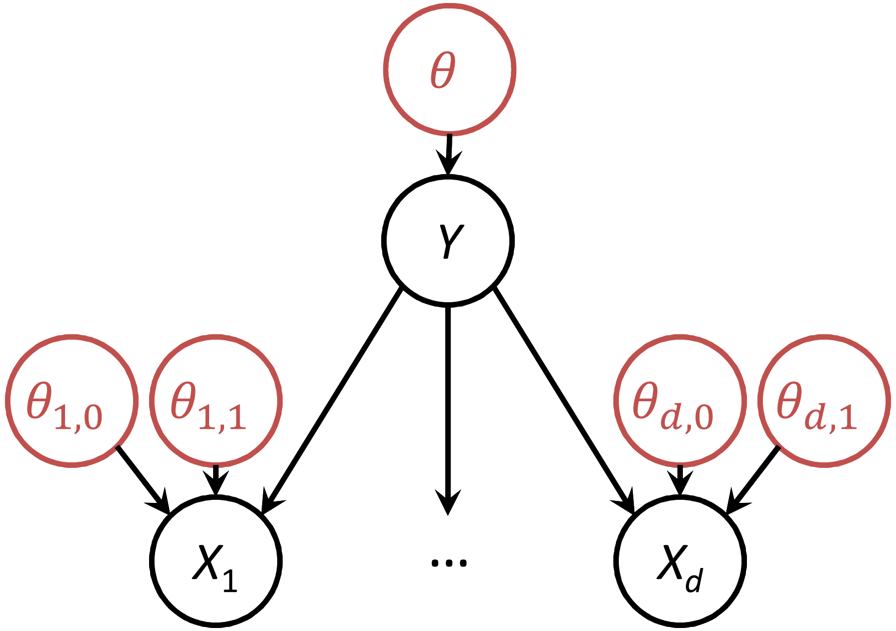
Short-hand for repeats: Plate notation





PGMs frequentist OR Bayesian…

- PGMs represent joints, which are central to Bayesian
- Catch is that Bayesians add: node per parameters, with table being the parameter’s prior



$Y \sim \mathrm{Bernoulli}(\theta)$

$X_j|Y \sim \mathrm{Bernoulli}(\theta_{j,Y})$

$\theta's \sim Beta$

Other types of PGMs…

Undirected PGM

- Graph
  - Edges undirected
- Probability
  - Each node a r.v.
  - Each clique C has "factor" $f_C\left(X_j : j \in C\right) \geq 0$
  - Joint $\propto$ product of factors

Directed PGM

- Graph
  - Edged directed
- Probability
  - Each node a r.v.
  - Each node has conditional $p\left(X_i | X_j \in parents(X_i)\right)$
  - Joint $=$ product of cond'ls

Summary

- Probabilistic graphical models
  - Motivation: applications, unifies algorithms
  - Motivation: ideal tool for Bayesians
  - Independence lowers computational/model complexity
  - PGMs: compact representation of factorised joints
  - Directed and Undirected PGMs