MATH 413 M 1/30 Lecture

$$\frac{n!}{n_1! \cdots n_k!} \qquad n = n_1 + \cdots + n_k$$

$$\frac{n!}{k!\, n_1! \cdots n_k!} \qquad \text{if the boxes are not labeled, and } n_1 = n_2 = \cdots = n_k$$

§ 3.5

Ex 1 What is # of nondecreasing sequences of length r whose terms are taken from 1, 2, ..., k?

Sol We count # of r-combinations from $\{\infty\cdot 1, \infty\cdot 2, \ldots \infty\cdot k\}$

$$= \binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

Ex 2 Let $S = \{10\cdot a, 10\cdot b, 10\cdot c, 10\cdot d\}$. What is # of 10-combinations of S which have the property that each of $\{a, b, c, d\}$ occurs at least once?

Sol Let x_1 = # of a's in the 10-combination.

x_2 = # of b's

x_3 = # of c's

x_4 = # of d's

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 10 \\ x_1 \geq 1,\ x_2 \geq 1,\ x_3 \geq 1,\ x_4 \geq 1 \end{cases}$$

$x_1 \geq 1 \Leftrightarrow x_1 - 1 \geq 0$

Let $y_1 = x_1 - 1$, $y_1 \geq 0$

$y_2 = x_2 - 1$, $y_2 \geq 0$

$y_3 = x_3 - 1$, $y_3 \geq 0$

$y_4 = x_4 - 1$, $y_4 \geq 0$

$y_1 + y_2 + y_3 + y_4 = 10 - 4 = 6$

# of solutions to

$$\begin{cases} y_1 + y_2 + y_3 + y_4 = 6 \\ y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0 \end{cases}$$

$$= \binom{6+4-1}{6} = \binom{9}{6} = \binom{9}{3} = 84$$

$S = \{n_1 \cdot a_1, n_2 \cdot a_2, n_3 \cdot a_3, \ldots, n_k \cdot a_k\}$

# of r-combinations of S = # of integral solutions of

$$\begin{cases} x_1 + x_2 + \cdots + x_k = r \\ 0 \le x_1 \le n_1,\ 0 \le x_2 \le n_2,\ \ldots,\ 0 \le x_k \le n_k \end{cases}$$

HW §2.7: 37, 39, 43

Ch 3

§3.1 Pigeonhole Principle: simple form

Thm 1 If $n+1$ objects are put into n boxes, then at least one box contains two or more of objects.

Proof Assume there is NO box ~~containing~~ containing two or more of the objects.
Then each box contains at most 1 object.
Since there are only n boxes, we have at most n objects.
But we have $n+1$ boxes. A contradiction.

Ex 1 Among 13 people there are two who have their birthdays in the same month.
There are 12 months in each year. By Thm 1, we get the conclusion.

Ex 2. There are n married couples. How many of $2n$ people must be selected in order to ~~get~~ guarantee that one has selected a married couple?

Sol Need to select $n+1$ persons.
Consider n couples as n boxes.
Put $n+1$ people into n boxes. By Thm. 1, there is a box containing two persons. Thus we get a couple.

Ex3 Given m integers $a_1, a_2, \dots, a_m$, there exist integers k and l with $0 \le k < l \le m$ such that $a_{k+1} + a_{k+2} + \cdots + a_l$ is divisible by m.

Consider $a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3, \dots,\ a_1 + a_2 + \cdots + a_m$

Case 1 If one of these sums is divisible by m,
then ~~we are done~~ DONE!

Case 2 None of these sums is divisible by m.
When we divide each of these sums by m,
the remainder is one of $1, 2, \dots, m-1$
By Pigeonhole Principle, there are two of ~~sums~~ the sums have the same remainder,
say $a_1 + \cdots + a_l$
$a_1 + \cdots + a_k$ with $l > k$.

then

$$m \mid (a_1 + \cdots + a_l) - (a_1 + \cdots + a_k)$$
$$\Leftrightarrow m \mid (a_{k+1} + a_{k+2} + \cdots + a_l)$$

MATH 413 F 2/3 Lecture

Strong form of Pigeonhole Principle:

Put $q_1 + \cdots + q_n - n + 1$ objects into n boxes

Then one box contains q_j objects for some $j \in \{1, \ldots, n\}$

Cor. 1. If $n(r-1)+1$ objects are put into n boxes, then at least one box contains at least r objects.

Ex 1 $m_1, \ldots, m_n$ are non-negative integers.

Suppose that $\frac{m_1 + \cdots + m_n}{n} > r-1$

Show that one of $m_1, \ldots, m_n$ is $\geq r$

Pf It follows from Cor 1.

Ex 2 A basket of fruit is being arranged out of apples, bananas and oranges. What is the smallest # of pieces of fruit in order to guarantee that either there are at least 7 a's or at least 8 b's or at least 6 o's.

Sol $q_1 = 7, q_2 = 8, q_3 = 6$

$q_1 + q_2 + q_3 - 3 + 1 = \boxed{19}$

Ex 3 Show that every seq $a_1, a_2, \ldots, a_{n^2+1}$ of n^2+1 real # contains either an increasing subseq of length $n+1$ or a decreasing subseq of length $n+1$

$\begin{pmatrix} a_1 \leq a_2 \leq \ldots \leq a_m \\ a_1 \geq a_2 \geq \ldots \geq a_m \end{pmatrix}$

Suppose that there is NO increasing subseq of length $n+1$.

We show that there is a decreasing subseq of length $n+1$

For any a_k in the seq $(1 \leq k \leq n^2+1)$

Let m_k be the length of the longest increasing subseq which begins with a_k.

$1 \leq m_1, m_2, \ldots, m_{n^2+1} \leq n$

By Pigeonholing, $m+1$ of $m_1, \dots, m_{n^2+1}$ must be equal!

Say, $m_{k_1} = m_{k_2} = \dots = m_{k_{n+1}}$

(Here $1 \le k_1 < k_2 < \dots < k_{n+1} \le n^2+1$)

Claim 1 $a_{k_1} \ge a_{k_2} \ge \dots \ge a_{k_{n+1}}$

Proof of claim 1 (by contradiction)

Assume that there is an integer $j \in \{1, \dots, n+1\}$ s.t.

$a_{k_j} < a_{k_{j+1}}$ (*)

Recall the definitions of m_{k_j}, $m_{k_{j+1}}$

First we know $m_{k_j} = m_{k_{j+1}}$

But by (*), $m_{k_j} > m_{k_{j+1}}$. A contradiction!

