Markov Chains and the *RISK* Board Game

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1. Introduction

Markov chains have been applied in a wide variety of areas, including production, linguistics, finance, marketing, computer science, and signal processing. The first applications of Andrei Andreyevich Markov (1856–1922) were in modeling the progress of consonants and vowels in the writings of Pushkin and Aksakov [**2**].

In this study, we answer two questions of interest to *RISK* players by using Markov chains. At each turn, a player must decide whether or not to attack a territory. The first question is the following: *If you attack a territory with your armies, what is the probability that you will capture this territory?* Of course, the probability that you will capture a territory could be high while the expected loss is also high. Therefore, our second question is the following: *If you engage in a war, how many armies should you expect to lose depending on the number of armies your opponent has on that territory?*

Markov chains have also been applied to other board games [**1**], [**2**]. Ash and Bishop [**2**] calculated the steady state probability of a player landing on any Monopoly square under the assumption that each Monopoly player who goes to Jail stays there until he or she rolls doubles or has spent three turns in Jail. This model leads to a very practical observation for people who play the game often: the Orange monopoly (Tennessee Ave., St. James Place, and New York Ave.) is the most profitable.

Games have been subject to numerous studies in applied mathematics, operations research, and economics. The major difference between mathematical modeling of games and mathematical modeling of *real* systems is in the approximation phase. A model builder approximates the real world, according to the intent of use of such a model, under some restrictive assumptions; then the model is analyzed with mathematical techniques. Thus the results obtained from the mathematical model are approximations to the real system. Games can be viewed as simplified versions of real life situations, so mathematical models of games can be developed under fewer restrictive assumptions. In our mathematical model, the results for a single war of the *RISK* board game are obtained under no restrictive assumptions. Therefore our results are exact for the game itself.

2. Rules of the Game

The *RISK* game has a world map separated into 42 territories. Each player has a different number of tokens that are used as armies. Each token represents one army. The number of tokens is determined at each turn according to the territories each player occupies. The objective of a player in the game is either to conquer the world or to fulfill a task assigned at the beginning of the game. This task can be destroying a specific enemy or conquering a specific area.

At each turn, a player may attack a territory from a neighboring territory occupied by the attacker. An attacker must leave at least one army on the attacking territory, and at least one army must be used in an attack. Thus a territory may be attacked by a player who has at least two armies (one to keep on the territory and one to attack) on a neighbor territory. A player who decides to attack a territory declares war on the opponent and the two players engage in a war. Dice are used in the war; successive outcomes of the defender's and the attacker's dice determine the outcome of the war. An attacker may withdraw before the war ends. An attacker who does not withdraw either destroys all the defender armies in that territory and occupies that territory, or loses all the attacking armies and fails to conquer the territory.

The number of armies an attacker or a defender has determines the number of dice each player rolls. A defender rolls two dice with two or more armies, and one die with one army. An attacker rolls three dice with three or more attacking armies, rolls two dice with two, and rolls only one die with only one army. According to the number of dice each opponent rolls, there are six different cases, given in Table 1.

At each turn, after the dice have been rolled, each side's dice are placed in descending order and the two sets of dice are paired off. The attacker loses one army for each die that is less than or equal to the corresponding defender's die. The defender loses one army for each die that is less than the corresponding attacker's die. After the armies lost at that turn are taken away, the dice are rolled again. This continues until one side loses all of its armies. An example of a war is given below. This war took four turns. At the end of the fourth turn, all of the attacking armies are lost and the defender wins the war.

