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ECE 310 Digital Signal Processing
Fall 2017

Lecture E
Professor Zhi-Pei Liang (z-liang@)
Professor Yoram Bresler (ybresler@)
MWF 3-3:50 pm
3017 ECEB

Recitation Professor
T 7-7:50 pm
1015 ECEB

Recitation TA
W 7-7:50 pm
3017 ECEB

ECE 311 Digital Signal Processing Lab
Fall 2017

Laboratory A
TA Ben Eng (bceng2@)
TA Yu-Jeh Liu (yliu233@)
T 8-10:50 am
2022 ECEB

ECE 310: Digital Signal Processing
Monday, 28 August 2017

See course website.

Tuesday recitation 1015 ECEB, 7-8pm
Wednesday recitation 7-8pm.

Applied Digital Signal Processing
Manolakis & Ingle, 2015] Text

Reading assignments on website, grading

One midterm: Oct 12.

Homework Thurs 1pm
due at Dropbox

2 quizzes dropped

Processing - emph. results over systems/devices

<u>Signals</u>

physical quantity that varies as a function of time, space or any variables
- speech signals - stock market index
- video signals - EKG
- WiFi / EM ...

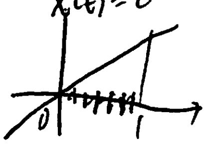
$x(t)$ will be considered in this course

<u>Processing</u>

* filtering (denoising)
- Recognition
- Enhancement
- Compression

Digital - reliable for range of tasks
rep. on computer

$x(t) = t$



sampling, discretization
info loss
Nyquist lossless

quantization

Image Inpainting Biomedical imaging

ECE 310 - W 30 Aug.

<u>Complex #'s</u>

$(z-a_1)(z-a_2)\cdots(z-a_n)=0$

$z^2+1=0$

$z^2=-1$

$z=\pm\sqrt{-1}=\pm j$

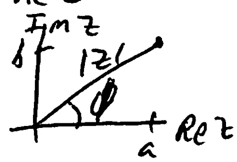
<u>Understand</u>

1. Role of complex #s
2. two representations - cartesian + polar
3. operations on complex #s
4. Functions of complex #s
5. Mag. + phase plots

Cartesian Representation

$z=a+jb$

$\mathrm{Re}\, z = a \qquad \mathrm{Im}\, z = b$



Polar Form

$z = re^{j\phi}$

$|z| = r$

$\angle z = \phi$

$a = |z|\cos\phi$

$b = |z|\sin\phi$

$e^{j\phi} = \cos\phi + j\sin\phi$

Euler's formula

<u>Complex Conjugate</u>

$z^*, \bar{z} = a - jb$

$|z|^2 = a^2 + b^2$

$|z| = \sqrt{a^2+b^2}$

$\angle z = ? \quad \tan\phi = \frac{b}{a}$

<u>Ex</u>

1. $z = 2 + j3$

Find polar form

$r = |z| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

$\phi = \mathrm{atan}(\frac{3}{2})$

$z = \sqrt{13}\, e^{j\,\mathrm{atan}(1.5)}$

ECE 310 W 30 Aug — Bresler

$j^2=-1$

$z_1+z_2=(a_1+a_2)+j(b_1+b_2)=(a_1+jb_1)+(a_2+jb_2)$

$z_1z_2=(a_1+jb_1)(a_2+jb_2)=\underbrace{a_1a_2-b_1b_2}_{Re(z_1z_2)}+j\underbrace{(a_1b_2+a_2b_1)}_{Im(z_1z_2)}$

$z_1=r_1e^{j\phi_1}\quad z_2=r_2e^{j\phi_2}$

$z_1z_2=r_1r_2e^{j(\phi_1+\phi_2)}$

$z_1+z_2=r_1e^{j\phi_1}+r_2e^{j\phi_2}=r_1\cos\phi_1+jr_1\sin\phi_1+r_2\cos\phi_2+jr_2\sin\phi_2$

$=(r_1\cos\phi_1+r_2\cos\phi_2)+j(r_1\sin\phi_1+r_2\sin\phi_2)$

$z^*=a-jb$

$z=re^{j\phi}$

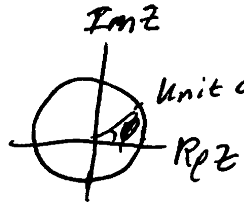
$z^*=re^{-j\phi}$

$zz^*=r^2\underbrace{e^{j(\phi-\phi)}}_{e^{j0}}=r^2=|z|^2$

Division

$$\frac{z_1}{z_2}=\frac{a_1-jb_1}{a_2+jb_2}=\frac{(a_1+jb_1)(a_2-jb_2)}{(a_2+jb_2)(a_2-jb_2)}=\frac{a_1a_2+b_1b_2+j(a_2b_1-a_1b_2)}{a_2^2+b_2^2}$$

$$=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+j\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}$$



$e^{j\phi}=e^{j(\phi+k2\pi)}$, k integer

$\phi\in[-\pi,\pi]$

$[0,2\pi]$

$z=1-j$

$|z|=\sqrt{1+1}=\sqrt{2}$

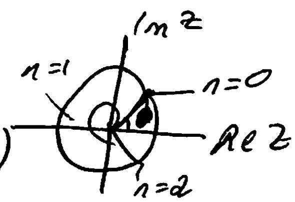
$\angle z=\tan^{-1}\frac{-1}{1}=\tan^{-1}(-1)=2\pi-\frac{\pi}{4}=\frac{7}{4}\pi$

$=-\frac{\pi}{4}$

3. $z = -1 - j$ $\quad \angle z \overset{?}{=} \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

$|z| = \sqrt{2}$ $\quad = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$ atan2 command

Solve the eqn



$e^{j0} = e^{jn2\pi} = 1$

$z^3 + 1 = 0$

$z^3 = -1 = e^{j(\pi + 2\pi n)}$

$\cos\pi + j\sin\pi = -1$, with $\sin\pi = 0$

$z = \left(e^{j(\pi + 2\pi n)}\right)^{1/3}$

$= e^{j\left(\pi/3 + \frac{2\pi n}{3}\right)}$

$e^{j\pi/3}$ $\quad n=0$

$n=1$: $e^{j\pi} = 1$

$n=2$: $e^{j\frac{5\pi}{3}} = e^{-j\frac{\pi}{3}}$

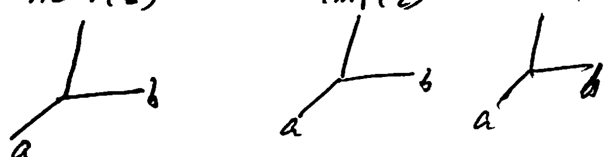
Functions of complex #s

Ex. $f(z) = z^2 + 1 \qquad z = a + jb$

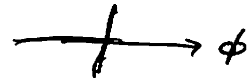
$f(z) = (a+jb)^2 + 1 = (a^2 - b^2 + 1) + j\,2ab$

$\mathrm{Re}\, f(z) = a^2 - b^2 + 1$

$\mathrm{Im}\, f(z) = 2ab \qquad f: \mathbb{C} \to \mathbb{C}$ mapping



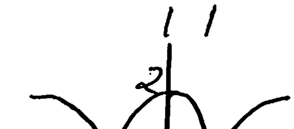
$z = e^{j\phi}$ $\quad$ Plot $\mathrm{Re} f(z), \mathrm{Im} f(z), |f(z)|, \angle f(z)$



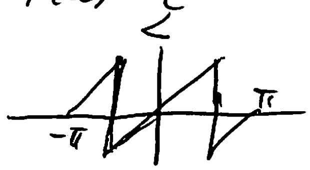
$f(z) = e^{j2\phi} + 1$

plot $|f(e^{j\phi})|$ $\quad \angle f(e^{j\phi})$

$f(z) = e^{j\phi}\left(e^{j\phi} + e^{-j\phi}\right) = 2\cos\phi\, e^{j\phi}$



$\angle$



ECE 310 F 1 Sept.

System Analysis

210   $x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t)$ (Analog)

$x(t) \rightarrow \boxed{A/D}$

$x[t] \rightarrow \boxed{\text{LTI}} \rightarrow y[t] \rightarrow \boxed{D/A} \rightarrow y(t)$ (Digital)   Discretization

if not LTI
- make LTI locally
- chaotic theory
- case-by-case basis

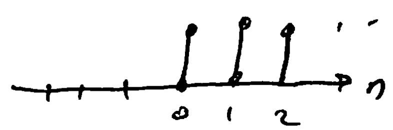
Lians

LSI

$x[n]$ takes integer values

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{else} \end{cases}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{else} \end{cases}$$



$x[n] = \{ \cdots x[-1], x[0], x[1], \ldots$

$= \{ \cdots \quad x_0, x_1, \ldots$

$\{ \cdots 10, 8, \ldots$

$10\delta[n] \quad x_1 \delta[n-1]$

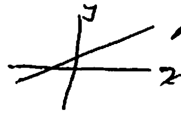
write as sum of deltas

each value is a "post"

$$x[n] = \sum_{\ell=-\infty}^{+\infty} x_\ell \, \delta[n-\ell]$$

Examples

	Linear	Shift inv
a) $y[n] + \cos(n)\, y[n-1] = x[n]$	Y	N
b) $y[n] = \cos^2(x[n])$	N	Y
c) $y[n] = x[\lvert n \rvert]$	Y	
d) $y[n] = 5x[n] + 3$	N	



System Properties

a) Linearity

b) Shift invariance

c) causality

d) Stability

causality not that important in digital world

since you can buffer the signal → time origin not important

~~Tas~~ Tasks

- def.
- criterion/a

Lineality

If $x_1[n] \to y_1[n]$, $x_2[n] \to y_2[n]$

$x_3[n] = a x_1[n] + b x_2[n] \to y_3[n]$. $\quad a, b$ const.

If linear, $y_3[n] = a y_1[n] + b y_2[n]$ superposition, decomposition

scale additive

~~Ex Line~~ Ex a)

Let $x_1[n] \to y_1[n]$: $y_1[n] + \cos(n) y_1[n-1] = x_1[n]$ — ①

$x_2[n] \to y_2[n]$: $y_2[n] + \cos(n) y_2[n-1] = x_2[n]$ — ②

Assume: $x_3[n] = a x_1[n] + b x_2[n] \to y_3[n]$

Test $y_3[n] \stackrel{?}{=} a y_1[n] + b y_2[n]$ $\quad y_3[n] + \cos(n) y_3[n-1] = x_3[n] = a x_1[n] + b x_2[n]$

① × a: $a y_1[n] + a\cos(n) y_1[n-1] = a x_1[n]$

② × b: $b y_2[n] + b\cos(n) y_2[n-1] = b x_2[n]$

$(a y_1[n] + b y_2[n]) + \cos(a y_1[n-1] + b y_2[n-1]) = a x_1[n] + b x_2[n]$

⇒ Yes, linear

shift variance:

If $x[n] \to y[n]$, then $x[n-n_0] \to y[n-n_0]$.

Causality

$y[n]$ is dependent on $x[n], x[n-1], \ldots$

Example a) $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n+1])$

b) $y[n] + 2y[n-1] = x[n] \Rightarrow y[n] = x[n] - 2y[n-1]$

$y[n] = \frac{1}{2}[y[n+1] + x[n+1])$

1. $y[n] = y[n-5] + x[n] + 10x[n-1]$

Let $x_1[n] \rightarrow y_1[n]$, $x_2[n] \rightarrow y_2[n]$

$x_3[n] = ax_1[n] + bx_2[n] \longrightarrow y_3[n]$

~~WTS~~ WTS $y_3[n] = ay_1[n] + by_2[n]$

$y_1[n] = y_1[n-5] + x_1[n] + 10x_1[n-1]$ — ①

— ②

$a① + b② =$

$ay_1 + by_2 = ay_1[n-5] + by_2[n-5] + a$

$y_3[n] = y_3[n-5] + x_3[n]$

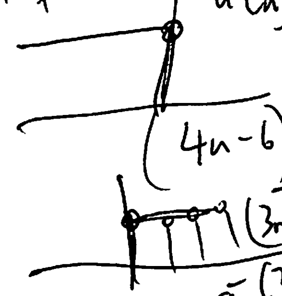
$x[n] = u[n]$

$h[n] = n(u[n] - u[n-4])$

$$\sum_{\ell=0}^{3} u[n-\ell]\, n(\ldots)$$

$u[n-\ell]$

$n-\ell$ $\quad u[n]$



$(4n-6)u[n]$

$(3n-6)\,0$

$-(2n-5)\,1$

$(n-3)\,2$

$n+n\ldots 1\ 4n-2$

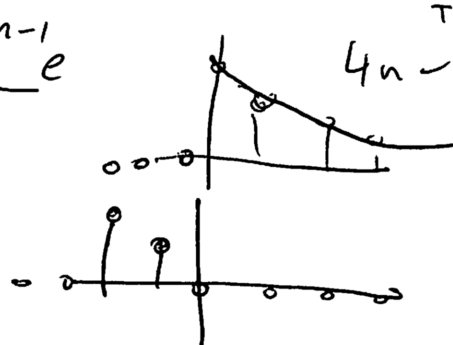
$7n-3$

$4n-6$

2. $\delta[n] = u[n] - u[n-1]$

$g[n] = h[n] * x[n]$

$$\frac{e^{-n} - (-1)^{-n-1}e}{1+e}$$



$$= h[n] * \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$= h[n] * \sum_{k=-\infty}^{\infty} x[k]\,(u[n-k] - u[n-k-1])$$

$$= \sum h[n] * x[k]\,u[n-k] + \sum h[n] * -x[k]\,u[n-k-1]$$

$$= \sum x[k]\,g[n-k] + \sum -x[k]\,g[n-k-1]$$

$$= \sum x[k]\,g[n-k] + \sum -x[k-1]\,g[n-k]$$

$$= \sum [x[k] + x[k-1]]\,g[n-k].$$

$$\frac{1-\left(\frac{r_1}{r_2}\right)^{n+1}}{r_2\left(\frac{r_1}{r_2}\right)}$$

$r_2 - r_1$

$$\frac{r_2 - \left(\frac{r_1}{r_2}\right)^{n+1} r_2}{r_2 - r_1}$$

$$\frac{1-(-e)^{n+1}}{1-(-e)} = \frac{1+e^{n+1}}{1+e}$$

$e^{-n} + e$

3. $x_2[n] = x[n] - \frac{1}{3}x[n-1]$

4. (b) $x[n] = 3^{(-n)}u[n]$, $h[n] = \{\underset{\uparrow}{0}, 1, 2\}$

$$x[n] * h[n] = \sum_{\ell=-\infty}^{\infty} x[\ell]\,h[n-\ell] = \sum_{\ell=0}^{\infty} 3^{-\ell}\, h[n-\ell]$$

$$= \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

$$= \sum_{\ell=0}^{\infty} 3^{-\ell}\,(\delta[n+1-\ell] + 2\delta[n+2-\ell])$$

$e^{-n} -$

$$e^{-n}\,\frac{1-(-e)^{n+1}}{1+e}$$

$$x[n] = 5 + 10e^{j(\frac{\pi}{4}n + 45^\circ)} + j^n$$

$$X(\omega) = 5 + 10e^{j(\frac{\pi}{4}n + 45^\circ)} + e^{j\frac{\pi}{2}n}$$

$$= 5\,H_d(0) + 10e^{j(\frac{\pi}{4}n + 45^\circ)}\,H_d(\tfrac{\pi}{4}) + e^{j\frac{\pi}{2}n}\,H_d(\tfrac{\pi}{2})$$

$$= 5 \cdot 0 + 10e^{j(\frac{\pi}{4}n + 45^\circ)} \cdot \frac{\pi}{4}e^{j\sin\frac{\pi}{4}} + e^{j\frac{\pi}{2}n}\,\frac{\pi}{2}e^{j\sin\frac{\pi}{2}}$$

$$= \underbrace{\frac{10\pi}{4}}_{\frac{5\pi}{2}}e^{j(\frac{\pi}{4}n + \frac{\pi}{4} + \frac{1}{\sqrt{2}})} + \frac{\pi}{2}e^{j(\frac{\pi}{2}n + 1)}$$

$$x[n] = 5 + 10\cos(\tfrac{\pi}{4}n + 45^\circ) + j^n$$

$$y[n] = \cancel{5H_d(0)+} \quad 5 + 5e^{j(\frac{\pi}{4}n + \frac{\pi}{4})} + 5e^{-j(\frac{\pi}{4}n - \frac{\pi}{4})} + e^{j\frac{\pi}{2}n}$$

$$y[n] = 5\,H_d(0) + 5e^{j(\frac{\pi}{4}n + \frac{\pi}{4})}\,H_d(\tfrac{\pi}{4}) + 5e^{-j(\frac{\pi}{4}n - \frac{\pi}{4})}\,H_d(-\tfrac{\pi}{4})$$
$$+ e^{j\frac{\pi}{2}n}\,H_d(\tfrac{\pi}{2})$$

$$= 0$$

$$5\frac{\pi}{4}, \quad -\frac{5\pi}{4}e^{-j(\frac{\pi}{4}n + \frac{\pi}{4} - \frac{1}{\sqrt{2}})}$$

---

3. $y[n] - \frac{1}{\sqrt{4}}\,y[n-1] = x[n], \quad -\infty < n < \infty$

$$Y(\omega) - \frac{1}{2}Y(\omega)\,e^{-j\omega} = X(\omega)$$

$$H_d(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$\frac{1}{j} = -j$$

$$\left(1 - \tfrac{1}{2}e^{-j\frac{\pi}{2}\omega}\right)\left(1 + \tfrac{1}{2}e^{-j\frac{\pi}{2}\omega}\right)$$

$$1 - \frac{1}{4}e^{-j\pi\omega}$$

Wed 6 Sept Lecture Liang

System properties

Ex $y[n] = \dfrac{x[n]}{x[1]}$ non-causal

$x_1[n] \to y_1[n]:\ y_1[n] = \dfrac{x_1[n]}{x_1[\phi]}$ ①

$x_2[n] \to y_2[n]:\ y_2[n] = \dfrac{x_2[n]}{x_2[1]}$ ②

Assume $x_2[n] = x_1[n-1]$ $\quad y_1[n-1] = \dfrac{x_1[n-1]}{x_1[1]}$

$$y_2[n] = \frac{x_1[n-1]}{x_1[0]}$$

$\Rightarrow$ not shift-invariant,
must be true for all shifts & $\forall$ inputs

Let $x_1 \to y_1:\ y_1[n] = \dfrac{x_1[n]}{x_1[1]}$

$x_2 \to y_2:\ y_2[n] = \dfrac{x_2[n]}{x_2[1]}$

$x_2 = a x_1$ $\qquad y_2[n] = \dfrac{\cancel{a}\, x_1[n]}{\cancel{a}\, x_1[1]}$

$\downarrow$

$y_2 = a y_1$

Stability

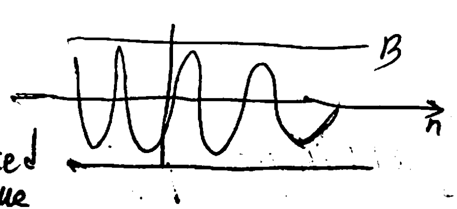
A "system" is BIBO stable if each output its
bounded input
bounded output

output is always bounded for any bounded input

any bounded

boundedness

$|x[n]| < B$



the value is bounded
by some finite value

Examples	Bounded	
$\delta[n]$	y	(from 310)
$e^{-3n}u[n]$	y	$\updownarrow$ diff
e^{-3n}	n	
$\delta(t)$	n	(from 210)

$y[n] = 3x[n] + 2x[n-1]$ $\quad$ $|x[n]| < B$

$|y[n]| < B_1$

$|y[n]| = |3x[n] + 2x[n-1]|$

$\leq 3|x[n]| + 2|x[n-1]|$

$\leq 5B$ bounded

$y[n] = n\,x[n]$ $\quad$ Let $x[n] = u[n]$ unbounded.

$y[n] = \dfrac{1}{x[n]}$ $\quad$ Let $x[n] = 0$ unbounded.

$x[n_0]\,\delta[n-n_0] \longrightarrow x[n_0]\,h[n-n_0]$

$\forall n_0$ $\quad$ $\{\delta[n-n_0]\} \longrightarrow h[n-n_0]$

$-\infty < n < \infty$ $\quad$ $\delta[n] \longrightarrow h[n]$ unit pulse response

$\longrightarrow \boxed{\text{LTI}} \longrightarrow$

$h[n]$ absolutely summerable

$\sum_{n=-\infty}^{+\infty} |h[n]| < B$

$x[n] = \{\ldots x_0, x_1, x_2 \ldots$

$$\boxed{x[n] = \sum_{l=-\infty}^{\infty} x_l\,\delta[n-l]}$$

$\delta[n] \rightarrow h[n] = 0 \quad n < 0$

Right-sided (=causal)

$l = 0 \rightarrow \infty$

$$\sum_{n_0=-\infty}^{\infty} x[n_0]\,\delta[n-n_0]$$

$$\sum_{l=-\infty}^{\infty} x[l]\,\delta[n-l] = x[n]$$

$\downarrow$

$$\boxed{y[n] = \sum_{l=-\infty}^{\infty} x[l]\,h[n-l]}$$

$$y[n] = \sum_{l=-\infty}^{+\infty} x[l]\,h[n-l]$$

$$= \sum_{l=-\infty}^{\infty} x[n-l]\,h[l]$$

$$= x[n] * h[n]$$

$$= h[n] * x[n]$$

Fri 8 Sept Lecture Liang

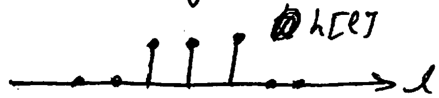
(Kaug. puzzle)

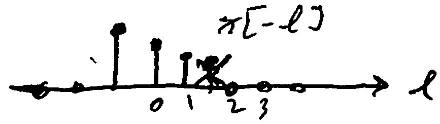
LSI properties

Example

Let $h[n] = [1, 1, 1]$ (L_1), $x[n] = [1, 2, 3]$ (L_2)

Evaluate $y[n] = x[n] * h[n]$





$x[n-l]$, $n<0$ shift to the left
$n>0$ shift to the right

$n \le -2, n \ge 4$, $h[l] * x[n-l] = 0$, $y[n] = 0$

$n=-1$ $y[-1] = 1 \times 1 = 1$

$n=0$ $y[0] = 1\times 2 + 1\times 1 = 3$

$n=1$ $y[1] = 3\times 1 + 2\times 1 + 1\times 1 = 6$

$n=2$ $y[2] = 1\times 3 + 2\times 1 = 5$

$n=3$ $y[3] = 3\times 1 = 3$

$y[n] = \{1, 3, 6, 5, 3\}$

$L_1 + L_2 - 1$

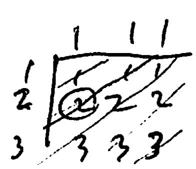
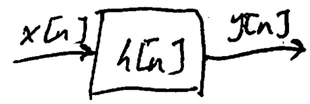


Table Method

	x_{-1}	x_0	x_1	x_2
h_{-1}	$h_{-1}x_{-1}$	$h_{-1}x_0$	$h_{-1}x_1$	$h_{-1}x_2$
h_0	h_0x_{-1}	h_0x_0	h_0x_1	h_0x_2
h_1	h_1x_{-1}	h_1x_0	h_1x_1	h_1x_2
h_2	h_2x_{-1}	h_2x_0	h_2x_1	h_2x_2
h_3	h_3x_{-1}	h_3x_0	h_3x_1	h_3x_2

time origin by x_0h_0 entry



$y[n] = x[n] * h[n]$

$$= \sum_{l=-\infty}^{+\infty} x[l]h[n-l] = \sum_{l=-\infty}^{+\infty} h[l]x[n-l]$$

$= h[n] * x[n]$

Properties

a) Commutative

$x_1 * x_2 = x_2 * x_1$

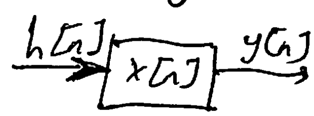
b) Associative

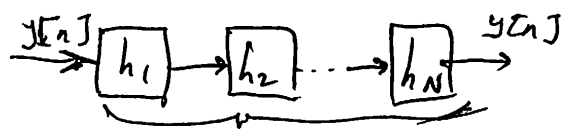
$x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$

c) distributive

$x_1 * (x_2 + x_3) = x_1 * x_2 + x_1 * x_3$

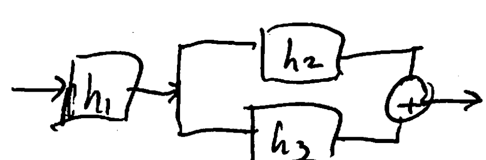
d) shifting

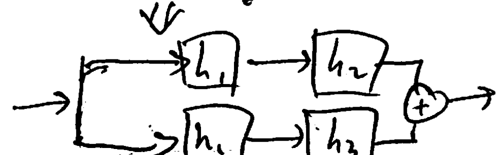




$h[n] = h_1 * h_2 * \cdots * h_n$

reorder boxes → same output





Shifting

$y[n] = x[n] * h[n]$

$y[n'] = x[n'] * h[n']$

$n' = n - n_0$

$y[n-n_0] = x[n-n_0] * h[n-n_0]$ ✗

$y[n-n_0] = x[n-n_0] * h[n]$ ✓

$y[n-n_1-n_2] = x[n-n_1] * h[n-n_2]$ ✓

$\delta[n] * x[n] = x[n]$

$\delta[n-n_0] * x[n] = x[n-n_0]$

$\delta[n] \cdot x[n] = x[0]\,\delta[n]$

$\delta[n-n_0] \cdot x[n] = x[n_0]\,\delta[n-n_0]$

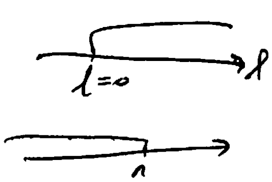
Example $r_1^n u[n] * r_2^n u[n] = \sum_{l=-\infty}^{+\infty} r_1^l u[l]\, r_2^{n-l} u[n-l]$

$$\sum_{n=0}^{N} r^n = \frac{1-r^{n+1}}{1-r}$$

$$= \sum_{l=-\infty}^{\infty} r_1^l r_2^{n-l} u[l] u[n-l]$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad (|r|<1)$$

$$= r_2^n \sum_{l=-\infty}^{+\infty} \left(\frac{r_1}{r_2}\right)^l u[l] u[n-l]$$



$$= r_2^n \sum_{l=0}^{\infty} \left(\frac{r_1}{r_2}\right)^l u[n-l]$$

$$= r_2^n \sum_{l=0}^{n} \left(\frac{r_1}{r_2}\right)^l u[n]$$

$$= \frac{r_2^{n+1} - r_1^{n+1}}{r_2 - r_1} u[n]$$

violates superposition

$y[n]$ total $= zi + zs$

$\boxed{\text{Zero-state response}}$

$$y[n] = -\underbrace{\sum_{k=1}^{N} a_k y[n-k]}_{zi} + \sum_{k=1}^{N} b_k x[n-k]$$

IC's $= y[-1], y[-2], \ldots, y[-N]$ zero-input response

$n-N$

Linear constant coefficient equation

$\to \boxed{h[n]} \to$

LCCD

$y[n] + a_1 y[n-1] + \ldots a_n y[0] = b_0 x[n] + b_1 x[n-1] + \ldots + b_N x[n-N]$

$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k]$

Geometric series
$\downarrow$
note sheet

Mon 11 Sept

Doesler

Z-Transform

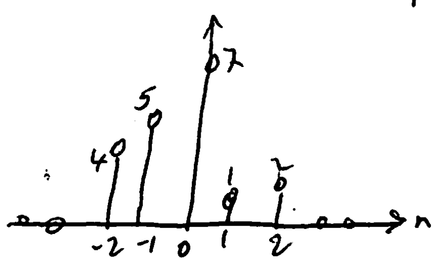
Motivation — Cont. $x(t) \xrightarrow{\mathcal{L}} X(s) = \int_{-\infty}^{\infty} x(t)e^{st}\,dt$

Definition $x[n] \rightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n]\,z^{-n}$

Computation

Complex Funct.

Ex 1 $x[n] = \{4, 5, \underset{\uparrow}{7}, 1, 2\}$

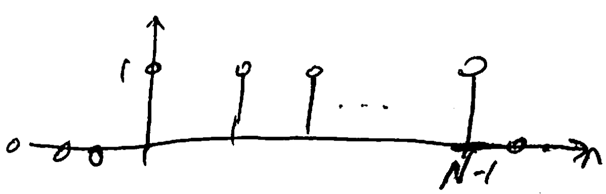


$$X(z) = 4\cdot z^{-(-2)} + 5\cdot z^{-(-1)} + 7\cdot z^{0} + 1\cdot z^{-1} + 2z^{-2}$$
$$= 4z^{2} + 5z + 7 + z^{-1} + 2z^{-2}$$

Ex 2 $x[n] = \delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$ Kronker delta = unit pulse.

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n]\,z^{-n} = 1\cdot 1 = 1$$ $\boxed{\text{z transform of unit pulse is 1}}$

Ex 3 $x[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{else} \end{cases}$



$$X(z) = \sum_{n=0}^{N-1} 1\cdot z^{-n}$$
$$= \sum_{n=0}^{N-1} (z^{-1})^{n}$$
$$= \left.\begin{cases} \dfrac{1-z^{-N}}{1-z^{-1}} & z \neq 1 \\ N & z = 1 \end{cases}\right\} z \neq 0$$

Geometric series

$$\sum_{n=0}^{N-1} a^{n} = \begin{cases} \dfrac{1-a^{N}}{1-a} & a \neq 1 \\ N & a = 1 \end{cases}$$

$$(1 + a + a^{2} + a^{3} + \cdots + a^{N-1})(1-a)$$
$$= 1 + a - a + a^{2} - a^{2} + a^{3} - a^{3} + \cdots - a^{N}$$
$$= 1 - a^{N}$$

Ex 4 $\quad x[n] = \begin{cases} a^n & 0 \le n \\ 0 & n < 0 \end{cases}$

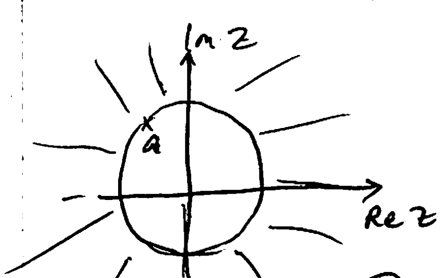
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \lim_{N\to\infty} \sum_{n=-N}^{N} x[n] z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N-1} \left(\frac{a}{z}\right)^n = \lim_{N\to\infty} \frac{1-\left(\frac{a}{z}\right)^N}{1-\left(\frac{a}{z}\right)}$$

$$= \frac{1}{1-\left(\frac{a}{z}\right)}, \quad \left|\frac{a}{z}\right| < 1$$

$$X(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$



ROC_x - Region of convergence of $X(z)$

right-sided sequence

$b^N \to 0$

$b = re^{j\theta}$

$b^N = r^N e^{jN\theta} \quad |b| < 1$

Convergence of series

$\cancel{\sum_{n=1}^{\infty} \frac{1}{n}} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

LTI

Linear

Time Invariant

eigenvector

$Ax = \lambda \cdot x$

$$x[n] \longrightarrow \boxed{LTI} \longrightarrow y[n] = h[n] * x[n]$$

$$\delta[n] \longrightarrow \boxed{LTI} \longrightarrow h[n]$$ – Impulse response

$$x[n] \longrightarrow \boxed{LTI} \longrightarrow \lambda \cdot x[n]$$

$$z^n \longrightarrow \boxed{LTI} \longrightarrow H(z)\, z^n$$ transfer function

$x[n] = z^n$

$y[n] = h[n] * x[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] * x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

$$z^{n-k} = z^n z^{-k}$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} = H(z) \cdot z^n$$

$\therefore$ z^n is an eigenfunction of LTI systems

eigen sequence

"eigen" = "self"

ECE 310 Recitation Tues 12 Sept Bresler

$y[n] = y[n-3] + x[n] + 5x[n-2]$

· Lin · Causal · Time invariant

Lin suppose $x_1[n] \to y_1[n]$ Claim system is linear

$x_2[n] \to y_2[n]$

Show $x_3[n] = a x_1[n] + b x_2[n] \to a y_1[n] + b y_2[n]$

If $y_5[n]$ and $y_6[n]$ satisfy the same LCDE then $y_5[n] = y_6[n]$

Linear Constant Coefficients Difference eqn

$y_1[n] = y_1[n-3] + x_1[n] + 5x_1[n-2]$

$y_2[n] = y_2[n-3] + x_2[n] + 5x_2[n-2]$

$y_3[n] = y_3[n-3] + a x_1[n] + b x_2[n] + 5(a x_1[n-2] + b x_2[n-2])$

$a y_1[n] + b y_2[n] = (a y_1[n-3] + b y_2[n-3]) + (a x_1[n] + b x_2[n])$

$+ 5(a x_1[n-2] + b x_2[n-2])$

$y_3[n] = a y_1[n] + b y_2[n] \Rightarrow$ linear

If system can be causal or non-causal $\Rightarrow$ choose causal

as written, system written in causal form ?

$y[n] + y[n-3] + x[n] = 0$

$x[n] \to y[n]$

$x[n-k] \to y[n-k]$

$x_2[n] = x[n-k]$

$y_2[n] = y_2[n-3] + x_2[n] + 5x_2[n-2]$

$= y_2[n-3] + x[n-k] + 5x[n-k-2]$

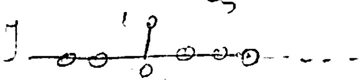
$y[n-k] = y[n-3-k] + x[n-k] + 5x[n-k-2]$

$y_2[n] = y[n-k]$ ✓

$y[n] = x[|n|] - 2y[n-2]$ $\qquad\qquad$ $y[n] = |x[n]|$

Linear ✓ Causal ✗ Time-Invariant ✗ $\qquad$ non-linear

Time-Invariant = Shift-Invariance $\quad TI = SI$

Suppose $x[n] = \delta[n]$ 

$y[n] = \delta[|n|] - 2y[n-2]$

$n < 0 \quad y[n] = 0$

$y[0] = 1$

$y[1] = 0$

$y[2] = -2$

$x[n] = \delta[n-1]$

$y[0] = 0$

$y[1] = 1$

$y[2]$

$\vdots$

Ex $y[n] = x[n^2]$

$\left(\frac{1}{2}\right)^n u[n-1] \longrightarrow \boxed{LTI} \longrightarrow \left(\frac{1}{3}\right)^n u[n-3]$

$x[n] \qquad\qquad\qquad y[n]$

Find zero state resp.

to $3\left(\frac{1}{2}\right)^n u[n-1] - \left(\frac{1}{2}\right)^{n-2} u[n-3]$

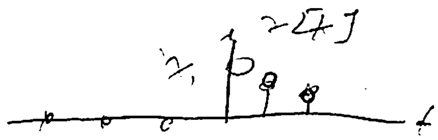
Convolution

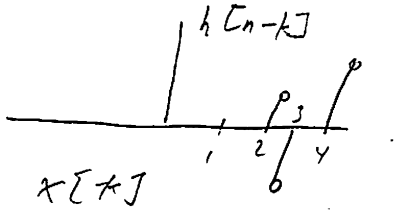
$x[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$

$h[n] = 4\delta[n] + \delta[n-2]\ 2\delta[n-3] + 3\delta[n-4]$

Find $y[n] = h[n] * x[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$





x[k]

$y[0] = 12$

$y[1] = 8$

$y[2] = 6+4 = 10$

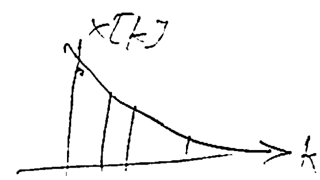
$$\boxed{\begin{array}{l} z[n] * \delta[n] = z[n] \\ z[n] * \delta[n-k] = z[n-k] \end{array}}$$

$= \sum_m z[m]\,\delta[n-k-m] = z[n-k]$

$x[n] * h[n] = (3\delta[n] + 2\delta[n-1] + \delta[n-2]) * (4\delta[n] + \delta[n-2] - 2\delta[n-3] + 3\delta[n-4])$

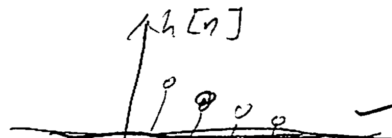
$x[n] = (0.9)^n u[n]$

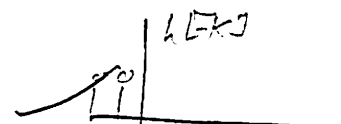
$h[n] = (0.7)^n u[n-1]$



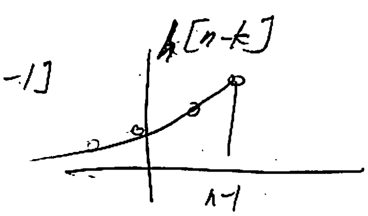
$n-1 \geq 0 \Rightarrow n \geq 1$

$y[n] =$





$$\sum_{k=0}^{n-1} (0.9)^k \cdot (0.7)^{n-k}\, u[n-1] = (0.7)^n \sum_{k=0}^{n-1} \left(\frac{9}{7}\right)^k u[n-1]$$

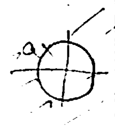


$$= (0.7)^n \frac{1-(9/7)^n}{1-9/7}\, u[n-1]$$

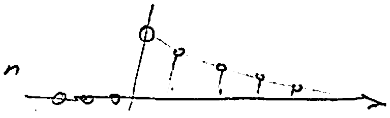
Breslar

Wed 13 Sept Lecture

z-transform



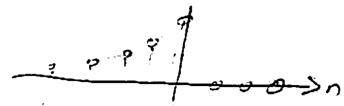
$x[n] \xrightarrow{z} X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$



right-handed sequence

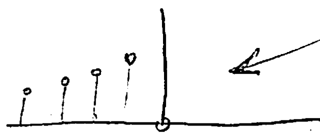
$x[n] = a^n u[n] \xrightarrow{z} \frac{z}{z-a}, \quad |z| > |a|$

Ex $y[n] = -b^n \cancel{\ } u[-n-1]$

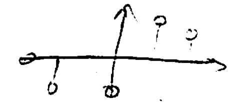


left-sided sequence

$|b| > 1$



$Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \cancel{\ } - \sum_{n=-\infty}^{-1} b^n z^{-n}$



two sided sequence

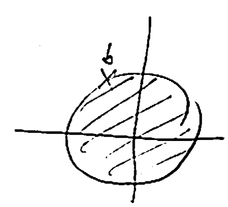
$= -\sum_{n=-\infty}^{-1} \left(\frac{b}{z}\right)^n$

$m = -n-1$

$Y(z) = -\sum_{m=0}^{\infty} \left(\frac{b}{z}\right)^{-(m+1)} = \cancel{\ } -\left(\frac{z}{b}\right) \sum_{m=0}^{\infty} \left(\frac{z}{b}\right)^m$

$= -\frac{z}{b} \cdot \frac{1}{1 - \frac{z}{b}}, \quad \left|\frac{z}{b}\right| < 1$

$= \boxed{\frac{z}{z-b}, \quad |z| < |b|}$



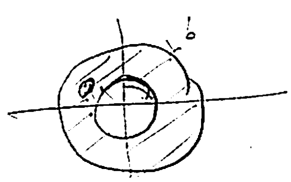
z-transform by itself does not fully describe sequence.

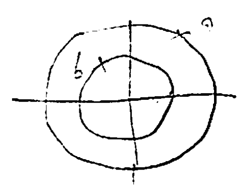
$w[n] = \underbrace{a^n u[n]}_{x[n]} + \underbrace{b^n u[-n-1]}_{y[n]}$

$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} + \sum_{n=-\infty}^{\infty} y[n] z^{-n}$

$\qquad |z| > |a| \qquad |z| < |b|$

$= \frac{z}{z-a} - \frac{z}{z-b} \qquad |a| < |z| < |b|$ strict ineq $\Rightarrow |a| \neq |b|$ cannot





z-transform does not exist in this scenario

Ex $x[n] = 1 \quad \forall n$

$\Rightarrow a = b = 1 \Rightarrow$ no z-transform

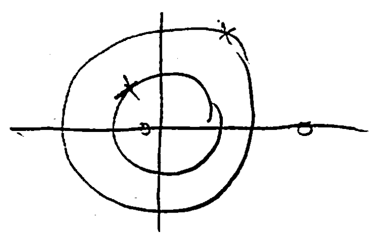
$$\frac{z}{z-a} - \frac{z}{z-b} = \frac{z^2 - bz - z^2 + az}{(z-a)(z-b)} = \frac{z(a-b)}{(z-a)(z-b)}$$

(b/c typically put everything in common denominator)

Poles: $z=a$, $z=b$ (customary to mark poles with x, zeros w/ o)

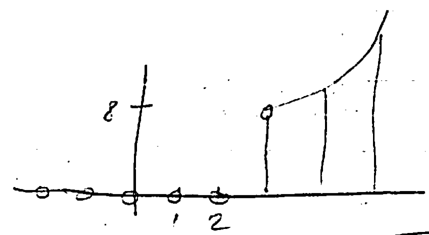
ROC cannot include poles, zeros say nothing about ROC

zeros ($z=0$, $z=\infty$)



Where can the sequence converge?

Ex $x[n] = 2^n u[n-3]$



$$x(z) = \sum_{n=3}^{\infty} 2^n z^{-n} = \sum_{n=3}^{\infty} \left(\frac{2}{z}\right)^n = \sum_{0}^{\infty} \cdot - \cdots = \sum_{m=0}^{\infty} \left(\frac{2}{z}\right)^{m+3}$$

subtract these terms $0 \ldots 2$

change of variables

$$\left(\frac{2}{z}\right)^m \left(\frac{2}{z}\right)^3$$

$$= \left(\frac{2}{z}\right)^3 \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n = \frac{1}{1-\frac{2}{z}}$$

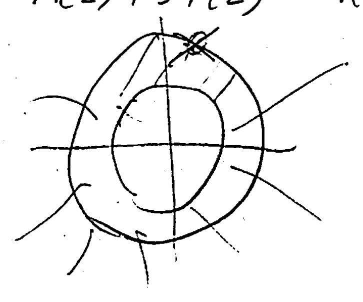
Properties of the Z-Transf.