HW: § 3.4: 14, 15, 16, 17, 18

§ 3.3 Ramsey Thm:

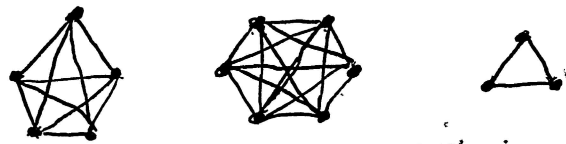
$n \in \mathbb{N}$ K_n — a graph with n points in the plane
(no 3 of which are collinear)
and all line segments connecting each pair of points

K_1 K_2 K_3

MATH 413 M 2/6 Lecture

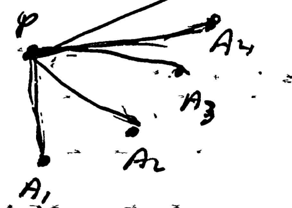
§3-3 Ramsey Thm

K_n. K_5 K_6 K_3



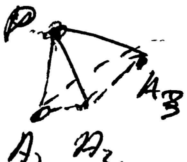
Thm 1 Suppose K_6 is colored with Red and Blue. Then there exists either a Red K_3 or a Blue K_3.

Proof



Let P be any point in K_6.
Let $A_1 \cdots A_5$ be other 5 points in K_6.
Then 5 edges, $PA_1, PA_2, PA_3, PA_4, PA_5$ in K_6.

By Pigeonhole Principle, at least 3 edges are colored by the same color.

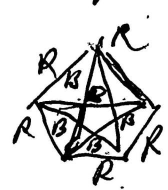


(WLOG), $PA1, PA2, PA3$ have the same color (red).
Consider K_3 formed by A_1, A_2, A_3

Case 1 If one of A_1A_2, A_1A_3, A_2A_3 is colored by Red, say A_1A_2, then we get a red K_3 formed by P, A_1, A_2.

Case 2 A_1A_2, A_1A_3, A_2A_3 are all blue.
Then we get a blue K_3, formed by A_1, A_2, A_3.

$K_5 \not\to K_2 K_3$?



Cannot form minimum 5 required

Thm 2 General form of Ramsey's Theorem
Let $m \geq 2$, $n \geq 2$ be integers. Then there is a positive integer p such that $K_p \to K_m, K_n$.
The Ramsey number $r(m,n)$ = the smallest p s.t. $K_p \to K_m, K_n$.
$r(3,3) = 6$. $r(m,n) = r(n,m)$.

$r(2,n) = n$ (pf in text)

$L(m, n) \le r(m, n) \le U(m, n)$

$k \in \mathbb{N}$ and $k \ge 2$

$K_p \to K_{n_1}, K_{n_2}, \ldots, K_{n_k}$

($\Longleftrightarrow$ If K_p is colored by k colors, then there is $i \in \{1, 2, \ldots, k\}$ such that K_{n_i} is colored with the same color.

$r(n_1, n_2, \ldots, n_k)$ = the smallest p s.t. $K_p \to K_{n_1}, \ldots, K_{n_k}$

Homework §3.4: #20.

$r(3, 3, 3) = 17$

m, n, l

§5.1 Pascal's Formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Thm 1. (Pascal's Formula) For any integers n, k with $1 \le k \le n-1$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k+1}$$

pf $S = \{a_1, a_2, \ldots, a_n\}$

of k-combinations of $S = \binom{n}{k}$

Case 1 k-combinations not containing a_1 $\binom{n-1}{k}$

Case 2 k-combinations containing a_1 $\binom{n-1}{k-1}$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$1 \quad 3 \quad 3 \quad 1$$

MATH 413 W 2/8 Lecture

§5.2. The Binomial Thm $n \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$(x+y)^0 = 1$

$(x+y)^1 = 1\cdot x + 1\cdot y$

$(x+y)^2 = 1\cdot x^2 + 2xy + 1\cdot y^2$

$(x+y)^3 = 1\cdot x^3 + 3x^2y + 3xy^2 + 1\cdot y^3$

$(x+y)^4 = 1\cdot x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1\cdot y^4$

Pf. Method 1.

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n}$$

$x^k y^{n-k}$. To get such a term,

Step 1 Find k factors from n factors so that each of k factors contributes x. $\binom{n}{k}$

Step 2. select $n-k$ factors from the remaining $n-k$ factors contributing y.

$\binom{n-k}{n-k} = 1$.

$\binom{n}{k}\binom{n-k}{n-k} = \binom{n}{k}$.

Method 2. Induction on n.

$n=1$. $(x+y)^1 = 1\cdot x + 1\cdot y = \binom{1}{0}x + \binom{1}{1}y$

Assume that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

Under this assumption, we need to show

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k$$

$$(x+y)^{n+1} = (x+y)^n (x+y) = \left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\right)(x+y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$$

$$= \binom{n}{0} x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n-1} \binom{n}{k} x^{n-k} y^{k+1} + \binom{n}{n} y^{n+1}$$

$$\sum_{k=0}^{n-1} \binom{n}{k} x^{n-k} y^{k+1} = \sum_{k=1}^{n} \binom{n}{k-1} x^{n-(k-1)} y^{(k-1)+1}$$

$$\rightarrow = \binom{n}{0} x^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] x^{n-k+1} y^k + \binom{n}{n} y^{n+1}$$

$$= \binom{n}{0} x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k + \binom{n}{n} y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k + \binom{n+1}{n+1} y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k$$

Homework: §5.7: 2, 4, 6, 7

§5.3 Identities $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

Let $x=y=1$. $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Let $x=1, y=-1$

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{k \text{ even}} \binom{n}{k} - \sum_{k \text{ odd}} \binom{n}{k}$$

$$\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k} = 2^n/2 = 2^{n-1}$$

Ex 1 Show that $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$

Proof $f(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

$$f'(x) = ((1+x)^n)' = n(1+x)^{n-1}$$

$$f'(x) = \left(\sum_{k=0}^{n} \binom{n}{k} x^k\right)' = \sum_{k=0}^{n} \binom{n}{k} (x^k)'$$

$$= \sum_{k=1}^{n} \binom{n}{k} k \cdot x^{k-1} \qquad n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}$$

Let $x=1$: $n \cdot 2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$

F 2/10 Lecture

Ex 1 Show that $\sum_{k=0}^{n} k^2\binom{n}{k} = n\cdot 2^{n-1} + n(n-1)\,2^{n-2}$.

Pf $\left[(1+x)^n\right]' = \left[\sum_{k=0}^{n}\binom{n}{k}x^k\right]'$

$\boxed{n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}}$

$n(1+x)^{n-1}x = \sum_{k=0}^{n} k\binom{n}{k}x^k$

$\left[n(1+x)^{n-1}x\right]' = \sum_{k=0}^{n} k^2\binom{n}{k}x^{k-1}$

$\quad\|$

$n(1+x)^{n-1} + n(n-1)\,x\,(1+x)^{n-2}$

Let $x = 1$

$\sum_{k=0}^{n} k^2\binom{n}{k} = n2^{n-1} + n(n-1)2^{n-2}$

Ex 2 Show that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$

Pf. $S = \{n \text{ men},\ n \text{ women}\}$

We count # of n-combinations from S.

$\# = \binom{2n}{n}$

On the other hand, let $U = \{n\text{-combinations from } S\}$.