Turn #	Number of armies		Number of dice rolled		Outcome of the dice		Number of losses	
	attacker	defender	attacker	defender	attacker	defender	attacker	defender
1	4	3	3	2	5, 4, 3	6, 3	1	1
2	3	2	3	2	5, 5, 2	5, 5	2	0
3	1	2	1	2	6	4, 3	0	1
4	1	1	1	1	5	6	1	0
5	0	1						

3. A State-Space Model

Before we can determine the probability of conquering a territory and the expected losses of both sides, we need to develop a state-space model of a single war. Let A be

TABLE 1 The number of dice each side rolls according to the number of attacker and defender armies

Case	Number of Armies		Number of Dice Rolled	
	Attacker	Defender	Attacker	Defender
I	1	1	1	1
II	2	1	2	1
III	≥ 3	1	3	1
IV	1	≥ 2	1	2
V	2	≥ 2	2	2
VI	≥ 3	≥ 2	3	2

the number of attacking armies and D the number of defending armies. The state of the system at a given time is characterized by A and D. Let X_n be the state of the system at the beginning of the nth turn:

$$X_n = (a_n, d_n),\ 0 \le a_n \le A,\ 0 \le d_n \le D$$

where a_n and d_n are the number of attacking and defending armies, respectively. The initial state of the system is $X_0 = (A, D)$.

If one side loses all its armies then that side loses the war; i.e., if $X_m = (0, d_m)$ with $d_m > 0$, then the attacker has lost the war at the end of the previous turn. Similarly, if $X_m = (a_m, 0)$, with $a_m > 0$, then the defender has lost the war at the end of the previous turn.

If we know the number of armies each side has at a given turn, then we can calculate the probability that each side wins or loses the war without knowing the states of the system prior to that turn. In other words, this process has the Markov property:

$$\begin{aligned} P[X_{n+1} &= (a_{n+1}, d_{n+1}) | X_n = (a_n, d_n), X_{n-1} = (a_{n-1}, d_{n-1}), \dots, X_0 = (A, D)] \\ &= P[X_{n+1} = (a_{n+1}, d_{n+1}) | X_n = (a_n, d_n)]. \end{aligned}$$

Thus $\{X_n\colon\ n = 0, 1, 2, \dots\}$ is a Markov chain, with state space $\{(a, d)\colon\ 0 \le a \le A, 0 \le d \le D\}$.

If the process starts at (A, D), it terminates either at $X_m = (0, d_m)$, with $d_m > 0$, or $X_m = (a_m, 0)$, with $a_m > 0$. In other words, one side either wins or loses. The states $(0, d_m)$, with $d_m > 0$, and $(a_m, 0)$, with $a_m > 0$ are called the absorbing states. The sum of the probabilities that the process terminates at $(0, d_m)$, $d_m > 0$ is the probability that the attacker loses (defender wins). Similarly, the sum of the probabilities that it terminates at $(a_m, 0)$, $a_m > 0$ is the probability that the attacker wins (defender loses).

If only the winning probabilities are of interest, then the states $(0, d_m)$, $d_m > 0$ can be lumped into a single state, say Z, denoting that the defender wins. Similarly the states $(a_m, 0)$, $a_m > 0$ can be lumped into a single state, say K, denoting that the attacker wins. Doing so reduces the dimensions of the state space. Since we are interested in the expected losses of each side, we have not lumped these states.

Because one side either wins or loses at the end of a war, state $(0, 0)$ cannot be reached from any other state. Therefore there are $A \cdot D + A + D$ states in the state space. Let the states of this system be ordered as $\{(1, 1), (1, 2), \dots, (1, D), (2, 1), (2, 2), \dots, (2, D), \dots, (A, 1), (A, 2), \dots, (A, D), (0, 1), (0, 2), \dots, (0, D), (1, 0), (2, 0), \dots, (A, 0)\}$, and let the states be indexed from 1 to $A \cdot D + A + D$. With this ordering, the first $A \cdot D$ states are transient and the remaining $A + D$ states are absorbing.

Let $\underline{P} = \{p_{ij}\}$ be the one-step state transition matrix; its elements p_{ij} denote the probability that the index of the state of the system at the beginning of the next turn is j given that the index of the state of the system at the beginning of this turn is i. We will find these probabilities in the next section. The matrix $\underline{P}$ has $A \cdot D + A + D$ rows and $A \cdot D + A + D$ columns, and has the following form:

$$\underline{P} = \begin{bmatrix} Q & R \\ \underline{0} & I \end{bmatrix}.$$

Here Q is an $A \cdot D \times A \cdot D$ matrix; its elements are the probabilities of transition only between the transient states for its elements. Similarly, R is an $A \cdot D \times (A + D)$

matrix whose elements are one-step transition probabilities from transient state to absorbing states, $\underline{0}$ is a $(A+D)\times A\cdot D$ matrix with all elements equal to zero, and I is the $(A+D)\times(A+D)$ identity matrix.