- Linearity: $Z\{\underbrace{ax[n] + by[n]}_{w[n]}\} = aZ\{x[n]\} + bZ\{y[n]\}$

$$= aX(z) + bY(z) \qquad ROC_w \supset ROC_x \cap ROC_y$$

$X(z) = \frac{z}{z-a}$

$Y(z) = \frac{z-c}{z-d}$



Ex $x[n] = r^n \cos(\omega_0 n) u[n]$

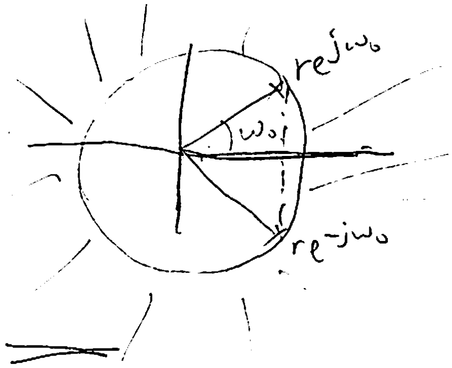
$$= \frac{1}{2} r^n \left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) u[n]$$

$$= \frac{1}{2} \left(r e^{j\omega_0}\right)^n u[n] + \frac{1}{2} \left(r e^{-j\omega_0}\right)^n u[n]$$

$a^n u[n]$ $\qquad$ $b^n u[n]$

$$X(z) = \frac{1}{2} \frac{z}{z - re^{j\omega_0}} + \frac{1}{2} \frac{z}{z - re^{-j\omega_0}}$$

$$= \frac{\cdots}{z^2 - 2r\cos\omega_0 + r^2}$$



Shift Property (Delay)

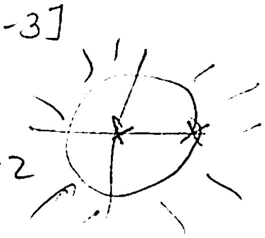
$y[n] = x[n-k]$ $\qquad$ k-integer

$Y(z) = z^{-k} X(z)$ $\qquad$ $ROC_y = ROC_x \pm$ poles at $z=0$

Ex $\quad 2^n u[n-3]$ $\qquad$ $2^n u[n] \Rightarrow 2^3 \cdot 2^{n-3} u[n-3]$

$2^n u[n] \longrightarrow \frac{z}{z-2}, \quad |z|>2$

$\rightarrow 2^3 z^{-3} \frac{z}{z-2} = 8 \frac{1}{z^2(z-2)}, \quad |z|>2$



Fri 15 Sept Lecture   Bresler

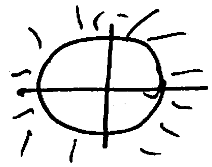
Z-Transform

Properties

$y[n]=x[n-k] \leftrightarrow z^{-k}X(z)$

$$Y(z)=\sum_{n=-\infty}^{\infty} x[\underbrace{n-k}_{m=n-k}]\, z^{-n} = \sum_{m=-\infty}^{\infty} x[m]\, \underbrace{z^{-(m+k)}}_{z^{-m}\cdot z^{-k}} = z^{-k}\underbrace{\sum_{m=-\infty}^{\infty} x[m]z^{-m}}_{X(z)}$$

$$2^n u[n-1000] = 2^{1000}\cdot 2^{n-1000}u[n-1000]$$
$$= 2^{1000}\,\frac{z\cdot z^{-1000}}{z-2},\ |z|>2$$



Ex $x[n]=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{else}\end{cases} = u[n]-u[n-N]$

$$X(z)=\frac{z}{z-1}-\frac{z^{-N}\cdot z}{z-1}=\frac{z(1-z^{-N})}{z-1},\ |z|>1$$

$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}$
$= \frac{z}{z-a}$
$|z|>|a|$

Convolution Property

$$y[n]=x_1[n]*x_2[n]=\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$
$$Y(z)=X_1(z)\,X_2(z)\quad ROC_Y \supset ROC_{x_1}\cap ROC_{x_2}$$

same convergence

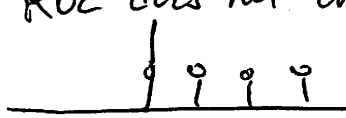
Derivative Property

ROC does not change.

$\sum x[n]\, n\, z^{-n}$

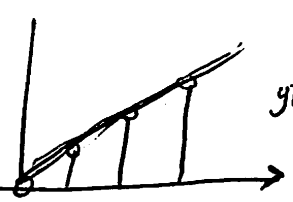
multiply by n still converge by exp. much faster than any polynomial

$y[n]= n\,x[n]$



$x[n]=u[n]$

$$Y(z)=\sum_{n=-\infty}^{\infty} n\,x[n]z^{-n}$$



$y[n]=nu[n]$

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

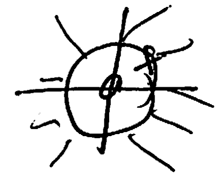
$y(z)=-z\,\frac{d}{dz}\,\frac{z}{z-1}=$

$$\frac{d}{dz}X(z)=\sum_{n=-\infty}^{\infty} x[n]\,n\,z^{-n-1}=-z^{-1}\underbrace{\sum_{n=-\infty}^{\infty} n\,x[n]z^{-n}}_{y(z)}$$

$\Rightarrow y(z)=-z\,\frac{d}{dz}X(z)$

Ex $x[n] = n a^n u[n]$

$$X(z) = -z \frac{d}{dz} \frac{z}{z-a} = \frac{az}{(z-a)^2}, \quad |z|>|a|$$



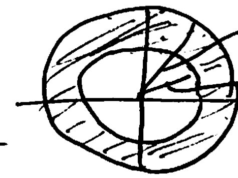
"Modulation" (Exponential Scaling)

$$y[n] = a^n x[n] = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right) \qquad ROC_x: R_+ < |z| < R_-$$

$$a \cdot ROC_x = ROC_y : R_+ < \left|\frac{z}{a}\right| < R_-$$

$$|a|R_+ < |z| < |a|R_-$$



★ ROC always takes the form of a ring.

$r^n \cos\omega_0 n \, u[n]$

$$\underbrace{\tfrac{1}{2}(re^{j\omega_0})^n}_{a^n} \underbrace{u[n]}_{u[n]} + \tfrac{1}{2}(re^{-j\omega_0})^n u[n]$$ use modulation property!

$$\frac{z}{z-1}$$

$$z^{-1}\frac{z}{z-1} \leftrightarrow (z_1)^{n-1} u[n]$$

Inverse z-transform

double pole also

$X(z)$ complex integral w/ contour integral

$$X(z) = \frac{B(z)}{A(z)} \text{ — Rational form}$$

Proper rational form: $Deg(B) < deg(A)$

(if geq, long division)

numerical methods required for > quintic

eqn for quintic & below

$$X(z) = z^{-1} + z^2$$

$$\delta[n-1] + \delta[n+2]$$

$$\frac{B(z)}{A(z)} = \frac{B(z)}{(z-z_1)(z-z_2)\cdots(z-z_n)} = \frac{A}{z-z_1} + \frac{A_2}{z-z_2} \cdots$$

one/two sided z-transform

Mon 18 Sept. Lecture

Liang

System Analysis

$$x[n] \xrightarrow{} \boxed{\text{LSI}} \xrightarrow{} y[n] \qquad h[n], H(z)$$

* determine system's stability
* determine causality
* predict system's resp. to $x[n]$

Tools

* convolution, $h[n]$, $y[n] = x[n] * h[n]$
* z-transform, $H(z)$, $Y(z) = X(z)H(z)$
* LCCDE

LCCDE:

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k]$$

$\underbrace{\text{IC's}}_{\text{initial conditions}}$ $\quad y[-1], y[-2], \ldots, y[-N]$

Causality $\quad h[n] = 0, \; n < 0, \quad$ causal, right-sided

$$y[n] = \sum_{l=-\infty}^{+\infty} h[l]x[n-l] = \sum_{l=0}^{\infty} h[l]x[n-l]$$

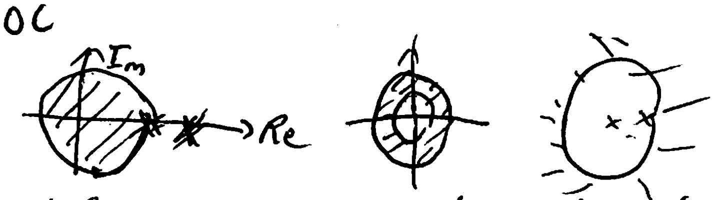
$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} h[n]z^{-n}$$

$Z\{h[n]\}$

$Z^+\{h[n]\}$

$x[n] \xrightarrow{Z} X(z)$ ROC

$$X(z) = \frac{z}{(z-1)(z-2)}$$



Left sided $\quad$ Two sided $\quad$ Right sided

ROC

$|z| > R$

$R = |z_p|$

~~Ex $k=2$~~

~~$Z^+\{y[n]\} = Y(z)$~~

~~$Z^+\{y[n-2]\} = Y(z) + \sum y[n-2]z^{-n}$~~

~~System Analysis~~

$x[n] \xrightarrow{Z^+} X(z)$

$y[n] \xrightarrow{Z^+} Y(z)$

$y[n-k] \xrightarrow{Z^+} z^{-k}X(z)$

$y[n-k] \xrightarrow{Z^+} z^{-k}Y(z)$

$$Z^+\{y[n-k]\} = z^{-k}Y(z) + \sum_{l=1}^{k} y[-l]z^{k-l}$$

Example $k=2$

$$Z^+\{y[n]\} = Y(z)$$

$$Z^+\{y[n-2]\} = \sum_{n=0}^{\infty} y[n-2]\, z^{-n}$$

$$= \boxed{y[-2] + y[-1]z^{-1}} + \sum_{n=2}^{\infty} y[n-2]\, z^{-n}$$

$$\hat{n} = n-2$$

$$z^{-2} \sum_{\hat{n}=0}^{\infty} y[\hat{n}]\, z^{-\hat{n}}$$

$$Y(z)\left(1+\sum_{k=1}^{N} a_k z^{-k}\right) = \sum_{k=0}^{N} b_k z^{-k})X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

Wed 30 Sept Lecture

Yang

__System Stability__

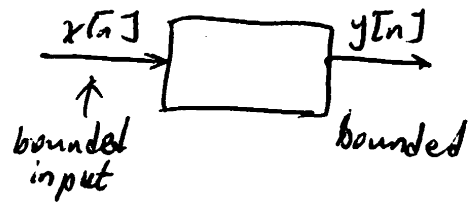
a) Definition (~~local~~ instability ok)

b) Criteria

c) Boundedness vs pole dist

(z domain $\leftrightarrow$ laplace domain continuous/discrete)

__Def__ $H(z)$, $h[n]$



__BIBO Stable__

$h[n] \leftarrow$ $y[n] = h[n] * x[n]$

$\downarrow z$ $\downarrow$

$Z\{h[n]\} = H(z)$ $\boxed{Y(z) = H(z)X(z)}$ LTI only

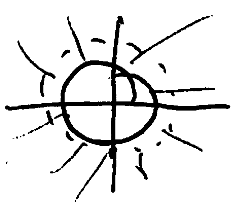
impossible to determine for non-LSI systems.

(must try all bounded inputs)

Criteria for determining the BIBO stability of LSI systems

a) T-domain: $\sum_{n=-\infty}^{\infty} |h[n]| < B < +\infty$ (iff)

b) Z-domain: $ROC_H \supset$ unit circle (circumference) bounding circle



$|z| > R$

$R < 1$

marginally unstable

resonance phenomenon - oscillatory system

<u>Proof</u> $x[n]$ $\quad \sum_{n=-\infty}^{\infty} |h[n]| < A$

<u>if</u> $|x[n]| < B \xrightarrow[LSI]{} y[n] = x[n] * h[n]$ $\quad |y[n]| < A,$

$$|y[n]| = \left|\sum_{\ell=-\infty}^{+\infty} h[\ell]\, x[n-\ell]\right| \le \sum_{\ell=-\infty}^{+\infty} |h[\ell]\, x[n-\ell]|$$

$$= \sum_{\ell=-\infty}^{\infty} |h[\ell]|\,\underbrace{|x[n-\ell]|}_{B} \le B \sum_{\ell=-\infty}^{\infty} |h[\ell]| \le AB = A,$$

<u>only if</u> $|x[n]| < B \xrightarrow[LSI]{\sum_{n=-\infty}^{\infty} |h[n]| \text{ unbounded}} y[n]$

pick one such bounded input $\to$ unbounded output

$x[n] = \mathrm{sgn}\{h[n_0 - n]\}$ $\quad h$ real-valued

$$y[n_0] = \sum_{\ell=-\infty}^{\infty} h[\ell]\, x[n_0-\ell] = \sum_{\ell=-\infty}^{\infty} h[\ell]\, \mathrm{sgn}\{h[n_0-(n_0-\ell)]\}$$

$$= \sum_{\ell=-\infty}^{\infty} h[\ell]\, \mathrm{sgn}\{h[\ell]\} = \sum_{\ell=-\infty}^{\infty} |h[\ell]| \quad \text{unbounded}$$

$$H(z) = \frac{B(z)}{A(z)} = \frac{B(z)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

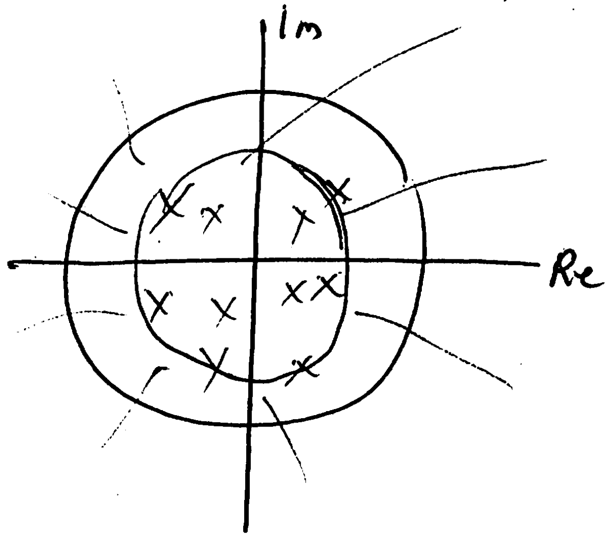
	bounded		
	outside uc	on uc	inside uc
p_ℓ single $\longrightarrow p_\ell^n u[n]$	✗	✓	✓
p_ℓ double $\longrightarrow n p_\ell^n u[n]$	✗	✗	✓
p_ℓ repeated k-times $\longrightarrow n^k p_\ell^n u[n]$	✗	✗	✓

$Y(z) = H(z)X(z)$

$\{p_\ell^Y\} = \{p_\ell^H\} \cup \{p_\ell^X\}$

resonance situation
for all inputs, single-single
$\Rightarrow$ repeated

		outside UC	on UC	inside UC
$\{p_\ell^H\}$	single		×	✓
	repeated		×	✓
$\{p_\ell^X\}$	single		✓	✓
	repeated			✓
$\{p_\ell^Y\}$	single		✓	✓
	repeated			✓



anticausal vs. non causal

Liang

Fri 22 Sept Lecture

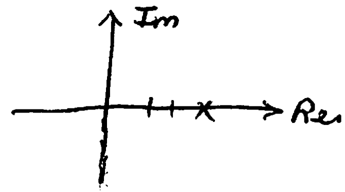
<u>System Stability</u> : LST Systems

- ~~Tim~~ T-domain: $\sum_{n=-\infty}^{\infty} |h[n]| < B < +\infty$

- z-domain: $ROC_H \supset U.C.$

<u>Example</u> $Y(z) = H(z)X(z) \leftrightarrow$ LSI, convolution relationship

a) $H(z) = \frac{z}{z-3}$



ROC_H : $|z| > 3$

causal, un stable

$\not\supset U.C$

$h[n] = 3^n u[n]$, $\sum |h[n]|$ unbounded

$|z| < 3$

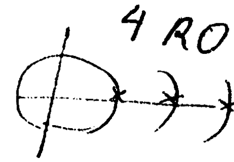
Anticausal, stable,

$h[n] = -3^n u[-n-1] \rightarrow \sum 3^n u[-n-1]$

bounded $= \sum_{n=-\infty}^{-1} 3^n = 0.5$

b) $H(z) = \frac{z}{z-\frac{1}{3}}$

4 ROCs



ROC_H : $|z| > \frac{1}{3}$, causal, unstable

$|z| < \frac{1}{3}$, anticausal, stable

<u>Example</u> $H(z) = \frac{z^2}{z^2+1}$ resonating system

both unstable

ROC_H : $|z| > 1$, causal, unstable

$H(z) = \frac{1}{2}\frac{z}{z+j} + \frac{1}{2}\frac{z}{z-j}$

$$h[n] = \frac{1}{2}(j)^n u[n] + \frac{1}{2}(-j)^n u[n] = \frac{1}{2}e^{j\frac{\pi}{2}n}u[n] + \frac{1}{2}e^{-j\frac{\pi}{2}n}u[n]$$

$$= \cos\left(\frac{\pi}{2}n\right)u[n]$$

$Y(z) = H(z)X(z)$ ← single

pole outside UC ✗

pole inside UC ✗

~~pole~~ repeated pole on UC ✓ → unbounded

$x[n] = e^{j\frac{\pi}{2}n}u[n]$ cos

$x[n] = e^{-j\frac{\pi}{2}n}u[n]$ sin

Liang

Mon 25 sept Lecture

Topics

* Discrete-Time Fourier Transform (DTFT)
* Discrete Fourier Transform (DFT) FFT
* Signal & System Analysis

$$
\begin{array}{ccc}
x_a(t) & \xrightarrow{\int_{-\infty}^{+\infty} x_a(t) e^{-j\Omega t}\, dt} & X_a(\Omega) \quad \text{Freq} \\
\Big\downarrow {\scriptstyle \text{discretization in time sampling}} & & \Big\uparrow \\
x[n] & \xrightarrow{\text{DTFT}} & X_d(\omega) \\
\Big\downarrow {\scriptstyle \text{Truncation in time}} & & \Big\uparrow {\scriptstyle \text{discritization in freq sampling}} \\
\{\tilde{x}[n]\}_{n=0}^{N-1} & \xrightarrow[\text{(FFT)}]{\text{DFT}} & \{X(k)\}_{k=0}^{N-1}
\end{array}
$$

under some assumptions, sampling can be lossless in terms of information content
$\Downarrow$
Nyquist criterion

Classes of Signals/sequences

a) Infinite length, arbitrary variations

b) Finite length/duration signals/systems

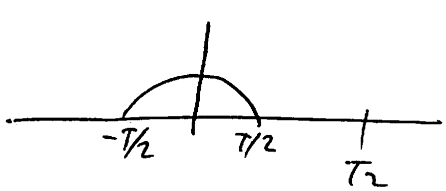
$x(t)\,(u(t) - u(t-T))$, length T

$x[n](u[n] - u[n-N])$, length N

c) Periodic signal/sequences

$\tilde{x}(t + lT) = \tilde{x}(t)$

$\tilde{x}[n + lN] = \tilde{x}[n]$, integer l



$T_2 \geq T_1$

otherwise aliasing (overlap)

finite duration → periodic signal

$x_1(t) = x(t)(u(t) - u(t-T))$

$\tilde{x}_1(t) = \sum_{l=-\infty}^{+\infty} x_1(t + lT)$

$x_1[n] = x[n](u[n] - u[n-N])$

$\tilde{x}_1[n] = x_1[\langle n \rangle_N]$

– Dirac δ-function

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{unbounded} & t = 0 \end{cases} \quad \text{s.t.} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

t

Def

$$\int_{-\infty}^{+\infty} \phi\, \delta(t)\,dt = \phi(0)$$

$f(t)$

$f(at)$

$\delta(-t)$? $\delta(at)$?

$$\delta(t-t_0): \int_{-\infty}^{\infty} \varphi(t)\, \delta(\overset{t'}{\widetilde{t-t_0}})\,dt = \phi(t_0)$$

t

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$a>0$

$$\int_{-\infty}^{+\infty} \phi(t)\, \underbrace{\delta(at)}_{t'}\,dt = \int_{-\infty}^{+\infty} \phi\left(\frac{t'}{a}\right) \delta(t')\frac{dt'}{a} \longrightarrow t' = at$$

$$= \frac{1}{a}\phi\left(\frac{t}{a}\right)\Big|_{t=0} = \frac{1}{a}\phi(0)$$

$$\frac{1}{|a|}\delta(t) \longrightarrow \int_{-\infty}^{+\infty} \phi(t)\frac{1}{a}\delta(t)\,dt = \frac{1}{a}\int_{-\infty}^{\infty}\phi(t)\delta(t)\,dt = \frac{1}{a}\phi(0)$$

$$\delta(t) = \delta(-t) \quad \text{even!}$$

$\cancel{\delta'(t)}$

$$\delta'(t-t_0)$$

derivative odd

$$\int_{-\infty}^{\infty}\phi(t)\,\delta'(t-t_0) = -\phi'(t_0)$$

$$\delta'(-t) = -\delta'(t)$$

$$\frac{d}{dt_0}\int_{-\infty}^{+\infty}\phi(t)\,\delta(t-t_0)\,dt = \frac{d}{dt_0}\phi(t_0)$$

$$-\int_{-\infty}^{+\infty}\phi(t)\,\delta'(t-t_0)\,dt = -\frac{d}{dt_0}\phi(t_0)$$

Friday 29 September Lecture

Discrete-Time Fourier Transform (DTFT)

FT

$$X(\Omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\Omega) e^{+j\Omega t}\, d\Omega$$

DTFT

$$X_d(\omega) \triangleq \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$-\infty < \omega < +\infty$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) e^{+j\omega n}\, d\omega$$

$$X_d(\omega + l 2\pi)$$
$$\shortparallel$$
$$X_d(\omega)$$

$$X_d(\omega + l 2\pi) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(\omega + l 2\pi) n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{-j l n 2\pi}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{m=-\infty}^{+\infty} x[m] e^{-j\omega m} \right) e^{+j\omega n}\, d\omega$$

$$\frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} x[m] \int_{-\pi}^{\pi} e^{+j\omega(n-m)}\, d\omega = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases} = \delta[n-m]$$

DTFT Pairs

$1 \leftrightarrow 2\pi\delta(\omega),\ |\omega| \le \pi$ or $2\pi \sum_{l=-\infty}^{\infty} \delta(\omega + 2l\pi)$

$e^{j\omega_0 n} \leftrightarrow 2\pi\delta(\omega - \omega_0)$

$\cos \omega_0 n \leftrightarrow \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$

$\sin \omega_0 n \leftrightarrow \frac{\pi}{j}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$

$\delta[n] \leftrightarrow 1$

$u[n] \leftrightarrow \frac{1}{1 - e^{-j\omega}} + \pi\delta(\omega)$

$a^n u[n] \leftrightarrow \frac{1}{1 - a e^{-j\omega}},\ |a| < 1$

Examples

$\delta[n] \leftrightarrow 1$ ✓

$1 \leftrightarrow 2\pi\delta(\omega),\ \underline{|\omega| < \pi}$

only interested in baseband btwn $-\pi$ and π

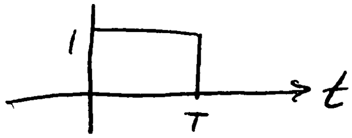
$a^n u[n]$

$$\sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{+\infty} (a e^{-j\omega})^n,\ |a| < 1$$

$= (1 - a e^{-j\omega})^{-1}$

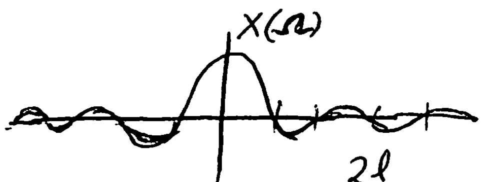
Tues 3 Oct ~~Dis~~ Recitation Liang

$x(t)$

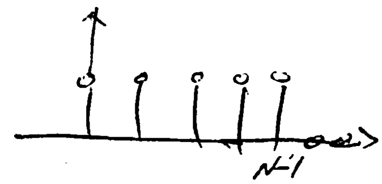


$\mathrm{sinc}(x) = \frac{\sin x}{x}$

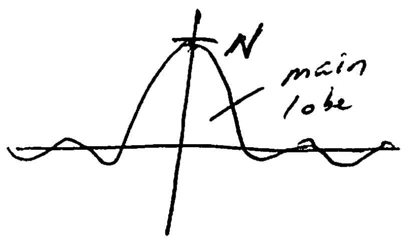
$T\,\mathrm{sinc}(\Omega \frac{T}{2})\, e^{-j\Omega T/2}$



$\frac{2l}{T}$



$\frac{\sin \omega N/2}{\sin \omega/2}\, e^{-j\omega \frac{N-1}{2}}$



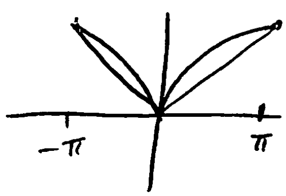
$\sum_{n=0}^{N-1} e^{-j\omega n}$

$\frac{\omega N}{2} = l\pi$

$\omega = \frac{2l\pi}{N}$

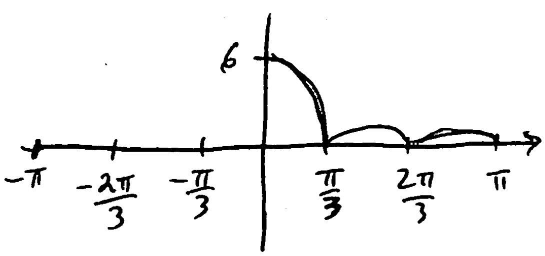
ratio of heights between main lobe and side lobes remains same

$\sin \frac{\omega}{2}$



As N increases, main lobe gets narrower and taller → squeezed to middle

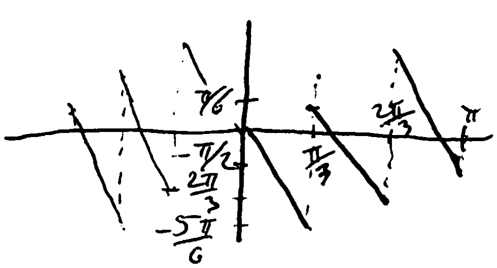
$N=6$



Real function

magnitude even symmetry

phase odd symmetry



jump of π in phase ⇒ zero crossing

but 2π natural

$\Omega = 2\pi F$

$\omega = 2\pi f$

DTFT

$1 \xleftrightarrow{FT} 2\pi\delta(\Omega)$ $\qquad$ $1 \longleftrightarrow 2\pi\delta(\omega) \quad |\omega| < \pi$

$e^{j\Omega_0 t} \longleftrightarrow 2\pi\delta(\Omega-\Omega_0)$ $\qquad$ $e^{j\omega_0 n} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$

$\cos(\Omega_0 t) \longleftrightarrow \pi(\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0))$ $\quad$ $\cos\omega_0 n \longleftrightarrow \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$

$\cos(\Omega_0 t+\phi_0)$

$= \cos\left(\Omega_0\left(t+\frac{\phi_0}{\Omega_0}\right)\right) \longleftrightarrow \pi(\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0))e^{j\Omega\frac{\phi_0}{\Omega_0}}$

$= \pi e^{j\Omega\frac{\phi_0}{\Omega_0}}\delta(\Omega-\Omega_0)+\pi e^{j\Omega\frac{\phi_0}{\Omega_0}}\delta(\Omega+\Omega_0)$

$= \pi e^{j\phi_0}\delta(\Omega-\Omega_0)+\pi e^{-j\phi_0}\delta(\Omega+\Omega_0)$

$\cos(\Omega_0 t+\phi_0)$

$= \frac{1}{2}\left(e^{j(\Omega_0 t+\phi_0)}+e^{-j(\Omega_0 t+\phi_0)}\right) = \frac{1}{2}\left(e^{j\phi_0}e^{j\Omega_0 t}+e^{-j\phi_0}e^{-j\Omega_0 t}\right)$

$= \frac{1}{2}\left(e^{j\phi_0}2\pi\delta(\Omega-\Omega_0)+e^{-j\phi_0}2\pi\delta(\Omega+\Omega_0)\right)$

$\cos(\omega_0 n+\phi_0) = \cos\left(\omega_0\left(n+\frac{\phi_0}{\omega_0}\right)\right)$

$\frac{1}{2}\left(e^{j\phi_0}e^{j\omega_0 n}+e^{-j\phi_0}e^{-j\omega_0 n}\right)$ $\qquad$ $\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))e^{j\omega\frac{\phi_0}{\omega_0}}$

$X_d(\omega) \longleftrightarrow x[n]$

$Y_d(\omega) = X_d(\omega)e^{-j\omega\frac{1}{2}} \longrightarrow y[n] = x[n-\frac{1}{2}]$

$x[n] \longrightarrow \boxed{e^{-j\omega\frac{1}{2}}} \xrightarrow{y[n]=x[n-\frac{1}{2}]}$ $\quad$ $\times$ must be integers

$x[n] \longrightarrow X_d(\omega)$, $\quad x[n-n_0] \longleftrightarrow X_d(\omega)e^{-j\omega n_0}$ (n_0 integer)

Send reason to take conflict → TA email
Will hw solns be posted before exam?
eqns given on Exam

Wed 4 October Lecture — Liang

Annoncement (next week)

a) Monday class canceled → TA office hrs
b) Review session Tues day evening (during recitation) HKN, TA review Wed recitation
c) Friday lecture uncanceled, no quiz
d) HW 6 due Thurs, covering exam 1 (5%)

$$x[n] \longrightarrow \boxed{LSI} \longrightarrow y[n]$$

$h[n]$, $H(z)$, DE (difference equation)

$$\underline{y[n] = x[n] * h[n]}$$

$$Y(z) = X(z)\,H(z)$$

DE

most universal in terms of computing power
- computational efficiency
- as an engineer

$$x[n] = \sum_{\ell=-\infty}^{+\infty} x[\ell]\,\delta[n-\ell]$$

pulse as basic constituent ⇛ convolution = transient analysis

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)\,e^{+j\omega n}\,d\omega$$

time as a summation of many many frequency components

$$e^{j\omega_0 n} \longrightarrow \boxed{H_d(\omega)} \xrightarrow{y[n]} \underbrace{e^{j\omega_0 n}}_{\text{frequency same}} \cdot \underbrace{H_d(\omega_0)}_{\text{amplitude scaling}}$$

any sinusoidal funct. is an eigenfunction of the system

$A v = \lambda v$

$H_d(\omega)$: frequency response
$|H_d(\omega)|$: magnitude response
$\angle H_d(\omega)$: phase response

$$= \underbrace{|H_d(\omega_0)|}_{\text{attenuate or amplify}}\, e^{j(\omega_0 n + \underbrace{\angle H_d(\omega_0)}_{\text{phase shift}})}$$

any new frequency component created
⇓
non-linearity

$$H_d(\omega) = H(z)\Big|_{z=e^{j\omega}}$$ for causal systems

$$\boxed{Y_d(\omega) = H_d(\omega)\,X_d(\omega)}$$

$$y[n] = \sum_{\ell=-\infty}^{+\infty} h[\ell]\,x[n-\ell] = \sum_{\ell=-\infty}^{+\infty} h[\ell]\,e^{j\omega_0(n-\ell)} = \underbrace{e^{j\omega_0 n}}_{\text{input funct.}} \underbrace{\sum_{\ell=-\infty}^{+\infty} h[\ell]\,e^{-j\omega_0 \ell}}_{\text{DTFT of unit pulse funct. } H_d(\omega_0)}$$

complex result of real input

$\cos(\omega_0 n) \longrightarrow \boxed{H_d(\omega)} \longrightarrow \cancel{H_d(\omega_0)\cos(\omega_0 n)}$

$|H_d(\omega_0)| \cos(\omega_0 n + \angle H_d(\omega_0))$

$\sin(\omega_0 n) \longrightarrow \boxed{H_d(\omega)} \longrightarrow |H_d(\omega_0)| \sin(\omega_0 n + \angle H_d(\omega_0))$

if system real

$\cancel{H_d(\omega_0)\cos(\omega_0 n)}$ ✗ wrong

$\cancel{H_d(\omega_0)\sin(\omega_0 n)}$ ✗

$\cos(\omega_0 n) = \frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n}) \longrightarrow \frac{1}{2}(H_d(\omega_0)e^{j\omega_0 n} + H_d(-\omega_0)e^{-j\omega_0 n})$

$\sin(\omega_0 n) = \frac{1}{2j}(\; \cdot\cdot \; - \; \cdot\cdot \;)$

$= \frac{1}{2}(|H_d(\omega_0)| e^{j(\omega_0 n + \angle H_d(\omega_0))} + \underbrace{|H_d(-\omega_0)|}_{|H_d(\omega_0)|} e^{-j(\omega_0 n \underbrace{- \angle H_d(-\omega_0)}_{\angle H_d(\omega_0)})})$

real $h[n]$ real valued

$|H_d(-\omega_0)| = |H_d(\omega_0)|$

$-\angle H_d(-\omega_0) = \angle H_d(\omega_0)$

Real $\frac{1}{2}|H_d(\omega_0)| \underbrace{(e^{j(\omega_0 n + \angle H_d(\omega_0))} + e^{-j(\omega_0 n + \angle H_d(\omega_0))})}_{2\cos(\omega_0 n + \angle H_d(\omega_0))}$

Wed 4 Oct Recitation

Yu-Jeh

What is DTFT?

- Eigensignals and the LSI system
- Fourier synthesis and Fourier Analysis
- CTFT, Laplace vs DTFT, z-transform

$$x[n] = Ae^{j\omega n} \longrightarrow \boxed{LSI} \longrightarrow y[n] = H(\omega)Ae^{j\omega n}$$

$$Ax = \lambda x$$

Fourier Synthesis

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$x[n] = \begin{array}{ccccccc} n=0 & n=1 & & & & & \\ [\,0 & 3 & 2 & 1 & 4 & \cdots &] \end{array}$$

$$e^{j\omega_0}[n] = [e^{j\omega_0 0}\ e^{j\omega_0 1}\ \cdots\ ]$$

$$e^{j\omega_1}[n] = [e^{j\omega_1 0}\ e^{j\omega_1 1}\ \cdots\ ]$$

$$\vdots$$

Fourier Analysis

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

CTFT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t}\, d\Omega$$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$\Omega = [Hz]$

$\omega = [rad/sample]$

Laplace

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

set $s = j\Omega \Rightarrow$ CTFT

z-transform

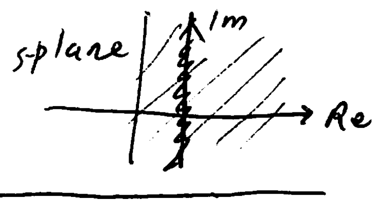
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

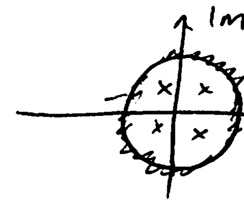
set $z = e^{j\omega} \Rightarrow$ DTFT

Why is ROC always a circle?

CTFT and DTFT

"special cases" for $\mathcal{L}$, Z





$z = e^{j\omega}$

Problem 1 A DTFT is said to not exist for a sequencee $x[n]$ whenever the DTFT $X(\omega)$ does not converge

$$|X(\omega)| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| < \infty$$

$\Rightarrow |\Sigma\, x[n]|$ converges, $X(\omega)$ exists

Determine if DTFT exists

(i) $x[n] = (0.2)^n \cos\left(\frac{\pi}{3}n\right)u[n]$

(ii) $x[n] = u[n]$

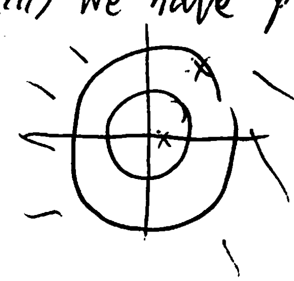
(iii) $X(z) = \dfrac{z}{(z-0.2)(z-2e^{j\pi/3})}$

(i) From observation $\Rightarrow$ yes!

(ii) No! $|\Sigma\, u[n]|$ does not converge

(iii) We have poles at $z = 0.2$ and $z = 2e^{j\pi/3}$

Why ROC outside? $\rightarrow$



$\Rightarrow x[n]$ does not converge

$\Rightarrow$ No DTFT!

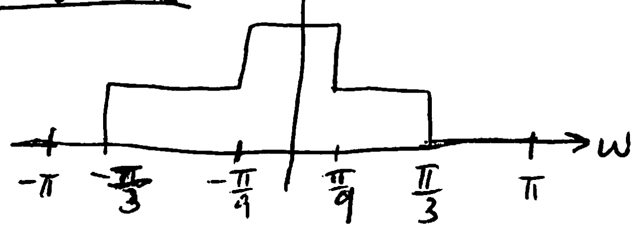
(iv) $x[n] = (h_1 * h_2)[n]$

$$\begin{cases} h_1[n] = (0.5)^n u[n] \\ h_2[n] = (e^{j\pi/4})^n u[n] \end{cases}$$

$$H_1(z) H_2(z) = \frac{z}{z-0.5} \frac{z}{z-e^{j\pi/4}}$$

$\Rightarrow$ No DTFT ROC: $z > |e^{j\pi/4}|$

<u>Problem 2</u> Given, $X(\omega)$

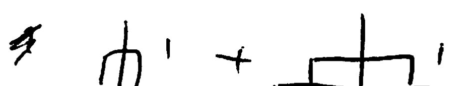


determine $x[n]$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \left[\int_{-\pi/3}^{-\pi/9} 1 e^{j\omega n} d\omega + \int_{-\pi/9}^{\pi/9} 2 e^{j\omega n} d\omega \right.$$

$$\left. + \int_{\pi/9}^{\pi/3} 1 e^{j\omega n} d\omega \right]$$

DTFT{x+y} = DTFT{x} + DTFT{y}



$x_1[n] = \frac{1}{9} \operatorname{sinc}(\frac{\pi}{9} n)$
$x_2[n] = \frac{1}{3} \sin(\frac{\pi}{3} n)$
$\Rightarrow x[n] = x_1[n] + x_2[n]$

<u>Problem 3</u> find DTFT of $x[n]$

(a) $x[n] = u[n] - u[n-6]$

(b) $x[n] = 2^n u[-n]$

stability?
unit circle?