For $0 \le k \le n$, $A_k = \{n\text{-combinations from } S \text{ with } n-k \text{ women and } k \text{ men}$

$U = A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{k=0}^{n} A_k$

$A_0, \dots, A_n$ form a partition of U

$|U| = \sum_{k=0}^{n}|A_k|$

$|A_k| = \binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$

$|U| = \sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$

Ex 3 Show that

$$\binom{n}{0} + \cancel{\binom{n}{1}} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \sum_{j=0}^{k}\binom{n+j}{j} = \binom{n+k+1}{k}. \quad (*)$$

$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Pf. $k=1$. $\binom{n}{0}+\binom{n+1}{1} = \binom{n+1}{0}+\binom{n+1}{1} = \binom{n+2}{1}$

$k=2$. $\left(\binom{n}{0}+\binom{n+1}{1}\right)+\binom{n+2}{2} = \binom{n+2}{1}+\binom{n+2}{2} = \binom{n+3}{2}$

Using Pascal Id (at most k times) we get (*)

Ex 4 $\sum_{m=0}^{n}\binom{m}{k} = \binom{n+1}{k+1}$ for $k \geq 1$, $n \geq 1$. $\left[\binom{n}{k} \cancel{k \leq n}\right]$

$\cancel{\binom{r}{k} = \begin{cases} \frac{r!}{k!(r-k)!} & \text{if} \end{cases}}$

$$\binom{r}{k} = \begin{cases} \overbrace{r\cdot(r-1)\cdots(r-k+1)}^{k} & \text{if } k \geq 1 \ \& \\ 1 & \text{if } k=0 \\ 0 & \text{if } k \leq -1 \end{cases}$$

$\binom{r}{k} = 0$ ~~for $k \geq$~~ if $r \in \mathbb{N}\cup\{0\}$, $k \in \mathbb{N}$ and $r<k$

$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$ for all $r \in \mathbb{R}$, all $k \in \mathbb{Z}$

Pf. $n=0$. $\binom{0}{k} = 0 = \binom{1}{k+1}$

$n=1$. $\binom{0}{k}+\binom{1}{k} = \binom{1}{k+1}+\binom{1}{k} = \binom{2}{k+1}$

$n=2$. $\binom{0}{k}+\binom{1}{k}+\binom{2}{k} = \binom{2}{k+1}+\binom{2}{k} = \binom{3}{k+1}$

Repeat this argument at most n times, we get

$$\sum_{m=0}^{n}\binom{m}{k} = \binom{n+1}{k+1} \quad \cancel{n,k \geq 1}$$

Homework: §5.7: 10, 13, 14, 15, 16

Exam I will be on Feb 24.

In how many ways can 2 red and 4 blue rooks be placed on an 8×8 board so that no two rooks can attack one another?

$\binom{8}{6}$

$\binom{8}{6} \cdot 6! = P(8,6)$

$\frac{6!}{4!\,2!}$

$\binom{8}{6}\binom{8}{6} 6! \frac{6!}{4!\,2!} = \frac{8!\,8!}{2!\,2!\,4!\,2!}$

M 2/13 Lecture

§5.3 Unimodality of Binomial Coefficients

Let $s_0, s_1, \ldots, s_n$ be a seq. of real numbers

if $0 \le t \le n$ s.t. $0 \le s_0 \le s_1 \le \cdots \le s_t$, $s_t \ge s_{t+1} \ge \cdots \ge s_n$

Such a seq is called unimodal.

Thm 1 $\left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right)$ is unimodal.

Pf Case 1 n even.

$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{n/2}$, $\binom{n}{n/2} > \binom{n}{n/2+1} > \cdots > \binom{n}{n}$

Case 2 n odd

$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\frac{n+1}{2}}$, $\binom{n}{\frac{n+1}{2}} > \binom{n}{\frac{n+1}{2}+1} > \cdots > \binom{n}{n}$

$x \in \mathbb{R}$

The floor of x $\lfloor x \rfloor$ = the greatest integer $\le x$

The ceiling $\lceil x \rceil$ = the smallest integer $\ge x$

$\lfloor 2.5 \rfloor = 2$, $\lceil 2.5 \rceil = 3$.

If $x \in \mathbb{Z}$, $x = \lfloor x \rfloor = \lceil x \rceil$

If n is even, $\lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$

If n is odd, then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$, $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$

$$\binom{n}{\lceil \frac{n}{2} \rceil} = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Let S be a set (a finite set)

Let $A_1, A_2, \ldots, A_k$ be subsets (combinations) of S.

If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k$

$\{A_1, \cdots, A_k\}$ is called a chain.

Def Let S be an n-set. Let $\{A_1, \ldots, A_k\}$ be a chain of S, such that 1) A_{j+1} has one more element than A_j, for any $1 \le j \le k-1$.

2) $|A_1| + |A_k| = n = |S|$

Then $\{A_1, \dots, A_k\}$ is called a symmetric chain.

<u>Def</u> Let $\mathcal{C}$ be a collection of subsets (combinations) of S.

If no combination (subset) of S ~~is contained~~ in $\mathcal{C}$ is contained in another, then $\mathcal{C}$ is called an <u>antichain</u> (clutter).

<u>Ex 1</u> $S = \{a, b, c, d\}$, $\mathcal{C} = \{\{a, b\}, \{a, b, c\}\}$

$\mathcal{C}$ is a chain.

$\mathcal{C} = \{\{a, b\}, \{b, c, d\}, \{a, d\}\}$

$\mathcal{C}_1$ is an <u>antichain</u> (clutter).

$\mathcal{C}_2 = \{\text{all 2-combinations}\}$, $\mathcal{C}_2$ is an antichain.

Let S be an n-set, $0 \le k \le n$

$\mathcal{C}_k = \{\text{all } k\text{-combinations of } S\}$, $\mathcal{C}_k$ is an antichain

<u>Ex 2</u> Let $S = \{1, 2, 3\}$.

$\mathcal{C} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$ is a symmetric chain.

<u>Def</u> The power set of S is a collection of all subsets of S.

<u>Lemma 1</u> Let S be an n-set. Then there exists a symmetric chain partition of the power set of S, that is, the power set of S $= \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_m$ where 1) $\mathcal{C}_j$'s are all symmetric chains

2) $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ if $i \ne j$, $1 \le j, j \le m$.

<u>Ex</u> $S = \{1, 2, 3\}$. Power set of $S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Let $\mathcal{C}_1 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$. $\mathcal{C}_2 = \{\{2\}, \{2, 3\}\}$, $\mathcal{C}_3 = \{\{3\}, \{1, 3\}\}$

$P(S) = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. is a symmetric chain partition.

F 2/17 Lecture

Ex A chess master will play at least 1 game every day for 11 weeks, but will NOT play more than 12 games during any week. Show that there is a consecutive days during which he will play exactly 22 games.