Let $f_{ij}^{(n)}$, $i=1,2,\ldots, A\cdot D$ and $j=1,2,\ldots, A+D$, be the probability that starting from a transient state with index i, the process enters an absorbing state with index $j+A\cdot D$ in n turns. Let $F^{(n)}$ be the matrix whose ijth element is $f_{ij}^{(n)}$. Note that if the process enters an absorbing state at the end of the nth turn, the transitions in the first $n-1$ turns must be among transient states and the transition at the nth turn must be from a transient state to an absorbing state. Therefore

$$F^{(n)}=Q^{n-1}R.$$

The process can terminate at the end of any turn, so the probabilities that the process eventually terminates at one of the absorbing states are calculated by summing $F^{(n)}$ from $n=0$ to infinity. That is,

$$F=\sum_{n=0}^{\infty}F^{(n)}=\sum_{n=0}^{\infty}Q^{n-1}R=(I-Q)^{-1}R, \tag{1}$$

where F is a $A\cdot D\times(A+D)$ matrix. The ith column of $F, F^{\langle i\rangle}$, contains the probabilities that the process eventually terminates at the absorbing state i, given the initial states (a,d), $0<a\leq A$, $0<d\leq D$, $i=1,2,3,\ldots, A+D$ for the states $(0,1),(0,2),\ldots,(0,D),(1,0),(2,0),\ldots,(A,0)$, respectively.

Since the defender wins the war if the process terminates at the states $(0,d)$, $0<d\leq D$, the vector P_K whose elements are the winning probabilities of the defender given an initial state (a,d), $0<a\leq A$, $0<d\leq D$, is obtained by adding the first D columns of the matrix F:

$$P_K=\sum_{i=1}^{D}F^{\langle i\rangle}.$$

Similarly, the vector P_Z whose elements are the winning probabilities of the attacker given an initial state (a,d), $0<a\leq A$, $0<d\leq D$, is obtained by adding the last A columns of the matrix F:

$$P_Z=\sum_{i=D+1}^{A+D}F^{\langle i\rangle}.$$

Since each side either wins or loses, P_K+P_Z is a vector with elements equal to one.

Expected losses of defender and attacker are also of interest: the probability of winning for one side could be high while the expected loss is also high. For the attacker, this may change the decision of whether or not to attack a territory.

When the defender wins, the state of the system is $(0,d)$, $d>0$, where d is the number of remaining defender armies. Multiplying d by the probability that the process terminates in state $(0,d)$ and summing over all d with $0<d\leq D$ gives the expected number of remaining defender armies. Expected remaining attacking armies can be determined in the same way.

Let E be the $(A+D)\times 2$ matrix defined as

$$E=\begin{bmatrix}1 & 2 & \cdot & D & 0 & 0 & \cdot & 0\\ 0 & 0 & \cdot & 0 & 1 & 2 & \cdot & A\end{bmatrix}^{T},$$

and let ER be the $(A\cdot D)\times 2$ matrix whose first and second columns are the expected number of remaining defender and attacker armies, respectively, given an initial state (a, d), $0 < a \leq A$, $0 < d \leq D$. Clearly,

$$ER = F \cdot E.$$

The expected losses of a defender or an attacker are found by subtracting the number of expected remaining armies from D and A, respectively.

For more detailed information about analysis of Markov chains, the reader is referred to **[3]**.

4. Determining the State Transition Probabilities

When two opponents engage in a war, an attacker rolls at most three dice and a defender at most two, as shown in Table 1. The state transition probabilities depend on the number of dice rolled. Before determining these probabilities, we consider the probability distributions of the maximum of one, two, and three dice and of the second largest number of two and three dice shown in Table 2.