(a) $X(\omega) = \sum_{n=-\infty}^{\infty} (u[n] - u[n-6]) e^{-j\omega n}$

$= \sum_{n=0}^{5} e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-j\omega n} - \sum_{n=6}^{\infty} e^{-j\omega n}$

$= \frac{1}{1-e^{-j\omega}} - \frac{e^{-j\omega 6}}{1-e^{-j\omega}}$

$\sum_{n=-\infty}^{0} 2^n e^{-j\omega n} = \sum_{n=0}^{\infty} 2^{-n} e^{j\omega n} \Rightarrow \left(\frac{e^{j\omega}}{2}\right)$

5% from HW6 on Exam

Fri 6 October lecture

Liang

must be stable system

$$e^{j\omega_0 n} \rightarrow \boxed{H_d(\omega)} \rightarrow |H_d(\omega_0)| e^{j(\omega_0 n + \angle H_d(\omega_0))}$$

$$\cos(\omega_0 n) \rightarrow \boxed{H_d(\omega)} \rightarrow \frac{1}{2}|H_d(\omega_0)| e^{j(\omega_0 n + \angle H_d(\omega_0))} + \frac{1}{2}|H_d(-\omega_0)| e^{-j(\omega_0 n - \angle H_d(-\omega_0))}$$

$$\sin(\omega_0 n) \rightarrow \boxed{H_d(\omega)} \rightarrow \frac{1}{2j}|H_d(\omega_0)| e^{j(\omega_0 n + \angle H_d(\omega_0))} - \frac{1}{2j}|H_d(-\omega_0)| e^{-j(\omega_0 n - \angle H_d(-\omega_0))}$$

special case, real system

$$\cos \omega_0 n \longrightarrow |H_d(\omega_0)| \cos(\omega_0 n + \angle H_d(\omega_0))$$

$$\sin \omega_0 n \longrightarrow |H_d(\omega_0)| \sin(\omega_0 n + \angle H_d(\omega_0))$$

* LSI test
* stability
* "realness" test

$h[n]$ — every value real

LCCDE: $y[n] + \sum_{\ell=1}^{N} a_\ell y[n-\ell] = \sum_{\ell=0}^{N} b_\ell x[n-\ell]$ — a, b real

$H(z)$: $$H(z) = \frac{\sum_{\ell=0}^{N} b_\ell z^{-\ell}}{1 + \sum_{\ell=0}^{N} a_\ell z^{-\ell}}$$ poles: real or conjugate pairs

$H_d(\omega)$ — magnitude even symmetry, phase odd symmetry

Example

$y[n] = x[n] + 2x[n-1]$

LSI

FIR filter

always stable

no poles

real

$x[n] = \cos\left(\frac{\pi}{2}(n-1)\right) + 1$

$\omega_1 = 0, \; \omega_2 = \frac{\pi}{2}$

$H(z) = 1 + 2z^{-1}$

$H_d(\omega) = 1 + 2e^{-j\omega}$

$y[n] = |H_d(\frac{\pi}{2})| \cos\left(\frac{\pi}{2}(n-1) + \angle H_d(\pi/2)\right) + 3$

Ex $\quad H_d(\omega) = \cos\omega \, e^{j\pi\cos\omega}$

$x[n] = \cos\left(\frac{\pi}{4}n + 5°\right) + e^{j\frac{\pi}{2}n}$

$y[n] = \frac{1}{2}|H_d(\frac{\pi}{4})| e^{+j(\frac{\pi}{4}n + 5° + \angle H_d(\frac{\pi}{4}))}$

$\quad + \frac{1}{2}|H_d(-\frac{\pi}{4})| e^{-j(\frac{\pi}{4}n + 5° - \angle H_d(\frac{\pi}{4}))}$

$\quad + \dots$

Ex $\quad H(z) = \frac{z}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2}$

$x[n] = \sin\left(\frac{\pi}{4}n + 23°\right) + \underbrace{(-1)^n}_{\cos n\pi} + j^n + e^{j\pi/3}$

$\omega_1 = \frac{\pi}{4}, \; \omega_2 = \pi, \; \omega_3 = \pi/2, \; \omega_4 = 0$

$H(z) = \frac{z}{z-2}, \quad |z| > 2.$

Same $x[n]$

$x[n] = \cos(\frac{\pi}{4}n), \; h[n] = 2^n u[n]$

$y[n] = x[n] * h[n]$

$= \sum_{l=0}^{\infty} 2^l \cos(\frac{\pi}{4}(n-l))$

Mon

ECE 310 9 October Lecture   Michael

Regular lecture cancelled. TA "office hour" (open)

\# 4, HW6: $x[n] = 3 + \cos\left(\frac{\pi}{4}n + 10°\right) + \sin\left(\frac{\pi}{3}n + 25°\right)$

Real LSI $y[n] = 9 + 2\sin\left(\frac{\pi}{4} + 10°\right)$

$\tilde{x}[n] = 5 + 2\sin\left(\frac{\pi}{4}n + 15°\right) + 10\cos\left(4\frac{\pi}{3}n - 25°\right)$

$\boxed{\tilde{y}[n] = ? = 15 + 4\sin\left(\frac{\pi}{4}n - 75°\right)}$

$\cos(\omega_0 n + \phi) \rightarrow \boxed{H_d(\omega)} \rightarrow |H_d(\omega_0)|\cos(\omega_0 n + \phi + \measuredangle H_d(\omega_0))$

$H_d\left(\frac{\pi}{3}\right) = 0$

$3 \rightarrow \boxed{H_d(\omega)} \rightarrow 9$

$H_d(0) = 3$

$\cos\left(\frac{\pi}{4}n + 10°\right) \rightarrow \boxed{H_d(\omega)} \rightarrow 2\sin\left(\frac{\pi}{4}n + 10°\right)$

$|H_d\left(\frac{\pi}{4}\right)| = 2$

$\measuredangle H_d\left(\frac{\pi}{4}\right) = -90°$

$e^{j\omega_0 n} \rightarrow \boxed{H_d(\omega)} \rightarrow H_d(\omega_0)e^{j\omega_0 n}$

$y(n) - y(n-1) = x(n)$

Real: $H_d(\omega) = H_d^*(-\omega)$

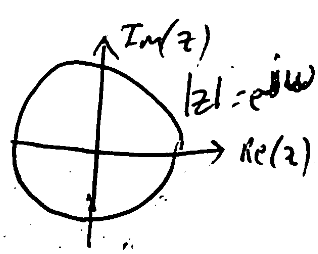
① $|H_d(\omega)| = |H_d(-\omega)|$

② $\measuredangle H_d(\omega) = -\measuredangle H_d(-\omega)$

$H_d(\omega) = \omega e^{j\sin(\omega)}$

$y(n) - jy(n-1) = x[n]$ $\quad$ $H_d^*(-\omega) = -\omega e^{-j\sin(\omega)} = -\omega e^{j\sin(\omega)} \neq H_d(\omega)$

$y(n) - je^{-j\omega}Y(\omega) = X(\omega) \Rightarrow H(\omega) = \frac{1}{1 - je^{-j\omega}}$ $\quad$ $H^*(-\omega) = \frac{1}{1 + je^{-j\omega}}$



$a^n u[n]$

$\sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}$

$|z| > a$

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

#2 $-1 = e^{j\pi} = e^{j\pi}$

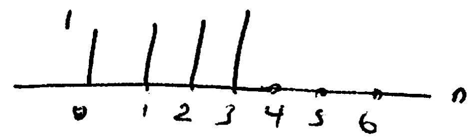
$(-1)^n = e^{-j\pi n}$

$H_d(-\pi)e^{-j\pi n}$

DTFT only if z-transform contains unit circle

→ substitute

$x[n] = u[n] - u[n-4]$



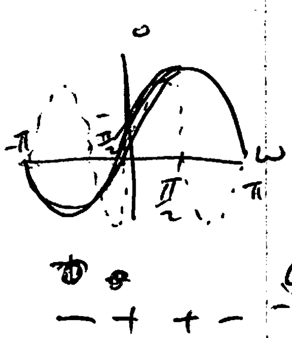
$$X_d(\omega) = \sum_{n=0}^{3} e^{-j\omega n} = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega}$$

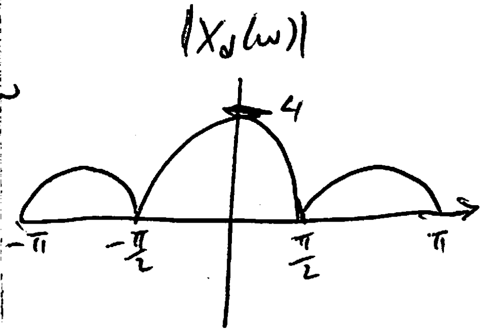
$$= \frac{1-e^{-j4\omega}}{1-e^{-j\omega}} = \frac{e^{-j2\omega}\left(e^{-j2\omega} - e^{-j2\omega}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} e^{-j\omega/2}\right)}$$

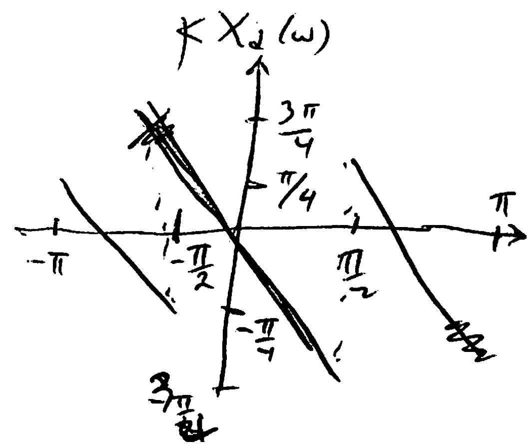
$$= e^{-j\omega 1.5} \frac{2j\sin(2\omega)}{2j\sin(\omega/2)} = \boxed{e^{-j\omega 1.5}\frac{\sin(2\omega)}{\sin(\omega/2)}}$$

$$X_d(\omega) = e^{-j\omega 1.5}\frac{\sin(2\omega)}{\sin(\omega/2)}$$

$$|X_d(\omega)| = \left|\frac{\sin(2\omega)}{\sin(\omega/2)}\right|, \quad \measuredangle X_d(\omega) = \begin{cases} -1.5\omega, & \frac{\sin(2\omega)}{\sin(\omega/2)} > 0 \\ -1.5\omega \pm \pi, & \frac{\sin(2\omega)}{\sin(\omega/2)} < 0 \end{cases}$$







scln wrong? Section G

Quiz 5, #2:

DTFT of $x[n] = \cos(\omega_0 n + \phi_0)$

$= \frac{1}{2}[e^{j\omega_0 n} e^{j\phi_0} + e^{-j\omega_0 n} e^{-j\phi_0}]$

$= \frac{1}{2} e^{j\phi_0} \mathcal{F}(e^{j\omega_0 n}) + \frac{1}{2} e^{-j\phi_0} \mathcal{F}[e^{-j\omega_0 n}]$

$= \frac{1}{2} e^{j\phi_0} \delta(\omega - \omega_0) + \frac{1}{2} e^{-j\phi_0} \delta(\omega + \omega_0)$

$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega) e^{j\omega n} \, d\omega$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\omega - \omega_0) e^{j\omega n} \, d\omega = \frac{1}{2\pi} e^{j\omega_0 n}$

$(2\pi) = e^{j\omega_0 n}$

$\cos(\Omega_0 t + \phi_0)$

$= \cos\left(\Omega_0\left(t + \frac{\phi_0}{\Omega_0}\right)\right)$

cheatsheet
definitions
transform pairs
geometric sums

HKN Review Session   Corey + Ben

Martin Vesserly textbook

LSIC systems

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

causal : $y[n] = y[n-1] + x[n]$

non-causal: $y[n-1] = -y[n] + x[n]$

BIBO stability

$h[n]$ absolutely summable

pz plots

Impulse Response

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{else} \end{cases}$$

$$\delta(t) = \begin{cases} \infty & t=0 \\ 0 & \text{else} \end{cases}$$

(Toeplix matrix)

→ Lab 2 convolutional sum algebraically

Table method

$n: L$, $x: M$, $y: L+M-1$

z-Transform

DTFT special case of z-transform

one-sided only: $\sum_{n=0}^{\infty}$

delay property

DTFT is only defined if the ROC contains the unit circle

$H_d(\omega) = H(z)\big|_{z=e^{j\omega}}$

DTFT ← repeats $-\pi$ to π

pole on unit circle marginally ~~un~~stable → unstable

symmetry

real: magnitude even, phase odd

Frequency Response

For real-valued systems only:

$x[n] = \cos(\omega_0 n + \theta) \Rightarrow y[n] = |H_d(\omega_0)| \cos(\omega_0 n + \theta + \measuredangle H_d(\omega_0))$

Magnitude and Phase Response

~~LSIC Examples~~ abs inside indexes

$y[n] = x^2[n]$ non-linear, shift-invariant, causal

$y[n] = x[|n|]$ linear, shift-varying, non-causal

$y[n] = 3^{-|n|} \log(\underbrace{|x[n]|}_{\text{outside abs}} + 1)$ non-linear, shift-varying, causal

Impulse Response and Convolution Examples

h \ x	6	12	-3	0	15	3	-9	0
$\frac{1}{3}$	2	4	-1	0	5	1	-3	0
$\frac{1}{3}$	2	4	-1	0	5	1	-3	0
$\frac{1}{3}$	2	4	-1	0	5	1	-3	0

$y[n] = [2, 6, 5, 3, 4, 6, 3, -2, -3]$

running avarage

$h[n] = \frac{1}{2} y_1[n] - y_2[n]$

$\left(\frac{1}{2}\right)^n u[n] * [\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}] \rightarrow \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$

BIBO Stability Example

unbounded outputs?

$$H(z) = \frac{1}{1+z^{-2}} \longrightarrow z = \pm j$$

$\cos(\frac{\pi}{2} n) u[n]$ ✓

$\delta[n]$ ✗

~~$u[n]$~~

$n[n]$ ✗

$e^{j\frac{\pi}{2}n} u[n]$ ✓

$$\frac{A}{(1-j z^{-1})} + \frac{B}{(1+j z^{-1})}$$

$$A + Aj z^{-1} + B - Bj z^{-1}$$

$A + B = 1, \quad (A-B)j = 1$

$A - B = -j$

$A = \frac{1-j}{2} \quad B = \frac{1+j}{2}$

phase splitting

$$1 - e^{-j\omega}$$

$$e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)$$

$$e^{-j\omega/2}\left(2j\sin\left(\frac{\omega}{2}\right)\right)$$

$$e^{-j\omega/2}\, j 2\sin\left(\frac{\omega}{2}\right)$$

$$e^{-j\omega/2} e^{j\pi/2}\, 2\sin(\omega/2)$$

$$e^{j(-\omega/2 + \frac{\pi}{2})}\, 2\sin\left(\frac{\omega}{2}\right)$$

$$\boxed{\begin{array}{ll} -\frac{\omega}{2} + \frac{\pi}{2} & 2\sin\omega/2 > 0 \\ -\frac{\omega}{2} + \frac{\pi}{2} + \pi & 2\sin\omega/2 < 0 \end{array}}$$

Wed 11 Oct Lecture — Liang

Review for Exam

Control systems - marginal stability

Signal processing — unstable

LSI causal BIBO

$h[n]$ unbounded

~~a) repeated poles in UC x~~

$H(z)$ b) outside UC, ✓

c) on UC, repeated pole ✓

$\sum |h[n]|$ unbounded

$h[n]$ bounded

single pole(s) on UC

$u[n], (e^{j\pi/2})^n u[n]$

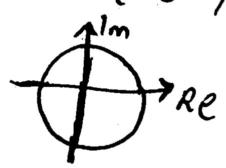
$\boxed{\begin{array}{l} H(z) \\ \text{DTFT} \end{array}}$ $X_d(\omega) = X(z)\big|_{z=e^{j\omega}}$ if ROC contains UC

convolution

but DTFT can still exist if ROC $\neq$ contain UC

$\cos(\omega_0 n) \longleftrightarrow \pi\left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right), |\omega| < \pi$

$x[n] \longrightarrow$



Sampling

Sampling

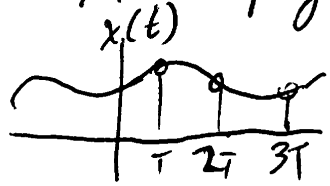
(A/D)

$\xrightarrow[X(\Omega)]{X_a(t)}$ T $\xrightarrow[X_d(\omega)]{x[n]}$

T: sampling interval

1/T: sampling frequency

T-domain: $x[n] = x_a(nT)$



how do we ensure information losslessness?

Under what condition is this a lossless process?

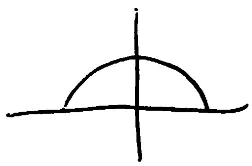
Freq-domain:

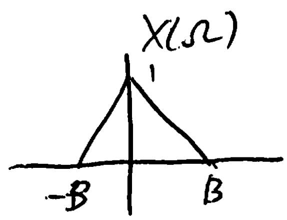
$$\boxed{X_d(\omega) = \frac{1}{T}\sum_{\ell=-\infty}^{+\infty} X\left(\frac{\omega+2\ell\pi}{T}\right)}$$

$$\ell=0: \; X(\Omega) = X\left(\frac{\omega}{T}\right)$$

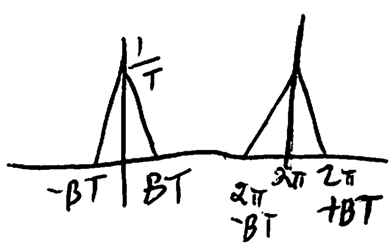
$$X(\Omega) \xrightarrow[\omega=\Omega]{\Omega=\omega} X(\omega) \longrightarrow X\left(\frac{\omega}{T}\right)$$

compressing: frequency scaling





Fourier transform
band-limited



no overlap

a) B is finite

b) $2\pi - BT \geq BT$

$\frac{1}{T} \geq \frac{B}{\pi}$

$2BT \leq 2\pi \rightarrow \boxed{T \leq \pi/B}$ $\frac{2\pi}{2B}$

Nyquist samplings criterion

causal/rightsided
acausal from Z-trans.
ROC
geometric seq.
defn.

Michael

W 11 October Recitation

T/F DTFT only for real-valued signals

T/F $x[n]$ is bandlimited, then $\exists\ \omega_{max}$ s.t. $|X_d(\omega)| = 0\ \forall \omega > \omega_{max}$

T/F DTFT of $x[n] = \cos(\frac{\pi}{2} n)$, $-\infty < n < \infty$ is $X_d(\omega) = \pi\left(\delta(\omega - \frac{\pi}{2}) + \delta(\omega + \frac{\pi}{2})\right)$

T/F For <u>any</u> system, the I/O relationship is completely determined by $h[n]$

$$X_d(\omega) = 2\pi\delta(\omega - \omega_0)$$

---

if given system stable

$$H(z) = \frac{z(z - \frac{3}{5})}{(z - \frac{1}{2})(z - \frac{1}{3})}, \quad \begin{cases} |z| > \frac{1}{2}, & \text{causal} \\ \frac{1}{3} < |z| < \frac{1}{2}, & \text{non causal} \\ |z| < \frac{1}{3}, & \text{non causal} \end{cases}$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{z(z - \frac{3}{5})}{(z - \frac{1}{2})(z - \frac{1}{3})}$$

$$Y(z)\left[(z - \tfrac{1}{2})(z - \tfrac{1}{3})\right] = X(z)\left[(z)(z - \tfrac{3}{5})\right]$$

$$Y(z)\left[z^2 - \tfrac{5}{6}z + \tfrac{1}{6}\right] = X(z)\left[z^2 - \tfrac{3}{5}z\right]$$

$$y[n+2] - \tfrac{5}{6}y[n+1] + \tfrac{1}{6}y[n] = x[n+2] - \tfrac{3}{5}x[n+1]$$

$m = n+2$: $\boxed{y[m] = x[m] - \frac{3}{5}x[m-1] + \frac{5}{6}y[m-1] - \frac{1}{6}y[n-2]}$

$x[n] \to X(z)$, $x[n] = 0\ \forall n < 0$

$$y[n] = \begin{cases} x[n], & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$Y(z) = \sum_{n=0}^{\infty} y[n] z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-(2n+1)} = z^{-1}\sum_{n=0}^{\infty} x[n] z^{-2n}$$

$$= z^{-1}\sum_{n=0}^{\infty} x[n](z^2)^{-n} = \boxed{z^{-1}X(z^2)}$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$X(z^2) = \sum_{n=0}^{\infty} x[n](z^2)^{-n}$$

Feedback
- HW due day of exam
- No exam $\widehat{\text{past}}$
- No HW drops

eigensequence property

relation btwn cos and sin
HW 6 questions

HW prob 2. → return sin not cosine for HW problem non-real part (maybe?)

$$H_d(\omega) = \omega e^{j\sin(\omega)} = H_d^*(-\omega) = -\omega e^{-j\sin(-\omega)}$$
$$= -\omega e^{j\sin(\omega)}$$

NOT REAL

$$e^{j\omega_0 n} \rightarrow \boxed{H_d(\omega)} \rightarrow \sum_{k=-\infty}^{\infty} e^{j\omega_0(n-k)} h[k]$$
$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} e^{-j\omega_0 k} h[k]$$
$$= \boxed{H_d(\omega)\, e^{j\omega_0 n}}$$

$$\cos(\omega_0 n) = \tfrac{1}{2} e^{j\omega_0 n} + \tfrac{1}{2} e^{-j\omega_0 n}$$

$$\rightarrow \boxed{H_d(\omega)} \rightarrow \tfrac{1}{2} H_d(\omega_0) e^{j\omega_0 n} + \tfrac{1}{2} H_d(-\omega_0) e^{-j\omega_0 n}$$
$$= \tfrac{1}{2}|H_d(\omega_0)|\, e^{j(\omega_0 n + \measuredangle H_d(\omega_0))}$$
$$+ \tfrac{1}{2}|H_d(\omega_0)|\, e^{-j(\omega_0 n - \measuredangle H_d(\omega_0))}$$
$$= \boxed{|H_d(\omega_0)| \cos(\omega_0 n + \measuredangle H_d(\omega_0))}$$

Fri 13 October Lecture — Liang

Sampling

(Ideal A/D converter)

$x_a(t) \xrightarrow{T} x[n]$

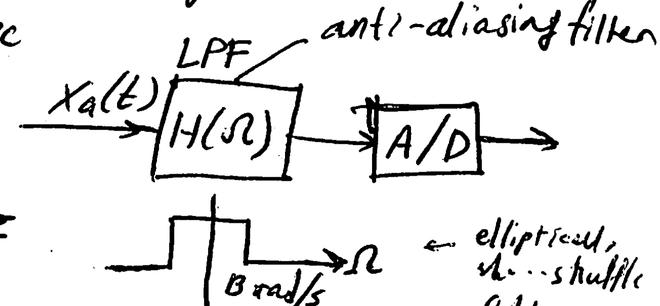
$X_d(\omega) \rightarrow X(\Omega)$

- $x[n] = x_a(nT)$
- $X_d(\omega) = \frac{1}{T} \sum_{\ell=-\infty}^{+\infty} X\left(\frac{\omega + 2\ell\pi}{T}\right)$

amplitude + frequency scaling
create many copies (∞)

a) $x_a(t)$ bandlimited (part of the signal)

$|X(\Omega)| = 0, \ |\Omega| > B$ rad/sec



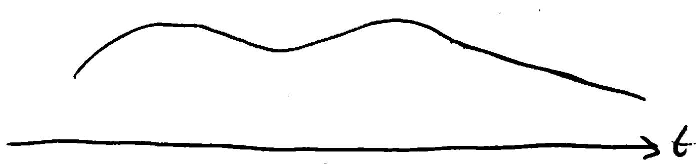
← elliptical, ... shuffle filter

oversampling A/D relaxes some of the burden of the LPF

b) $T \le \frac{\pi}{B} = \frac{2\pi}{2B}$ (requirement of the system)

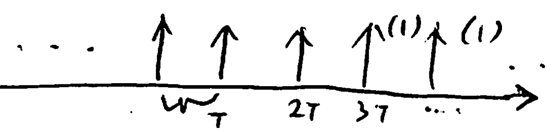
$1/T \ge \frac{2B}{2\pi} = 2F_{max}$

$x_a(t)$



x

$x_a(t)\delta(t-t_0) = x_a(t_0)\delta(t-t_0)$



$x_a(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t-nT) = \sum_{n=-\infty}^{+\infty} x_a(t)\delta(t-nT) = \sum_{n=-\infty}^{+\infty} x_a(nT)\delta(t-nT)$

↓FT

$= \sum_{n=-\infty}^{+\infty} x[n]\delta(t-nT)$

↓FT

LHS

$$x_1(t)\cdot x_2(t) \longrightarrow \frac{1}{2\pi} X_1(\Omega) * X_2(\Omega)$$

$$\frac{1}{2\pi} X(\Omega)$$

$$\sum c_n e^{jn\Omega_0 t}$$

$$= \frac{1}{T}\sum_{\ell=-\infty}^{+\infty} e^{j\ell\frac{2\pi}{T}t}$$

$$\downarrow FT$$

$$\frac{2\pi}{T}\sum_{\ell=-\infty}^{+\infty} \delta\left(\Omega - \ell\frac{2\pi}{T}\right)$$

$$\Omega_0 = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\Omega_0 t}\, dt = \frac{1}{T}$$

$$1 \longrightarrow 2\pi\delta(\Omega)$$

$$e^{j\Omega_0 t} \longrightarrow 2\pi\delta(\Omega - \Omega_0)$$

$$\frac{1}{2\pi} X(\Omega) * \frac{2\pi}{T}\sum_{\ell=-\infty}^{+\infty} \delta\left(\Omega - \frac{2\ell\pi}{T}\right)$$

$$\frac{1}{T}\sum_{\ell=-\infty}^{+\infty} X(\Omega) * \delta\left(\Omega - \frac{2\ell\pi}{T}\right)$$

$$= \frac{1}{T}\sum_{\ell=-\infty}^{+\infty} X\left(\Omega - \frac{2\ell\pi}{T}\right)$$

RHS:

$$\sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n T} = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(\underbrace{\Omega T}_{\omega})n} = X_d(\Omega T)$$

$$= X_d(\omega)$$

$$\delta(t - t_0) \rightarrow 1 e^{-j\Omega t_0}$$

$$\frac{1}{T}\sum_{\ell=-\infty}^{\infty} X\left(\Omega - \frac{2\ell\pi}{T}\right) = X_d(\underbrace{\Omega T}_{\omega})$$

$$\uparrow \frac{\omega}{T}$$

$$\omega = \Omega T$$

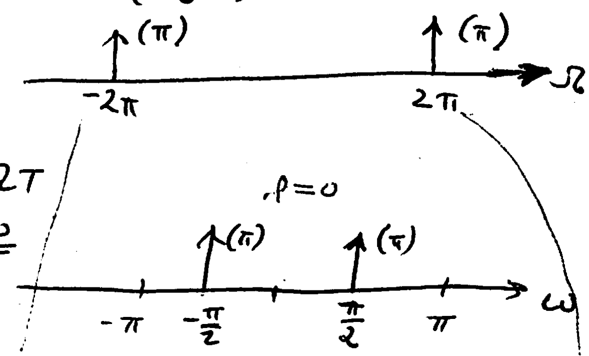
$$\Omega = \omega/T$$

$$\frac{1}{T}\sum_{\ell=-\infty}^{\infty} X\left(\frac{\omega - 2\ell\pi}{T}\right) = X_d(\omega)$$

Example $x_a(t) = \cos(2\pi t)$

$$T < \frac{\pi}{2\pi} = \frac{1}{2}$$

say $T = \frac{1}{4}$, $x[n] = \cos\left(2\pi \frac{n}{4}\right) = \cos\left(\frac{\pi}{2} n\right)$

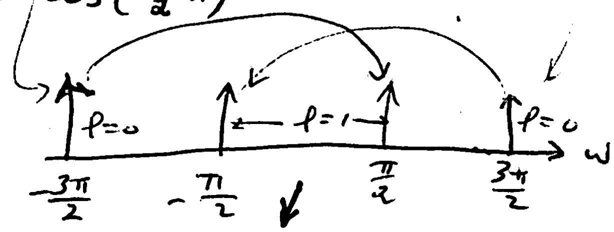


$\omega = \Omega T$

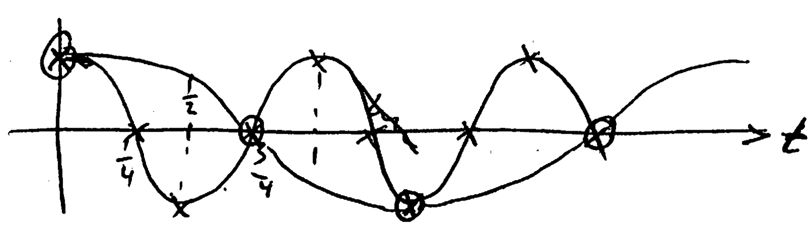
$\Omega = \frac{\omega}{T}$

$T = \frac{3}{4}$. $x[n] = \cos\left(2\pi \frac{3}{4} n\right) = \cos\left(\frac{3\pi}{2} n\right)$

$= \cos\left(\frac{\pi}{2} n\right)$



$$\frac{\frac{\pi}{2}}{\frac{3}{4}} = \frac{2\pi}{3}$$



Liang

Digital (?) filter

Mon 16 Oct Lecture

Discrete Fourier Transform (DFT)

a) Definition

b) Properties

c) Its relation to DTFT

$m \rightarrow \omega \rightarrow \Omega$

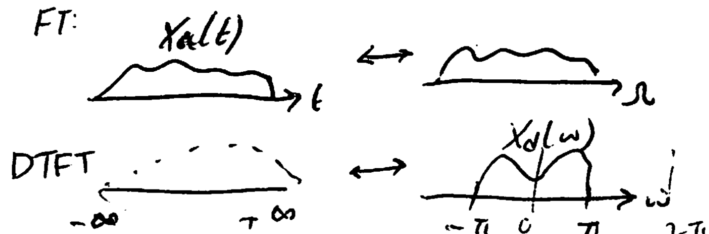
$\Omega \rightarrow \omega \rightarrow m$

DTFT

$x[n] \longleftrightarrow X_d(\omega)$

$$X_d(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) e^{+j\omega n} d\omega$$



$$\omega_n = \frac{2\pi}{N} m$$

$$\{x[n]\}_{n=0}^{N-1}$$

$$\sum_{n=0}^{N-1} x[n] e^{-j\omega n} = X_d(\omega)$$

$$X[m] \triangleq X_d(\omega)\Big|_{\omega = \frac{2\pi}{N} m} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} mn}$$

$$\{x_n\}_{n=0}^{N-1} \longrightarrow \boxed{DFT} \longrightarrow \{X[m]\}_{m=0}^{N-1}$$

DFT $X[m] \triangleq \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} mn}$

$m = 0 \dots N-1$

$n = 0 \dots N-1$

DFT^{-1}: $x[n] = \frac{1}{N} \sum_{m=0}^{N-1} X[m] e^{+j\frac{2\pi}{N} mn}$

$$\frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{\ell=0}^{N-1} x[\ell]e^{-j\frac{2\pi}{N}m\ell}\right)e^{+j\frac{2\pi}{N}mn}$$

$$= \frac{1}{N}\sum_{\ell=0}^{N-1} x[\ell]\underbrace{\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}m(n-\ell)}}_{\frac{1-e^{j\frac{2\pi}{N}(n-\ell)N}}{1-e^{j\frac{2\pi}{N}(n-\ell)}}} =$$

$$\begin{cases} N & n=\ell \\ 0 & n\neq \ell \end{cases} = N\delta[n-\ell]$$

$$= \frac{1}{N}\sum_{\ell=0}^{N-1} x[\ell]N\delta[n-\ell] = x[n]$$

N can be greater than minimum possible value

$$x[n+\ell N] = \frac{1}{N}\sum_{m=0}^{N-1} X[m]e^{+j\frac{2\pi}{N}m(n+\ell N)}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X[m]e^{+j\frac{2\pi}{N}mn}e^{+j\frac{2\pi}{N}m\ell N} = 1$$

$$= x[n]$$

$$x[n+\ell N] = x[n]$$

Liang

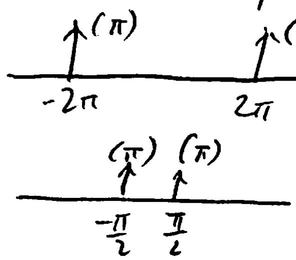
Tues 17 Oct Recitation

$\cos 2\pi t \rightarrow \pi(\delta(\Omega - 2\pi) + \delta(\Omega + 2\pi))$

$t = n\frac{1}{4} \rightarrow x[n] = \cos(\frac{\pi}{2}n) \xrightarrow{DTFT} \pi(\delta(\omega - \frac{\pi}{2}) + \delta(\omega + \frac{\pi}{2}))\ |\omega| < \pi$

$X_d(\omega) = \frac{1}{T}\sum_{\ell=-\infty}^{+\infty} X\left(\frac{\omega + 2\ell\pi}{T}\right)$

$\ell = 0 \rightarrow \frac{1}{T}X\left(\frac{\omega}{T}\right) \rightarrow 4\pi(\delta(4\omega - 2\pi) + \delta(4\omega + 2\pi))$



compress frequency axis
– be careful when doing pictorially

$\boxed{\omega = \Omega T}$

~~Exam~~ HW Question

$x[n] = \cos(\frac{\pi}{4}n)$

$x_a(t) = \cos(\Omega_0 t)$

$\Omega_0 \rightarrow 2\pi$

$t = n\frac{1}{4}, n\frac{3}{4}$

$x[n] = x_a(nT) = \cos(\Omega_0 nT)$

$2\ell\pi + \Omega_0 T = \frac{\pi}{4}$

$\Omega_0 = \frac{\pi}{4}/T$

$X_d(\omega) = \frac{1}{T}\sum_{\ell=-\infty}^{+\infty} X\left(\frac{\omega + 2\ell\pi}{T}\right)$

Exam Question

$x[n]$

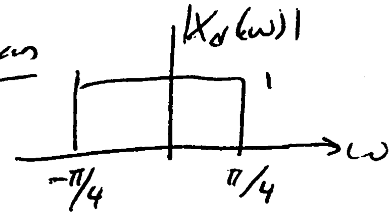
$Y_d(\omega) = X_d(\omega)e^{-j2.5\omega}$

shift must be integer!

$H_d(\omega) = e^{-j2.5\omega}$

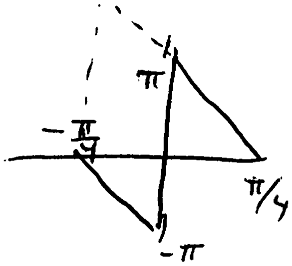
$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j2.5\omega} e^{j\omega n}\, d\omega$

Exam



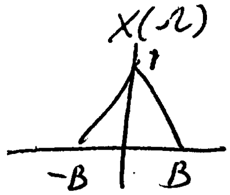
$\to \frac{1}{4}\mathrm{sinc}\left(\frac{\pi}{4}n\right)$

$-\frac{1}{4}\mathrm{sinc}\left(\frac{\pi}{4}(n-4)\right)$

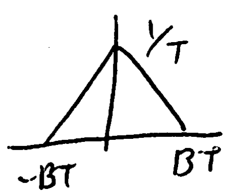


$e^{j(\pi-4\omega)} = -e^{j(\pi-4\omega)}$

HW



$p=0$



$BT < \pi$

Liang

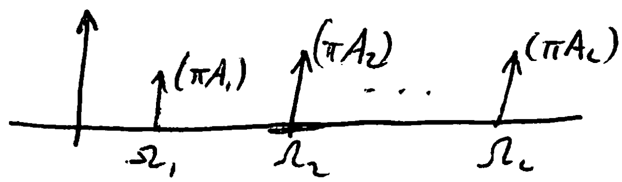
Friday 20 October Lecture

Spectral Estimation

Task: $x_a(t) \longrightarrow \boxed{FT} \longrightarrow X(\Omega)$

$$x_a(t) = \sum_{\ell=1}^{L} A_\ell \cos(\Omega_\ell t)$$

Determine L, Ω_ℓ, A_ℓ



DSP spectral estimation

$$x_a(t) \longrightarrow \boxed{A/D} \longrightarrow \otimes \xrightarrow{\{x[n]\}_{n=0}^{N-1}} \boxed{DFT} \xrightarrow{\{X[m]\}_{m=0}^{N-1}}$$

$$\uparrow w[n]$$

a) Nyquist

b) What is windowing effects?

c) spectral resolution limit

d) $m \to \omega \to \Omega$

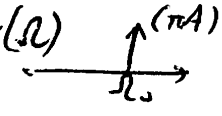
$x_a(t) = A\cos(\Omega_0 t) \longrightarrow \pi A\left(\delta(\Omega+\Omega_0) + \delta(\Omega-\Omega_0)\right)$

$x[n] = A\cos(\Omega_0 nT)$

$= A\cos(\Omega_0 T n)$

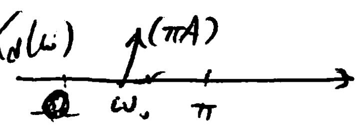
$= A\cos(\omega_0 n) \longrightarrow \pi A\left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right)$

$X(\Omega)$



$\omega_0 = \Omega_0 T$

$X_d(\omega)$

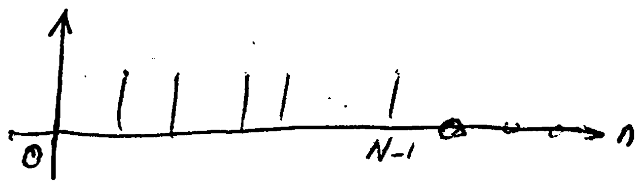


$\omega_\ell \to \Omega_\ell = \frac{\omega_\ell}{T}$

$\hat{x}[n] = x[n]w[n]$

$\hat{X}_d(\omega) = \cancel{X_d(\omega)}\ \frac{1}{2\pi} X_d(\omega) \circledast W_d(\omega)$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\tau)\, W_d(\omega-\tau)\, d\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_d(\tau)\, X_d(\omega-\tau)\, d\tau$



$W_d(\omega) = e^{-j\omega\frac{N-1}{2}} \dfrac{\sin \omega \frac{N}{2}}{\sin \frac{\omega}{2}}$

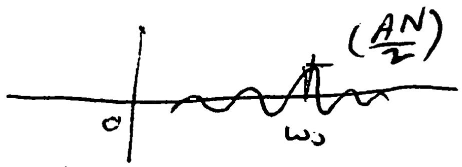
$\sim \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(e^{-j\tau\frac{N-1}{2}} \dfrac{\sin \tau \frac{N}{2}}{\sin \frac{\tau}{2}}$

$\big(\pi A\, \delta(\omega-\tau-\omega_0)\big)d\tau\Big)$

$+\pi A\, \delta(\omega+\tau+\omega_0)$

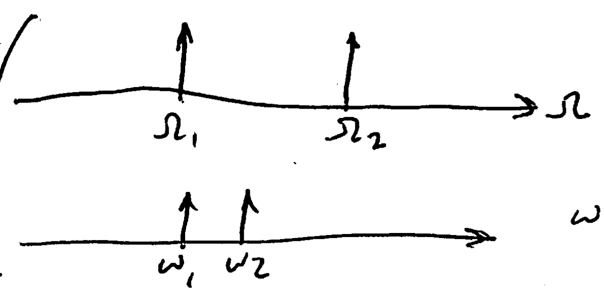
$= \Big(\frac{A}{2} e^{-j(\omega-\omega_0)\frac{N-1}{2}} \cdot \dfrac{\sin(\omega-\omega_0)\frac{N}{2}}{\sin \frac{\omega-\omega_0}{2}}\Big)$

also $-j(\omega+\omega_0)\frac{N-1}{2}$ copy



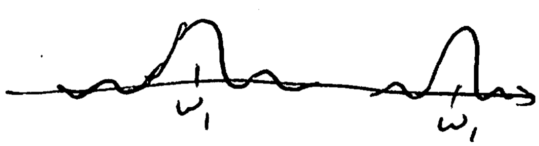
unlimited resolution

no resolution limit



~~FT~~ FT

$\omega_2 - \omega_1 = (\Omega_2 - \Omega_1)T$ DTFT

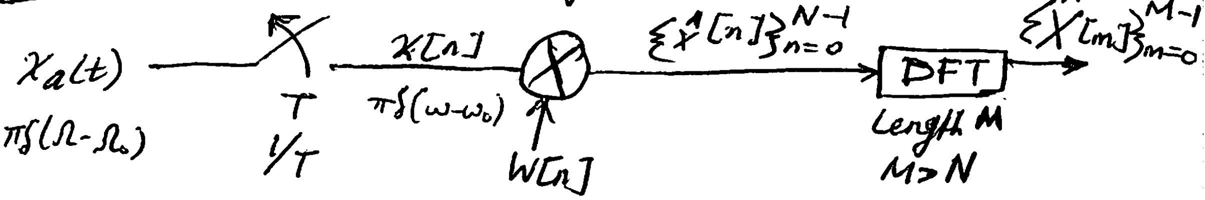


DFT domain

Liang

Mon 23 Oct Lecture

DFT Spectral ~~Estimation~~ Analysis



$\hat{x}[n] = x[n]\,w[n]$

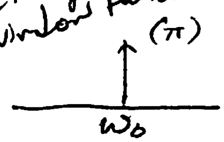
L, A_ℓ, Ω_ℓ

a) $\hat{X}_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\tau)\, W_d(\omega-\tau)\, d\tau$

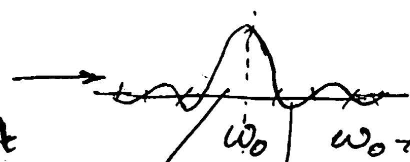
$\pi\delta(\omega-\omega_0) \longrightarrow \frac{1}{2} W_d(\omega-\omega_0)$

$= \frac{1}{2} e^{-j(\omega-\omega_0)\frac{N-1}{2}} \frac{\sin\left((\omega-\omega_0)\frac{N}{2}\right)}{\sin\frac{\omega-\omega_0}{2}}$

Rectangular window function



$\longrightarrow$



rect $\to$ narrowest main lobe

b) Resolution: $\frac{4\pi}{N}$, $\frac{2\pi}{N}$

your choice of metric

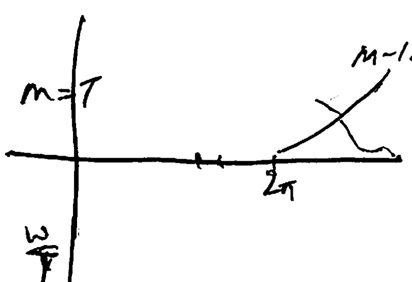
full width half maximum

Hamming window

L, Ω_ℓ, A_ℓ $\quad m_\ell \quad m = 0, \ldots, M-1$

$m_\ell \Rightarrow \omega_\ell = \frac{2\pi}{N} m \to \Omega_\ell : \omega/T = \frac{2\pi}{TM} m_\ell$

$= \frac{2\pi}{MT}(m_\ell - M)!$



$$\omega_0 = \frac{2\pi}{M} m_\ell$$

$\Uparrow$

interleave $\{ x[0], \ldots, x[N-1], \ldots \}$

Liang

T 24 October Recitation

$\{y_n\}_{n=0}^{15}$ $\overbrace{\{x_0, x_1, \ldots, x_{10}, 0, 0, \ldots, 0\}}^{16} \xrightarrow{DFT_{16}} \{Y_m\}_{m=0}^{15}$

$\{z_n\}_{n=0}^{15}$ $\underbrace{\{x_0, x_1, \ldots, x_{10}, 0, 0, \ldots 0, \ldots\}}_{32} \xrightarrow{DFT_{32}} \{Z_m\}_{m=0}^{31}$

$$Y_d(\omega) = \sum_{n=0}^{10} x[n]\, e^{-j\omega n} + \sum_{n=11}^{15} 0 e^{-j\omega n}$$

$$Z_d(\omega) = \sum_{n=0}^{10} x[n]\, e^{-j\omega n} + \sum_{n=11}^{31} 0 e^{-j\omega n}$$

$Y_d(\omega) = Z_d(\omega)$

$Y_m = Y_d\left(\frac{2\pi}{16} m\right)$ $\qquad Z_m = Z_d\left(\frac{2\pi}{32} m\right)$

$\{w_n\}_{n=0}^{31} = \{0, 0 \ldots 0, x_0, x_1, \ldots, x_{10}\}$

$\xrightarrow{DFT_{32}} \{W_m\}_{m=0}^{31}$

$\{Y_0 \ldots Y_{15}\}$

$\{Z_0 \ldots Z_{32}\}$

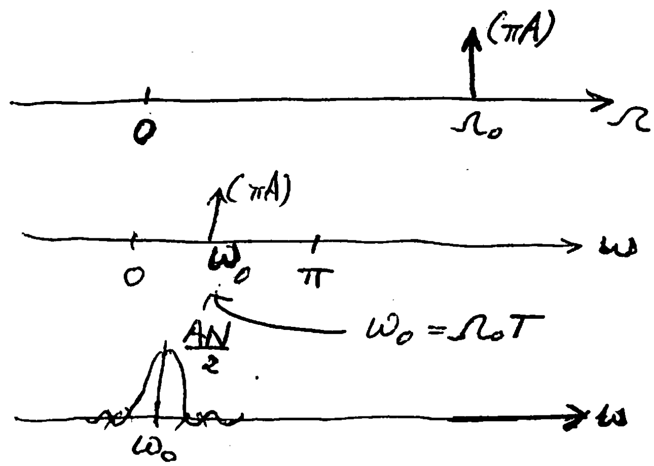
$$\frac{2\pi}{16} m_1 = \frac{2\pi}{32} m_2$$

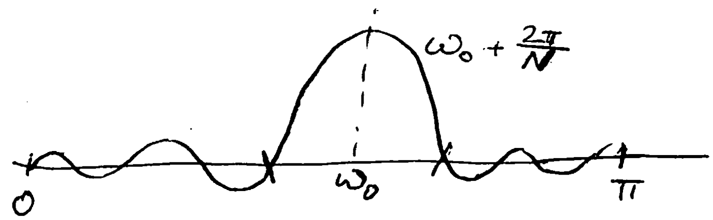
$$\frac{m_1}{16} = \frac{m_2}{32} \Rightarrow m_1 = \frac{m_2}{2}, \quad m_1, m_2 \text{ integers}$$

$W_d(\omega) = Y_d(\omega)\, e^{-j\omega 21}$

$e^{-j\frac{2\pi}{32} \cdot 21}$

$A \cos \Omega_0 t$





$$Y[m] = \frac{1}{2} X[m] e^{-j\frac{2\pi}{4} m 1}$$

$$x[n] = [1, -1, 2, 3, -3, 0, 0, 0]$$

$y[\,]=$ $\quad\downarrow$ DFT

$X[m]$

$$Y[m] = X[m] e^{-j\frac{2\pi}{6} m n_0}$$

$$= X[m] e^{-j\frac{2\pi}{8} m \frac{8}{6} n_0} = e^{-j\frac{2\pi}{8} m}$$

$$y[n] = x[\langle n-4 \rangle_8]$$

$$x[\langle n \pm k \rangle_N] \rightarrow x[m] e^{\pm j \frac{2\pi}{N} m k}$$

$\Omega_0 \rightarrow \omega_0 = \Omega_0 T$

$$\omega_0 = \frac{2\pi}{m} m_0$$

Liang

Wed 25 October Lecture

Fast Fourier Transform (FFT)

$$\{x_n\}_{n=0}^{N-1} \xleftrightarrow{DFT} \{X_m\}_{m=0}^{N-1}$$

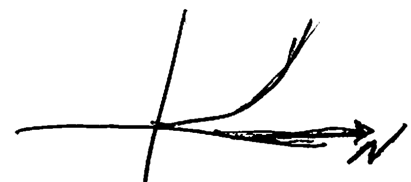
$$X_m = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}mn} \qquad m = 0, \ldots N-1$$