M 2/20 Lecture

Ex 1 Show that

$$1^2+2^2+\cdots+n^2=\sum_{j=1}^{n} j^2=\frac{n(n+1)(2n+1)}{6}$$

Pf. $j^2=2\binom{j}{2}+\binom{j}{1}$

$$\sum_{j=1}^{n} j^2=2\sum_{j=1}^{n}\binom{j}{2}+\sum_{j=1}^{n}\binom{j}{1}$$

Recall $\sum_{m=0}^{n}\binom{m}{k}=\binom{n+1}{k+1}$

Here $k\geq 1$, $n\geq 1$.

$$\sum_{j=1}^{n}\binom{j}{2}=\sum_{j=0}^{n}\binom{j}{2}=\binom{n+1}{2+1}=\binom{n+1}{3}$$

$$\sum_{j=1}^{n}\binom{j}{1}=\sum_{j=0}^{n}\binom{j}{1}=\binom{n+1}{2}$$

$$\sum_{j=1}^{n} j^2=2\binom{n+1}{3}+\binom{n+1}{2}=\frac{n(n+1)(2n+1)}{6}$$

HW: §5.7 : 17, 20, 37, 40

Exam 1: Ch 2, 3, 5.1-5.3

§5.4 The Multinomial Thm

$(x_1+x_2+\ldots+x_t)^n$

Multinomial Coefficient: Let $n=n_1+n_2+\cdots+n_t$

$$\frac{n!}{n_1!\,n_2!\cdots n_t!}=\binom{n}{n_1,n_2,\ldots,n_t}$$

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}=\binom{n}{k,\,n-k}$$

Thm 1 (Pascal)

$$(*)\quad \binom{n}{n_1, n_2, \ldots, n_t} = \binom{n-1}{n_1 - 1, n_2, \ldots, n_t} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_t} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_t - 1}$$

Pf. RHS of $(*)$

$$= \frac{(n-1)!}{(n_1-1)!\,n_2!\cdots n_t!} + \frac{(n-1)!}{n_1!(n_2-1)!\cdots n_t!} + \cdots + \frac{(n-1)!}{n_1!\,n_2!\cdots n_{t-1}!(n_t-1)!}$$

$$= \frac{(n-1)!\,n_1}{n_1!\,n_2!\cdots n_t!} + \frac{(n-1)!\,n_2}{n_1!\,n_2!\cdots n_t!} + \cdots + \frac{(n-1)!\,n_t}{n_1!\,n_2!\cdots n_t!}$$

$$= \frac{(n-1)!\overbrace{(n_1 + n_2 + \cdots + n_t)}^{=n}}{n_1!\,n_2!\cdots n_t!} = \frac{n!}{n_1!\,n_2!\cdots n_t!} = \binom{n}{n_1, n_2, \ldots, n_t}$$

Multinomial Thm

$$(x_1 + x_2 + x_3 + \cdots x_t)^n = \sum \binom{n}{n_1, n_2, \ldots, n_t} x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$$

Where the summation is taken over all nonnegative integral solutions $n_1 \cdots n_t$ of $n_1 + \cdots + n_t = n$

Pf. $(x_1 + \cdots x_t)^n = \underbrace{(x_1 + \cdots + x_t) \cdots (x_1 + \cdots x_t)}_{n}$

Count how many terms in the form of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$

To get $x_1^{n_1}$, we choose n_1 factors from n factors. $\binom{n}{n_1}$

To get $x_2^{n_2}$, we choose n_2 factors from $n - n_1$ factors $\binom{n-n_1}{n_2}$

To get $x_t^{n_t}$, we choose n_t factors from $n - n_1 - n_2 - \cdots - n_{t-1}$ factors

$$\binom{n - n_1 - n_2 - \cdots - n_{t-1}}{n_t}$$

\# of $x_1^{n_1} \cdots x_t^{n_t}$

$$= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1 - \cdots - n_{t-1}}{n_t} = \frac{n!}{n_1! \cdots n_t!} = \binom{n}{n_1, \ldots, n_t}$$

Ex1 When $(x_1 + x_2 + x_3 + x_4)^{10}$ is expanded, find the coeff. of $x_1^2 x_2^2 x_3^3 x_4^3$.

Sol $\binom{10}{2,2,3,3} = \frac{10!}{2!2!3!3!}$

Ex2 When $(2x_1 - 3x_2 + 4x_3)^6$ is expanded, find the coeff. of $x_1^3 x_2 x_3^2$

Sol The coeff of $(2x_1)^3(-3x_2)(4x_3)^2$

$\binom{6}{3,1,2} = \frac{6!}{3!1!2!}$

The coeff of $x_1^3 x_2 x_3^2 = \frac{6!}{3!2!} 2^3(-3)4^2$

Ex3 In the expansion of $(x_1 + \cdots + x_t)^n$ by the Multinomial Thm, there are $\binom{n+t-1}{n}$ different terms in the RHS.

Sol # diff. terms = # of solutions to $n_1 + n_2 + \cdots + n_t = n$

$n_1 \geq 0, \ldots, n_t \geq 0$

§5.5 Newton's Binomial Thm

$\alpha \in \mathbb{R}$, $k \in \mathbb{Z}$

$$\binom{\alpha}{k} = \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k=0 \\ 0 & \text{if } k \leq -1 \end{cases}$$

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}$$

Thm 1 $(x+y)^\alpha$ analytic [?] function, letting $\alpha \in \mathbb{R}$

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}$$

holds if for all x and y with $|x|<|y|$.

1) $\alpha = n \in \mathbb{N} \cup \{0\}$.

Thm 1 is the binomial thm.

2) $\cancel{y^\alpha}\left(\frac{x}{y}+1\right)^\alpha = \cancel{y^\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{-k}$

$$\left(\frac{x}{y}+1\right)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k}\left(\frac{x}{y}\right)^k$$

Let $z = \frac{x}{y}$.

$$(z+1)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k$$

Here $|z|<1$.

3) Let $\alpha = -n$

$$(z+1)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} z^k \text{ for } |z|<1.$$

$$\binom{-n}{k} = \frac{-n(-n-1)\cdots(-n-k+1)}{k!} = \frac{(-1)^k \overbrace{n(n+1)\cdots(n+k-1)}^{k}}{k!}$$

$= (-1)^k \binom{n+k-1}{k}$

(*) $\boxed{(1+z)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} z^k \quad \text{if } |z|<1.}$ power series (Taylor series is special case of power series)

<u>Ex 1</u> Prove that (*) without using Thm 1.