TABLE 2 The probability distributions of the maximum of one, two, three dice and the second largest number of two and three dice

Number of dice		1	2	3	4	5	6
1	outcome of the die	1/6	1/6	1/6	1/6	1/6	1/6
2	maximum	1/36	3/36	5/36	7/36	9/36	11/36
	second largest	11/36	9/36	7/36	5/36	3/36	1/36
3	maximum	1/216	7/216	19/216	37/216	61/216	91/216
	second largest	16/216	40/216	52/216	52/216	40/216	16/216

With the information in Table 2, the state transition probabilities can be calculated for the six different cases given in Table 1. The maximum number of armies an attacker or a defender loses is 2. Therefore from a state (a, d), the possible transitions are to states $(a-2, d)$, $(a-1, d)$, $(a, d-1)$, $(a, d-2)$, and $(a-1, d-1)$, provided that $a \geq 2$ and $d \geq 2$.

As an example of the state transition probability calculations, consider case 6, where the attacker rolls three dice and the defender rolls two. Let Z_1 and Z_2 be the random variables denoting the outcomes of the defender's dice, $Z^{(1)}$ be the maximum and $Z^{(2)}$ be the second largest of Z_1 and Z_2. Similarly, let Y_1, Y_2 and Y_3 be the random variables denoting the outcomes of the attacker's dice, $Y^{(1)}$ be the maximum and $Y^{(2)}$ be the second largest of Y_1, Y_2 and Y_3. If $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ then the defender loses two armies. In the opposite case, i.e., if $Y^{(1)} \leq Z^{(1)}$ and $Y^{(2)} \leq Z^{(2)}$, the attacker loses two armies. In all the other cases, both attacker and defender lose one army. Thus the state transition probabilities from the state (a, d) (with $a \geq 3$ and $d \geq 2$) to states $(a-2, d)$, $(a-1, d-1)$, and $(a, d-2)$ are as follows:

$$P[X_{n+1} = (a, d-2) | X_n = (a, d)] = P[Y^{(1)} > Z^{(1)}] \cdot P[Y^{(2)} > Z^{(2)}]$$

$$P[X_{n+1} = (a-2, d) | X_n = (a, d)] = P[Y^{(1)} \leq Z^{(1)}] \cdot P[Y^{(2)} \leq Z^{(2))}]$$

$$\begin{aligned} P[X_{n+1} = (a-1, d-1) | X_n = (a, d)] &= 1 - P[X_{n+1} = (a, d-2) | X_n = (a, d)] \\ &\quad - P[X_{n+1} = (a-2, d) | X_n = (a, d)] \end{aligned}$$

Since $Y^{(1)}$ and $Z^{(1)}$, $Y^{(2)}$ and $Z^{(2)}$ are independent of each other, by using the probabilities given in Table 2 we obtain:

$$\begin{aligned} P[Y^{(1)} > Z^{(1)}] &= \sum_{y=2}^{6} \sum_{z=1}^{y-1} P[Y^{(1)} = y, Z^{(1)} = z] \\ &= \sum_{y=2}^{6} \sum_{z=1}^{y-1} P[Y^{(1)} = y] \cdot P[Z^{(1)} = z] = 0.471 \end{aligned}$$

$$\begin{aligned} P[Y^{(2)} > Z^{(2)}] &= \sum_{y=2}^{6} \sum_{z=1}^{y-1} P[Y^{(2)} = y, Z^{(2)} = z] \\ &= \sum_{y=2}^{6} \sum_{z=1}^{y-1} P[Y^{(2)} = y] \cdot P[Z^{(2)} = z] = 0.551 \end{aligned}$$

Therefore,

$$P[Y^{(1)} \leq Z^{(1)}] = 1 - P[Y^{(1)} > Z^{(1)}] = 0.529$$

$$P[Y^{(2)} \leq Z^{(2)}] = 1 - P[Y^{(2)} > Z^{(2)}] = 0.449$$

Thus the state transition probabilities for the case $a \geq 3$ and $d \geq 2$ are as follows:

$$P[X_{n+1} = (a, d-2) | X_n = (a, d)] = P[Y^{(1)} > Z^{(1)}] \cdot P[Y^{(2)} > Z^{(2)}] = 0.259$$

$$P[X_{n+1} = (a-2, d) | X_n = (a, d)] = P[Y^{(1)} \leq Z^{(1)}] \cdot P[Y^{(2)} \leq Z^{(2))}] = 0.237$$

$$\begin{aligned} P[X_{n+1} = (a-1, d-1) | X_n = (a, d)] &= 1 - P[X_{n+1} = (0, d) | X_n = (a, d)] \\ &\quad - P[X_{n+1} = (a, d-2) | X_n = (a, d)] = 0.504 \end{aligned}$$

The state transition probabilities for all other cases can be obtained in the same way; the results are given in Table 3.