Divide and Conquer strategy

$$x_n = \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{+j\frac{2\pi}{N}mn}$$

$\times$: N^2 complexity ~~time~~

$+$: $(N-1)N$



quadratic

10^9 ~~speed~~ improvement in past 4 decades for computer

FFT: (Decimation-in-time, DIT, Radix-2) $N = 2^r$

$N^2 \longrightarrow N \log_2 N$

$N = 2^{14}$

~~$N^2 = 2^{28}$~~ saving factor > 1100

Cooley-Tukey 1965

$\{x_0, x_1, \ldots, x_{N-1}\}$

$\hookrightarrow \{x_0 \ldots x_{\frac{N}{2}-1}\}$ $\{x_{\frac{N}{2}} \ldots x_{N-1}\}$

DIF

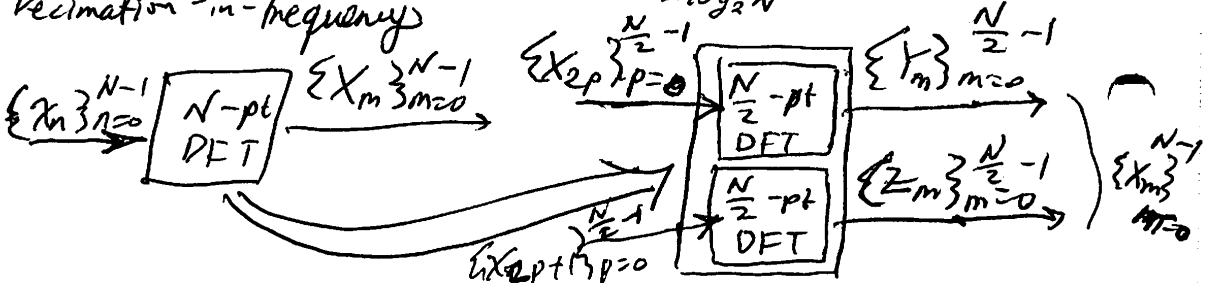
Decimation-in-frequency

$\hookrightarrow \{x_0, x_2, \ldots x_{N-2}\}$ $\{x_1, x_3, x_5, \ldots, x_{N-1}\}$

DIT

$\frac{N}{16} \leftarrow \frac{N}{8} \leftarrow \frac{N}{4} \leftarrow \frac{N}{2} \leftarrow 1$

$r = \log_2 N$



$$X_m = \sum_{p=0}^{\frac{N}{2}-1} x_{2p}\, e^{-j\frac{2\pi}{N}m(2p)} + \sum_{p=0}^{\frac{N}{2}-1} x_{2p+1}\, e^{-j\frac{2\pi}{N}m(2p+1)}$$

for $m = 0 \dots N-1$

$$= \sum_{p=0}^{\frac{N}{2}-1} x_{2p}\, e^{-j\frac{2\pi}{(\frac{N}{2})}mp} + e^{-j\frac{2\pi}{N}m} \sum_{p=0}^{\frac{N}{2}-1} x_{2p+1}\, e^{-j\frac{2\pi}{(\frac{N}{2})}mp}$$

$$= \sum_{p=0}^{\frac{N}{2}-1} x_{2p} W_{\frac{N}{2}}^{mp} + W_N^m \sum_{p=0}^{\frac{N}{2}-1} x_{2p+1} W_{\frac{N}{2}}^{mp}$$

for the twiddle factor $W_N = e^{-j\frac{2\pi}{N}}$

$$X_m = Y_m + W_N^m Z_m \qquad m = 0, \dots \tfrac{N}{2}-1$$

$$X_{m+\frac{N}{2}} = Y_{m+\frac{N}{2}} + W_N^{m+\frac{N}{2}} Z_{m+\frac{N}{2}}$$ DFT repeating

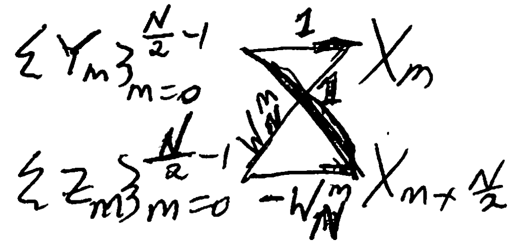
$$= Y_m + W_N^{m+\frac{N}{2}} Z_m$$

$$W_{m+\frac{N}{2}} = e^{-j\frac{2\pi}{N}(m+\frac{N}{2})} = e^{-j\frac{2\pi}{N}m}\, e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}}$$

$$= -e^{-j\frac{2\pi}{N}m} = -W_N^m$$

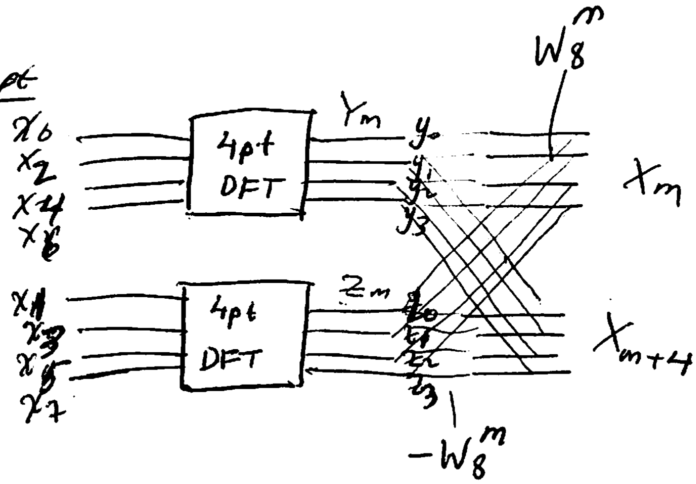
$$X_{m+\frac{N}{2}} = Y_m - W_N^m Z_m$$

Butterfly structure

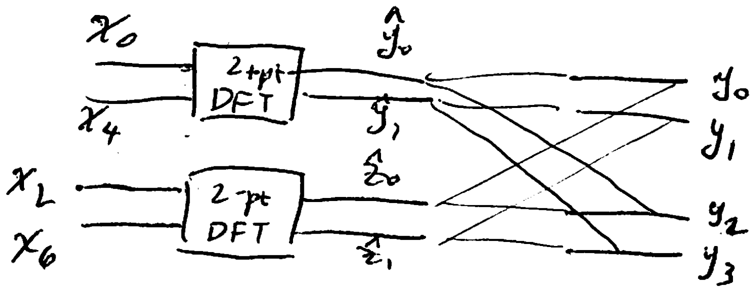


$N \log_2 N$

8 pt



x0
x2
x4
x6
x1
x3
x5
x7
4pt DFT
4pt DFT
Ym
Zm
y0
y1
y2
y3
z0
z1
z2
z3
W8^m
-W8^m
Xm
Xm+4



x0
x4
x2
x6
2-pt DFT
2-pt DFT
ŷ0
ŷ1
ẑ0
ẑ1
y0
y1
y2
y3

Iam
Yuanheng

Wed 25 Oct Recitation

DTFT: $X_d(\omega) = \sum_{m=-\infty}^{\infty} x[n] e^{-j\omega n}$ $x[n]$ any length
$X_d(\omega)$ continuous, periodic by 2π

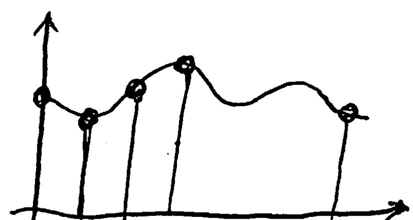
DFT: $X[m] = \sum_{m=0}^{N-1} x[n] e^{-j\frac{2\pi mn}{N}}$ $x[n]$ finite length
$X[m]$ discrete, finite length N

Truncated DTFT: $X_d'(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$

$X[m]$ related $X_d'(\omega)$ by sampling of $\frac{2\pi m}{N}$

$\frac{2\pi}{N} n \cdot 0$ $\quad \frac{2\pi}{N} n \cdot 1$ $\quad \frac{2\pi}{N} n \cdot 2 \cdots$ $\quad \frac{2\pi}{N} n (N-1)$

$X_d'(\omega)$



$\omega = \frac{2\pi m}{N} \in [0, 2\pi]$

$X[m+N] = X[m]$ since $X_d(\omega)$ is 2π periodic ~ modularity

$x[n] e^{j\frac{2\pi}{N}nk} \longleftrightarrow X[\langle m-k\rangle_N]$ freq property

$x[\langle n-k\rangle_N] \longleftrightarrow e^{-j\frac{2\pi k}{N}} X[m]$ modular time shift

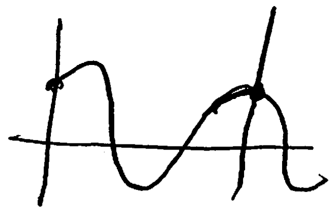
$x[\langle -k\rangle_N] = X[\langle N-k\rangle_N]$

$x[\langle -n\rangle_N] \longleftrightarrow X[\langle N-k\rangle_N]$

HW Prob #8

$x[n] = \cos(\omega_0 n)$

$\{x[n]\}_{n=0}^{15}$

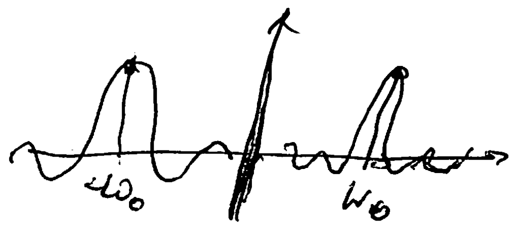


DFT $\longrightarrow$ truncated DTFT

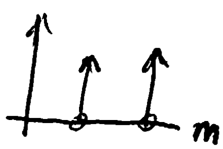
$\text{rect}(\cdots) \cdot x[n]$

$\updownarrow$

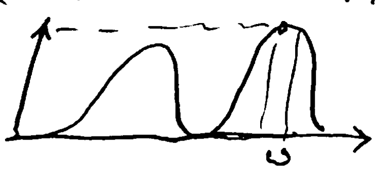
$\text{sinc}(\omega) * X(\omega) \qquad \delta(\omega - \omega_0) * \cdots$



$\sum_{n=0}^{15} x[n]\, e^{-j\omega n} \qquad \frac{\sin}{\sin} + \frac{\sin}{\sin}$



#9 $|X_d(k_0)| \geq |X_d(k)|$



$\frac{2\pi}{N} k$

nearest integer to the peak

#8 $\sum_{n=0}^{15} x[n]\, e^{-j\omega n}$

$\cos(\omega_0 n)\, e^{-j\omega n}$

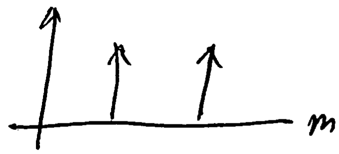
$\frac{1}{2} \sum \left(e^{j\frac{\omega_0}{2} n} + e^{-j\frac{\omega_0 n}{2}} \right) e^{-j\omega n}$

$\sum e^{j\frac{\omega_0}{2}} e^{-j\omega n} + \sum e^{-j\frac{\omega_0}{2}} e^{-j\omega n}$

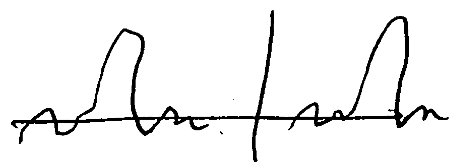
$\frac{0}{0}$ L'hospital rule $\quad \frac{1 - e^{j\frac{\omega_0}{2}\omega N}}{1 - e^{j\frac{\omega_0}{2}\omega}}$

$$e^{j\frac{\omega_0}{4}\omega N} e^{\frac{-j\omega_0}{4}\omega N} - e^{j\frac{\omega_0}{4}\omega N} \cdot e^{j\frac{\omega_0}{4}\omega N}$$

$$e^{-j\frac{\omega_0}{4}\omega N} - e^{j\frac{\omega_0}{4}\omega N}) \cdot \frac{2j}{2j} = 2e^{j\frac{\pi}{2}}$$



DFT



DTFT

Butterfly

$\sum x[n] e^{-j\frac{n\pi}{N}m}$ $\log_2 N \in \mathbb{Z}$

$= \sum x[n] W_N^{nm}$ $W_N = e^{-j\frac{\pi}{N}}$

split into even and odd parts

$e^{-j\frac{(2k+1)\pi}{N}}$ . . . odd

$N = 2^k$

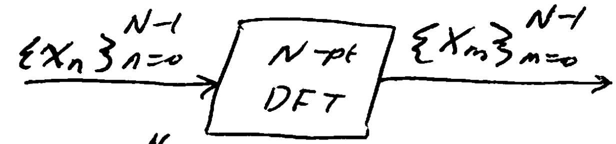
$X[m] = Y[m] + W_N^m Z[m]$

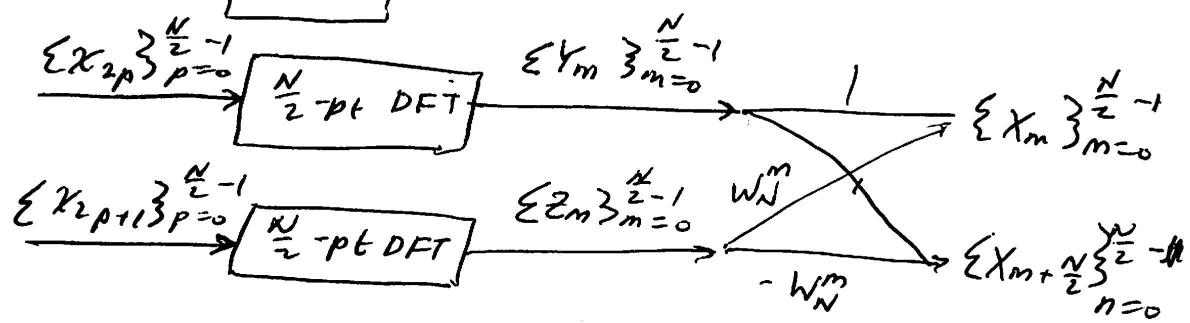
Liang

Fri 27 October Lecture

FFT

FFT, Radix-2, DIT
↑
$N=2^r$





$$\begin{cases} X_m = Y_m + W_N^m Z_m \\ X_{m+\frac{N}{2}} = Y_m - W_N^m Z_m \end{cases}$$

Fast Convolution

$$y[n] = x[n] * h[n] = \sum_{l=-\infty}^{+\infty} x[l]\, h[n-l]$$

Consider $\{y[n]\}_{n=0}^{L-1} = \{x[n]\}_{n=0}^{N-1} \circledast \{h[n]\}_{n=0}^{M-1}$

$L = M+N-1$

$Y(z) = X(z)\, H(z)$

$Y_d(\omega) = X_d(\omega)\, H_d(\omega)$

$\text{DFT}\{x[n]\}_{n=0}^{N-1}$ $\qquad$ $\text{DFT}\{h[n]\}_{n=0}^{M-1}$

$\text{DFT}\{y[n]\}_{n=0}^{L-1}$ $\qquad$ $\frac{2\pi}{L} m$

$$\{y[n]\}_{n=0}^{N-1} = \{x[n]\}_{n=0}^{N-1} \circledast \{h[n]\}_{n=0}^{N-1}$$

$$\{y[n]\}_{n=0}^{N-1} = \sum_{l=0}^{N-1} x[l]\, h[\langle n-l \rangle_N]$$

$$\{Y[m]\}_{m=0}^{N-1} = \{X[m]\cdot H[m]\}_{m=0}^{N-1}$$

$$\{Y[m]\}_{m=0}^{L-1} = \text{DFT}\underbrace{\{\{x[n]\}_{n=0}^{N-1}, \underbrace{0,0,0,\ldots}_{M-1}\}}_{L} \cdot \text{DFT}\underbrace{\{\{h[n]\}_{n=0}^{N-1}, \underbrace{0,0,0}_{N-1}\}}_{L}$$

DFT^{-1}

Procedure

a) Pick a length for the FFT algorithm

$L = M + N - 1 \qquad \hat{L} = 2^r$

b) $\{x[n]\}_{n=0}^{N-1} \rightarrow \{\hat{x}[n]\}_{n=0}^{\hat{L}-1}$, $\{h[n]\}_{n=0}^{M-1} \rightarrow \{\hat{h}[n]\}_{n=0}^{\hat{L}-1}$

c) $FFT\{\hat{x}[n]\}_{n=0}^{\hat{L}-1} = \{\hat{X}[m]\}_{m=0}^{\hat{L}-1}$

$FFT\{\hat{h}[n]\}_{n=0}^{\hat{L}-1} = \{\hat{H}[m]\}_{m=0}^{\hat{L}-1}$

d) $\{\hat{Y}[m]\}_{m=0}^{\hat{L}-1} = \{\hat{X}[m] \cdot \hat{H}[m]\}_{m=0}^{\hat{L}-1}$

e) $FFT^{-1}\{\hat{Y}[m]\}_{m=0}^{\hat{L}-1} \longrightarrow$ remove zeros at end if padded at end

$N \log_2 N \downarrow N^2$

Liang

Mon 30 Oct Lecture

Digital Filter ~~Design~~ Structures

a) FIR filters (Finite Impulse Response)

b) IIR filters (Infinite Impulse Response)

3 things: no feedback, all zeros, length finite

$x[n] \rightarrow \boxed{LSI} \rightarrow y[n]$

$h[n]$

z-transform $\downarrow$ $\quad$ DTFT $\downarrow$ $H_d(\omega)$

$H(z)$ $\quad H_d(\omega) = H(z)\big|_{z=e^{j\omega}}$

BD $\rightarrow H(z)$

BD $\rightarrow$ LCCDE

Block diagram

LCCDE:

$$y[n] = \sum_{k=1}^{N} a_k\, y[n-k] = \sum_{k=0}^{N} b_k\, x[n-k]$$

standard delay form

advance form: $[n-k] \rightarrow [n+k]$

Example:

$$2y[n+3] - \frac{1}{2}y[n] + 3y[n-2] = 4x[n] - \frac{1}{4}x[n-2]$$

order: 5

$\hat{n} = n+3 \rightarrow n = \hat{n}-3$

$$2y[\hat{n}] - \frac{1}{2}y[\hat{n}-3] + 3y[\hat{n}-5] = 4x[\hat{n}-3] - \frac{1}{4}x[\hat{n}-5]$$

Ex

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

$\downarrow$

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$

Transfer

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$\boxed{\text{pulse zeros}}$

FIR (of n-th order) $H(z) = \sum_{k=0}^{N} b_k z^{-k}$

$h[n] = \{b_0, b_1, \dots, b_N\}$ length: $N+1$

FIR: length finite, zeros, feed forward

numb of errors:
lattice
Dirac 1,2

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

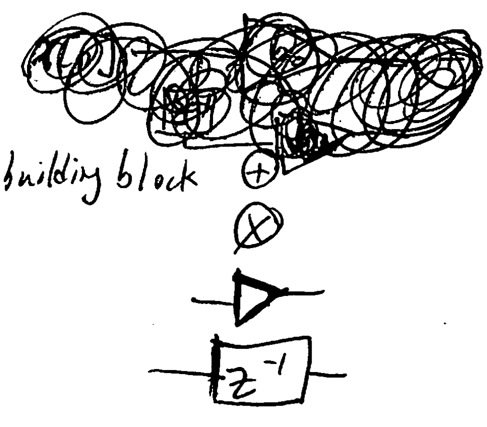
$$h[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

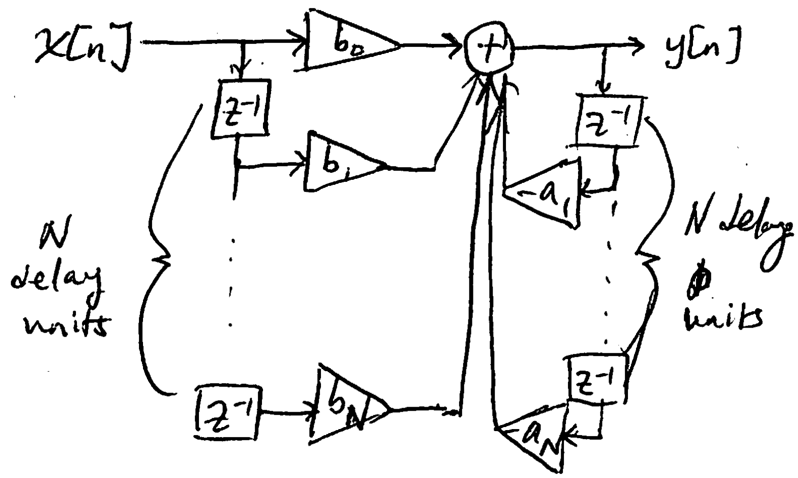
IIR: Infinite length/long, poles, feedback

transfer function in minimum form: no cancel[l]ing

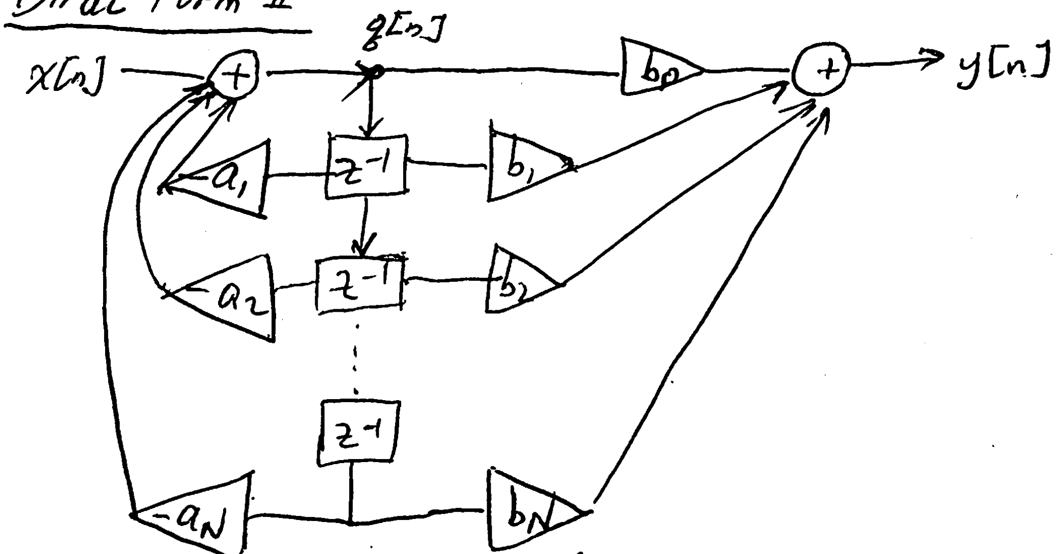
Dirac Form I

building block





Dirac Form II



$$y[n] = \sum_{k=0}^{N} b_k\, g[n-k] = \sum_{\ell=0}^{N} b_\ell\, g[n-\ell]$$

$$g[n] = -\sum_{k=1}^{N} a_k\, g[n-k] + x[n]$$

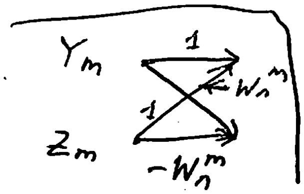
$$y[n] = -\sum_{\ell=0}^{N} b_\ell \sum_{k=1}^{N} a_k\, g[n-\ell-k] + \sum_{\ell=0}^{N} b_\ell\, x[n-\ell]$$

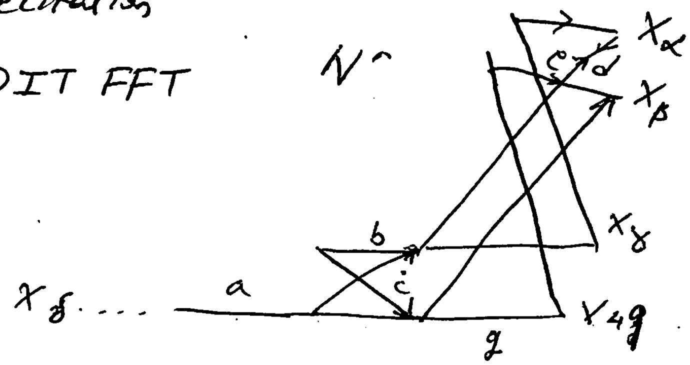
VSLI implementation

$$= -\sum_{k=1}^{N} a_k \underbrace{\sum_{\ell=0}^{N} b_\ell\, g[n-\ell-k]}_{y[n-k]} + \sum_{k=0}^{N} b_k\, x[n-k]$$

Tues 31 Oct Recitation Liang

Radix-2, 64-pt DIT FFT





$\beta = 49 - \frac{N}{2} = 49-32 = 17$!

$\gamma = 49 - \frac{N'}{2} = 49-16 = 33$!

$\alpha = \gamma - \frac{N}{2} = 33-32 = 1$!

$49 \to (110001)_2 \Rightarrow (100011)_2 = 35$

bit reverse

$g = -W_{64}^{17} = -e^{-j\frac{2\pi}{64}\cdot 17}$

$e = 1$

$d = W_{64}^{1}$

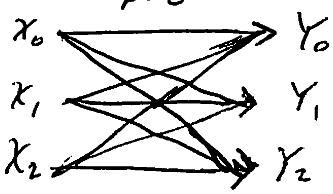
$b = 1, c = 1$

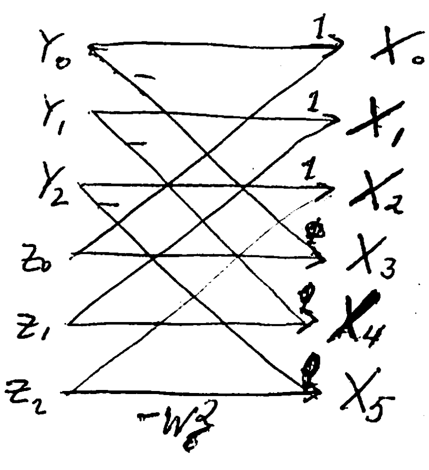
$\{x_0, x_1, x_2, x_3, x_4, x_5\}$

$\{x_0, x_2, x_4\} \xrightarrow{\text{3-pt DFT}} \{Y_0, Y_1, Y_2\}$

$\{x_1, x_3, x_5\} \xrightarrow{\text{3-pt DFT}} \{Z_0, Z_1, Z_2\}$

$Y_m = \sum_{p=0}^{2} x_{2p} W_3^{mp}$

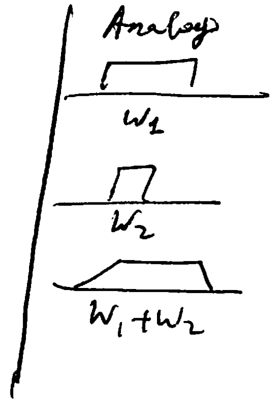




$$\{y_n\}_{n=0}^{L-1} = \{x_n\}_{n=0}^{N-1} * \{h_n\}_{n=0}^{M-1}$$

Length property : $\boxed{L = M + N - 1}$

$$\boxed{Y_d(\omega) = X_d(\omega) \cdot H_d(\omega)}$$



$H_d\left(\frac{2\pi}{M} m\right)$ $\quad \frac{2\pi}{L} m$ minimum sampling

$$Y_d\left(\frac{2\pi}{L} m\right) = X_d\left(\frac{2\pi}{L} m\right) H_d\left(\frac{2\pi}{L} m\right)$$

$$Y[m] = X[m] \cdot H[m]$$

$$\mathrm{DFT}\{y_n\}_{n=0}^{L-1} = \mathrm{DFT}\{\overbrace{\{x_n\}_{n=0}^{N-1} \underbrace{0, 0, 0, \cdots 0}_{L-N}}^{\hat{x}_n}\} \cdot \mathrm{DFT}\{\underbrace{\{h_n\}_{n=0}^{M-1} \underbrace{0, 0, 0, 0}_{L-M}}_{\hat{h}_n}\}$$

$$\{y_n\}_{n=0}^{L-1} = \mathrm{DFT}^{-1}\{ \quad \}$$

$$y_n = \hat{x}_n \circledast \hat{h}_n$$

$$L \rightarrow \hat{L} \text{ (power of 2)} = 2^n$$

Ex $N = 8, M = 17$

$L = 8 + 17 - 1 = 24 \rightarrow \hat{L} = 32$

For DFT-based: pad x_n with 16 zeros
h_n with 7 zeros

For FFT-based: pad x_n with 24 zeros
h_n with 15 zeros

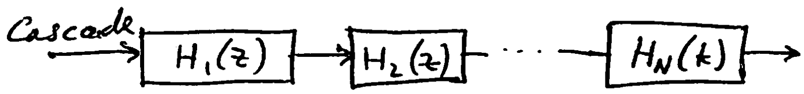
Wed 1 November Lecture Liang

Digital Filter Structures

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$ standard delay form / canonical form

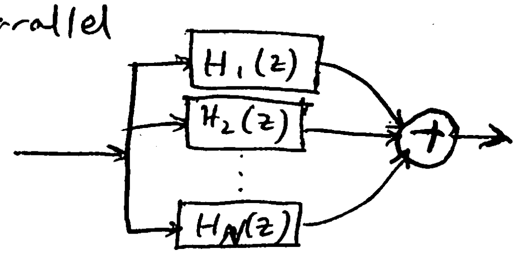
$$= C\,\frac{\prod_{k=1}^{N}(z - z_k^{o})}{\prod_{k=1}^{N}(z - z_k^{p})} = C \prod_{k=1}^{N} \frac{z - z_k^{o}}{z - z_k^{p}}$$

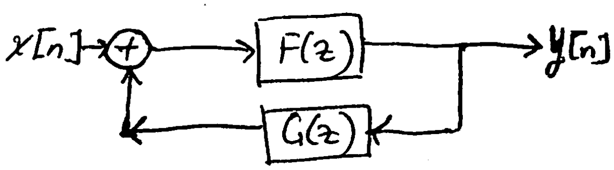
Cascade



$$= \sum_{k=1}^{N} \frac{c_k z}{z - z_k^{p}}$$

parallel





$$Y(z) = F(z)\big(X(z) + G(z)Y(z)\big)$$

$$Y(z)\big(1 - F(z)G(z)\big) = F(z)X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{F(z)}{1 - F(z)G(z)} = H(z)$$

Eg $F(z) = \frac{z}{z-3}$, $|z| > 3$

$G(z) = -g$

$$H(z) = \frac{\frac{z}{z-3}}{1 + g\frac{z}{z-3}} = \frac{z}{z-3+gz} = \frac{z}{(1+g)z - 3}$$

$z^p = \frac{3}{1+g}$

$\left|\frac{3}{1+g}\right| < 1$

$3 < |1+g|$

$g > 2$!

$y[n] = \frac{1}{x[n]}$ not LSI

FIR Filters

$$H(z) = \sum_{k=0}^{N} b_k z^{-k} = \sum_{k=0}^{N} h_k z^{-k}$$

a) All zeros

b) Always stable (ROC entire z-plane, except for $z=0$)

c) Length $(N+1)$, while the order is N!

d) Can implement generalized linear phase (GLP)

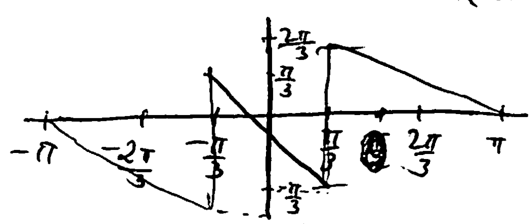
Example

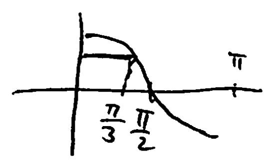
$h[n] = \{1, -1, 1\}$ symmetry

$$H_d(\omega) = 1 - e^{-j\omega} + e^{-j2\omega} = e^{-j\omega}\left(e^{+j\omega} - 1 + e^{-j\omega}\right)$$
$$= e^{-j\omega}(2\cos\omega - 1)$$

$$\angle H_d(\omega) = \begin{cases} -\omega & \{\omega : 2\cos\omega - 1 > 0\} \\ -\omega + \pi & \{\omega, 2\cos\omega - 1 < 0\} \end{cases}$$

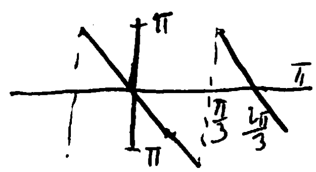
$\cos\omega = \frac{1}{2}$
$\rightarrow \omega = \frac{\pi}{3}$





$H_d(\omega) = |H_d(\omega)| e^{-j\omega m}$

π jump $\Rightarrow$ amplitude making a sign change



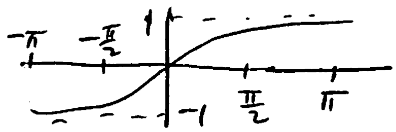
$\theta(\omega) = -3\omega$

linear phase

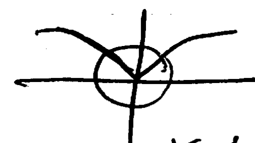
Eg. $h[n] = \{1, -1\}$

$$H_d(\omega) = 1 - e^{-j\omega} = e^{-j\frac{\omega}{2}}\left(e^{+j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)$$

$$= e^{-j\frac{\omega}{2}}\, 2j \sin\frac{\omega}{2} = e^{j\left(\frac{\pi}{2} - \frac{\omega}{2}\right)}\, 2\sin\frac{\omega}{2}$$



$$\angle H_d(\omega) = \begin{cases} \frac{\pi}{2} - \frac{\omega}{2} & 0 \le \omega < \pi \\ -\frac{\pi}{2} - \frac{\omega}{2} & -\pi < \omega \le 0 \end{cases}$$



X LPF

$H_d(\omega) = |H_d(\omega)|\, e^{-j\omega m}$ in practice, difficult to do

$H_d(\omega) = R(\omega)\, e^{-j\omega n}$ Type 1 this is easier

Type 2: $H_d(\omega) = R(\omega)\, e^{j\left(\frac{\pi}{2} - \omega m\right)}$

band notch filter

$$H_d(\omega) = |H_d(\omega)|\, e^{j\angle H_d(\omega)}$$

FIR filters reduce independence

lan

Wed 1 Nov Recitation

FFT

Conv FFT — Fast Fourier Transform $O(N\log N)$

Cyclic Conv. — DFT: $X[m] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi nm}{N}}$ $O(N^2)$

Radix-2 FFT $\{x[n]\}_{2^N}^{0}$

N-point DFT $\rightarrow$ $\frac{N}{2}$ point DFT $\rightarrow \frac{N}{4}$ . . . 2-point DFT

$\frac{N}{2}$ point DFT $\rightarrow \frac{N}{4}$ . . . 2-point DFT

Derivation:
$$X[m] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi mn}{N}}$$

$$= \sum_{\substack{n\in\text{even}\\ 0,2,4,6,\ldots N-2}} x[n] e^{\frac{-j2\pi nm}{N}} + \sum_{\substack{n\in\text{odd}\\ 1,3,5\ldots N-1}} x[n] e^{\frac{-j2\pi nm}{N}}$$

$2r=n$: $\quad 2r+1=n$
$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r] e^{-j\frac{2\pi(2r)m}{N}} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] e^{\frac{-j2\pi(2r+1)m}{N}}$$

$W_N = e^{\frac{-j2\pi}{N}}$
$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_N^{2rm} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_N^{2rm} W_N^{m}$$

$$= Y_m + W_N^m Z_m \qquad m\in[0, \tfrac{N}{2}-1]$$

$X[m+\frac{N}{2}] \longrightarrow \in[\frac{N}{2}, N-1]$

$= Y[m+\frac{N}{2}] + W_N^{m+\frac{N}{2}} Z[m+\frac{N}{2}] = Y[m] - W_N^m Z[m]$

Y_m: $\frac{N}{2}$ point DFT

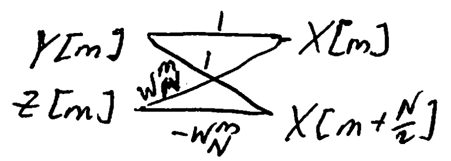
$Y[\langle m+\frac{N}{2}\rangle_{N/2}] = Y[m]$

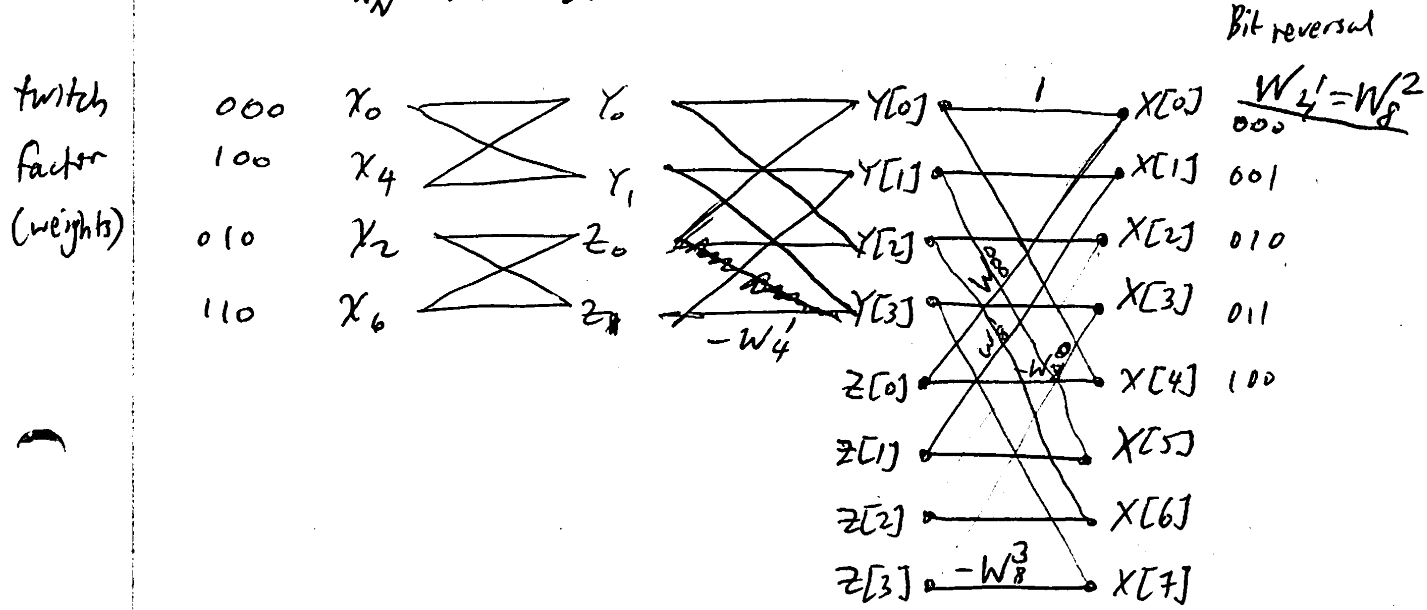
$X[\langle m+N\rangle_N] = X[\langle m\rangle]$

$W_N^{m+\frac{N}{2}} = \left(e^{\frac{-j2\pi}{N}}\right)^{m+\frac{N}{2}} = e^{\frac{-j2\pi m}{N}} e^{\frac{-j2\pi}{N}\frac{N}{2}} = -W_N^m$

$X[m] = Y[m] + W_N^m Z[m]$

$X[m+\frac{N}{2}] = Y[m] - W_N^m Z[m]$





$Z = \{x\}_0^{N-1} * \{Y\}_0^{M-1}$

$Z_d(\omega) = \underline{X_d(\omega) \cdot Y_d(\omega)}$

$\underline{X[m] \cdot Y[m]}$

1. zeropad to length $L = N+M-1$
2. zeropad to length $K = 2^{\lceil \log_2 L \rceil}$
3. $\mathrm{IFFT}\{\mathrm{FFT}\{x_n\} \cdot \mathrm{FFT}\{Y_n\}\}_0^{K-1}$
4. Take first L elements

$N > M$

$$(x \circledast y)[n] = \sum_{k=0}^{N} x[k]\, y[\langle n-k \rangle_M]$$

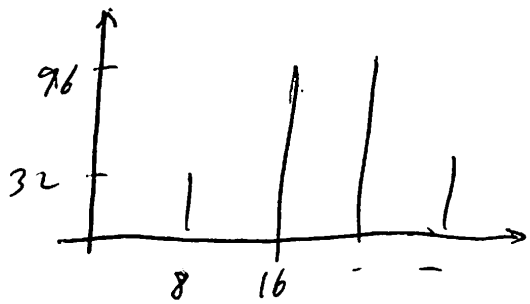
$\frac{AN}{2}$

x | 1, 2, 3, 4, 5, 6 | flip about spot 0

y | 1, 0, 0, 1, 0, 0 | n=0, n=1 ... n=6

0, 0 | 1, 0, 0, 1, 0, 0 | 1

HW prob 1



$A\cos(\Omega t)$ $\quad \frac{AN}{2} = 32$

$\frac{2\pi m}{N} = \omega_\ell \qquad \Omega_\ell = \frac{\omega_\ell}{T}$

$\frac{1}{N}\left(32\delta[m-8] + 92\delta[m-16] + \cdots\right) e^{\frac{j2\pi n m}{2}}$

$32 e^{\frac{j2\pi n 8}{N}} + 96 e^{\frac{j2\pi n 16}{N}} + 96 \cdots + 32 \cdots$

Fri 3 November Lecture

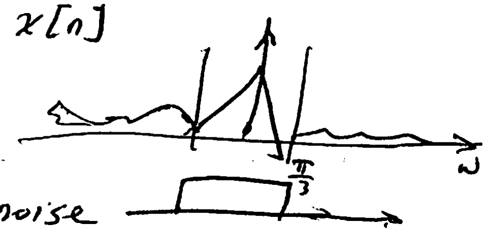
Bresler

Phase of Freq. Response

Filters

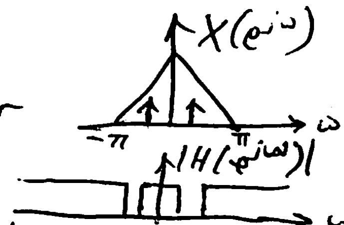
$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$

$H(e^{j\omega})$



Low Pass Filter (LPF) — reject noise

telephone calls] music recording
band stop filter



mono — music/speech

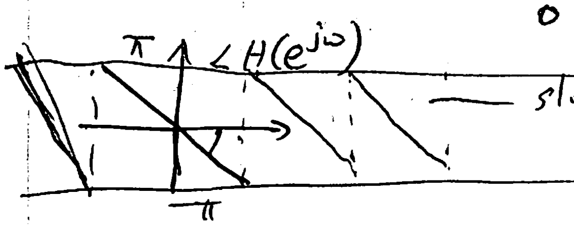
one ear only sensitive to frequency — Helmholtz
phase doesn't matter

phase encodes direction
└ stereo — or more channel

(strictly)
Linear Phase: $|H(e^{j\omega})| e^{-j\omega k}$

≥ 0

$\phi(\omega) = -\omega k$



— slope $-k$, 2π jumps

must be negative slope for causal filters: k corresponds to delay

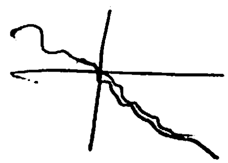
Non linear phase

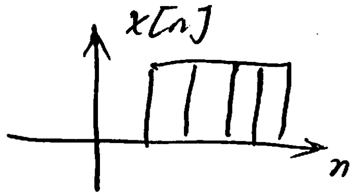
$\phi(\omega)$ — arbitrary

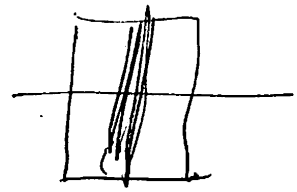
$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\phi(\omega)}$

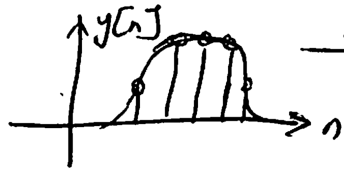
$x[n] \longleftrightarrow X(e^{j\omega})$

$x[n-k] \longleftrightarrow X(e^{j\omega}) e^{-j\omega k}$









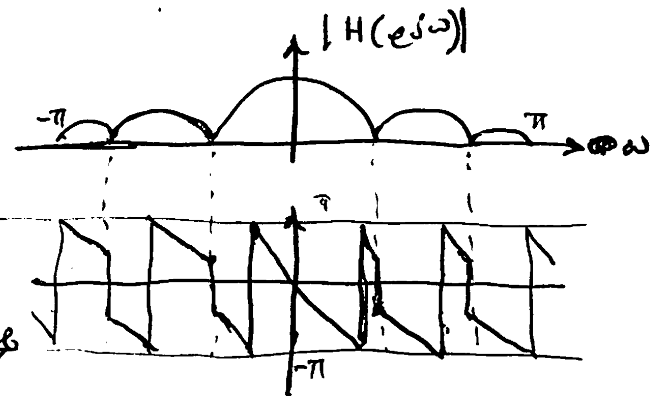
shifted to right

Thm No causal IIR filter can have linear phase

Generalized Linear Phase (GLP) (allows π jumps)

$$H(e^{j\omega}) = \underbrace{A(e^{j\omega})}_{\text{Real}} e^{j(\beta-\omega k)}$$

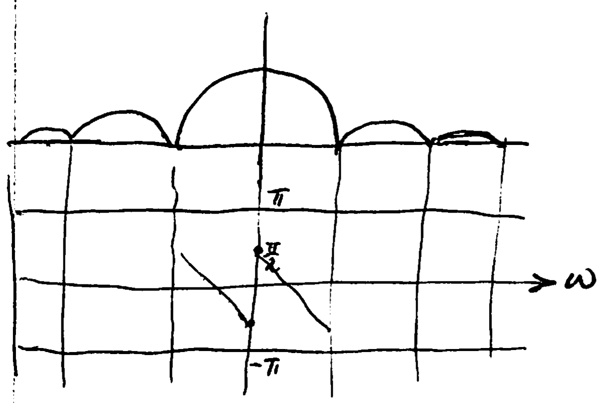
$$|H(e^{j\omega})| = |A(e^{j\omega})|$$



Rolle's Thm:
there must be a zero crossing
between + and −

Group Delay / — Delay

$$\tau_g \triangleq -\frac{d}{d\omega} \angle H(e^{j\omega})$$

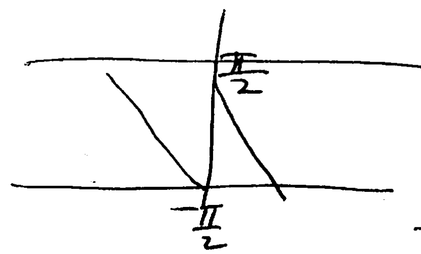


$\beta = \pm \frac{\pi}{2}, 0$

phase must be odd

$\beta \neq \frac{\pi}{2}$

impossible for this filter



Is this a LPF?