<u>Pf</u> Replace z by $-z$

(*) becomes

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k = [(1-z)^{-1}]^n = \left(\frac{1}{1-z}\right)^n$$

$$\frac{1}{1-z} = (1-z)^{-1} = 1+z+z^2+\cdots z^m+\cdots = \sum_{m=0}^{\infty} z^m \text{ if } |z|<1.$$

$$\left(\frac{1}{1-z}\right)^n = (1+z+z^2+z^3+\cdots)^n = \underbrace{(1+z+z^2+\cdots)\cdots(1+z+z^2+\cdots)}_{n} = \sum_{k=0}^{\infty} a_k z^k. \text{ Find } a_k.$$

To get z^k, choose z^{k_1} from 1-st factor, z^{k_2} from 2-nd factor ... z^{k_n} from n-th factor s.t. $z^{k_1} z^{k_2} \cdots z^{k_n} = z^k$

$\boxed{k_1+k_2+\cdots+k_n = k, \ k_1 \geq 0, k_2 \geq 0, \ldots, k_n \geq 0}$

$a_k = \#$ of solutions to

$$\begin{cases} k_1+\cdots+k_n = k \\ k_1, \ldots, k_n \geq 0 \end{cases}$$

$$= \binom{n+k-1}{n-1} = \binom{n+k-1}{k} = (1+z+z^2+\cdots)\cdots(1+z+z^2+\cdots)$$

$\sqrt{2} \approx 1.414$

<u>Ex 2</u> $\sqrt{1+z} = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2^{2k-1}k} \binom{2k-2}{k-1} z^k$

$$\sqrt{2} = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2^{2k-1}k} \binom{2k-2}{k-1} \approx \sum_{k=0}^{M} \frac{(-1)^{k-1}}{2^{2k-1}} \binom{2k-2}{k-1} + O_M$$

$$\sqrt{1+z} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} z^k$$

Claim $\binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}}{2^{2k-1}} \binom{2k-2}{k-1}$

$$\binom{\frac{1}{2}}{k} = \frac{\overbrace{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-k+1\right)}^{k}}{k!} = \frac{(-1)^{k-1}}{2^k} \frac{\overbrace{1\times 3\times 5\times\cdots\times(2k-3)}^{k-1}}{k!}$$

$$= \frac{(-1)^{k-1}}{2^k} \frac{(2k-2)!}{\underbrace{2\times 4\times\cdots\times(2k-2)}_{k-1 \text{ factors}}\; k!}$$

$$= \frac{(-1)^{k-1}}{2^k} \frac{1}{2^{k-1}} \frac{(2k-2)!}{(k-1)!\,k!} = \frac{(-1)^{k-1}}{2^{2k-1}} \binom{2k-2}{k-1} \frac{1}{k}$$

$$\sqrt{30} = \sqrt{25+5} = \sqrt{25\left(1+\frac{5}{25}\right)} = 5\sqrt{1+\frac{1}{5}} = 5\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2^{2k-1}k} \binom{2k-2}{k-1} \frac{1}{5^k}$$

HW §5.7 : 43, 47

W 3/1 Lecture

89-100 A

72 - 88 B

60 - 71 C

49-59 D

0-48 F

§5.6. Partial Order.

Let X be a set. A partial order "$\leq$" on X. is a relation satisfying

$x, y \in X$, $x \leq y$.

1) $x \leq x$ for all $x \in X$ (reflexivity)

2) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry)

3) if $x \leq y$, $y \leq z$, then $x \leq z$ (transitivity)

Ex 1. "$\leq$" in $\mathbb{R}$. $x \leq y$, $x, y \in \mathbb{R}$. "$\leq$" (less than equal) is a partial order

Ex 2. "$\subseteq$" in $\mathcal{P}(X)$ = the collection of all subsets of X.

$A \subseteq B \iff A$ is a subset of B, $A, B \in \mathcal{P}(X)$

"$\subseteq$" is a partial order on $\mathcal{P}(X)$.

Let "$\leq$" be a partial order on X. If $x \leq y$, then we say x, y are comparable.

Let $A \subseteq X$. If there is NO two elements in A that are comparable, then A is an antichain.

If each pair of elements in A is comparable, then A is called a chain

$|A|$ = # of elements in A.

Def Let S be a finite set. Let "$\leq$" be a partial order on X.

A minimal element of S is an element $a \in S$ such that no element x different from a satisfies $x \leq a$.

A maximal element of S is an element $a \in S$ s.t no element x different from a satisfies $a \leq x$.

<u>Ex2</u> $X = \{2, 3, 4, 5, 6, 7, 8\}$.

$a | b \Leftrightarrow b$ is divisible by a.

The relation "$|$" is a partial order.

2 is a min. in X w.r.t. "$|$".

3 is a min. in X ————

5 is a min. ————

7 is a min. ————

8 is a max. in X w.r.t. "$|$".

7 is max.

6 is max.

5 is max.

"$|$" $A_1 = \{2, 3, 5, 7\}$ is an antichain.

$X \setminus A_1 = \{4, 6, 8\}$

$A_2 = \{4, 6\}$ is an antichain.

$X \setminus (A_1 \cup A_2) = \underbrace{\{8\}}_{=A_3}$ is ~~ant~~ an antichain.

$X = A_1 \cup A_2 \cup A_3$.

antichain's

<u>Thm 1.</u> Let X be a finite set. "$\leq$" be a partial order on X.

Let r be the largest size of a chain (in X).

Then X can be partitioned into r but no fewer antichains.

F 3/3 Lecture

<u>Thm 1</u> Let X be a finite set with partial order "$\leq$". Let r be the size of the largest chain. Then X can be partitioned into r (but no fewer) antichains.

<u>Pf.</u> First, we prove that X can NOT be partitioned into fewer than r antichains.

Assume that $X = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_m$. (antichain partition) with $1 \leq m < r$.

There is a chain from X whose size $= r$.

$\mathcal{C} = \{x_1, x_2, \dots, x_r\}$. Here $x_1 \leq x_2 \leq \cdots \leq x_r$.

By pigeonholing, $\exists A_j$ $(1 \leq j \leq m)$ s.t. A_j consists of <u>two elements</u> in $\mathcal{C}$. It is a contradiction!

$x_{i_1} \leq x_{i_2}$

Let $X_1 = X$.

A_1 = the set of all min. elements of X_1 w.r.t. "$\leq$"

Let $X_2 = X \setminus A_1 = X_1 \setminus A_1 = X \cap \overline{A_1} = X_1 \cap \overline{A_1}$.

$X = \underline{A_1} \cup X_2$.

Let A_2 = the set of all min. of X_2 w.r.t. "$\leq$"

$X_3 = X_2 \setminus A_2$.

$X = \underline{A_1 \cup A_2} \cup X_3$

Continue like this until the first integer p s.t.

$X_p \neq \emptyset$ but $X_p = \emptyset$.

Then $X = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_p$.

antichain partition.

Claim $\underline{p = r}$

<u>Pf</u> $p \geq r$. Known by pigeonholing.

It remains to show $p \leq r$.

Note that for each $a_{i+1} \in A_{i+1}$

$\exists\, a_i \in A_i$ s.t. $a_i \le a_{i+1}$

$a_1 \le a_2 \le a_3 \le \cdots \le a_p$

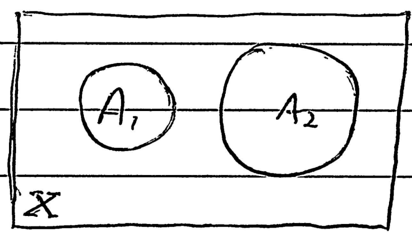
$C = \{a_1, a_2, \ldots, a_p\}$ is a chain.