FIGURE 1 shows the state transition diagram in the specific situation where the attacker has 6 and the defender has 4 armies. Each case from I to VI given in Tables 1 and 3 is depicted with a rectangle. The state transition probabilities in each case are shown only for a representative state.

TABLE 3 The state transition probabilities

Case	a	d	From state	To state	Transition Probability
I	1	1	$(1,1)$	$(1,0)$ $(0,1)$	0.417 0.583
II	2	1	$(2,1)$	$(2,0)$ $(1,1)$	0.578 0.422
III	≥ 3	1	$(a,1)$	$(a,0)$ $(a-1,1)$	0.659 0.341
IV	1	≥ 2	$(1,d)$	$(0,d)$ $(1,d-1)$	0.254 0.746
V	2	≥ 2	$(2,d)$	$(2,d-2)$ $(0,d)$ $(1,d-1)$	0.152 0.373 0.475
VI	≥ 3	≥ 2	(a,d)	$(a,d-2)$ $(a-2,d)$ $(a-1,d-1)$	0.259 0.237 0.504

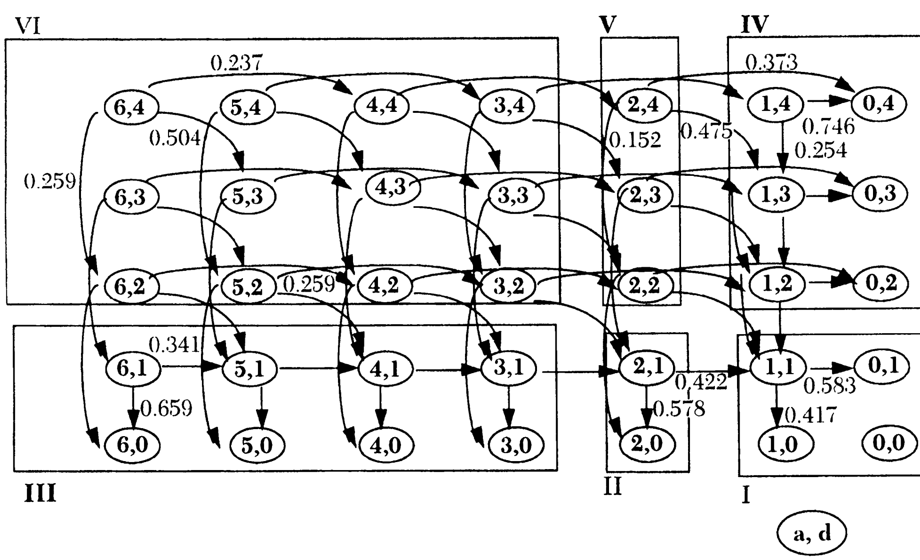

FIGURE 1
The state-transition diagram when the attacker has 6 and the defender has 4 armies.

5. Numerical Results

A computer program was written to generate the state-transition probability matrix and to determine the winning probabilities and expected losses for a given number of initial armies of defender and attacker. In this section, we report the results for each initial state (a, d), $0 < a \leq 30$, $0 < d \leq 30$.

The increase in the dimensions of the state transition matrix is $O(a^2 d^2)$, and the increase in the number of elements is $O(a^4 d^4)$. For example, if the attacker and defender each has 30 armies, the state space has $31 \times 31 - 1 = 960$ states. Thus the state transition matrix has dimension 960×960, and the number of elements in $\underline{P}$ is 921,600,000. This rapid increase in dimension may create computational difficulties in calculating equation (1). However, since the state transition matrix has only a few non-zero elements, numerical techniques for sparse matrices can be exploited to increase computational efficiency and stability.