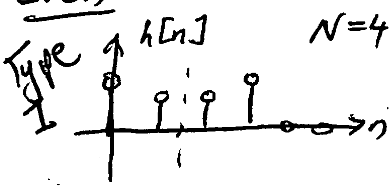
No.

Thm A causal filter has GPP if and only if it is an FIR filter - (say of length N), and its impulse response has one of the following symmetries: (1) $h[n] = h[N-1-n]$

(2) $h[n] = -h[N-1-n]$

Even

Type I

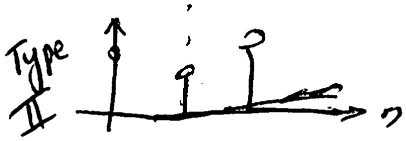


$N=4$

$h[0] = h[4-1-0] = h[3]$

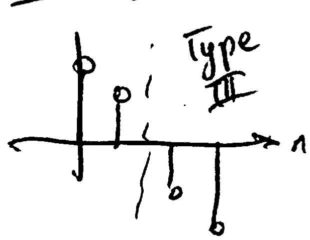
$h[1] = h[4-1-1] = h[2]$

Type II

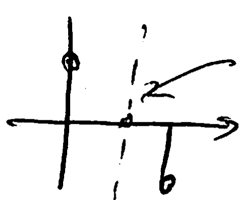


odd symmetry

Type III



Type IV



must be zero (odd length filter)

Type III, Type IV

β non-zero

$\Downarrow$

cannot make LPF

Bresler

Nov Recitation

Tues 7 ~~Oct Rect~~

FIR/IIR:

<u>If FIR</u>

(1) Is $h[n]$ even or odd length?

(2) Does $h[n]$ have odd or even sym

(3) Does $H_d(\omega)$ have LP? GLP?

what type?

If GLP, $\alpha = ?$, $k = ?$

$H_d(\omega) = A(\omega)\, e^{j(\alpha - k\omega)}$

(a) $y[n] = x[n] + x[n-1]$

$Y(z) = (1+z^{-1})X(z) \qquad z^k \longleftrightarrow \delta[n+k]$

$H(z) = \dfrac{Y(z)}{X(z)} = 1 + z^{-1} \Rightarrow h[n] = \delta[n] + \delta[n-1]$

$N = 2 \Rightarrow$ even

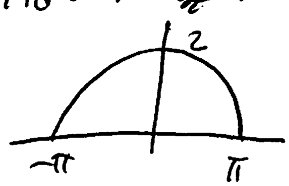
even symmetry  Type II

<u>GLP</u>

$k = \frac{N-1}{2} = \frac{2-1}{2} = 1/2$

$\alpha = 0$

$H_d(\omega) = 1 + e^{-j\omega} = e^{-j\omega/2}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right) = \underbrace{2\cos\tfrac{\omega}{2}}_{A(\omega)}\, e^{-j\frac{\omega}{2}}$



$2\cos\frac{\omega}{2}$

No sign change $\Rightarrow$ LP

(b) $y[n] = \frac{1}{3}x[n] + x[n-1] - x[n-2] - \frac{1}{2}x[n-3]$

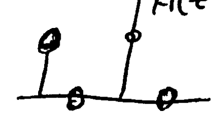
$Y(z) = [\frac{1}{3} + z^{-1} - z^{-2} - \frac{1}{3}z^{-3}]X(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{3} + z^{-1} - z^{-2} - \frac{1}{3}z^{-3}$

$y[n] = x[n+2] + x[n]$

$Y(z) = (z^2+1)X(z)$

$H(z) = z^2+1$



FIR

$h[n] = \frac{1}{3}\delta[n] + \delta[n-1] - \delta[n-2] - \frac{1}{3}\delta[n-3]$

$\{h[n]\}_{n=0}^{3} = \{\frac{1}{3}, 1, -1, -\frac{1}{3}\}$

even length, odd symmetry $\Longrightarrow$ GLP, Type IV filter

$\alpha = \frac{\pi}{2}$, $k = \frac{4-1}{2} = 3/2$

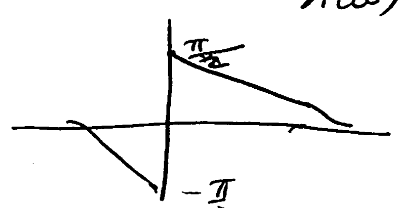
$(-A(\omega))e^{j(+\frac{\pi}{2}-k\omega)}$

$H_d(\omega) = \frac{1}{3} + e^{-j\omega} - e^{-j2\omega} - \frac{1}{3}e^{-j3\omega}$

$= e^{-j\frac{3}{2}\omega}[\frac{1}{3}e^{j\frac{3}{2}\omega} + e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} - \frac{1}{3}e^{-j\frac{3}{2}\omega}]$

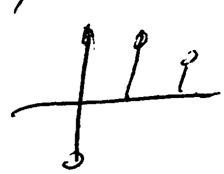
$H_d(\omega) = e^{-j\frac{3}{2}\omega}[\frac{2}{3}j\sin\frac{3}{2}\omega + 2j\sin\frac{\omega}{2}]$

$= e^{j(\frac{\pi}{2} - \frac{3}{2}\omega)}\underbrace{[\frac{2}{3}\sin\frac{3}{2}\omega + 2\sin\frac{\omega}{2}]}_{A(\omega)}$



Not LP

(c) $y[n] = -x[n] + x[n-1] + \frac{1}{3}x[n-2]$, FIR, $N=3$



(d) $y[n] = x[n] - 0.76y[n-1]$

$Y(z) = X(z) - 0.76z^{-1}Y(z)$

$(1+0.76z^{-1})Y(z) = X(z)$

$H(z) = \frac{1}{1+0.76z^{-1}}$ IIR $|z| > 0.76 \Rightarrow (-0.76)^n u[n] = h[n]$

$$\frac{1-z^{-1}}{z^2-2z+1} = \frac{z^{-1}(z-1)}{(z-1)^2} = \frac{z^{-1}}{z-1} = \frac{1}{z(z-1)}$$

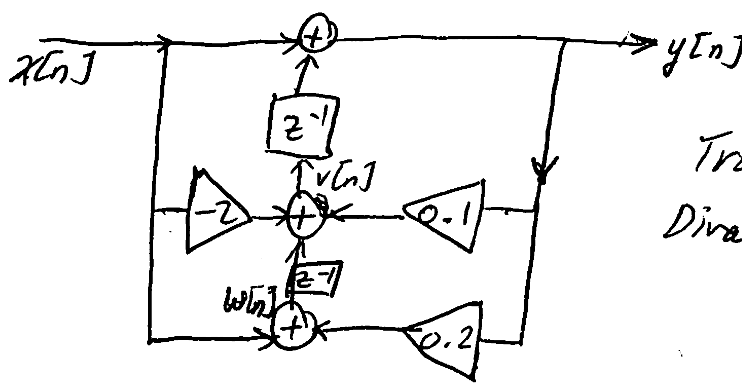
$$\frac{z^2-2z+1}{1-z^{-1}} = z(z-1)$$

$$\frac{1-z^{-1}}{z^2-2z+1}$$

$$H(z) = \frac{1-2z^{-1}+z^{-2}}{1-z^{-1}} = \frac{Y(z)}{X(z)} = 1-z^{-1}$$

$y[n] = y[n-1] + x[n] - 2x[n-1] + x[n-2]$

$y[n] = x[n] - x[n-1]$



Transpose
Dirac Form I

(1) Find D.E. relating $y[n]$ to $x[n]$

(2) Draw a filter structure of an equivalent system as a cascade of two first order systems

$v[n] = -2x[n] + 0.1y[n] + w[n-1]$

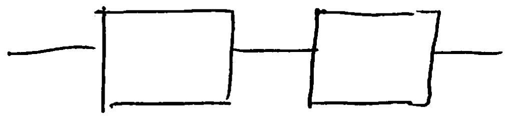
$w[n] = x[n] + 0.2y[n]$

$y[n] = x[n] + v[n-1]$

$y[n] = x[n] - 2x[n-1] + x[n-2] + 0.1y[n-1] + 0.2y[n-2]$

$Y(z)[1-0.1z^{-1}-0.2z^{-2}] = X(z)[1-2z^{-1}+z^{-2}]$

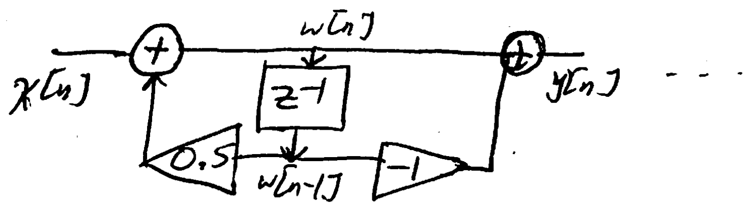
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}+z^{-2}}{1-0.1z^{-1}-0.2z^{-2}} = \frac{(z-1)^2}{(z-0.5)(z+0.4)}$$



$H(z) = H_1(z)\, H_2(z)$

$H_1(z) = \dfrac{z-1}{z-0.5}$ $\quad H_2(z) = \dfrac{z-1}{z-0.4}$

$= \dfrac{1-z^{-1}}{1-0.5z^{-1}}$ $\quad = \dfrac{1-z^{-1}}{1-0.4z^{-1}}$



Dirac form II:

delays = order of system

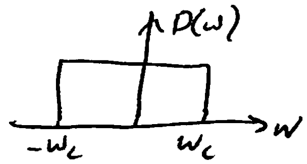
$y[n] = w[n] - w[n-1]$ $\qquad Y(z) = W(z)[1 - z^{-1}]$

$w[n] = x[n] + 0.5\, w[n-1]$ $\qquad W(z)[1 - 0.5z^{-1}] = X(z)$

Bresler

Wed 8 November Lecture

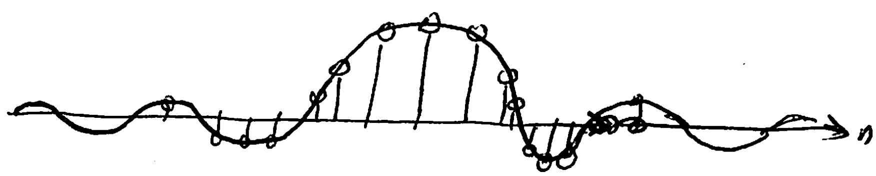
FIR Filter Design - GLP



$$d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} D(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\, e^{j\omega n}\, d\omega$$

$$= \frac{\omega_c}{\pi}\,\mathrm{sinc}(\omega_c n)$$



IIR

Non causal

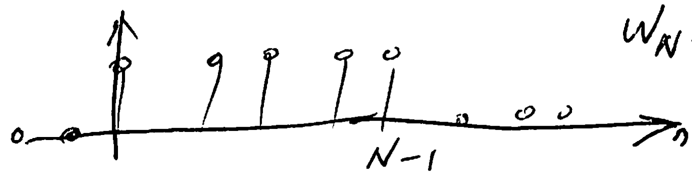
$$\sum_{n=-\infty}^{\infty} |d_n|$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \text{falls off as } 1/t$$

$$\frac{\sin t}{t} = \mathrm{sinc}(t)$$

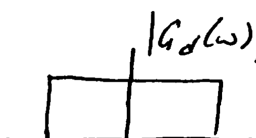
$$g_n = d_{N-k} \longleftrightarrow D(\omega)\, e^{-j\omega k}$$

$h_n = g_n W_n$



$$W_N = \begin{cases} 0, & n \notin [0, N-1] \\ 1 \end{cases}$$

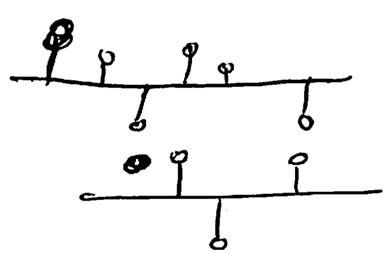
$g_n = d_{n-k}$



$h_n = g_n W_n$

$$H_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_d(\theta)\, W_d(\omega-\theta)\, d\theta$$

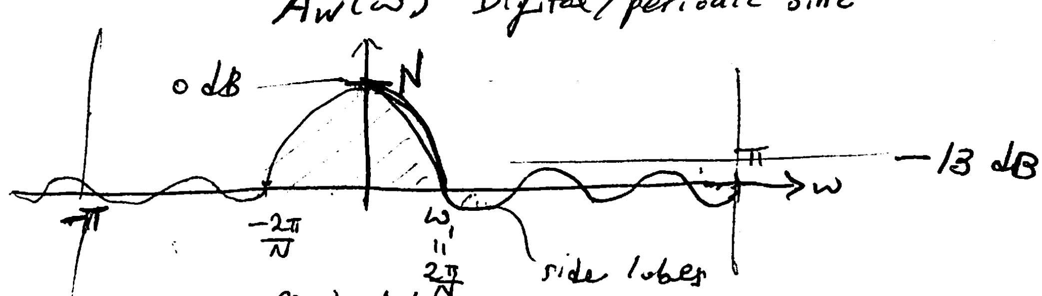
$N = \text{odd} \qquad k = \frac{N-1}{2}$



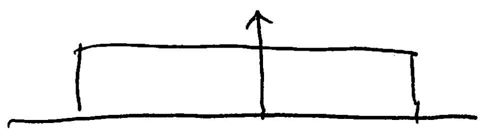
$$\boxed{W_d(\omega) = \sum_{n=0}^{N-1} w_n e^{-j\omega n} = \underbrace{\left(\frac{\sin\left(\omega \frac{N}{2}\right)}{\sin \frac{\omega}{2}}\right)}_{A_w(\omega)} e^{-j\omega \frac{N-1}{2}}}$$

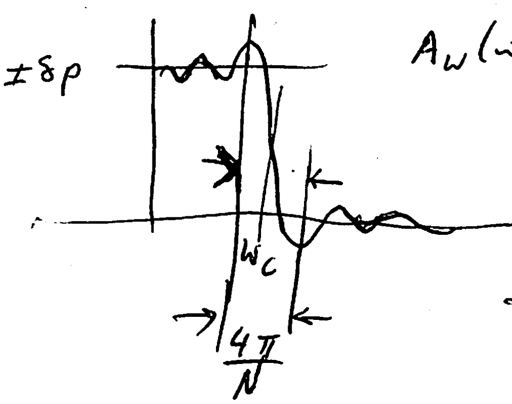
$$H_d(\omega) = G_d(\omega) \circledast W_d(\omega) \cdot \frac{1}{2\pi}$$

$A_w(\omega)$ Digital/periodic sinc



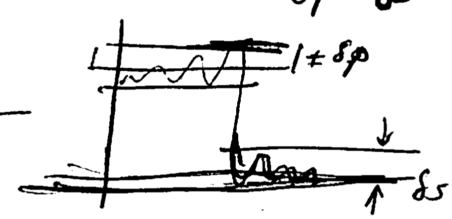
$$\frac{\omega N}{2} = \pi \Rightarrow \omega_1 = \frac{2\pi}{N}$$





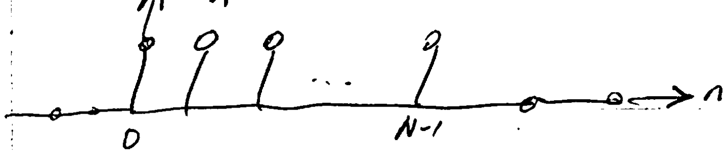
$A_w(\omega) \circledast D(\omega)$ Window based design

$\delta p = \delta s$



$\delta s \sim -20$ dB

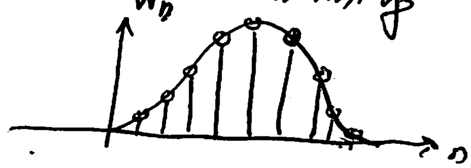
Rectangular Window



$$20 \log_{10} \frac{\delta s}{1}$$

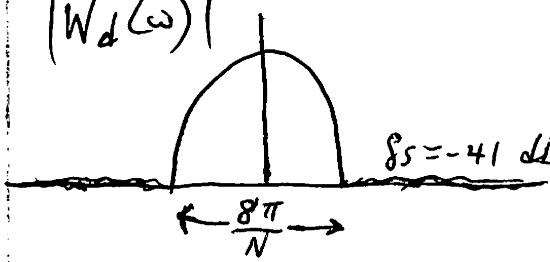
humans logarithmic senses — wide dynamic range

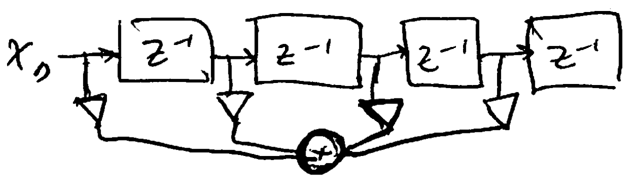
W_n – Hamming window



$$W_n = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) & 0 \le n \le N-1 \\ 0 & \text{else} \end{cases}$$

$|W_d(\omega)|$



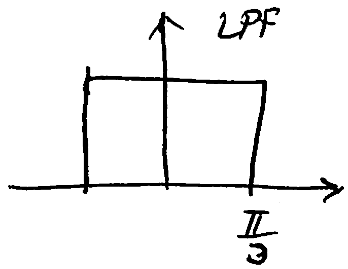


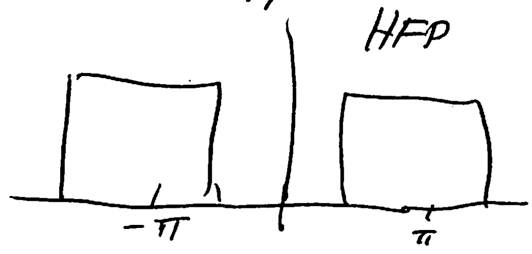
larger N
more delays
more multipliers
more adder
} more hardware

more computations on software side

Kaiser window I_0 . . .

- adjustable to achieve specification depending on trade off
- works best for amount of energy in side lobes





low pass convolved
w/ cosine in time

Wed ~~0~~8 November Recitation

Michael

~~y[n+1]~~ $y[n+1] - 2y[n] - 5y[n-1]$
$= x[n] + 8x[n+1] - 2x[n-1]$

$n \triangleq n+1$

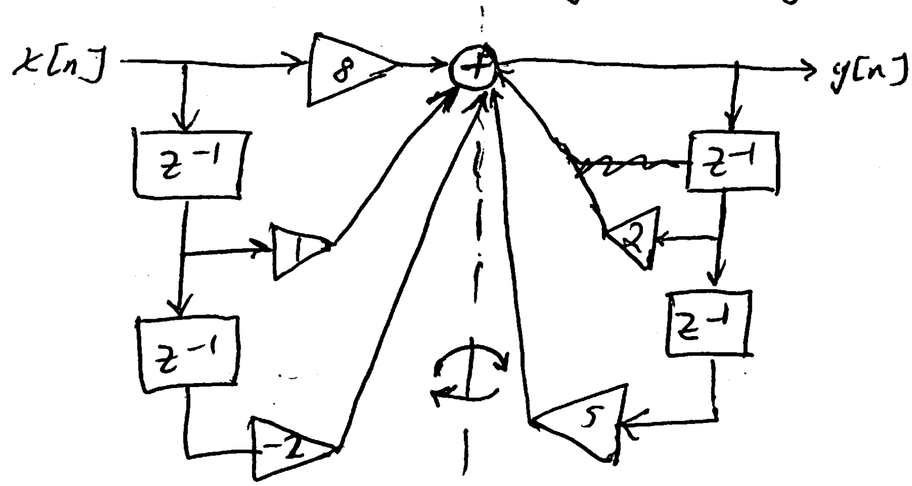
$y[n] - 2y[n-1] - 5y[n-2] = 8x[n] + x[n-1] - 2x[n-2]$

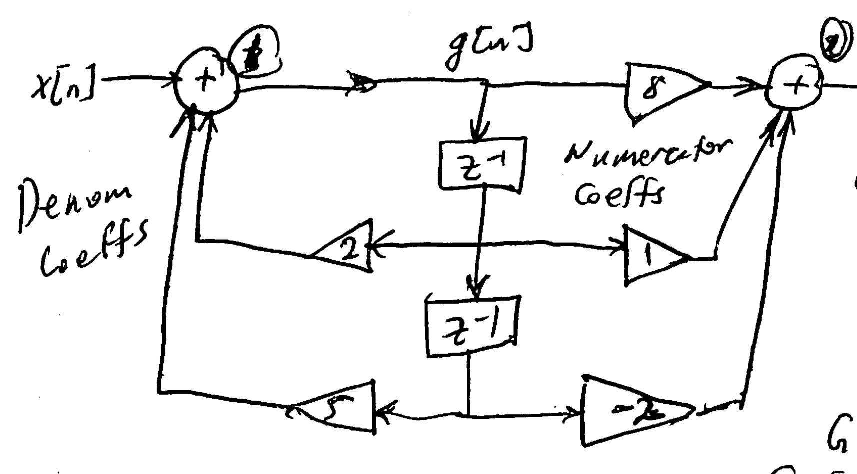
$Y(z)[1 - 2z^{-1} - 5z^{-2}] = X(z)[8 + z^{-1} - 2z^{-2}]$

$\Rightarrow H(z) = \dfrac{8 + z^{-1} - 2z^{-2}}{1 - 2z^{-1} - 5z^{-2}}$

Direct Form I

$y[n] = 8x[n] + x[n-1] - 2x[n-2] + 2y[n-1] + 5y[n-2]$





① $g[n] = x[n] + 2g[n-1] + 5g[n-2]$

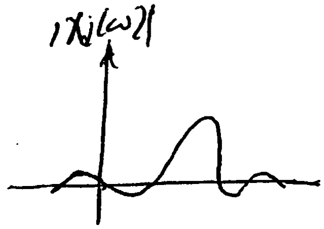
$G(z) = X(z) + 2z^{-1}G(z) + 5z^{-2}G(z)$

$G(z)(1 - 2z^{-1} - 5z^{-2}) = X(z)$

② $y[n] = 8g[n] + g[n-1] - 2g[n-2]$

$Y(z) = G(z)(8 + z^{-1} - 2z^{-2})$

$H(z) = \dfrac{Y(z)}{H(z)} = \dfrac{8 + z^{-1} - 2z^{-2}}{1 - 2z^{-1} - 5z^{-2}}$



$$X(\omega)\,e^{-j\omega n_0} \longleftrightarrow x[n-n_0]$$

$H_d(\omega) = R(\omega)e^{j(\alpha - m\omega)}$, $R(\omega)$ is <u>real</u>

Type 1 $\alpha = 0\ (\pm\pi)$ even symmetry → Type I odd length (N odd)
→ Type II Even length

Type 2 $\alpha = \pm\frac{\pi}{2}$ odd symmetry → Type III odd length
→ Type IV Even length

Type I → Any filter

Type II → <u>NOT</u> HPF

Type III → <u>only</u> BPF

Type IV → <u>NOT</u> LPF

$$\boxed{e^{-j\omega m},\ m=\frac{N-1}{2}}$$

$\{a, b, c, d, c, b, a\}$ $N=7$

$$H_d(\omega) = a + be^{-j\omega} + ce^{-j2\omega} + de^{-j3\omega} + ce^{-j4\omega} + be^{-j5\omega} + ae^{-j6\omega}$$

$$= e^{-j3\omega}\left(ae^{-j3\omega} + be^{j2\omega} + ce^{j\omega} + d + ce^{-j\omega} + be^{-j2\omega} + ae^{-j3\omega}\right)$$

$$= e^{-j3\omega}\left(2a\cos(3\omega) + 2b\cos(2\omega) + 2c\cos(\omega) + d\right)$$

↑ can increase → LP

Type II: $e^{-j2.5\omega}\left(2a\cos(2.5\omega) + 2b\cos(1.5\omega) + 2c\cos(.5\omega)\right)$

$|H_d(0)| = 2a + 2b + 2c$

$|H_d(\pi)| = 0$ $\forall\, a,b,c \rightarrow |H_d(\pi)| = 0$

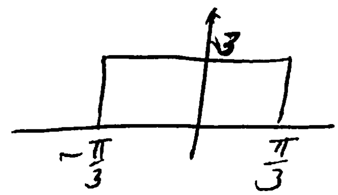
<u>No HPF</u>

$\{1, 2, 3, -2, -1\}$ freqz

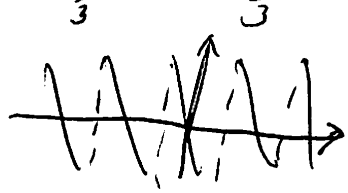
$\{3, 2, 2, 3\}$

$e^{-j20.5} \dfrac{\sin(21\omega)}{\sin(\frac{\omega}{2})}$

$h[n] = (-1)^n \operatorname{sinc}\left(\frac{\pi}{3}\left(n - \frac{41}{2}\right)\right), \quad 0 \le n \le 41$



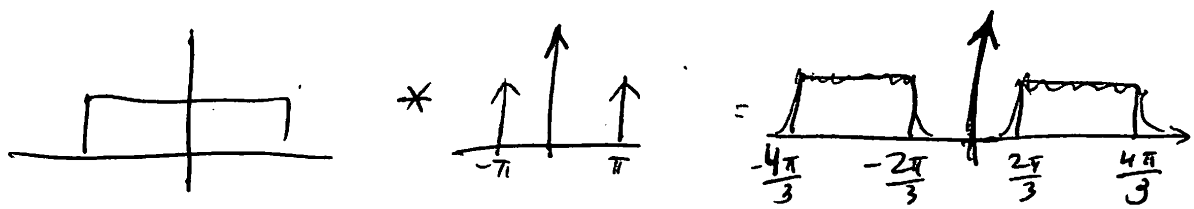
$\frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} e^{-j\omega n} d\omega = \frac{3e^{-j\omega n}}{2\pi - jn}\Bigg|_{-\pi/3}^{\pi/3} = \operatorname{sinc}\left(\frac{\pi}{3} n\right)$



$e^{-j\omega 41/2}$ factor $\rightarrow \sin\left(\frac{\pi}{3}\left(n - \frac{41}{2}\right)\right)$

$(-1)^n = \left(e^{j\pi}\right)^n = e^{-jn\pi}$

$\frac{1}{2\pi}\int e^{-j\omega n} d\omega = e^{-jn\pi}$



$H_d(\omega) = \underbrace{\frac{2\pi}{3} \operatorname{rect}(3(\omega - \pi))}_{R(\omega)} e^{j\left(\frac{\pi}{2} - \frac{41}{2}\omega\right)}$

$\alpha \quad M$

Gibb's phenomenon

$h[0] = h[5]$ $\quad h[n] = h[5-n], \; |H_d(\omega)| > 0, \; |\omega| < \pi$

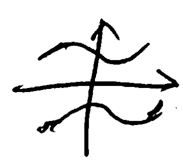
$h[1] = h[4]$

$h[2] = h[3]$ $\quad H_d(\omega) = R(\omega) e^{j(-2.5\omega)}$ $\quad \measuredangle H_d(\omega) = -2.5\omega$

OR

$\pm\pi - 2.5\omega$

$N = 6$



Friday 10 November Lecture

<u>FIR GLP Filter Design</u> (Windowing

(1) spec $\rightarrow$ N, type (sym), w_n, $D(\omega)$

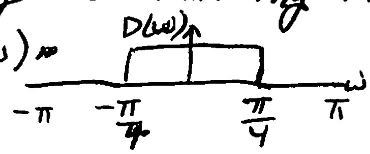
(2) $G_d(\omega)$ – GLP

(3) g_n – $DTFT^{-1}\{G_d(\omega)\}$

(4) $h_n = g_n \cdot w_n$

(5) Evaluate $H_d(\omega)$

<u>Ex1</u> Design a LPF with $\omega_c = \frac{\pi}{4}$, length $N = 30$
using the Hamming window

(1) $D(\omega)$ 

, Even sym

(2) $G_d(\omega) = \begin{cases} 1 \cdot e^{-j\omega k} & |\omega| \le \frac{\pi}{4}, \quad k = \frac{30-1}{2} = 14.5 \\ 0 & \text{else} \end{cases}$

$$g_n = \frac{1}{2\pi}\int_{-\pi/4 = -\omega_c}^{\pi/4 = \omega_c} 1 \cdot e^{-j\omega k} e^{j\omega n}\, d\omega = \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c(n-k))$$

(3) $h_n = g_n \cdot w_n = \begin{cases} \frac{1}{4}\operatorname{sinc}\left[\frac{\pi}{4}(n-14.5)\right]\left[0.54 - 0.46\cos\left(\frac{2\pi n}{30}\right)\right] & 0 \le n \le 29 \\ 0 & \text{else} \end{cases}$

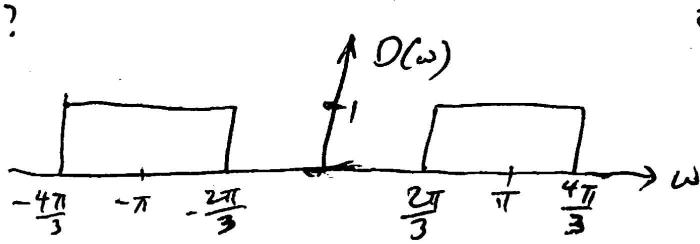
$$H_d(\omega) = \sum_{n=0}^{29} h_n e^{-j\omega n}$$

Ex 2 Design a HPF with $\omega_c = \frac{2\pi}{3}$, length $N = 61$ using the truncation window

Type ?

$$\overset{0}{\{}1, \overset{1}{0}, \overset{2}{-1}\}$$

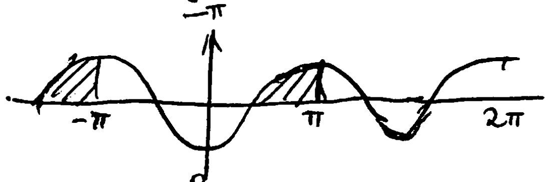
$$H_d(\omega) = 1 - e^{-j2\omega}$$
$$= e^{-j\omega}(e^{+j\omega} - e^{-j\omega})$$
$$= 2j \sin\omega \, e^{-j\omega}$$



$\Rightarrow$ must be even symmetry Type

$$G_d(\omega) = \begin{cases} 1 \cdot e^{-j\omega k} & \frac{2\pi}{3} \le |\omega| \le \pi \\ 0 & \text{else} \end{cases}$$

$$g_n = \frac{1}{2\pi} \int_{-\pi}^{-\frac{2\pi}{3}} e^{-j\omega k} e^{j\omega n} \, d\omega + \frac{1}{2\pi} \int_{\frac{2\pi}{3}}^{\pi} \cdots$$



shift limits of integration by symmetry

$$g_n = \frac{1}{2\pi} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} e^{-j\omega k} e^{j\omega n} \, d\omega = \left. \frac{e^{j\omega(n-k)}}{2\pi j(n-k)} \right|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}, \; k = 30$$

$$= \underbrace{(-1)^n \frac{1}{3} \operatorname{sinc}\left(\frac{\pi}{3}(n-30)\right)}$$

$$h_n = g_n w_n = \begin{cases} & 0 \le n \le 60 \\ 0 & \text{else} \end{cases}$$

check $n = 30 = k$ for integral to make sure actually sinc

Check GLP.

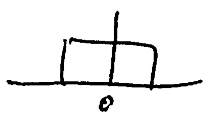
$$h_n = +h_{N-1-n}$$
$$+h_{60-n}$$

<u>HPF</u> $\omega_c = \frac{2\pi}{3}$, $N = 62$

<u>Type = ?</u>

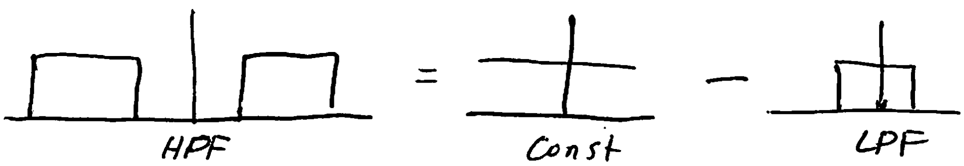
Try even symmetry $h_n = \{1, 1\}$

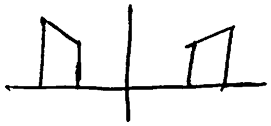
$\Rightarrow H_d(\omega) = 2e^{j\frac{\omega}{2}}\cos\frac{\omega}{2}$



$x_n \longleftrightarrow X_d(\omega)$

$x_n \cos\omega_0 n \longleftrightarrow \frac{1}{2}X_d(\omega-\omega_0) + \frac{1}{2}X_d(\omega+\omega_0)$



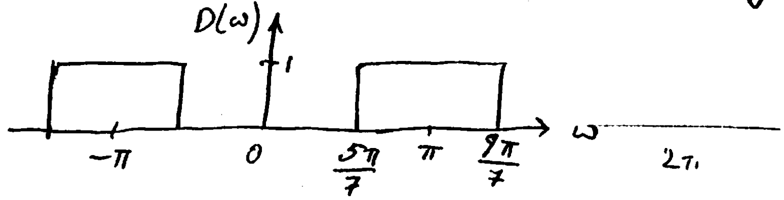


e.g. fix attenuation of high frequencies across some audio channel

Mon 13 Nov Lecture Breslen

FIR Design

Design a GLP HPF of length $N=62$, cutoff $\omega_c = \frac{5\pi}{7}$. Use the Hamming window.



Even sym

$\{h_n\} = \{1, 1\}$ $\Rightarrow$ Not HPF

$H_d(\omega) = 1 + e^{-j\omega} = e^{-j\omega/2} \cdot 2\cos\frac{\omega}{2}$

Odd sym

$\{1, -1\} \rightarrow 1 - e^{-j\omega} = e^{-j\omega/2}\, 2j\sin\frac{\omega}{2}$

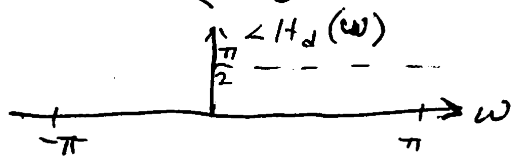
$\downarrow$

Pick odd sym. Type IV

$$G_d(\omega) = \begin{cases} 1 e^{j(\frac{\pi}{2} - \frac{61}{2}\omega)} & \frac{5\pi}{7} \le \omega \le \pi \\ 0 & |\omega| \le \frac{5\pi}{7} \end{cases}$$

$G_d(-\omega) = G_d^*(\omega)$

$G_d(\omega) = \{ 1 e^{j(-\frac{\pi}{2} - \frac{61}{2}\omega)} \quad -\pi \le \omega \le -\frac{5\pi}{7}$



$A(\omega)e^{-j\omega\frac{N-1}{2}}$

$$G_d(\omega) = \begin{cases} 0 & 0 \le \omega < \frac{5\pi}{7},\ \frac{9\pi}{7} < \omega \le 2\pi \\ 1 \cdot e^{j(\frac{\pi}{2} - \frac{61}{2}\omega)} & \frac{5\pi}{7} \le \omega \le \frac{9\pi}{7} \end{cases}$$

$G_d(\omega - 2\pi) = G_d(\omega)$

$G_d(\omega - 2\pi) = 1 e^{j(-\frac{\pi}{2} - \frac{61}{2}(\omega - 2\pi))} \quad \pi \le \omega \le \frac{9\pi}{7}$

$= e^{j(-\frac{\pi}{2} - \frac{61}{2}\omega + 61\pi)} = e^{j(\frac{\pi}{2} - \frac{61}{2}\omega)}$

Bartlet window

$$g_n = \frac{1}{2\pi}\int_{\frac{5\pi}{7}}^{\frac{9\pi}{7}} 1 e^{j\left(\frac{\pi}{2} - \omega\frac{61}{2}\right)} e^{j\omega n}\, d\omega$$

$$= \frac{e^{j\omega n\left(n-\frac{61}{2}\right)}}{2\pi\left(n-\frac{61}{2}\right)}\Bigg|_{\frac{5\pi}{7}}^{\frac{9\pi}{7}} = \frac{1}{2\pi\left(n-\frac{61}{2}\right)}\left[e^{j\frac{9\pi}{7}\left(n-\frac{61}{2}\right)} - e^{j\frac{5\pi}{7}\left(n-\frac{61}{2}\right)}\right]$$

$$e^{j\theta_1} - e^{j\theta_2} = e^{j\frac{\theta_1+\theta_2}{2}}\Big[\underbrace{e^{j\frac{\theta_1-\theta_2}{2}} - e^{j\frac{\theta_2-\theta_1}{2}}}_{2j\sin\left(\frac{\theta_1-\theta_2}{2}\right)}\Big]$$

$$g_n = \frac{e^{j\pi\left(n-\frac{61}{2}\right)}}{2\pi\left(n-\frac{61}{2}\right)}\left[e^{j\frac{2\pi}{7}\left(n-\frac{61}{2}\right)} - e^{-j\frac{2\pi}{7}\left(n-\frac{61}{2}\right)}\right]$$

$$= e^{j\pi n}\,\cancel{e^{-j\pi 60}}\,\cancel{e^{-j\frac{\pi}{2}}}\cdot \cancel{2j}\,\frac{\sin\left[\frac{2\pi}{7}\left(n-\frac{61}{2}\right)\right]}{\frac{2\pi}{7}\left(n-\frac{61}{2}\right)}\cdot\frac{1}{7}$$

$$= \underbrace{e^{j\pi n}}_{(-1)^n}\,\frac{2}{7}\,\mathrm{sinc}\left[\frac{2\pi}{7}\left(n-\frac{61}{2}\right)\right] = g_n$$

$$h_n = g_n W_n$$

$$W_n = 0.54 - 0.46\cdot\cos\left(\frac{2\pi n}{\underset{N-1}{61}}\right) \qquad 0\le n\le 61$$

Hamming

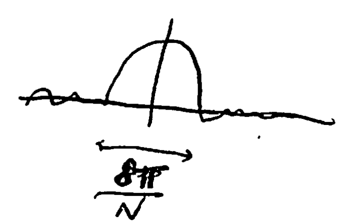
54 46

$$h_n = \begin{cases}(-1)^n\,\frac{2}{7}\,\mathrm{sinc}\left[\frac{2\pi}{7}\left(n-\frac{61}{2}\right)\right]\underbrace{\left[0.54-0.46\cos\frac{2\pi n}{61}\right]}_{\text{raised cosine}} & 0\le n\le 61\\ 0 & \text{else}\end{cases}$$

Truncation window

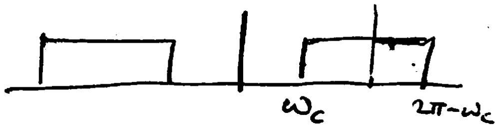
Hamming Window

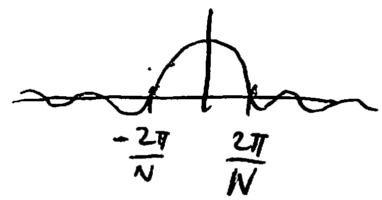
$|W_d(\omega)| =$

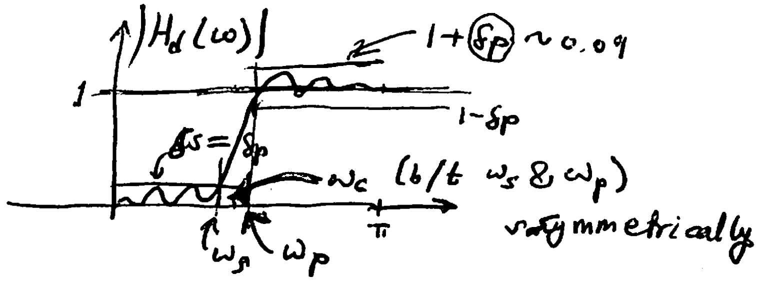


$h[n] = \pm h[N-1-n]$

freqz



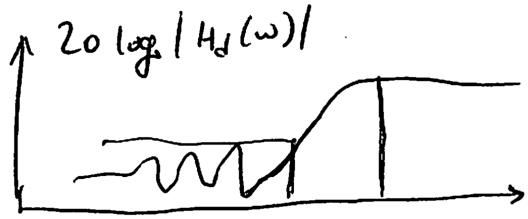




$\Delta\omega$ — width of transition band

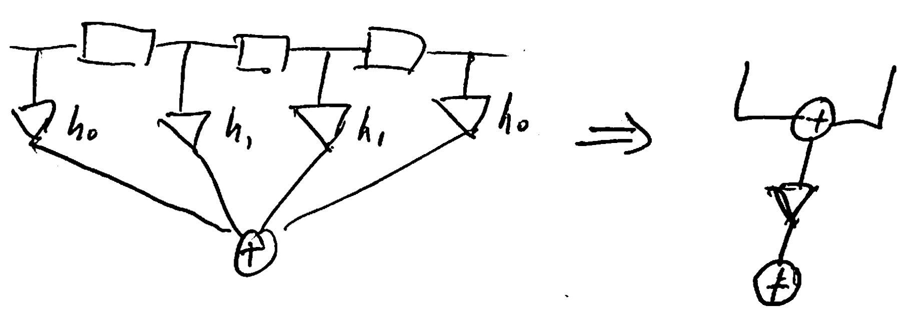
$\Delta\omega \sim \frac{4\pi}{N} \rightarrow$ Truncation

$\frac{8\pi}{N} \rightarrow$ Hamming.