$p \le r$.

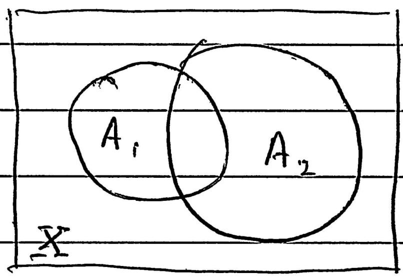
Dual form of thm: swap chain & antichain in statement

HW §5.7: 48, 49, 50.

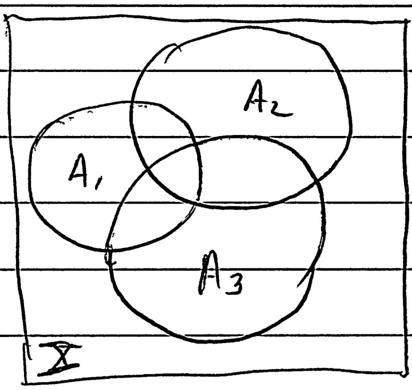
Ch6 §6.1 The inclusion-exclusion principle



$$|A_1 \cup A_2| = |A_1| + |A_2| \quad \text{if } A_1 \cap A_2 = \phi$$



$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



$$\begin{aligned}|A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &\quad - |A_1 \cap A_2| - |A_1 \cap A_3| \\ &\quad - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|\end{aligned}$$

Thm 1 (Inclusion - Exclusion) $2 \le m \in \mathbb{N}$

$$|A_1 \cup A_2 \cup \cdots \cup A_m|$$

$$= \sum_{i=1}^{m} |A_i| + \sum_{1 \le i_1 < i_2 \le m}^{(-1)^1} |A_{i_1} \cap A_{i_2}|$$

$$+ \sum_{1 \le i_1 < i_2 < i_3 \le m}^{(-1)^2} |A_{i_1} \cap A_{i_2} \cap A_{i_3}|$$

$$+ \cdots + (-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

$$+ \cdots + (-1)^{m-1} |A_1 \cap A_2 \cap \cdots \cap A_m|$$

$$= \sum_{k=1}^{m} \left[(-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \right]$$

Thm 2 a) $\overline{A_1 \cup A_2 \cup \cdots \cup A_m} = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \cdots \cap \overline{A_m}$

$$\Downarrow$$

$$A_1 \cup A_2 \cup \cdots \cup A_m = \bigcup_{j=1}^{m} A_j$$

$$A_1 \cap A_2 \cap \cdots \cap A_m = \bigcap_{j=1}^{m} A_j \qquad \overline{\bigcup_{j=1}^{m} A_j} = \bigcap_{j=1}^{m} \overline{A_j}$$

b) $\overline{\bigcap_{j=1}^{m} A_j} = \bigcup_{j=1}^{m} \overline{A_j}$ (a), (b) DeMorgan's Laws

Pf of a) $x \in \overline{\bigcup_{j=1}^{m} A_j} \Leftrightarrow x \notin \bigcup_{j=1}^{m} A_j \Leftrightarrow x \notin A_j$ for all $1 \le j \le m$

$\Leftrightarrow x \in \overline{A_j}$ for all $1 \le j \le m \Leftrightarrow x \in \bigcap_{j=1}^{m} \overline{A_j}$.

M 3/6 Lecture

§6.1 Thm 1

$$\left|\bigcup_{j=1}^{m} A_j\right| = \sum_{k=1}^{m} (-1)^{k-1} \sum_{1\le i_1<i_2<\cdots<i_k\le m} |A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}|$$

Ex 1 Find # of integers between 1 and 1,000 that are not divisible by 5, 6, 8.

Sol $S=\{1,\dots,1000\}$

$A_1=\{n\in S: 5|n\}$

$A_2=\{n\in S: 6|n\}$

$A_3=\{n\in S: 8|n\}$

Count $\overline{A_1}\cap\overline{A_2}\cap\overline{A_3} = \overline{\bigcup_{j=1}^{3} A_j}$

$$\left|\overline{\bigcup_{j=1}^{3} A_j}\right| = |S| - \left|\bigcup_{j=1}^{3} A_j\right|$$

$$\left|\bigcup_{j=1}^{3} A_j\right| = \sum_{k=1}^{3} (-1)^{k-1} \sum_{1\le i_1<\cdots<i_k\le 3} |A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}|$$

$$= |A_1|+|A_2|+|A_3| - (|A_1\cap A_2|+|A_1\cap A_3|+|A_2\cap A_3|) + |A_1\cap A_2\cap A_3|$$

$|A_1| = \left\lfloor \frac{1000}{5}\right\rfloor = 200$

$|A_2| = \left\lfloor \frac{1000}{6}\right\rfloor = 166$

$|A_3| = \left\lfloor \frac{1000}{8}\right\rfloor = 125$

$|A_1\cap A_2| = \left\lfloor \frac{1000}{30}\right\rfloor = 33$

$|A_1\cap A_3| = \left\lfloor \frac{1000}{40}\right\rfloor = 25$

$|A_2 \cap A_3| = |\#$ of integers in S divisible by 6 and 8$|$
$= |\#$ of integers in S divisible by l.c.m. of 6 and 8$|$

$\text{lcm}(6,8) = 24$

$|A_2 \cap A_3| = \left\lfloor \frac{1000}{24} \right\rfloor = 41$

$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{1000}{\text{lcm}(5,6,8)} \right\rfloor = \left\lfloor \frac{1000}{5 \times 24} \right\rfloor = 8$

$\left|\bigcup_{j=1}^{3} A_j\right| = 400$

$\left|\overline{\bigcup_{j=1}^{3} A_j}\right| = 1000 - 400 = 600$

Ex 2 How many permutations of the letters
M, A, T, H, I, S, F, U, N are there s.t. none of the
words MATH, IS, FUN occur as consecutive letters?