FIGURE 2 depicts the probabilities that attacker wins with 5, 10, 15, 20, 25, and 30 armies as a function of the number of defender armies. FIGURE 2 shows that when both attacker and defender have the same number of armies, the probability that the attacker wins is below 50%. (This is because in the case of a draw, the defender wins.) When there are twice as many attackers as defenders, the winning probability exceeds 80%.

FIGURE 3 shows expected losses of an attacker with 5, 10, 15, 20, 25, and 30 armies, as a function of the number of defending armies. If the number of attacking armies is twice as many as the defending armies, then the expected loss of an attacker is slightly less than the number of defending armies. For example, if an attacker has 20 and a defender has 10 armies, then attacker wins the war with a probability of 98%, with

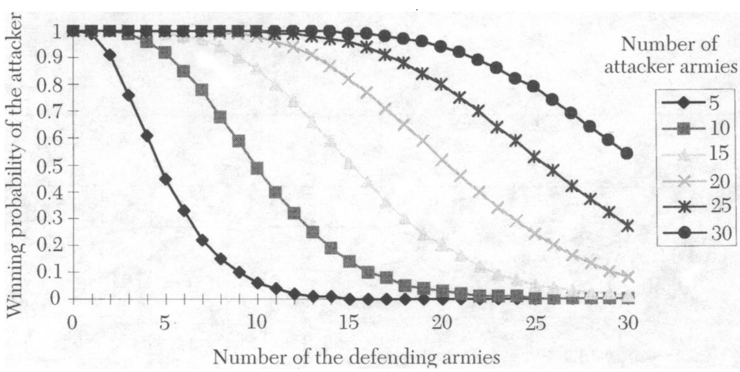

FIGURE 2
The winning probabilities of the attacker.

FIGURE 3 shows expected losses of an attacker with 5, 10, 15, 20, 25, and 30 armies, as a function of the number of defending armies. If the number of attacking armies is twice as many as the defending armies, then the expected loss of an attacker is slightly less than the number of defending armies. For example, if an attacker has 20 and a defender has 10 armies, then attacker wins the war with a probability of 98%, with expected losses of about 9 armies.

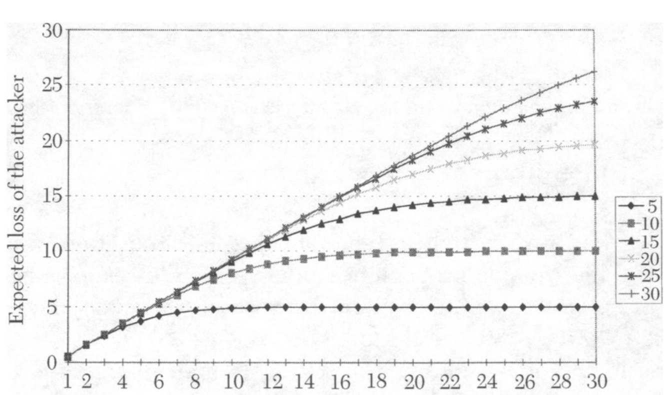

FIGURE 3
Expected loss of the attacker.

In devising a strategy for the game, one considers how many armies will be left in a territory that has just been occupied, since another opponent may attack that territory in the same turn. A simple rule of thumb can be stated as follows: Based on how many defending armies you want to leave on a territory that you want to conquer, attack if you have twice as many armies on a neighbor territory and also if the number of the armies your opponent has is at most half of the number of your armies.

6. Conclusions

One should note that the results above will be useful only over a sequence of many wars. However, since at each turn a player can attack any number of territories and there are many turns in a game, the results may help in devising a useful strategy.

As mentioned in the rules of the game, an attacker may withdraw at any turn. In particular, an attacker who sustains big losses in the course of a war may consider doing so. Determining when to withdraw is another interesting problem; expected loss and winning probability may be used together to devise a strategy. In doing so, one may include the risk characteristics of an individual, leading to different strategies for risk-averse and risk-neutral players. This can be accomplished by using utility functions.

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