On HW prob

Efficient implementation of FIR filter



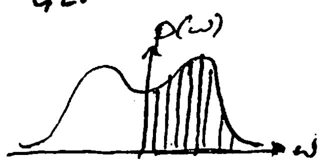
add then multiply

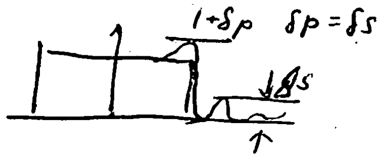
GLP symmetry

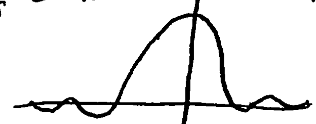
Wed 15 Nov Lecture   Bresler

FIR Filter Design by Windowing

+	−
• Easy (piecewise ~~linear~~ constant)	($\delta_p = \delta_s$) no separate control
• GLP	Overperformance (δ_p must equal δ_s)
	Change ratio of ripples in pass/stop band
	← from convolution







even pass/stop band errors

how to make one larger than other error in multiband filter?

• Not optimal − Not equiripple
(ripples same height, but diminishing)

• Transition width

IIR

No window → no convolution → no issue w/ transitions

+	−
Can have sharp transitions with low order	Lose GLP

Classical IIR Filters

(1) convert digital spec ⇒ spec on analog filter

(2) Design analog filter

(3) Convert analog filter to a digital filter

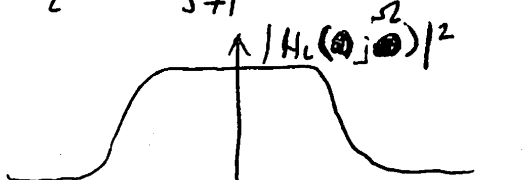
$h(t) \xleftrightarrow{\text{Laplace}}$

$H_c(s) = \dfrac{1}{s+1}$

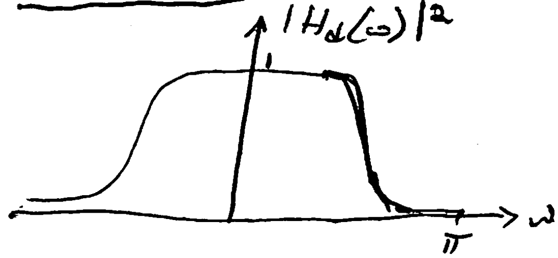
$s \rightarrow \alpha \cdot \dfrac{1-z^{-1}}{1+z^{-1}}$ $(\Omega \rightarrow \omega)$

Bilinear transformation

BLT

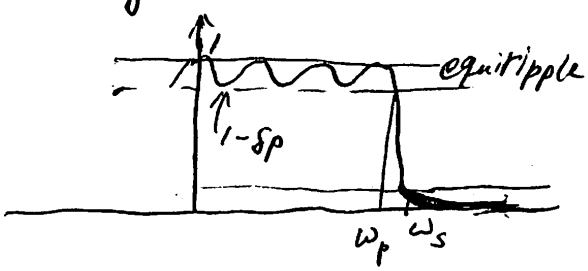


Butterworth



Maximally flat filter

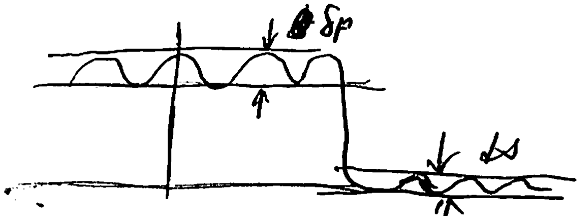
Chebychev I



for agreeing to have ripples, we can have a sharper transition

Chebychev II — ripples are in stopband instead

Elliptical ("cauer")

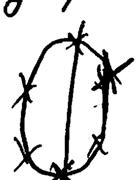


$\delta_s \neq \delta_p$ necessarily

optimal — smallest order

fixed order stop & pass band errors	⇒ smallest transition band

Analog: poles on ellipse



BUT phase ugliest
— differs the most from GLP

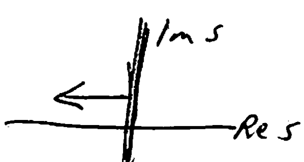
Bessel filter — best phase response

$\frac{1}{\Omega^2+1}$ simple formula for $|H_d(\omega)|^2$

Matlab signal processing toolbox

FD Tool graphical

$\Omega = \alpha \tan \frac{\omega}{2}$

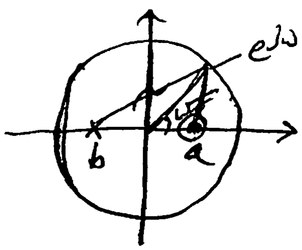


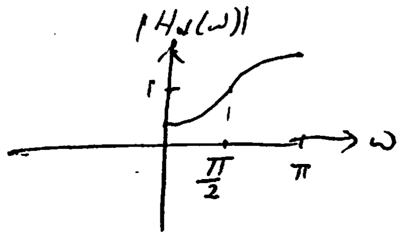
analog $\infty \rightarrow$ digital Ω since vertical not warped

Pole/zero location $\Rightarrow$ effect on frequency response

$H(z) = \frac{z-a}{z-b}$ $\quad |H_d(\omega)| = |H(e^{j\omega})|$

$|H_d(\omega)| = \frac{|e^{j\omega} - a|}{|e^{j\omega} - b|}$

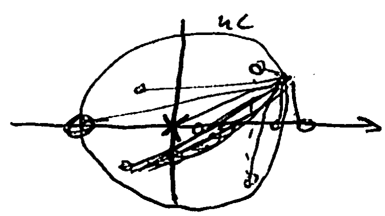




FIR

$H(z) = h_0 + h_1 z^{-1} + \dots + h_{N-1} z^{-(N-1)}$

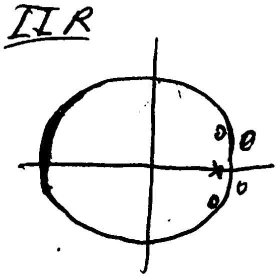
$= c(1-b_1 z^{-1})(1-b_2 z^{-1}) \dots (\quad)$



$z = e^{j\omega}$

$|1 - b_1 e^{-j\omega}| = |e^{j\omega} - b_1|$

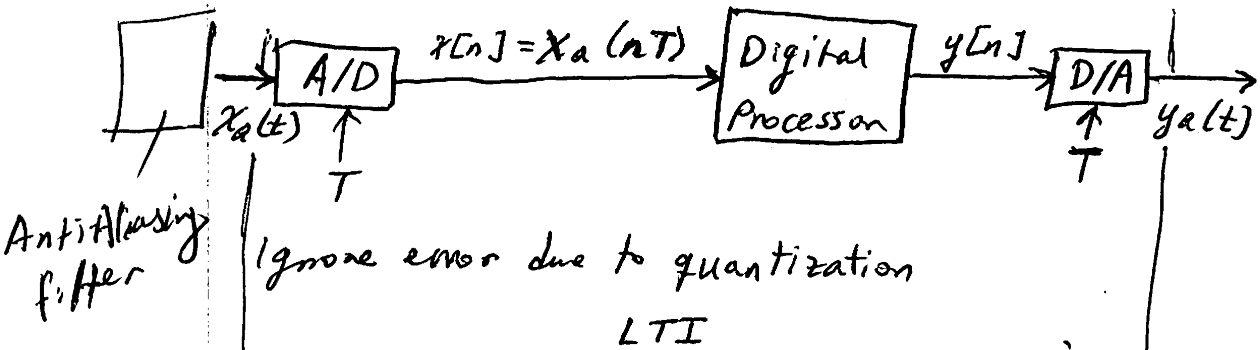
IIR



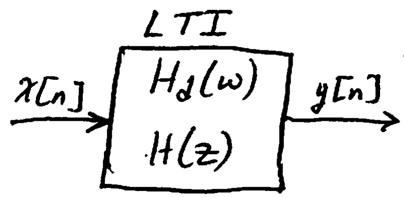
clever about
pole locations
- sharp responses

Bresler

Fri 17 Nov Lecture



Ignore error due to quantization



Analog LTI system

$$Y_a(\Omega) = H_a(\Omega)\, X_a(\Omega)$$

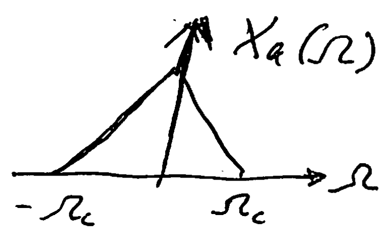
$$y_a(t) = h_a(t) * x_a(t)$$

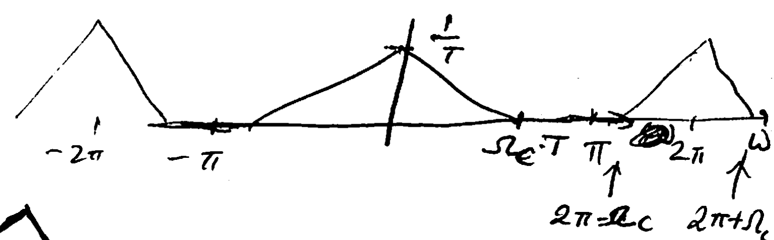
$$x[n] = x_a(nT)$$

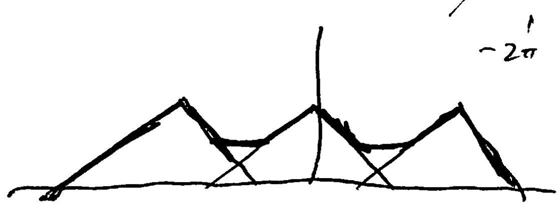
$\downarrow$ DTFT

$$\boxed{X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi k}{T}\right)}$$

$$X_a(\Omega) = FT\{x_a(t)\}$$







20kHz upper bound speech & music b/c ears can't hear higher

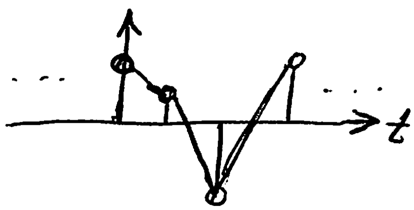
$$\Omega_c T < \pi$$

$$\frac{\pi}{T} > \Omega_c$$

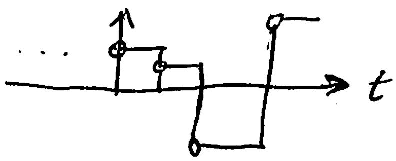
$$\boxed{X_d(\omega) = \frac{1}{T} X_a\left(\frac{\omega}{T}\right) \quad \text{for } |\omega| \le \pi}$$ under condition of no aliasing

$$Y_d(\omega) = H_d(\omega)\, X_d(\omega)$$

<u>D/A</u>



1st order hold



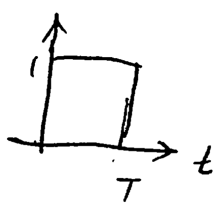
zero order hold
typically used b/c easy to build, inexpensive, practical

Resistor divided network, MOSFETS, switches → change voltage

$$y_a(t) = \sum_{n=-\infty}^{\infty} y[n]\, g_a(t - nT)$$

Reconstruction formula

$g_a(t)$

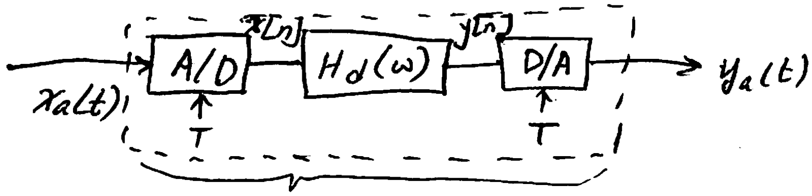


$g_a(t)$ - ZOH

perfect recovery - sinc fxn

Boresler

Mon 27 Nov lecture



LTI $\Leftrightarrow$ $y_a(t) = h_a(t) * x_a(t)$

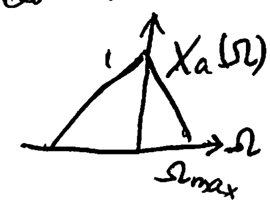
$\Leftrightarrow$ $Y_a(\Omega) = H_a(\Omega)\, X_a(\Omega)$

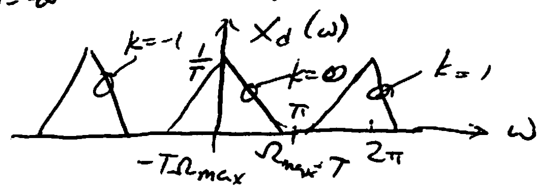
$x[n] = x_a(nT)$

$y[n] = h[n] * x[n]$

$$y_a(t) = \sum_{n=-\infty}^{\infty} y[n]\, g_a(t-nT)$$

$$X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega-2\pi k}{T}\right)$$

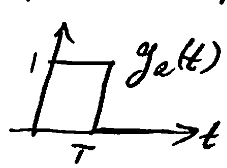




$T\Omega_{max} < \pi \Rightarrow \frac{\pi}{T} > \Omega_{max}$ Nyquist

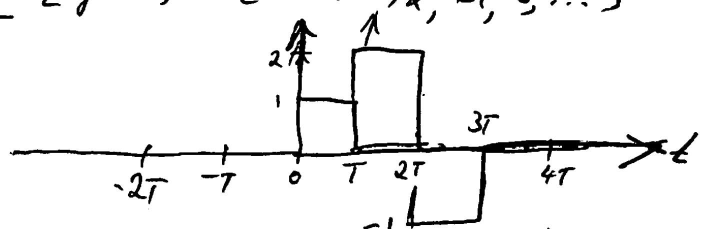
$X_d(\omega) = \frac{1}{T} X_a\left(\frac{\omega}{T}\right)$ $|\omega| < \pi$ – No aliasing

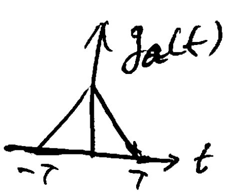
$Y_d(\omega) = X_d(\omega) \cdot H_d(\omega)$

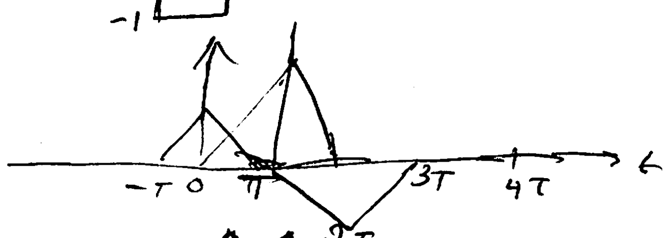


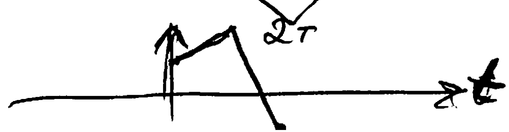
D/A In time domain

Ex $\{y[n]\} = \{0, 0, 1, 2, -1, 0, \ldots\}$

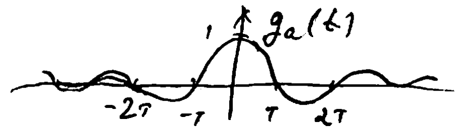




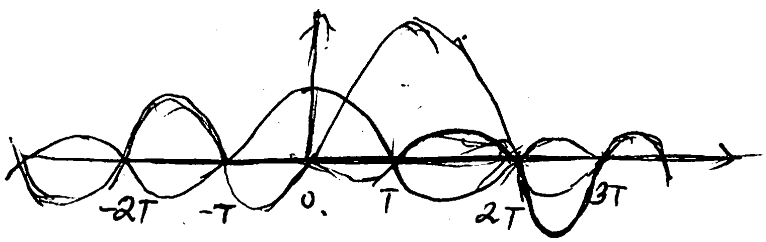




$g_a(t) = \text{sinc}\left(\frac{\pi}{T} t\right)$



non causal



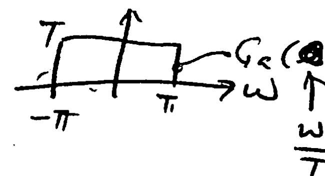
Ideal D/A

D/A : In the Frequency domain

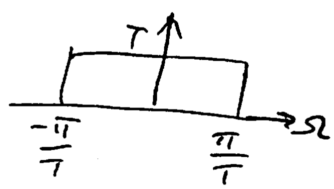
$$Y_a(\Omega) = FT\left\{\sum_{n=-\infty}^{\infty} y[n]\, g_a(t-nT)\right\} = \sum_{n=-\infty}^{\infty} y[n]\, FT\underbrace{\{g_a(t-nT)\}}_{G_a(\Omega)\cdot e^{-j\Omega Tn}}$$

$$= G_a(\Omega)\sum_{n=-\infty}^{\infty} y[n] e^{-j(\Omega T)n}$$

$$Y_a(\Omega) = G_a(\Omega)\, Y_d(\Omega T)$$



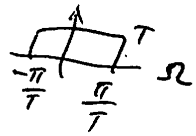
$$G_a(\Omega) = \begin{cases} T & |\Omega| < \frac{\pi}{T} \\ 0 & \text{else} \end{cases}$$



$\xleftrightarrow{FT} g_a(t) = \text{sinc}\left(\frac{\pi}{T} t\right)$

$$Y_a(\Omega) = G_a(\Omega)\underbrace{\underbrace{\frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\Omega - \frac{2\pi k}{T}\right)}_{X_d(\Omega T)} H_d(\Omega T)}_{Y_d(\Omega T)}$$

- No aliasing



- Ideal D/A

$$X_d(\omega) = \frac{1}{T}\sum_k X_a\left(\frac{\omega - 2\pi k}{T}\right)$$

$\omega \to \Omega T$

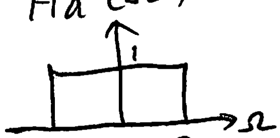
$$Y_a(\Omega) = \begin{cases} X_a(\Omega)\cdot H_d(\Omega T) & |\Omega| \le \pi/T \\ 0 & \text{else} \end{cases}$$

$$\frac{Y_a(\Omega)}{X_a(\Omega)} = \left.\begin{cases} H_d(\Omega T) & |\Omega| \le \pi/T \\ 0 & \text{else} \end{cases}\right\} H_a(\Omega)$$

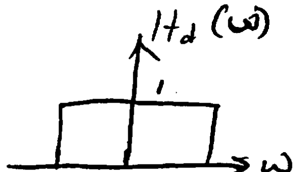
Design the "system" to act as a LPF w cutoff 1kHz
for inputs BL to 20 kHz

$$\frac{\pi}{T} > \underbrace{20 \cdot 10^{3} \cdot 2\pi}_{\Omega_{max}} \Rightarrow T \leq \frac{1}{40 \cdot 10^{3}} = 25\,\mu sec$$

$T = 25\,\mu sec$



Ω_c
$= 2\pi \cdot 10^3$



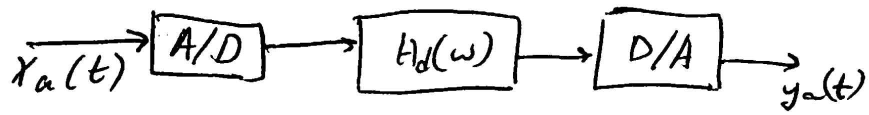
ω_c
$= \Omega_c \cdot T$
$= T \cdot 2\pi \cdot 10^3$
$= 2\pi \cdot 10^3 \cdot 25 \cdot 10^{-6} = 0.05\pi$

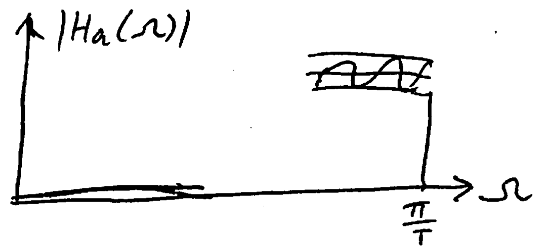
$x_a(t) = \cos(30 \cdot 10^3 \cdot 2\pi \cdot t)$

$x[n] = \cos(30 \cdot 10^3 \cdot 2\pi \cdot n \cdot 25 \cdot 10^{-6})$

Tues 28 Nov. Recitation

Bresler

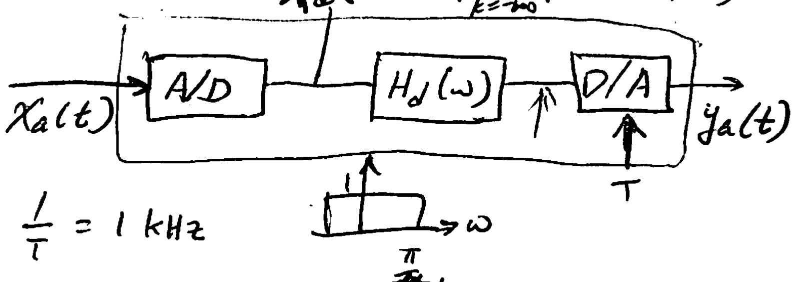




$\Omega_s = 8 \cdot 10^3 \cdot 2\pi$

$\Omega_p = 9 \cdot 10^3 \cdot 2\pi$

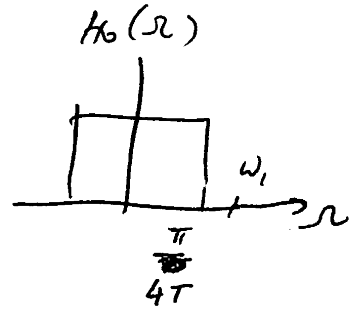
$X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi k}{T}\right)$



$H_a(\Omega) = \begin{cases} H_d(\Omega T) & |\Omega| < 10^3\pi \\ 0 & \text{else} \end{cases}$

$\frac{1}{T} = 1\ \text{kHz}$

$x_a(t) = \cos(\underbrace{2\pi \cdot 300 t}_{\omega_1}) + \cos(\underbrace{2\pi \cdot 600 t}_{\omega_2})$



$\omega_1 < 10^3\pi = \frac{\pi}{T}$

$< \omega_2 = 1200\pi$

$\frac{10^3\pi}{4} = 250\pi$

(1) Find $X_d(\omega)$

(2) Find $Y_d(\omega)$

(3) Find $Y_a(t) = G_a(\Omega) Y_d(\Omega T)$

(4) Find $y_a(t)$

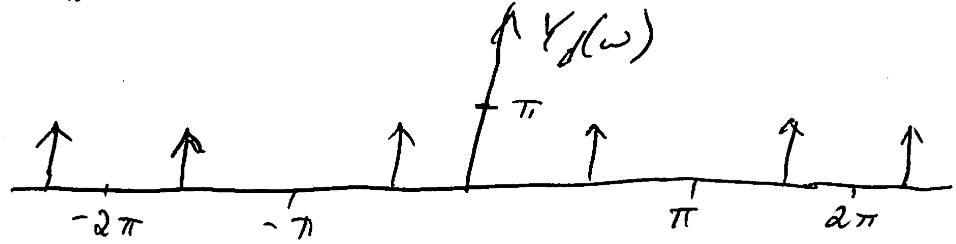
$x_a(t) = \cos(1200\pi t)$

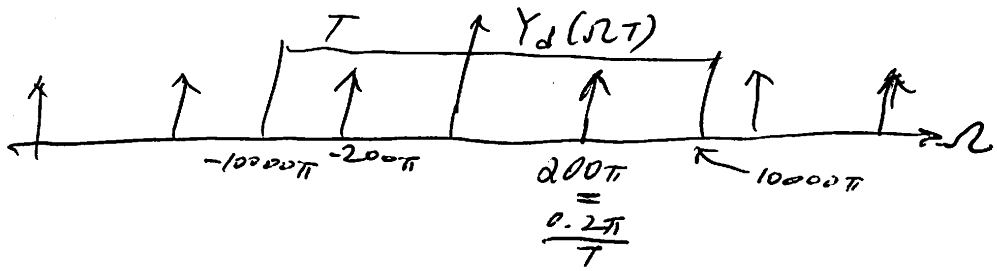
$x[n] = x_a(nT) = \cos(1.2\pi n) = \cos((1.2\pi - 2\pi)\cdot n)$

$= \cos(0.8\pi n) = y[n] \Rightarrow y_a(t) = \cos(800\pi t)$

$y[n] = \cos(0.2\pi n)$

$$Y_d(\omega) = \sum_{k=-\infty}^{\infty} \pi\delta(\omega - 0.2\pi - 2\pi k) + \pi\delta(\omega + 0.2\pi - 2\pi k)$$





$$Y_d(\Omega T) = \sum_k \pi\,\delta(\Omega T - 0.2\pi - 2\pi k) + \pi\delta(\quad)$$

$$Y_d(\Omega T)\cdot G_a(\Omega) = \pi\delta(\Omega T - 0.2\pi) + \pi\delta(\Omega T + 0.2\pi)$$

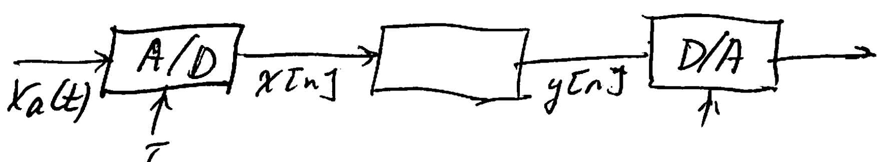
G_a ?

$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)$$

$$Y_d(\Omega T)\cdot G_a(\Omega) = T\left(\frac{1}{T}\pi\delta\left(\Omega - \frac{0.2\pi}{T}\right) + \frac{1}{T}\pi\delta\left(\Omega + \frac{0.2\pi}{T}\right)\right)$$

$\cos(\cancel{1}800\pi t) \rightarrow x[n] = \cos(0.2\pi n) \leftarrow \cos(200\pi t)$

$T = 10^3\ Hz$



Digital system described by

$y[n] = \frac{1}{2}y[n-1] + x[n]$

Find the freq response of the entire system for inputs of freq less than 500 Hz

$$H_a(\Omega) = \begin{cases} H_d(\Omega T), & |\Omega| < \frac{\pi}{T} \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{1-\frac{1}{2}e^{-j10^{-3}\Omega}} & |\Omega| < 1000 \text{ Hz} \\ 0 & \text{else} \end{cases}$$

(1) Find $H_d(\omega)$

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + X(z)$$

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$H_d(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

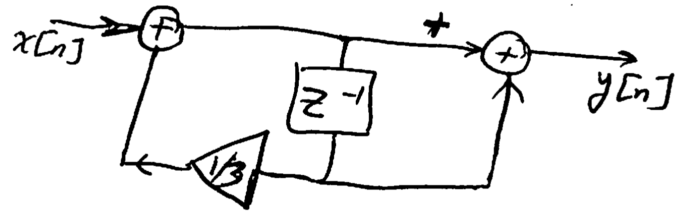
Given $x_a(t) = \cos(2\pi\, 300\, t) + \cos(2\pi\, 900\, t)$

Find $y_a(t)$

Wed 29 Nov Lecture

DSP of Analog signals

$T = 10^{-3}$

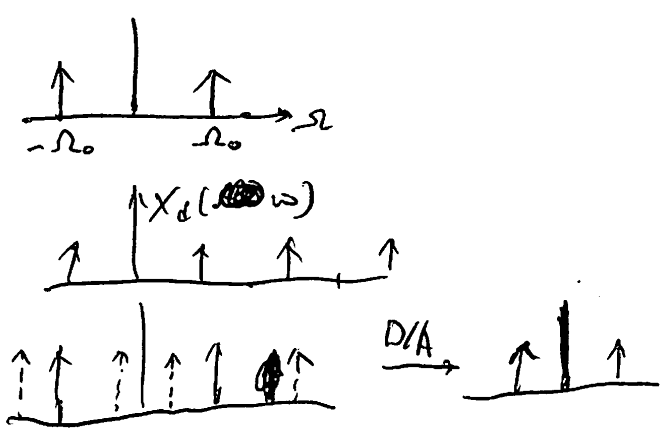


① Does the system behave as an LTI system, and if so, for what kind of signals?

$x[n] = x_a(nT)$ A/D

linear ✓

$H_a(\Omega)$ {



• A: Yes, only for signals BL to 500 Hz.

$\frac{f_s}{2} = \frac{\pi}{2\pi T} > \frac{\omega_{max}}{2\pi} = F_{max}$

$$\text{Given: } x_a(t) = \overbrace{\cos(2\pi\cdot 300t)}^{x_{a_1}(t)} + \overbrace{\cos(2\pi\cdot 800t)}^{x_{a_2}(t)}$$

Find $y_a(t)$

$$y_{a_1}(t) = |H_a(600\pi)| \cos(2\pi 300t + \angle H_a(600\pi))$$

Find $H_a(\Omega)$

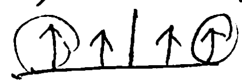
$$H_a(\Omega) = \begin{cases} H_d(\Omega T) & |\Omega| \le \frac{\pi}{T} \\ 0 & \text{else} \end{cases}$$

$$y_{a_1}(t) = |H_d(0.6\pi)| \cos(600\pi t + \angle H_d(0.6\pi))$$

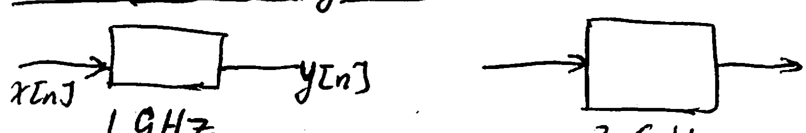
$$H(z) = \frac{1 - z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

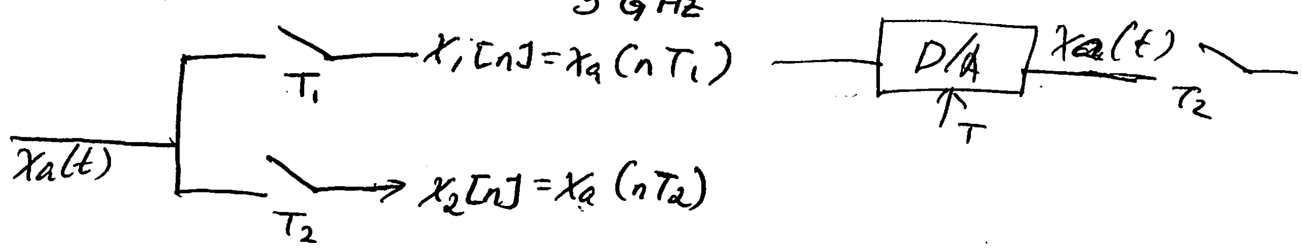
$$x_2[n] = x_{a_2}(nT) = \cos(1.6\pi n) = \cos(.4\pi n)$$

$$\tilde{x}_a(t) = \cos(400\pi t)$$



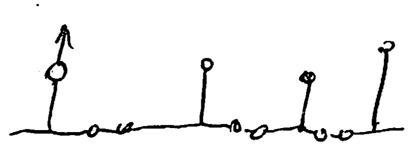
Multi-Rate Systems



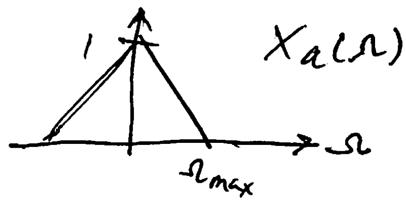


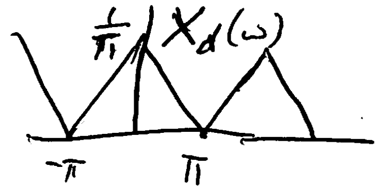
$$T_2 = \frac{T_1}{L}$$

$$v[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = kL \quad \underset{\text{integer}}{\uparrow} \\ 0 & \text{else} \end{cases}$$

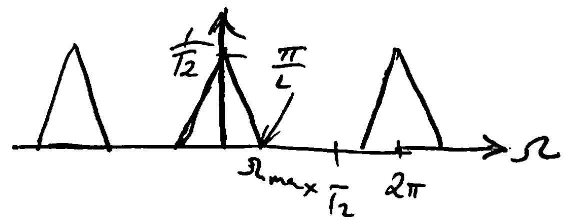


upsampling/
Rate Expansion





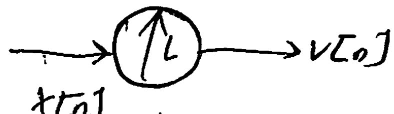
$$T = \frac{\pi}{\Omega_{max}}$$

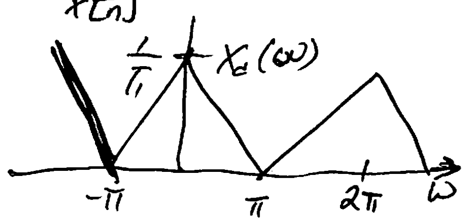


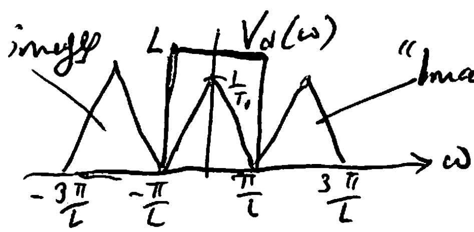
$$\frac{\Omega_{max} T_1}{L} = \frac{\pi}{L}$$

$$V_d(\omega) = \sum_{n=-\infty}^{\infty} v[n]\, e^{-jn\omega} = \sum_{\substack{n=kL \\ k=-\infty}}^{\infty} x\left[\tfrac{n}{L}\right] e^{-jn\omega}$$

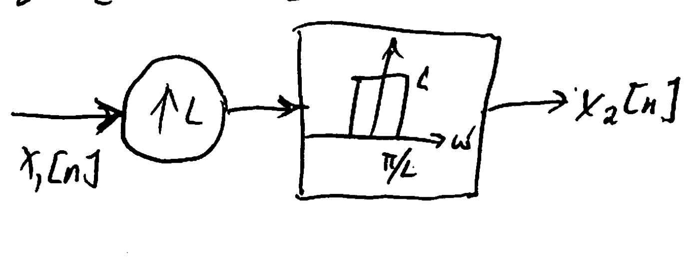
$$= \sum_{k=-\infty}^{\infty} x[k]\, e^{-jk(L\omega)} = X_d(L\omega)$$







"Images" phantom images



this is the ideal filter. You may not like it

Wed 29 Nov Recitation

Ian

$x_a(t) \rightarrow$ [A/D] $\xrightarrow{x[n]}$ [Digital Processing] $\xrightarrow{y[n]}$ [D/A] $\xrightarrow{y_a(t)}$

equiripple

monotone decreasing

$Y_a(\Omega) = H_a(\Omega) \cdot X_a(\Omega)$

$X_d(\omega) = \frac{1}{T} \sum_{l=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi l}{T}\right)$

For BL $x_a(t)$, $X_d(\omega) = X_a\left(\frac{\omega}{T}\right) \cdot \frac{1}{T}$

$\frac{1}{T} H_d(\omega) \cdot X_d(\omega)$

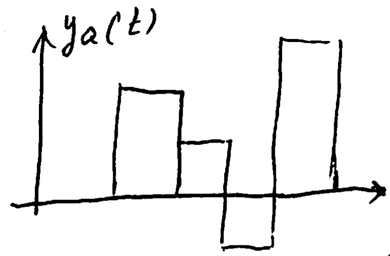
$y[n] \rightarrow$ [D/A] $\rightarrow y_a(t)$

Interpolation $\quad y_a(t) = \sum_{n=-\infty}^{\infty} y[n]\, g_a(t - nT)$

opposite of sampling

$\{y[n]\} = \{0, 2, 1, -1, 4\}$

$g_a(t) = \begin{cases} 1 & t \in [0, 1] \\ 0 & \text{else} \end{cases}$



$G_a e^{-j\Omega(nT)}$

$Y_a(\Omega) = FT\{y_a(t)\} = \sum_{n=-\infty}^{\infty} y[n]\, FT\{g_a(t - nT)\}$

$= \sum_{n=-\infty}^{\infty} y[n]\, G_a(\Omega) e^{-j\Omega(nT)} = G_a(\Omega)\, Y_d(\Omega T)$

$Y_d(\Omega T) = \frac{1}{T} X_d(\Omega T) \cdot H_d(\Omega)$

$Y_a(\Omega) = G_a(\Omega) \cdot Y_d(\Omega T) = \frac{1}{T} G_a(\Omega) X_d(\Omega T) \cdot H_d(\Omega T)$

$Y_a(\Omega) = H_a(\Omega)\, X_a(\Omega)$

$G_a(\Omega) = \begin{cases} T & \Omega \in \{\Omega \mid H_a(\Omega) \neq 0\} \\ 0 & 0 \text{ else} \end{cases}$

Advanced sampling theory

$$x_n = \int_{-\infty}^{\infty} x_t g^*(t-nT)\,dt = \int_0^1 x_t\, g(t-nT)$$

$$x_t = \sum_{k\in\mathbb{Z}} x_k g_{n-kT}$$

$\Rightarrow \langle x_n, g\rangle$

HW #2 $y_a(t) = x_a(t) + \alpha x_a(t-\tau) + \beta x_a(t-3\tau)$

$Y_a(\Omega) = X_a(\Omega) + \alpha X_a(\Omega)e^{-j\Omega\tau} + \beta X_a(\Omega)e^{-j3\Omega\tau}$

$$H_a(\Omega) = \frac{Y_a(\Omega)}{X_a(\Omega)}$$

$H_d(\omega) = H_a(\frac{\Omega}{T})$

$Y_a(\Omega) = \frac{1}{T}G_a(\Omega)H_d(\Omega T)\,X_d(\Omega T)$

#4) $x_a(t) = \cos(1500\pi t) + \cos(3000\pi t) + \cos(3600\pi t)$

$f_s = 3000$

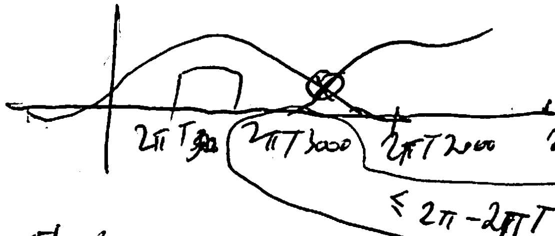
$y[n] = x[n] + x[n-1] + \cdots$

$x[n] = \sum_{l=} x_l \cos \omega\pi n$

$H_d(\omega)$

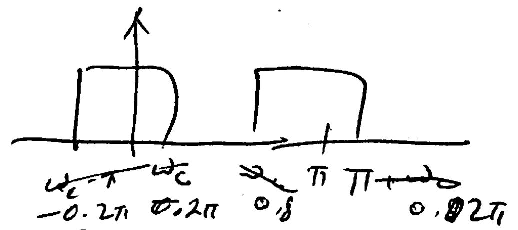
$y[n] = A_l |H_d(\omega)| \cos(\omega\pi n + \angle H_d(\omega_d))$

$y(t) = y[\frac{n}{T}]$



5] See last quiz solution.