Sol Let $A_1 = \{$permutations containing MATH as consecutive letters$\}$
$A_2 = \{$permutations containing IS$\}$
$A_3 = \{$permutations containing FUN$\}$

Count $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} = \overline{A_1 \cup A_2 \cup A_3}$

$|\overline{A_1 \cup A_2 \cup A_3}| = |S| - |A_1 \cup A_2 \cup A_3| = 9! - |A_1 \cup A_2 \cup A_3|$

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3|$
$- |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$|A_1| = 6!$

$|A_2| = 8!$ $\quad A_3 = 7!$

$|A_1 \cap A_2| = \{$permutations containing MATH and IS$\}| = 5!$

$|A_1 \cap A_3| = 4! \quad |A_2 \cap A_3| = 6!$

$|A_1 \cap A_2 \cap A_3| = 3!$

$= 6! + 8! + 7! - 5! - 4! - 6! + 3!$

$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = 9! - |A_1 \cup A_2 \cup A_3| = 317,658$

§6.2 Combinations with repetition

$S = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$

\# of r-combinations from S = \# of solutions to $\begin{cases} x_1 + x_2 + \cdots + x_k = r \\ x_1 \geq 0, \ldots, x_k \geq 0 \end{cases}$

$= \binom{r+k-1}{k-1}$

$S = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$

\# of r-combinations of S = \# of solutions to $\begin{cases} x_1 + x_2 + \cdots + x_k = r \\ n_1 \geq x_1 \geq 0, n_2 \geq x_2 \geq 0, \ldots, n_k \geq x \geq 0 \end{cases}$

HW §6.7: 2

W 3/8 Lecture

$\#$ of r-combinations of $S=\{n_1\cdot a_1, \dots, n_k\cdot a_k\}$

$$= \# \text{ of solutions to } \begin{cases} x_1+x_2+\dots+x_k = r \\ n_1 \geq x_1 \geq 0 \\ \vdots \\ n_k \geq x_k \geq 0 \end{cases}$$

$\mathbb{X} = \{\infty\cdot a_1, \infty\cdot a_2, \dots, \infty\cdot a_k\}$, $S \subseteq \mathbb{X}$

$A_1 = \{r\text{-combinations of } \mathbb{X} \text{ containing } > n_1 \text{ many } a_1\text{'s}\}$

$A_2 = \{r\text{-combinations of } \mathbb{X} \text{ containing } > n_2 \text{ many } a_2\text{'s}\}$

$\vdots$

$A_k = \{r\text{-combinations of } \mathbb{X} \text{ containing } > n_k \text{ many } a_k\text{'s}\}$

$$1 \leq j \leq k, \ |A_j| = \# \text{ of sol. to } \begin{cases} x_1+x_2+\dots+x_k = r \\ x_j \geq n_j+1 \\ x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k \geq 0 \end{cases}$$

$$\overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k} = \{r\text{-combinations of } S\} = \overline{\bigcup_{j=1}^{k} A_j}$$

$$\left|\overline{\bigcup_{j=1}^{k} A_j}\right| = |\mathcal{C}| - \left|\bigcup_{j=1}^{k} A_j\right| = \binom{k+r-1}{k-1} - \left|\bigcup_{j=1}^{k} A_j\right|$$

$\mathcal{C} = \{r\text{-combinations of } \mathbb{X}\}$

Ex1 Find $\#$ of 10-combinations of $\{3\cdot a, 4\cdot b, 5\cdot c\}$

Sol
$$\# = \# \text{ of sol. to } \begin{cases} x_1+x_2+x_3 = 10 \\ 3 \geq x_1 \geq 0 \\ 4 \geq x_2 \geq 0 \\ 5 \geq x_3 \geq 0 \end{cases}$$

$\mathbb{X} = \{\infty\cdot a, \infty\cdot b, \infty\cdot c\}$

$A_1 = \{10\text{-combinations containing} \geq 4 \ a\text{'s}\}$

$A_2 = \{ \text{———} \geq 5 \ b\text{'s}\}$

$A_3 = \{ \text{———} \geq 6 \ c\text{'s}\}$

$$\# = |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = \left|\overline{\bigcup_{j=1}^{3} A_j}\right| = \binom{3+10-1}{10} - \left|\bigcup_{j=1}^{3} A_j\right|$$

$$\binom{12}{2}$$

HW § 6.7

2, 6, 9

$|A_1| = \#$ of sol. to $\begin{cases} x_1 + x_2 + x_3 = 10 \\ \begin{cases} x_1 \geq 4 \Rightarrow x_1 - 4 = y_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{cases} \end{cases}$

$= \#$ of sol. to $\{ y_1 + x_2 + x_3 = 10 - 4 = 6$

$= \binom{3+6-1}{6} = \binom{8}{2}$

$|A_2| = \binom{7}{2}$ $\quad |A_3| = \binom{6}{2}$

$|A_1 \cap A_2| = \#$ of sol. to $\begin{cases} x_1 + x_2 + x_3 = 10 \\ x_1 - 4 \geq 0 \Rightarrow \\ x_2 - 5 \geq 0 \\ x_3 \geq 0 \end{cases}$ Let $y_1 = x_1 - 4$, $y_2 = x_2 - 5$

$= \#$ of sol. to $\begin{cases} y_1 + y_2 + x_3 = 10 - 9 = 1 \\ y_1 \geq 0, y_2 \geq 0, x_3 \geq 0 \end{cases}$

$= \binom{3+1-1}{2} = \binom{3}{2} = 3$

$|A_1 \cap A_3| = \binom{2}{2} = 1$

$|A_2 \cap A_3| = \#$ of sol. to $\begin{cases} x_1 + x_2 + x_3 = 10 \\ x_1 \geq 0 \\ x_2 \geq 5 \\ x_3 \geq 6 \end{cases}$

$= 0$

$|A_1 \cap A_2 \cap A_3| = 0$

$\left| \bigcup_{j=1}^{3} A_j \right| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$= \binom{8}{2} + \binom{7}{2} + \binom{6}{2} - \binom{3}{2} - \binom{2}{2} - 0 + 0 = 60$

$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = \binom{12}{2} - 60 = 66 - 60 = 6$

§ 6.3 Derangements

n gentlemen check their hats.

In how many ways can their hats be returned so that no gentlemen gets his own hat?

$\{1, 2, \dots, n\}$

A derangement is a permutation $i_1, i_2, \dots, i_n$ of $\{1, 2, \dots, n\}$ such that $i_1 \neq 1$, $i_2 \neq 2, \dots, i_n \neq n$

Let D_n = # of derangements of $\{1, 2, \dots, n\}$

F 3/10 Lecture

D_n - # of derangements from $\{1, 2, \ldots, n\}$

<u>Thm</u> For $n \geq 1$,

$$D_n = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

<u>Pf</u> Let $S = \{$permutations of $\{1, 2, \ldots, n\}\}$

$|S| = n!$

$A_1 = \{$permutations $i_1, i_2, \ldots, i_n$ in S s.t. $i_1 = 1\}$

$A_2 = \{$permutations $i_1, i_2, \ldots, i_n$ in S s.t. $i_2 = 2\}$

For $1 \leq j \leq n$ $A_j = \{$permutations $i_1, i_2, \ldots, i_n$ in S s.t. $i_j = j\}$

$\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} = \{$all derangements$\}$

$$D_n = |\overline{A_1} \cap \cdots \cap \overline{A_n}| = \left|\overline{\bigcup_{j=1}^{n} A_j}\right| = n! - \left|\bigcup_{j=1}^{n} A_j\right|$$

$$\left|\bigcup_{j=1}^{n} A_j\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

$A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap \cdots \cap A_{i_k}$

$= \{$permutations $j_1 j_2 \cdots j_n$ in S s.t. $j_{i_1} = i_1, j_{i_2} = i_2, \ldots j_{i_k} = i_k\}$

$$\begin{array}{ccccccccc} & & & i_1 & & i_2 & & i_k & & \\ 1 & 2 & 3 & i_1 & & i_2 & & i_k & n-1 & n \end{array}$$

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = P(n-k, n-k) = (n-k)!$$

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap \cdots \cap A_{i_k}| = \sum_{i_1 \cdots i_k} (n-k)! = (n-k)!\binom{n}{k} = \frac{n!}{k!}$$

$$\left|\bigcup_{j=1}^{n} A_j\right| = \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!}$$

$$D_n = n! - \left|\bigcup_{j=1}^{n} A_j\right| = n! - \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} = n!\left(\sum_{k=0}^{n} (-1)^k \frac{1}{k!}\right)$$

$D_1, D_2, D_3, D_4, \dots, D_n, D_{n+1}, \dots$

Thm 2 For $n \geq 3$, $\boxed{D_n} = (n-1)(D_{n-2} + D_{n-1})$

Pf Let S_n be the collection of all derangements from $\{1, 2, \dots, n\}$.