Method 2 shift LPF to get HPF



$G_{LPF}(\omega)$

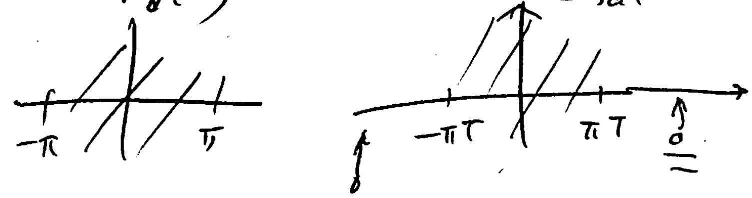
$G_{HPF}(\omega) = G_{LPF}(\omega - \pi)$

$g_{LPF} = \frac{\omega_c}{\pi} \text{sinc}\left(\omega_c \left(n - \frac{L-1}{2}\right)\right)$

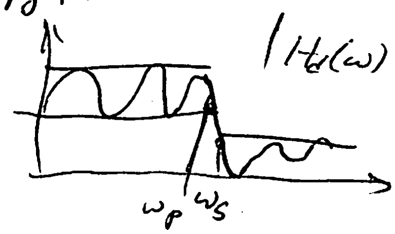
$g_{HPF} = (e^{-j\pi})^n \, G_{LPF}$

c) If signal not BL, $\rightarrow$ aliasing $\rightarrow$ NOT GLP

$\angle G_d(\omega)$ $\qquad$ $\angle G_a(\Omega)$



d) copy fr. book



e) $y[n] = \sum_{n=0}^{N-1} h[n]\, x[n]$

$N \cdot N$

$Y_d(\omega) = \underbrace{X_d(\omega)}_{\substack{FFT \\ N \log N}}^{N} \; \underbrace{H_d(\omega)}_{\text{precompute}}^{N \; hn}$

$y[n] = \text{IFFT}\{Y_d(\omega)\}$

$\min\{N \cdot N,\ 2N \log N + N\}$ for N inputs

$\frac{1}{TN} \min \{N \cdot N,\ 2N' \log N' + N'\}$ Nsamples, T sec

N' next highest power of 2 ~~for N inputs~~

f) $h_n = g_n \cdot w_n$

$y[n] = |\ \ | \cos(\cdots + \angle\ \ )$

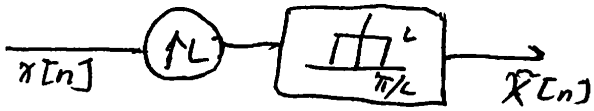
$(-1)^n \frac{\omega_c}{\pi} \operatorname{sinc}$

$L = 133$ $\quad \frac{2\pi}{133}, \frac{2\pi}{133} \cdot 2, \ldots$

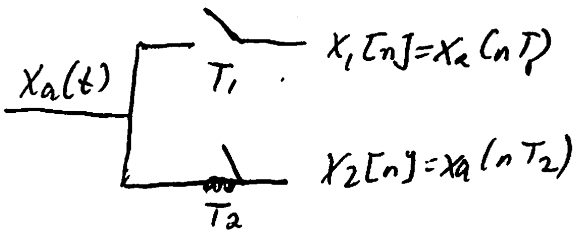
Friday 1 December Lecture

Bresler

Interpolation (Rate increase)



Decimation (Rate decrease)

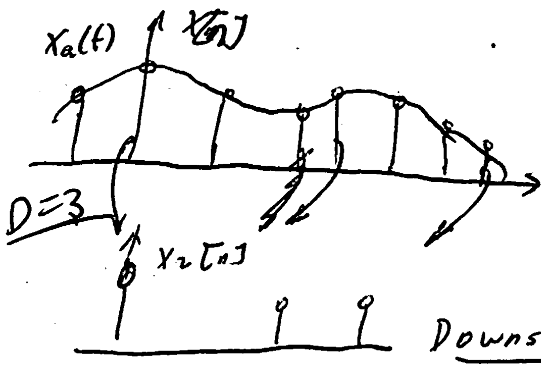


Integer

$T_2 = DT_1$

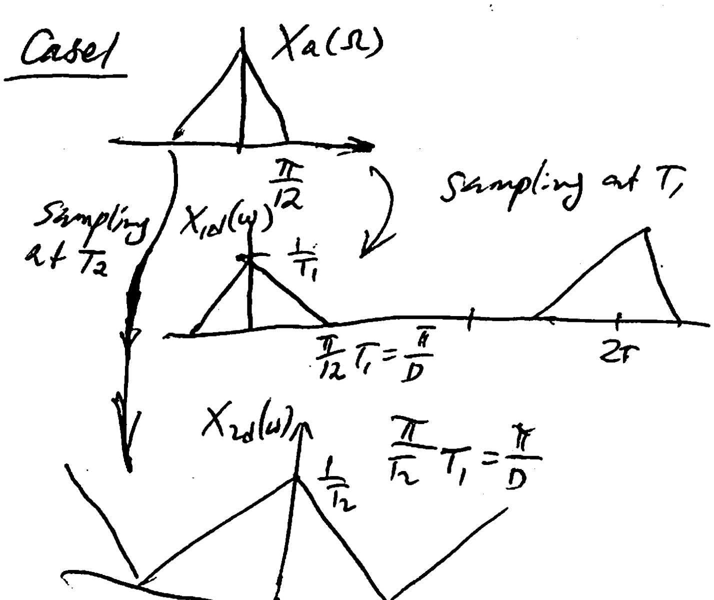
$$x_2[n] = x_a(nT_2) = x_a(nDT_1) = x_1[Dn]$$

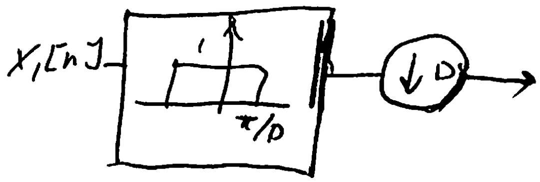
$$x_2[n] = x_1[Dn]$$



Downsampling

Case1





$x[n] \rightarrow \boxed{\downarrow D} \rightarrow v[n] = x[nD]$

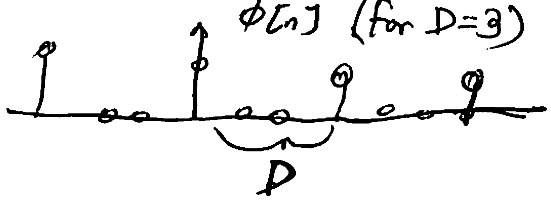
$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$V_d(\omega) = \sum_{n=-\infty}^{\infty} v[n] e^{-j\omega n}$$

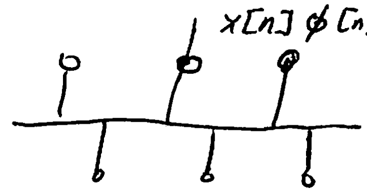
$$= \sum_{n=-\infty}^{\infty} x[nD] e^{-j\omega n} = \sum_{\substack{k=nD \\ n=-\infty}}^{\infty} x[k] e^{-j\frac{\omega}{D}k}$$

$$\phi[n] = \frac{1}{D}\sum_{\ell=0}^{D-1} e^{j\frac{2\pi}{D}n\ell} = \begin{cases} \frac{1}{D}\,\dfrac{1-e^{j\frac{2\pi}{D}n\cdot D}}{1-e^{j\frac{2\pi}{D}n}} & n \neq rD \quad \text{n not multiple of D} \\ 1 & n = rD \end{cases}$$

$\phi[n]$ (for $D=3$)



$x[n]\,\phi[n]$

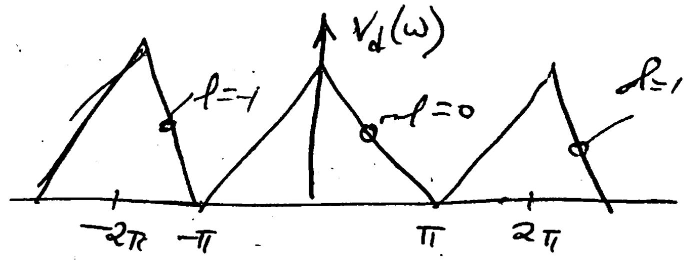


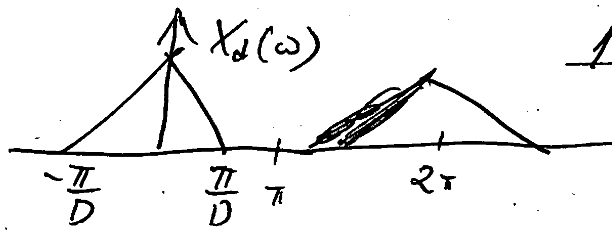
$$\sum_{k=-\infty}^{\infty} x[k]\phi[k] e^{-j\frac{\omega}{D}k} = \sum_{k=-\infty}^{\infty} x[k] \cdot \frac{1}{D}\sum_{\ell=0}^{D-1} e^{j\frac{2\pi}{D}k\ell} e^{-j\frac{\omega}{D}k}$$

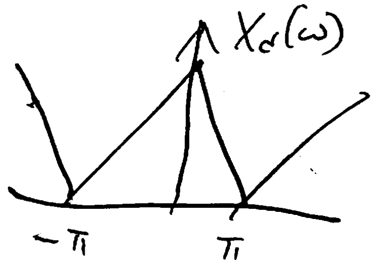
$$= \frac{1}{D}\sum_{\ell=0}^{D-1} x(k) e^{-j\underbrace{\left(\frac{\omega-2\pi\ell}{D}\right)}_{\omega'} k}$$

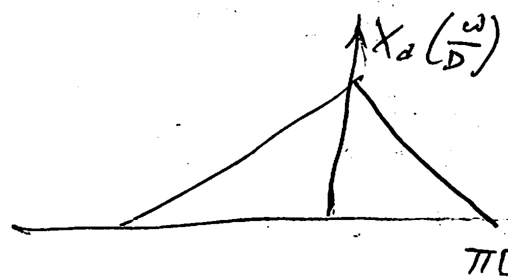
$$= \frac{1}{D}\sum_{\ell=0}^{D-1} X_d\left(\frac{\omega-2\pi\ell}{D}\right)$$

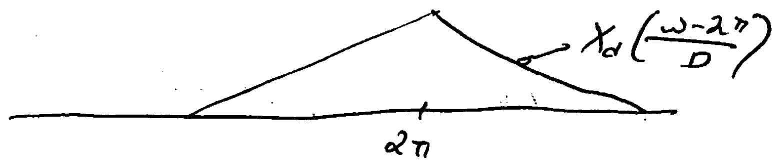
$$V_d(\omega) = \frac{1}{D}\sum_{l=0}^{D-1} X_d\left(\frac{\omega - 2\pi l}{D}\right)$$

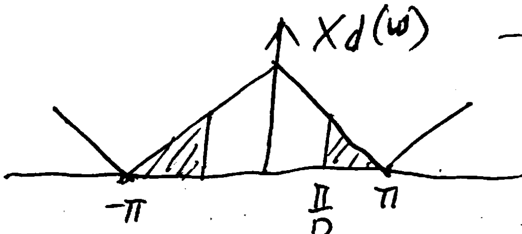


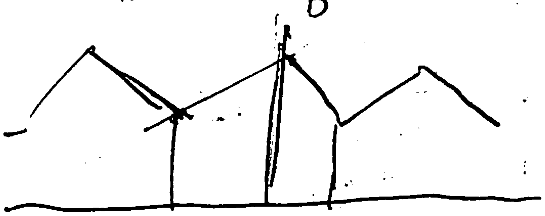










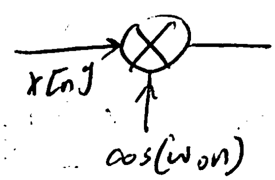


Decimation

Is this an LTI system?
✓ ✗

$x[n] = e^{j\omega_0 n}$

$x[Dn] = e^{j(D\omega_0)n}$



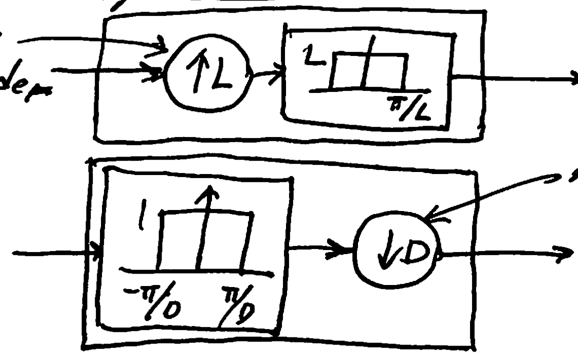
$X[n] \longrightarrow (\;)^2 \longrightarrow$

Bresler

Mon 4 Dec Lecture

Spectral Gymnastics (Multirate DSP)

Rate expander



Interplaton

rate compressor (Downsampler)

Decimator

Cont. time

$x_a(t) \longleftrightarrow X_a(\Omega)$ Duality

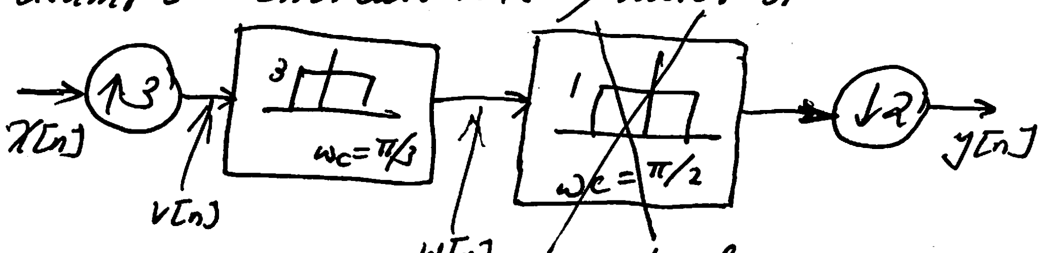
$y_a(t) = x_a(\alpha t)$

$\alpha >$

$Y_a(\Omega) = \frac{1}{|\alpha|} X_a\left(\frac{\Omega}{\alpha}\right)$

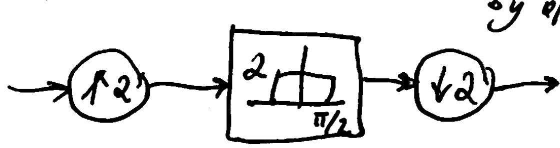
Fractional Rate Change

Example: Increase rate by factor of 1.5

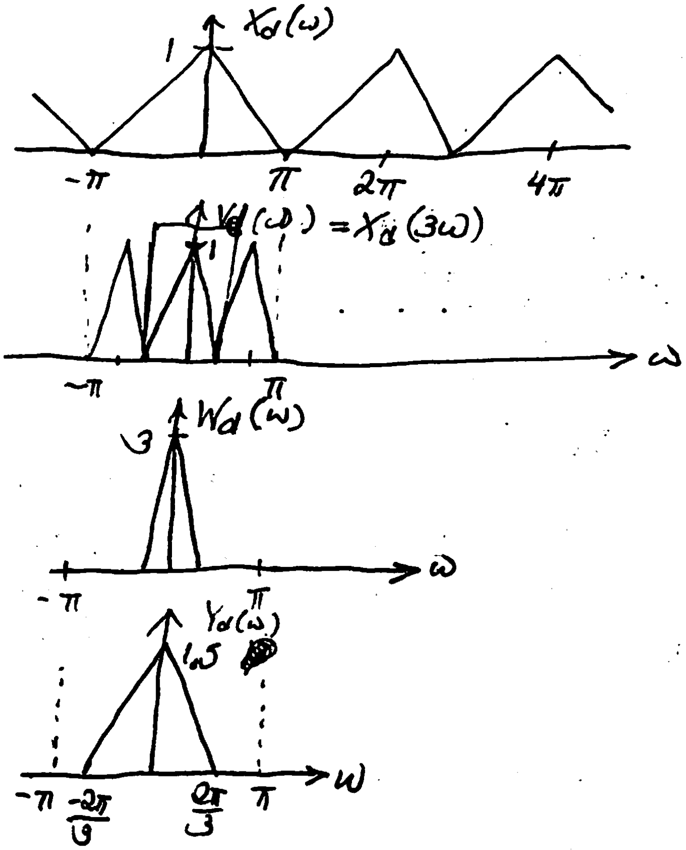


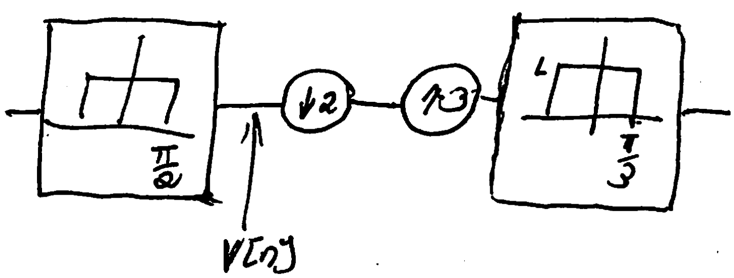
cascade of two filters ⇒ superceeded by other filter

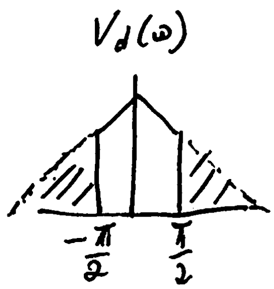
LTI? No, for all signals

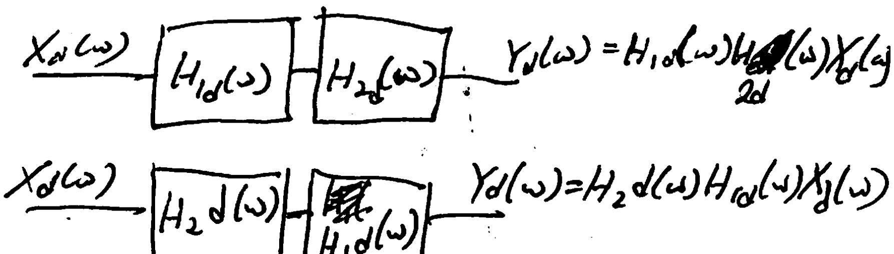


$\frac{\pi}{3} \cdot \frac{1}{1.5} = \frac{\pi}{4.5}$ X not TI







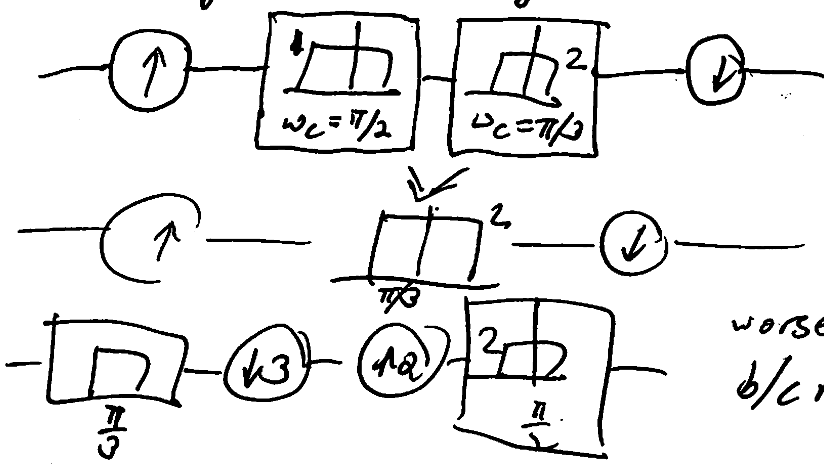


LTI

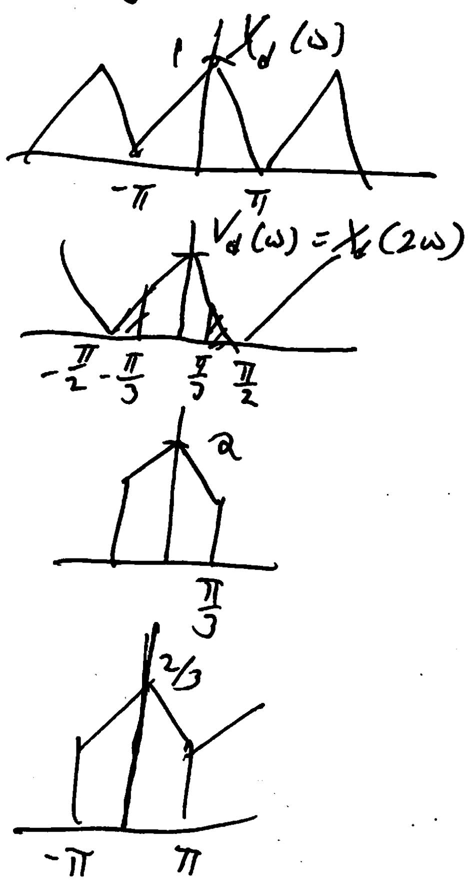
Not LTI – upsampler and downsampler in cascade
– order matters!
– down then up → req. BL signal!

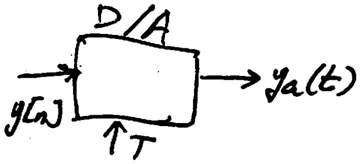
Decrease rate by 1.5x

Increase by 2x, decrease by 3x



worse scheme b/c req. BL signals more at beginning

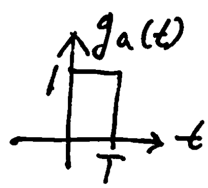


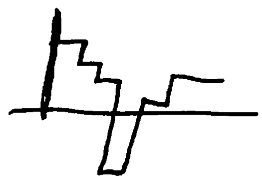


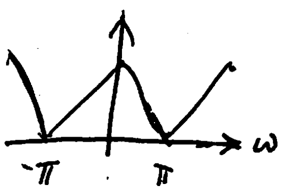
$$y_a(t) = \sum_{n=-\infty}^{\infty} y[n]\, g_a(t - nT) \longleftrightarrow Y_a(\omega) = G_a(\Omega) \cdot Y_d(\Omega T)$$

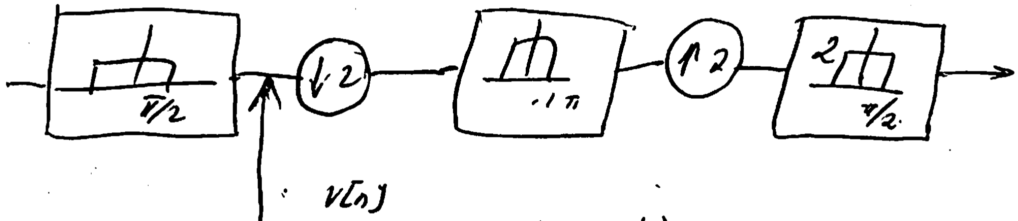
(1) Ideal D/A. $g_a(t) = \mathrm{sinc}\left(\frac{\pi}{T} t\right)$

(2) Zero Order Hold (ZOH)









$X_d(\omega)$ $\qquad V_d(\omega) = \frac{1}{2} X_d\left(\frac{\omega}{2}\right)$

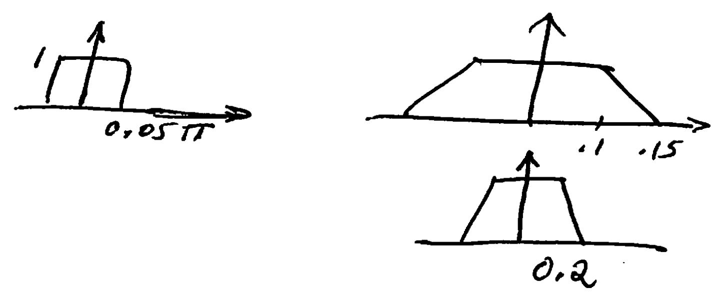
$W_d(\omega) = V_d(\omega)\, G_d(\omega)$

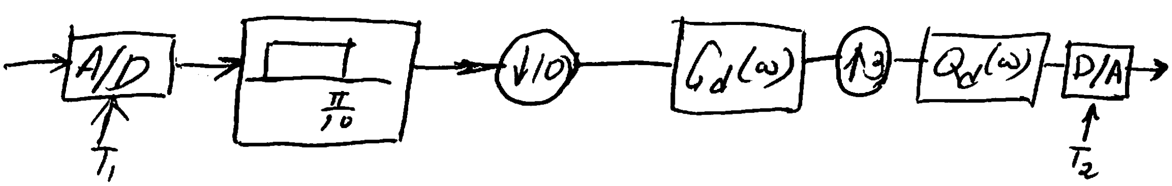
$$= \begin{cases} \frac{1}{2} X_d\left(\frac{\omega}{2}\right), & |\omega| \le .1\pi \\ 0 & \text{else} \end{cases}$$

$$Y_d(\omega) = \begin{cases} 2 W_d(2\omega), & |\omega| < \frac{\pi}{2} \\ 0, & \text{else} \end{cases}$$

$$= \begin{cases} X_d(\omega) & |\omega| < 0.05\pi \\ 0 & \text{else} \end{cases}$$

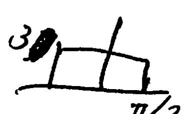
$H_d(\omega)\, X_d(\omega)$





Find T_2 and $Q_d(\omega)$ of largest bandwidth such that system is LTI?

Input BL to $\Omega_{max} \le \frac{\pi}{T_1}$

$T_2 = \frac{10}{3} T_1 \qquad Q_d(\omega) =$ 

1) Check that $Y_a(\Omega) = H_a(\Omega)\, X_a(\Omega)$

$X_a(\Omega) \rightarrow X_d(\omega) = \frac{1}{T_1} X_a\left(\frac{\omega}{T_1}\right), \ |\omega| \le \pi$

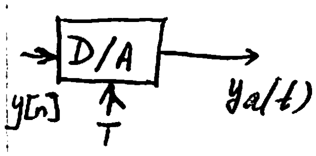
after downsampler, $\frac{1}{10}\frac{1}{T_1} X_a\left(\frac{\omega}{10T_1}\right)$

$H_a(\Omega) = G_d(3\Omega T_2)$ $\cancel{X_d(\Omega)}$

?

Boesler

Wed 6 Dec lecture

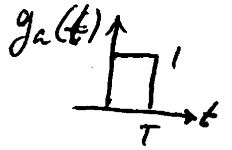


$$y_a(t) = \sum_n y[n]\, g_a(t-nT)$$

$$Y_a(\Omega) = G_a(\Omega)\, Y_d(\Omega T)$$

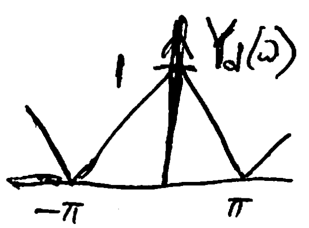
$$g_a(t) = \text{sinc}\left(\frac{\pi}{T} t\right) \text{ – Ideal D/A}$$

$$G_a(\Omega) = \begin{cases} T & |\Omega| \le \frac{\pi}{T} \\ 0 & \text{else} \end{cases} \text{ – Ideal D/A}$$

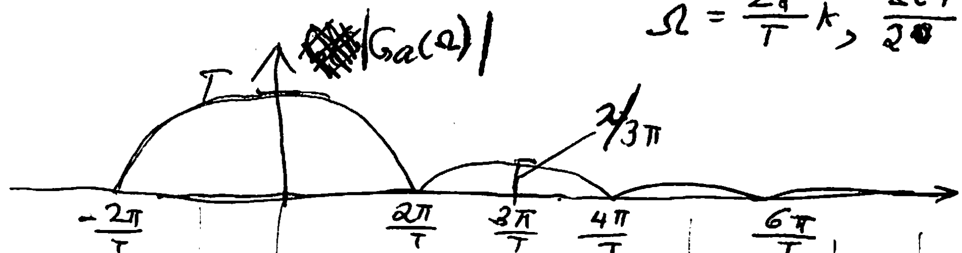


ZOH

$$G_a(\Omega) = T\underbrace{e^{-j\frac{\Omega T}{2}}}_{\text{shift by } T/2} \text{sinc}\left(\frac{\Omega T}{2}\right) \text{ – ZOH}$$

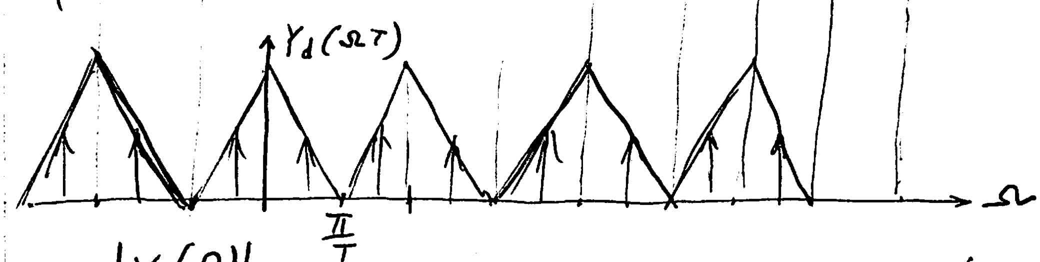


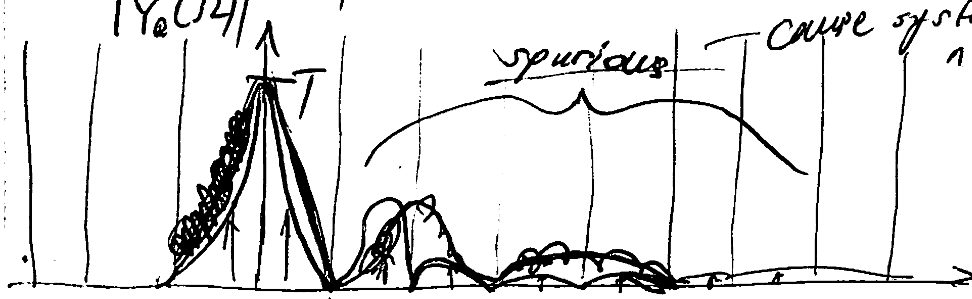
$$\Omega = \frac{2\pi}{T} k, \quad \frac{\Omega T}{2} = k\pi$$



$$\left|\frac{\sin \frac{3\pi}{2}}{\frac{3\pi}{2}}\right| = \frac{1}{\frac{3\pi}{2}} = \frac{2}{3\pi}$$

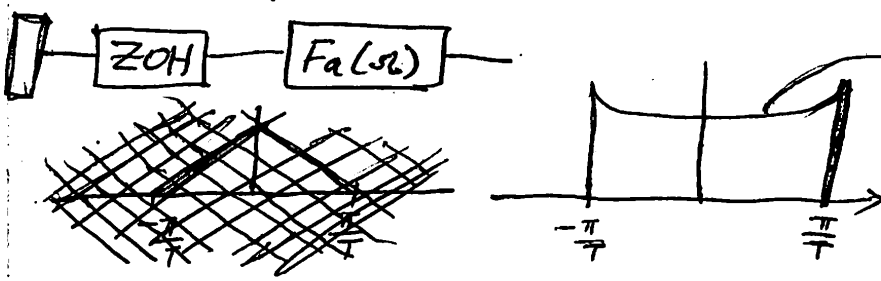
$y[n] = \cos \omega_0 n$





cause system to behave not LTI

ZOH- Compensation Filter

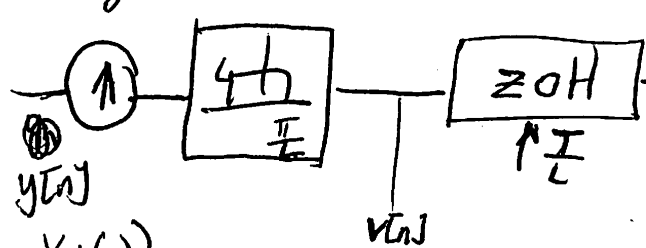


$\frac{1}{\text{sinc}\,\frac{\Omega T}{2}}$

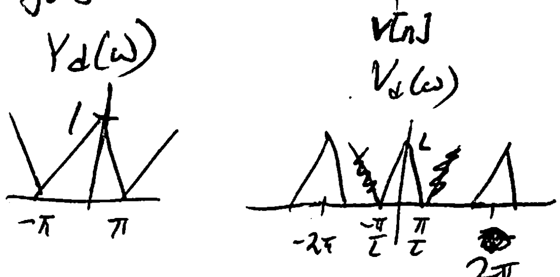
$Fa(\Omega)\cdot Ga(\Omega) = Ga(\Omega)$

↑ZOH ↑ideal

delay of T/2 in phase $\Omega = \frac{2\pi}{T}k \quad \frac{\Omega T}{2} = k\pi$

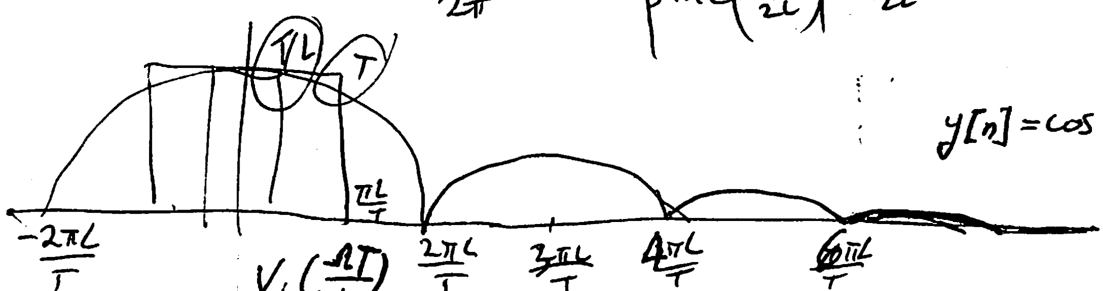


$\to Ga(\Omega)V_d(\frac{\Omega T}{L})$

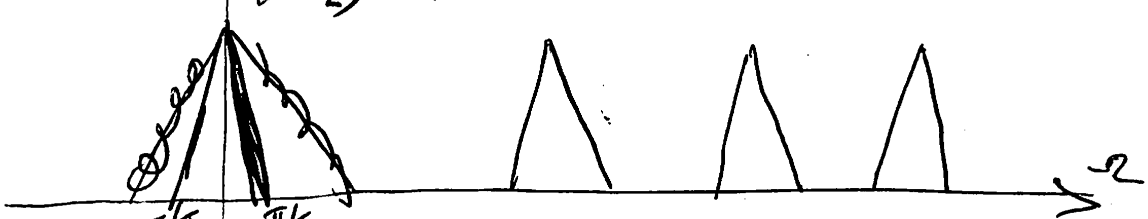


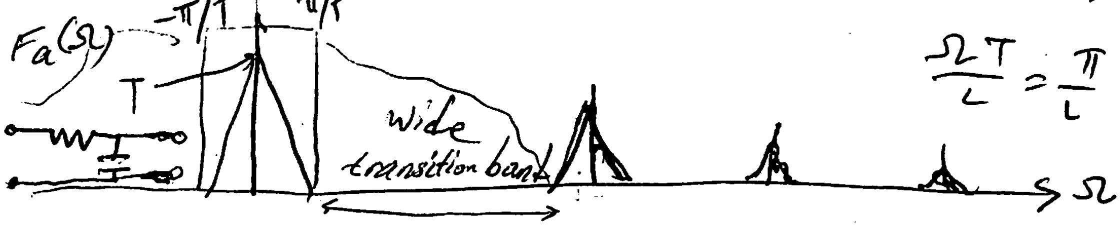
$\left|\text{sinc}\left(\frac{\Omega T}{2L}\right)\right| \quad \frac{\Omega T}{2L} = kT \quad \Omega = \frac{2k\pi L}{T}$

$y[n] = \cos \omega_0 n$



$V_d(\frac{\Omega T}{L})$

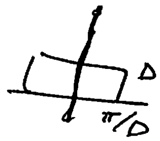




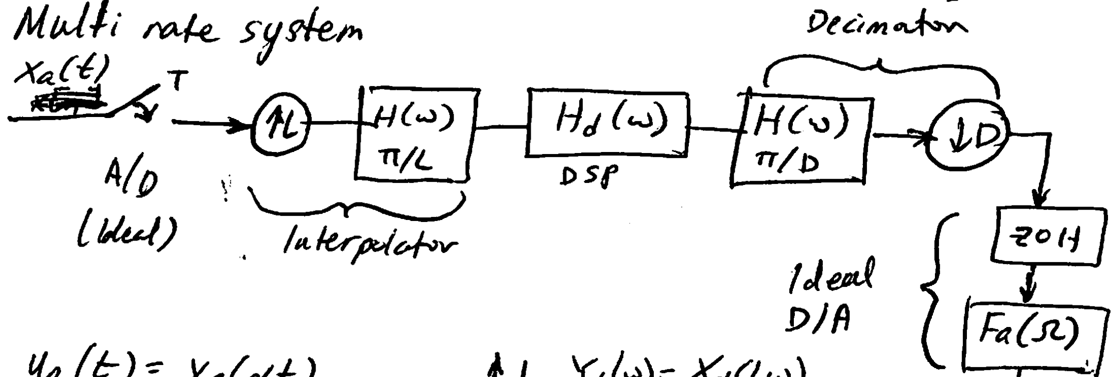
$\frac{\Omega T}{L} = \frac{\pi}{L} \quad \Omega \leftrightarrow \frac{\omega}{T}$

Ian

Wed 6 December Recitation



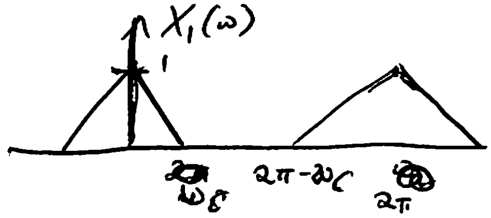
Multi rate system

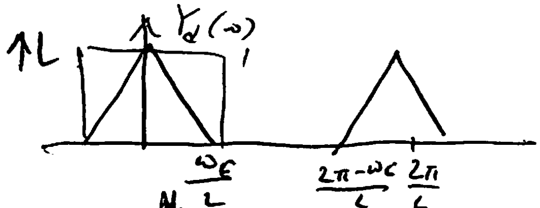


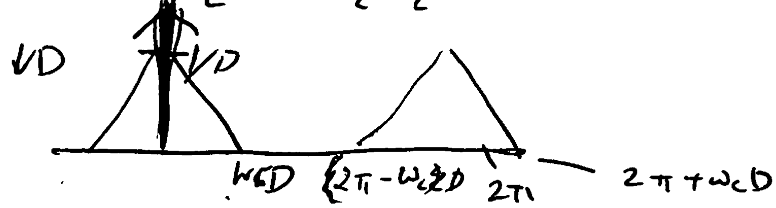
$y_a(t) = x_a(\alpha t)$ $\uparrow L \quad Y_d(\omega) = X_d(L\omega)$

$Y_a(\Omega) = \frac{1}{|\alpha|} X_a\left(\frac{\Omega}{\alpha}\right)$ $\downarrow D \quad Y_d(\omega) = \frac{1}{D}\sum_{\ell=-\infty}^{\infty} X_d\left(\frac{\omega - 2\pi \ell}{D}\right)$ $y_a(t)$

$\frac{1}{T} \sum X_a\left(\frac{\omega - 2\pi l}{T}\right)$ $\uparrow L$







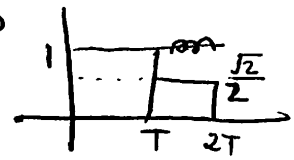
$y_a(t) = \sum_{n \in \mathbb{Z}} y[n] \cdot g(t - nT)$

$Y_a(\Omega) = Y_d(\Omega T) G_a(\Omega)$

$g(t) = \begin{cases} 1 & t \in [0, T) \\ 0 & \text{else} \end{cases}$ $y[n] = \cos\left(\frac{\pi}{4} n\right)$
$y_a(t)$

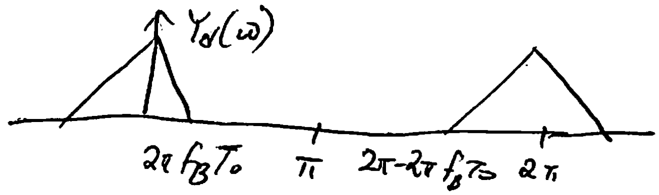
$n = 0: \; y[0] \cdot g(t - 0)$

$n = 1: \; y[1] \cdot g(t - T)$

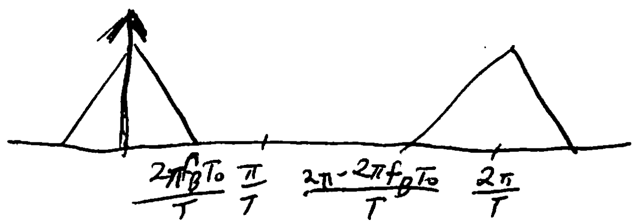


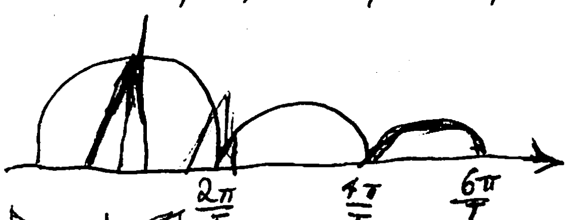
defined on → $y_0 \mathbb{I}[0,T] + y_1 \mathbb{I}[T,2T] + y_2 \mathbb{I}[2T,3T] + \dots$

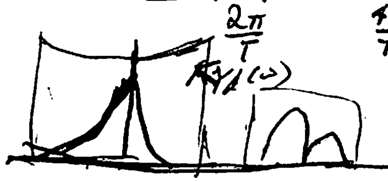
$Y_a(\Omega) = \boxed{Y_d(\Omega T)}\, G_a(\Omega)$ $\quad f_B$ Hz

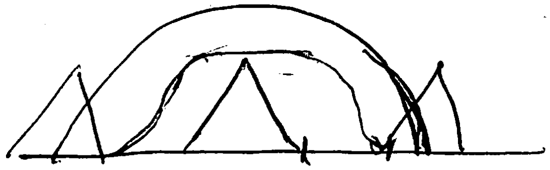


$G_a(\omega) = Te^{-\frac{j\Omega}{2}} \operatorname{sinc}\left(\frac{\Omega T}{2}\right)$



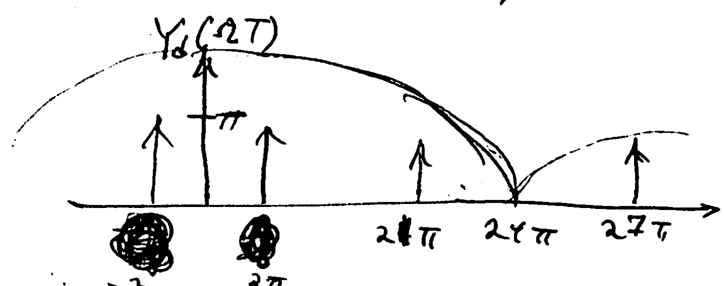






BL signal

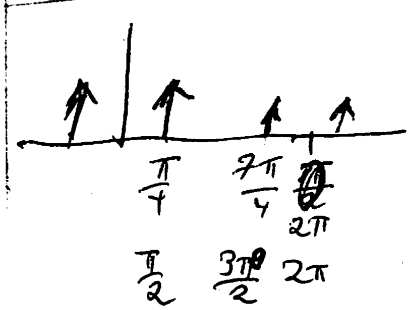
vs in lecture Full Band $[-\pi, \pi]$



HW#6 Q $\quad T = 12$ kHz

$y[n] = \cos(n\pi/4)$

HW#2 Q $\quad x[n]$



$\downarrow 2 \ \uparrow 2 \rightarrow H_d(\omega) \rightarrow y[n]$

HW Q3 $\quad f \quad 5$ kHz

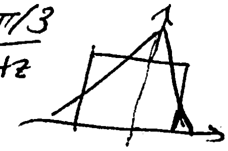
10^3 $\downarrow 5$

S $\quad 10$ kHz $\quad \frac{\omega_c}{3\text{kHz}} = \frac{\pi/3}{5\text{kHz}}$

$x[n]$

$y[n]$ $f \quad 3$ kHz

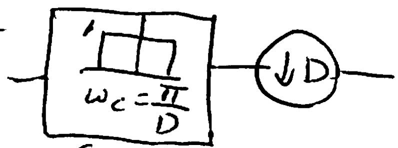
s $\quad 6$ kHz



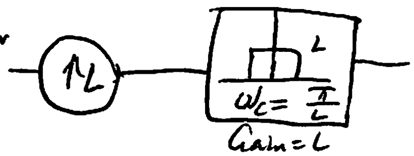
Bresler

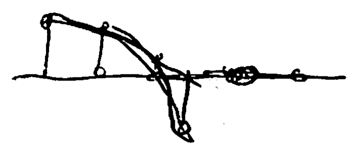
Fri 8 Dec Lecture

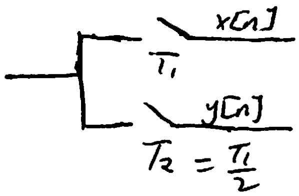
Decimator



Interpolator







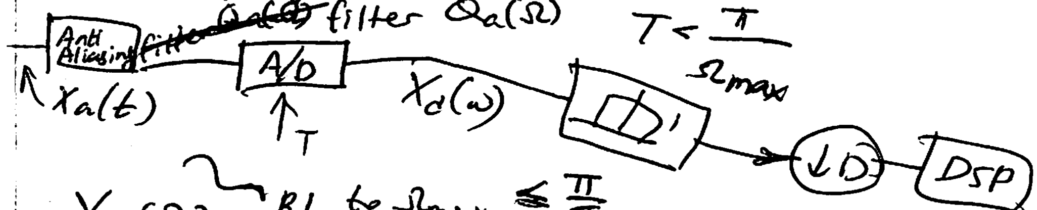
$X_d(\omega) = \frac{1}{T_1} \sum_k X_a\left(\frac{\omega - 2\pi k}{T_1}\right)$

$Y_d(\omega) = \frac{1}{T_2} \sum_k X_a\left(\frac{\omega - 2\pi k}{T_2}\right)$

$\frac{L}{T_1}$

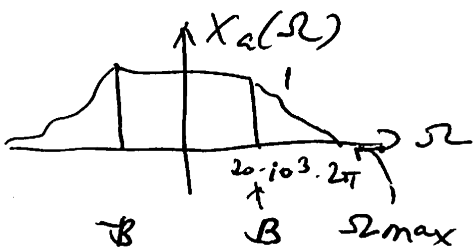
Oversampling A/D

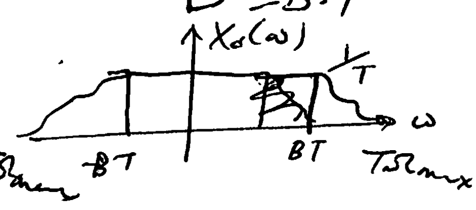
~~$Q_a(\Omega)$~~ filter $Q_a(\Omega)$ $\quad T < \frac{\pi}{\Omega_{max}}$



$X_a(\Omega)$ BL to $\Omega_{max} \leq \frac{\pi}{T}$

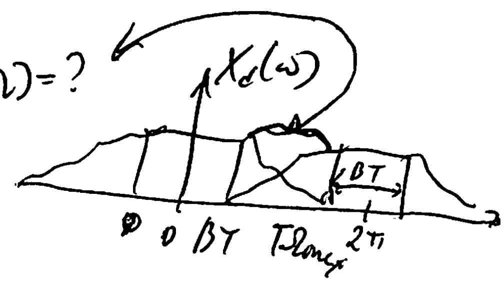
$\frac{\pi}{D} \geq B \cdot T$





Transition band for $Q_a(\Omega) = ?$

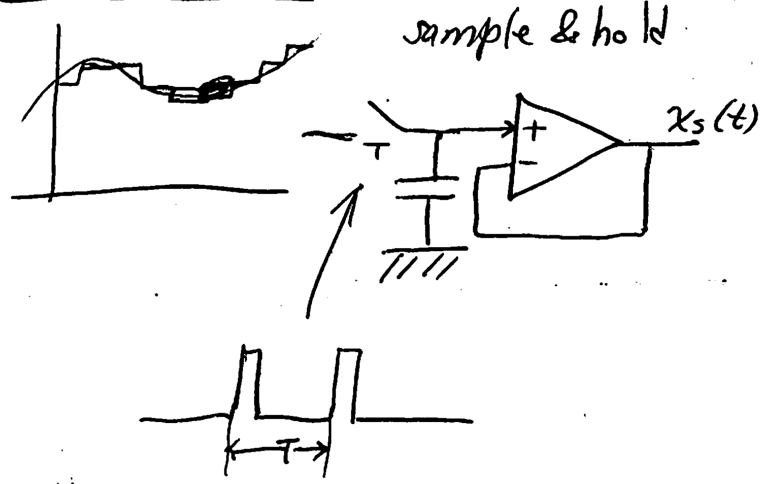
$\frac{2\pi - 2BT}{T} = 2\left(\frac{\pi}{T} - B\right)$



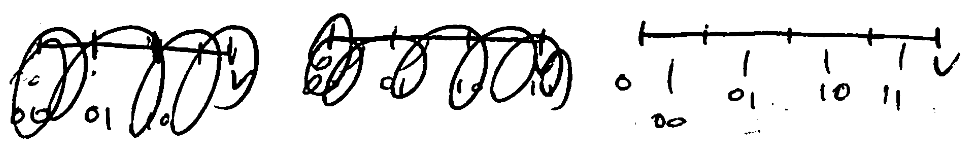
$T = \frac{\pi}{DB}$ $\quad \therefore$ Transition band is

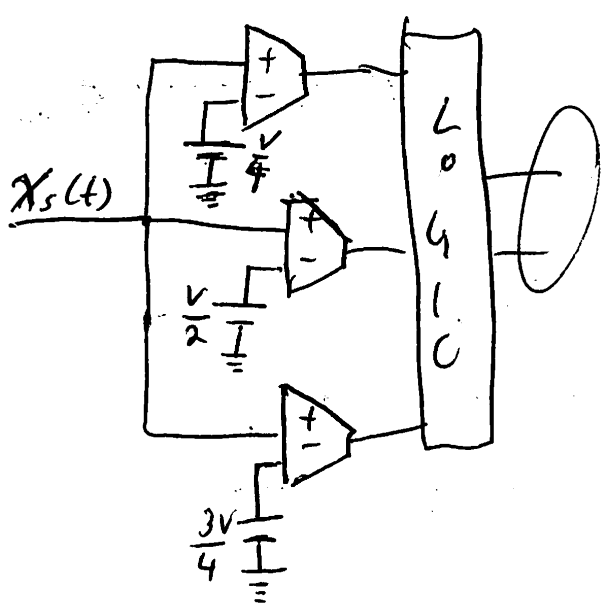
$2(DB - B) = 2B(D-1)$ max transition band

Practical A/D



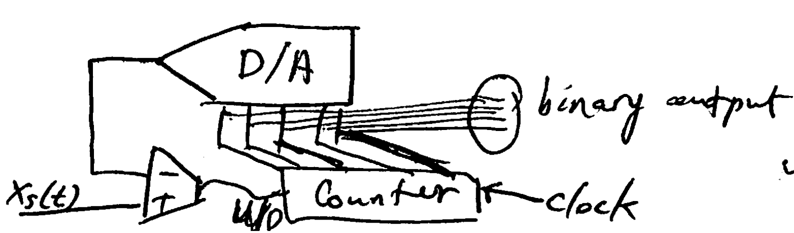
Flash A/D





Successive Approximation A/D

~ linear search

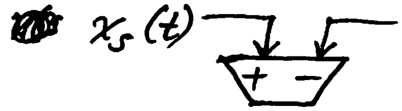


up-down counter

Bresler

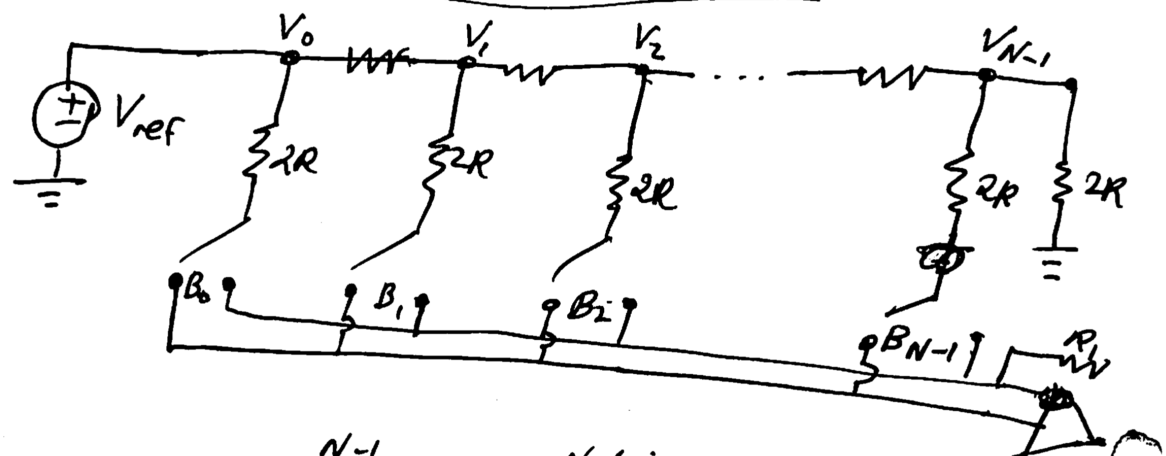
Mon 11 December lecture

Successive Approximation A/D



Flash A/D.

Resistive ladder network & switches



$$\Rightarrow V_{out} = -\frac{R_f}{2R}\sum_{i=0}^{N-1} B_i V_{N-1} 2^{N-1-i}$$

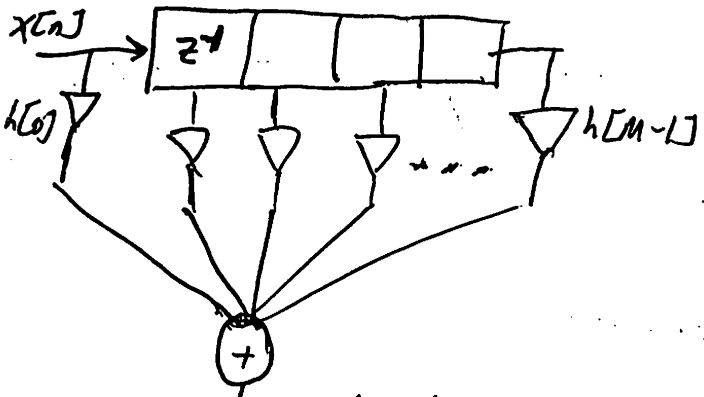
$$= -\frac{R_f}{2R} V_{N-1}\left[2^{N-1}B_0 + 2^{N-2}B_1 + \cdots + 2B_{N-2} + B_{N-1}\right.$$

$$\frac{V_{N-1}}{2R} + \frac{V_{N-1}}{2R} + \frac{V_{N-1} - V_{N-2}}{R} = 0$$

$$\Rightarrow V_{N-1} + V_{N-1} - V_{N-2} = 0 \Rightarrow V_{N-2} = 2V_{N-1}$$

Similarly: $V_{n-1} = 2V_n$, $n = 1, 2, \ldots, N-1$

FIR



$y[n] = h[n] * x[n]$

M ⊗ per input sample
$M-1$ ⊕ " " "

Total: $M \cdot N$ for length N input block

Fast Convolution

Block N - $\{x[n]\}_0^{N-1}$

zero-pad to $k = N + M - 1$ (filter length)

$\{\tilde{x}[n]\}_{n=0}^{k-1}$

FFT $\{\tilde{x}[n]\}$ ⊗ IFFT $\{y[n]\}_0^{N-M-1}$
FFT $\{\tilde{h}[n]\}$

$\left.\begin{array}{l} 3 \cdot \frac{k}{2} \log_2 k + k \ \otimes \\ 3 \cdot k \log_2 k \ \oplus \end{array}\right\}$ complex

Overlap & Add

$\{x[n]\}_{n=0}^{\infty}$ ⊛, $\{h[n]\}_{n=0}^{M-1}$

$x_k(l) = x[kL + l]$

L L L L

$\{x_0(l)\}_{l=0}^{L-1}$ $\{x_1(l)\}_{l=0}^{L-1}$ . . . $\{x_k(l)\}_{l=0}^{L-1}$ . . .

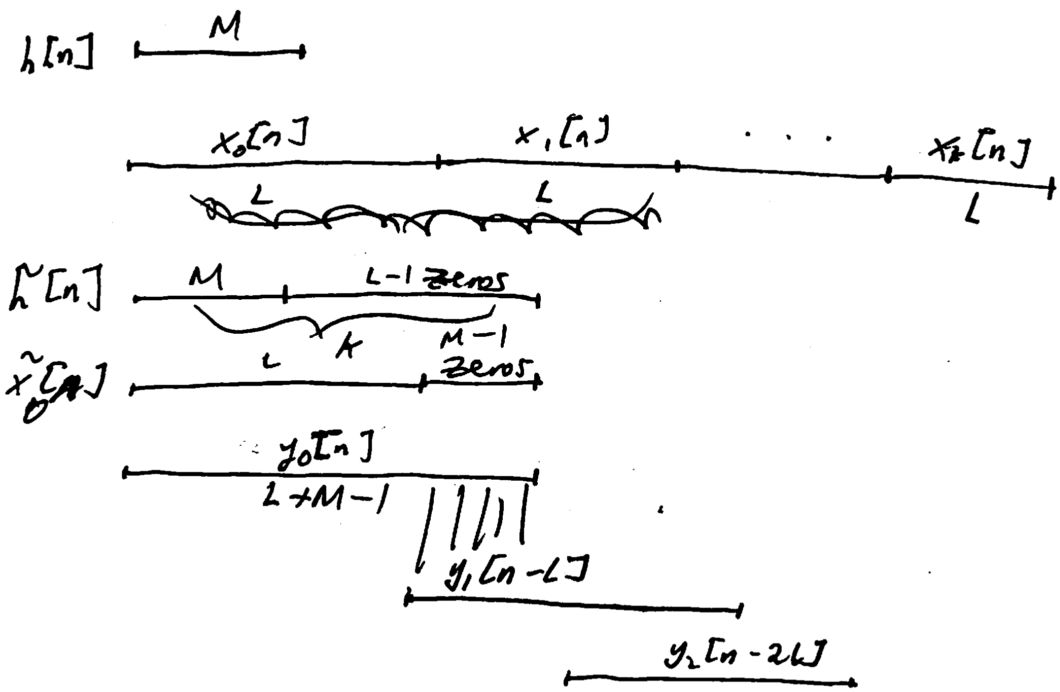
$x[n] = \sum_{k=0} x_k(n - kL)$

$y[n] = x[n] * h[n]$

$y[n] = h[n] * \sum_{k=0} x_k(n-kL) = \sum_{k=0} h[n] * x_k(n-kL)$

$x_k[n] * h[n] = y_k[n]$

$\therefore\ y[n] = \sum_{k=0} y_k[n-kL]$



<u>Example</u>

$\{h[n]\}_{n=0}^{249}$, $M = 250$

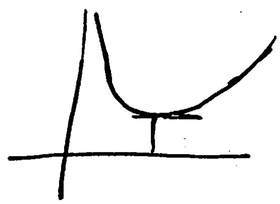
K- FFT length $= M + L - 1$

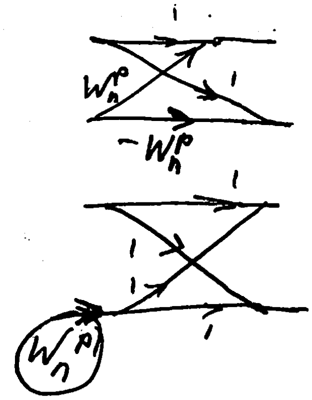
power of 2 for radix-2 FFT

$L = K - M + 1 = K - 249$

$$2\cdot\left(\frac{k}{2}\right)\log_2 k + k \ \otimes = \frac{k(\log_2 k+1)}{k-249}$$

reduce butterfly





K	L	comp. $\otimes$ / input
256	7	621.7
512	263	37.0
1024	775	27.7
2048	1799	26.2
4096	3847	26.6

× 4

Direct form: 250 $\otimes$ Input

$(a+bj)(c+dj)$

4 $\otimes$ real per complex $\otimes$

but you can do w/ 3 $\otimes$ real per complex $\otimes$

streaming

Bresler — Wed 13 December Lecture

Applications — IEEE Signal Processing Magazine

jpeg wavelets 8x8

video frame shifting of 8x8

Digital $\underbrace{\text{Processing}}_{\text{systems}}$ of signals

- Discrete time — $\cos(\omega_0 n)$ - may or may not be periodic
- cont. time signals
- periodic/aperiodic
- left-sided, right-sided, two-sided
- frequency
- $\delta[n]$, $u[n]$

$\sum_{n=-\infty}^{\infty} \delta[n-7]$, $x[n+3]$

D.T.	C.T.
z-transform,	Laplace T
DTFT	CTFT
DFT	

systems, stability ⟸ z-transform

DTFT — frequency domain

analyze LTI systems w/ products & not convolutions

filters in freq.

2π-periodic

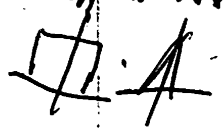
$D(\omega) \rightarrow H_d(\omega)$

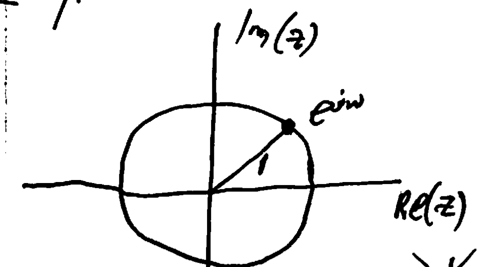
$\downarrow$

$H(z)$

Implementation

$h[n] * x[n]$





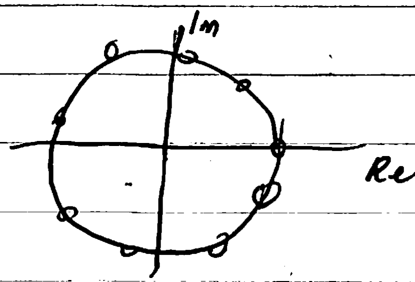
If ROC $\supset$ u.c.

$$H(z)\Big|_{z=e^{j\omega}} \overset{\Downarrow}{=} H_d(\omega)$$

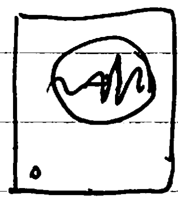
$$\sum_{n=-\infty}^{\infty} h[n] z^{-n} \qquad \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

if ROC includes the unit circle $\Leftrightarrow$ stable

DFT



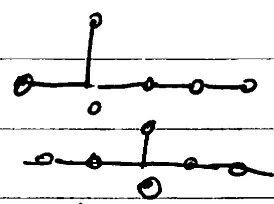
- computation
- Discrete
- DTFT ω—
- FFT!
- overlap and add on an infinite length input
- Spectral Analysis



— AAF — D/A — window — FFT — P/A

Digital Display

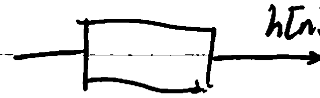
L
TI
Causality
Stability



LTI genie

cos($\omega_0 n$)) LTI

$\left(\frac{1}{2}\right)u[n]$



Causal

$h[n] \leftrightarrow H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$

$|z| > \frac{1}{2}$

$\cancel{h[n]} \leftrightarrow Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$

$|z| > \frac{1}{2}$

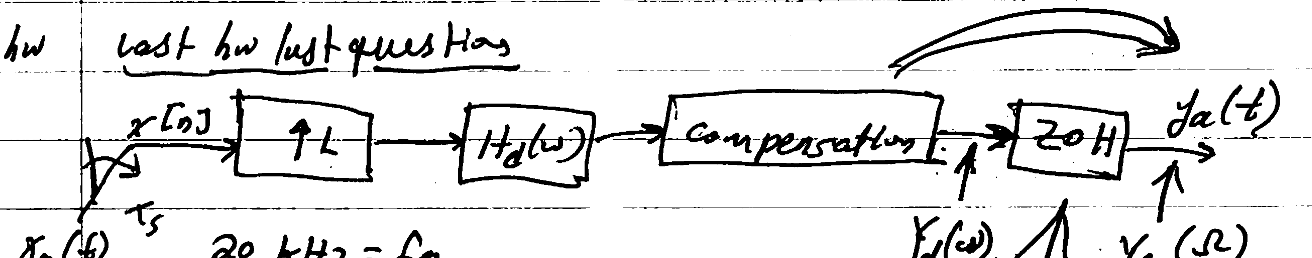
$\frac{Y(z)}{X(z)} = H(z) \ \forall X$ then LTI anti aliasing filter

lap

Wed 13 December Recitation

Newest hw

last hw last questions

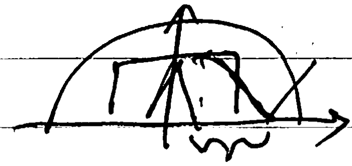


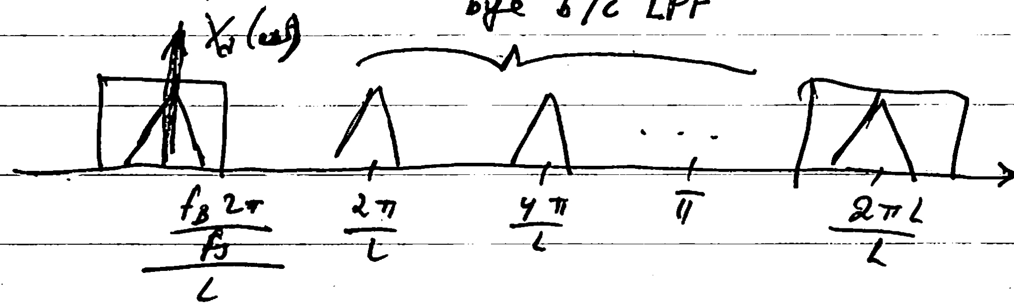
$20\ kHz = f_B$

$72\ kHz = f_s$

$Y_a(\Omega) = G(\Omega) Y_d(\Omega T)$

$T = \frac{T_s}{L}$





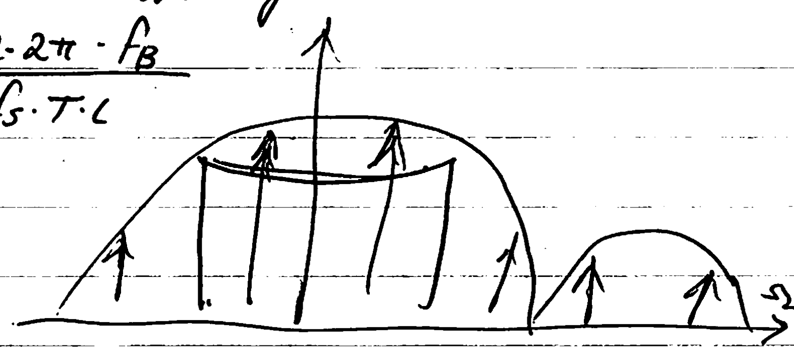
Compensation filter is analog.

$\Omega_T \le \frac{2\pi}{T} - \frac{2 \cdot 2\pi \cdot f_B}{f_s \cdot T \cdot L}$

$f_T \le L \cdot f_s - 2 f_B$

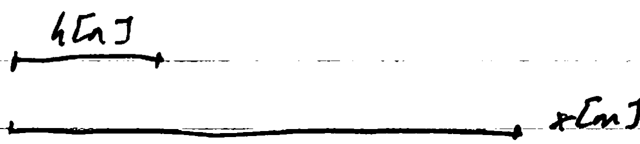
$f_s - 2 f_B$

$4 f_s - 2 f_B$



Overlap and add

$FFT^{-1}\{ FFT\{h[n]\} \cdot FFT\{x[m]\}\}$



FFT length $k = 9$

$M = 7 \qquad k = M + L - 1 \Rightarrow L = 3$

$y = x * h$

$\{x_0[n]\}$ $x[0]$ $x[1]$ $x[2]$ 0 0 0 0 . . .

$\{x_1[n]\}$ $x[3]$ $x[4]$ $x[5]$ 0 0 0 0 . . .

$IFFT\{H_d(\omega) X_0(\omega)\} + IFFT\{H_d(\omega) X_1(\omega)\}$

$\underbrace{y_0 \quad y_1 \quad y_2 \quad y_3 \quad \cdots \quad y_8}$

$\underbrace{y_{2,0} \quad y_{2,1} \quad \cdots \quad y_{2,8}}$

Spectrogram SFFT

HW 14 #2

$x = 40{,}000$

h 250

2a) real multiplications & additions

$$y[1] = \sum_{k=1}^{250} h[n] \cdot x[n-k]$$

2b) real mults & adds by fast convolution

$M+L-1 \nearrow 2^?$

$40{,}249$ 2^k

$\frac{k}{2}\log_2 k + k\log_2 k + k$

Important to know zero pad, FFT implementation w/ butterfly

$FFT\{x\}$

$FFT\{y\}$ IFFT

2c) mult & additions for overlap & add

$L = k - M + 1 = k - 249$

$\lceil \frac{N}{L} \rceil$

Michael & Ian

Thurs 14 Dec Final Review Session

Butterfly FFT (FFTB)

DIT in time
DIF in frequency

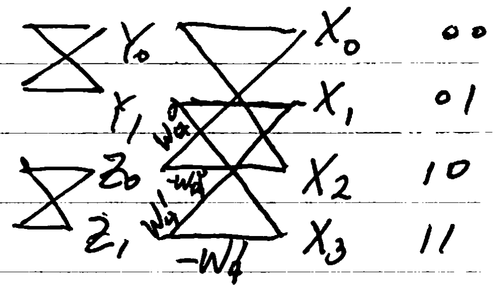
$X_k = Y_k + W_N^k Z_k$

$X_{k+\frac{N}{2}} = Y_k - W_N^k Z_k$

Twiddle factors

$W_N^k = e^{j\frac{2\pi k}{N}}$

x_0 00
x_2 10
x_1 01
x_3 11



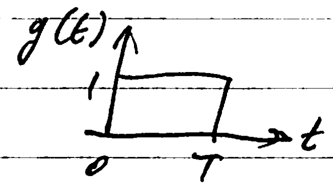
Topics
- FFT B
- Compensation Filter
- HW 14
- Overlap & Add
- Circle Convolution
- GLP

ZOH - Zero Order Hold

Approximation to the Ideal D/A

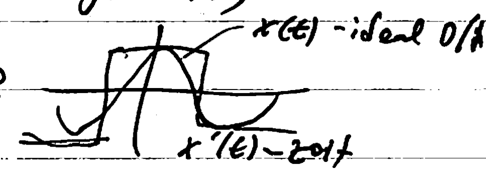
$x[n] \rightarrow \boxed{D/A} \rightarrow x(t)$
ideal

$x[n] \rightarrow \boxed{ZOH} \rightarrow x'(t) \neq x(t)$



$x[n] \rightarrow \boxed{ZOH} \rightarrow y(t) = \sum_{n=-\infty}^{\infty} x[n]\, g(t-nT)$

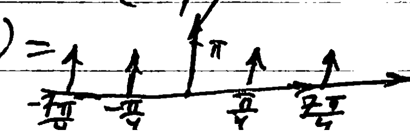
$x[n] = \cos\left(\frac{\pi n}{2}\right) = \{1, 0, -1, 0\}$

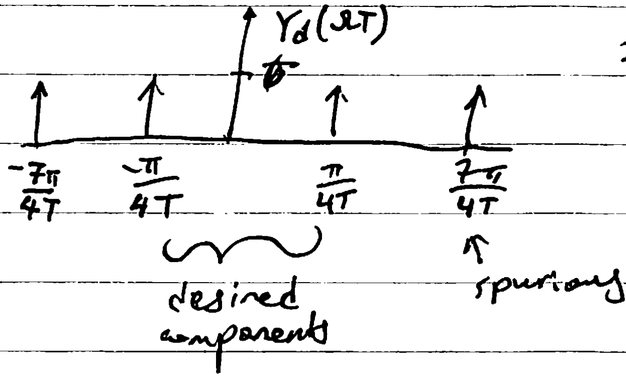


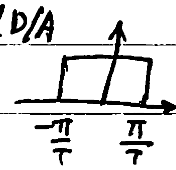
$Y_a(\Omega) = G_a(\Omega)$
CTFT of ZOH
$= T e^{-j\frac{\Omega T}{2}} \operatorname{sinc}\left(\frac{\Omega T}{2}\right)$

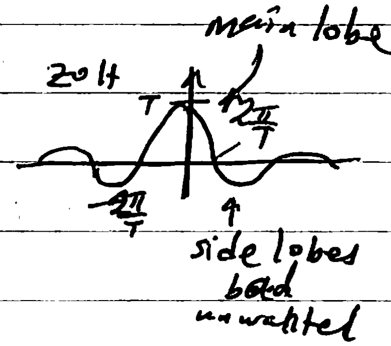
$g[n] = \cos\left(\frac{\pi n}{4}\right)$ $\qquad \omega = \Omega T$

$Y_d(\omega) =$









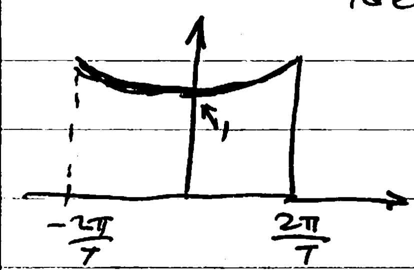
HW #13 & Quiz Solns for ZOH

ZOH = $G_a(\Omega)\ Y_d(\Omega T)$

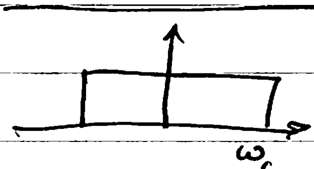
Compensation Filter

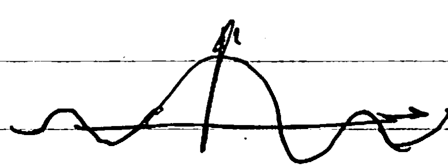
- ZOH is not ideal

$$y[n] \to \underbrace{\boxed{ZOH} \to \boxed{H_a(\Omega)}}_{\text{Ideal D/A}} \to y(t)$$



GLP Filter Design

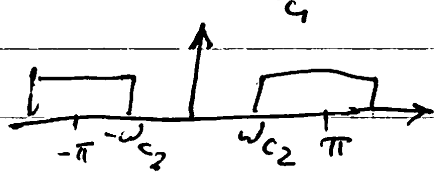


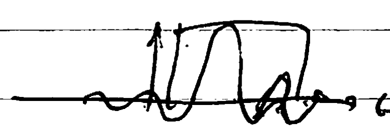


Ideal not implementable

Non causal

Infinite length



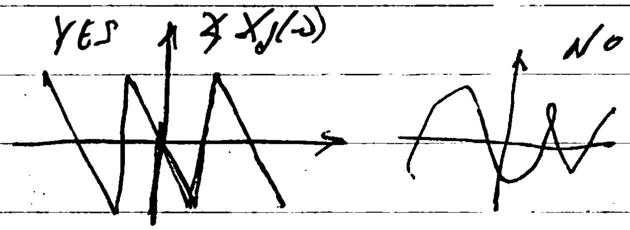


$x[n] \to X_d(\omega)$

$x[n-n_0] \to X_d(\omega)e^{-j\omega n_0}$

WRITE PROPERTIES & TABLES ON NOTE SHEET

YES $\measuredangle X_d(\omega)$ NO

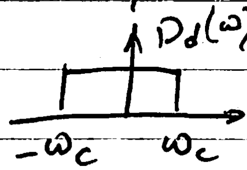


Type,

Symmetry	Length	LPF?	HPF?	BPF?	BSF?
Even	odd	Y	Y	Y	Y
Even	Even	Y	X	Y	X
odd	odd	X	X	Y	X
odd	Even	X	Y	Y	X

$h[n] = \pm h[N-1-n]$

LPF



~~Exact~~

$\frac{6.6\pi}{L} \le \Delta\omega$

Approximate

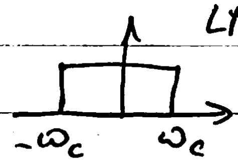
$\frac{8\pi}{L} \le \Delta\omega$

specify which one to use

Filter Design

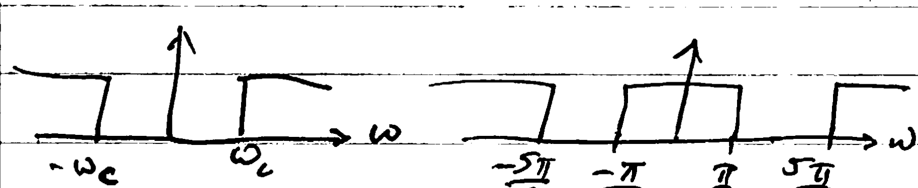
1. Find $D_d(\omega)$ - desired response
2. Determine symmetry & length + window type
3. Create $G_d(\omega) = D_d(\omega) e^{j(\alpha - M\omega)}$ $\quad M = \frac{N-1}{2}$
4. Find $g[n] = DTFT^{-1}\{G_d(\omega)\}$
5. Apply window $w[n]$

LPF



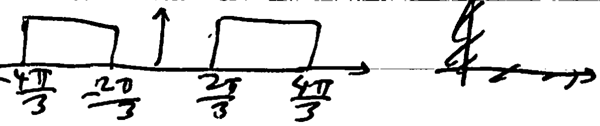
$g[n] = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\omega_c \left(n - \frac{N-1}{2}\right)\right)$

HPF

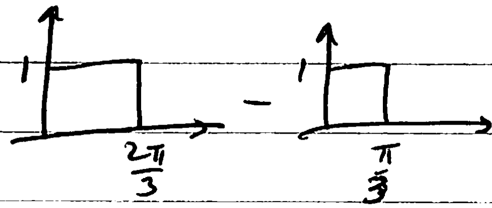


$X_d(\omega - \pi) \leftrightarrow x[n] e^{j\pi n}$

$e^{-j\pi n} = (e^{-j\pi})^n = (-1)^n$

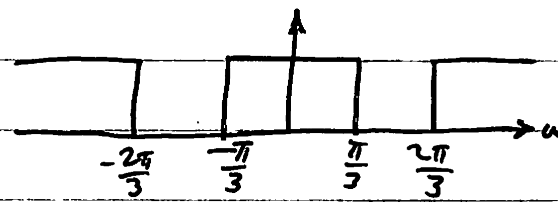


BPF



$$y[n] = \frac{\omega_2}{\pi}\,\mathrm{sinc}\left(\omega_2\left(n - \frac{N-1}{2}\right)\right) - \frac{\omega_1}{\pi}\,\mathrm{sinc}\left(\omega_1\left(n - \frac{N-1}{2}\right)\right)$$

BSF



Add LPF and HPF

$$g[n] = \frac{\omega_1}{\pi}\,\mathrm{sinc}\left(\omega_1\left(n - \frac{N-1}{2}\right)\right) + (-1)^n \frac{\pi-\omega_2}{\pi}\,\mathrm{sinc}\left(\underbrace{(\pi-\omega_2)}\left(n - \frac{N-1}{2}\right)\right)$$

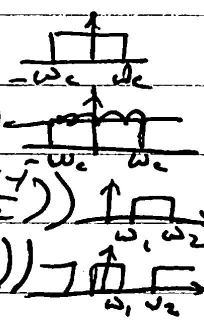
Before adding window

LPF $g[n] = \frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\omega_c\left(n - \frac{N-1}{2}\right)\right)$

HPF $g[n] = (-1)^n \frac{\pi-\omega_c}{\pi}\,\mathrm{sinc}\left((\pi-\omega_c)\left(n - \frac{N-1}{2}\right)\right)$

BPF $g[n] = \frac{\omega_2}{\pi}\,\mathrm{sinc}\left(\omega_2\left(n - \frac{N-1}{2}\right)\right) - \frac{\omega_1}{\pi}\,\mathrm{sinc}\left(\omega_1\left(n - \frac{N-1}{2}\right)\right)$

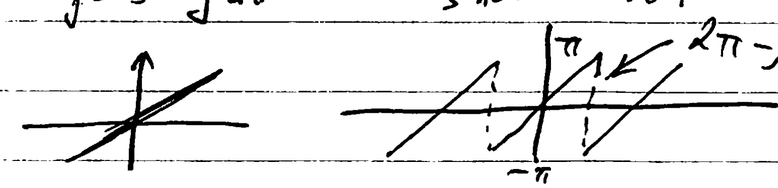
BSF $g[n] = \frac{\omega_1}{\pi}\,\mathrm{sinc}\left(\omega_2\left(n - \frac{N-1}{2}\right)\right) - \frac{\omega_1}{\pi}\,\mathrm{sinc}\left(\omega_1\left(n - \frac{N-1}{2}\right)\right)$



$\rightarrow h[n] = w[n]\, g[n]$

$g[n] = \int G_d(\omega)$ show work!

GLP (LP)



2π-jump OK for LP

ake strictly LP

$\angle H_d(\omega) = k\omega$

$R_d(\omega) \in [-\pi, \pi]$

Angle brackets & subscript $y[n] = \sum x[\langle k \rangle_N]\, h[\langle n-k \rangle_N]$ $\quad$ x, h of len N

$$y[n] = \sum_{k=-\infty}^{\infty} x[n]\, h[n-k] \quad L+M-1$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$x[n] * h[n] \rightarrow k^2$ mult

k multiplications $\frac{3k}{2}\log_2 k + k \ll k^2$

$y[n] = \mathrm{IFFT}\{\mathrm{FFT}\{x[n]\}\,\mathrm{FFT}\{h[n]\}\}$ by fast convolution

$\frac{k}{2}\log_2 k$ $\quad$ $\frac{k}{2}\log_2 k$ $\quad$ $\frac{k}{2}\log_2 k$

Overlap and Add

- $x[n]$, length M (large)
- $h[n]$, length N

$$y[n] = [h[0] \cdots \cancel{h}[N-1]] \begin{bmatrix} x[n] \\ \vdots \\ x[n-N-1] \end{bmatrix}$$

$x[n+1]$

history vector

N^2 mult / input sample
but real-time

- Break up $x[n]$ into "frames" of length L

$x[n] = [\; \underbrace{\overset{L}{\quad}}_{x_0[n]} \;|\; \underbrace{\quad}_{x_1[n]} \;|\; \quad | \quad | \quad]$ $\;0 \ldots m-1$

$h[n]$: zero-pad to length K (power of 2)

$x[n]$: " "

Take fast ~~X~~ of x, h.

Table method

$x[n] = \{ \cancel{0}\, 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$ $\quad N\cancel{0} = 9$

$h[n] = \{ 0, 1, 2 \}$ $\quad M = 3$

```
      1  2  3  4  5  6  7  8  9
 2  1  0                           → 0   y[0]
    2  1  0
       2  1  0
          2  1  0
```

power of 2, FFT size

$K = \cancel{0} N + M - 1$ ← filter len

$8 = \cancel{0} + 3 - 1$

L ← block length

$\{0, 1, 4, 7, 10, 13, 16, 19, 22, 25, 18\}$

$K = 8 \rightarrow h[n] = \{0, 1, 2, 0, 0, 0, 0, 0\}$

Fast Convolution / Circular convolutions

radix-3
definitions,
tables,
properties

① Take a block of $x[n]$

$\hookrightarrow x_0[n] = \{1, 2, 3, 4, 5, 6\}$

② Zero-pad to length $K=8$

$\hookrightarrow x_0[n] = \{1, 2, 3, 4, 5, 6, 0, 0\}$

③ Circularly convolve $x_0[n]$ with $h[n]$

1 2 3 4 5 6 0 0

0 0 0 0 0 2 1 | 0 0 0 0 0 0 2 1 | 0 → 0

2 | 1 0 → 1

→ 7

→ 10

→ 13

→ 16

→ 12

$y_0[n] = \{0, 1, 7, 10, 13, 16, 12\}$

Block 0

7 8 9 0 0 0 0 0

$y_1[n] = \{0, 7, 22, 25, 18, 0, 0, 0\}$ overlap Block 1

$y_0[n]$: 0 1 4 7 10 13 $\underbrace{16 \quad 12}_{\text{overlap}}$

0 7 22 25 18

---

0 1 4 7 10 13 16 19 22 25 18

\# of multiplications:

$x[n]$: $[_0 - L - - L - - L -]_{m-1}$

\# blocks: $\left\lceil \frac{M}{L} \right\rceil$

<u>On each block</u>

FFT of block: $\frac{K}{2}\log_2(K)$

Multiply w/ FFT of filter K

IFFT of result: $\frac{K}{2}\log_2(K)$

For each block: $K\log_2 K + K$

~~Overall~~ →

Overlap & Save

Real vs complex multiplies

FIR/IIR benefits!

quiz/hw prob.

mult/block # of blocks FFT of h[n]

Overall: $$\frac{(k\log_2 k + k)\left(\left\lceil \frac{M}{L} \right\rceil\right) + \frac{k}{2}\log_2(k)}{M}$$

(per input sample)

FFT length power of 2

of input samples

Addition: $2k\log_2 k$

Eigenfunctions (of LTI systems)

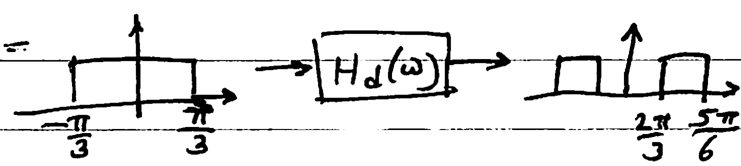
LTI system $H_d(\omega)$

$$e^{j\omega_0 n} \rightarrow \boxed{H_d(\omega)}\ e^{j\omega_0 n} H_d(\omega_0)$$

for $H_d(\omega)$ real $\leftrightarrow H_d(\omega) = H_d^*(-\omega)$

$$\cos(\omega_0 n + \phi) \rightarrow \boxed{H_d(\omega)} \rightarrow |H_d(\omega_0)| \cos(\omega_0 n + \phi + \measuredangle H_d(\omega_0))$$

$X_d(\omega) =$



For non-real systems, break up cosine:

$\cos(\omega_0 n + \phi)$

$$\Rightarrow \frac{e^{j\omega_0 n} e^{j\phi} + e^{-j\omega_0 n} e^{-j\phi}}{2}$$

Geometric series

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$