$D_n = |S_n|$

$i_1 = k \neq 1$ $\qquad$ $i_1, i_2, \dots i_n \in S_n$

$\underset{1}{\underline{\;i_1\;}}\ \underset{2}{\underline{\quad}}\ \underset{3}{\underline{\quad}} \dots \underset{k}{\underline{\;\square\;}} \dots \underset{n}{\underline{\quad}}$

Step 1 Choose i_1,

$(n-1)$ ways.

Step 2 Assume that $i_1 = k$.

Count # of arrangements of the remaining $n-1$ numbers.

Case 1 1 is in the k-th position.

Case 2 1 is NOT in the k-th position

For **Case 1** $\quad \underset{1}{\underline{\;k\;}}\ \underset{2}{\underline{\quad}} \dots \underset{k}{\underline{\;1\;}} \dots \underset{n}{\underline{\quad}}$

Derange $n-2$ numbers D_{n-2}

For **Case 2**, derange $n-1$ numbers D_{n-1}

$D_{n-2} + D_{n-1}$ ways to finish step 2.

HW §6.7 : 15, 16, 17, 20, 21

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Thm 2 $D_n = (n-1)(D_{n-2} + D_{n-1})$ for any $n \geq 3$.

Thm 3 $D_n = nD_{n-1} + (-1)^{n-2}$ if $n \geq 2$

Pf. $D_n = (n-1)D_{n-2} + (n-1)D_{n-1}$

$= (n-1)D_{n-2} + nD_{n-1} - D_{n-1}$

$D_n - nD_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2})$

Let $b_n = D_n - nD_{n-1}$ for $n \in \mathbb{N}$

$b_n = (-1)b_{n-1}$ if $n \geq 2$

$b_n = (-1)^{n-2} b_2$

$b_2 = D_2 - 2D_1$

$D_1 = 0, \ D_2 = 1$

$b_2 = 1 - 0 = 1$

$b_n = (-1)^{n-2} = D_n - nD_{n-1}$

Thm 3 $\Rightarrow$ Thm 1

Thm 1 $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Pf. Induction on n..

Step 1 $n = 1$.

$$D_1 = 0 = 0! \frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} = 1 - 1 = 0$$

Assume $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Show that $D_{n+1} = (n+1)! \sum_{k=0}^{n+1} \frac{(-1)^k}{k!}$.

$$D_{n+1} \overset{\text{Thm3}}{=} (n+1)D_n + (-1)^{n-1}$$

$$= (n+1)n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + (-1)^{n-1}$$

$$= \cancel{(n+1)! \sum_{n=0}^{n} \frac{(-1)^k}{k!} + (-1)^{n-1} \frac{(n+1)!}{(n+1)!}}$$

$$= \cancel{(n+1)! \left[\sum_{k=0}^{n} \frac{(-1)^k}{k!} + \frac{(-1)^{n+1}}{(n+1)!} \right] = (n+1)! \sum_{k=0}^{n+1} \frac{(-1)^k}{k!}}$$

§6.4 Permutations with forbidden position

Let $X_1, \dots, X_n$ be subsets of $\{1, 2, \dots, n\}$

$P(X_1, \dots, X_n) = \{$all permutations $i_1 i_2 \dots i_n$ of $\{1, \dots, n\}$ s.t. $i_1 \notin X_1, \dots, i_n \notin X_n\}$

$X_1 = \{1\}, X_2 = \{2\}, \dots, X_n = \{n\}$

$P(\{1\}, \dots \{n\}) = \{$derangements$\}$.

$|P(X_1, \dots, X_n)| = \#$ of ways to put non-attacking rooks on an $n \times n$ board with the forbidden positions

$(1, i_1)$ for all $i_1 \in X_1$

$(2, i_2)$ for all $i_2 \in X_2$

$\vdots$

(n, i_n) for all $i_n \in X_n$

Forbidden positions in an $n \times n$ board

$(1, i_1)$, $i_1 \in X_1$

$(2, i_2)$, $i_2 \in X_2$

$\vdots$

(n, i_n), $i_n \in X_n$

For $k \in \mathbb{N}$

$r_k = \#$ of ways to place k non-attacking rooks on the $n \times n$ board where each of k rooks is in a forbidden position

<u>Thm</u> $|P(X_1, \dots, X_n)| = n! + \sum_{k=1}^{n} (-1)^k r_k (n-k)!$

W 3/29 Lecture

Midterm 2 Review

Topics

<u>Ch 5</u>

- Use Binomial Thm to prove the identity
- Partial order, chain, antichain

<u>Ch 6</u> I-E principle

Examples

<u>Ex 1</u> In the expansion of $(2x_1+3x_2-4x_3)^{10}$ find the coeff of $x_1^3 x_2^3 x_3^4$

$$\frac{10!}{3!\,3!\,4!} \cdot 2^3 \cdot 3^3 (-4)^4$$

<u>Ex 2</u> Show $1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$

<u>Sol</u> $(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$

$$\int_0^1 (1+x)^n\,dx=\int_0^1\sum_{k=0}^{n}\binom{n}{k}x^k\,dx=\sum_{k=0}^{n}\int_0^1\binom{n}{k}x^k\,dx$$

$$=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}=1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}$$

$$\int_0^1 (1+x)^n\,dx=\left.\frac{(1+x)^{n+1}}{n+1}\right|_{x=0}^{1}=\frac{2^{n+1}}{n+1}=\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}$$

<u>Ex 3</u> Prove that the only antichain of $S=\{1,2,3,4\}$ of size 6 is the antichain of all 2-combinations of S.

<u>Pf</u> $A\subseteq B \Leftrightarrow A\le B$

0-combination: $\emptyset$

1-combinations: $\{1\}, \{2\}, \{3\}, \{4\}$

2-combinations: $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$

3-comb: $\{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\}$

4-comb.: $\{1,2,3,4\}$

$\mathcal{C} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

is an antichain of size 6.

Let $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}_*$ be an antichain.

1) $\mathcal{C}$ does NOT contain $\emptyset$, $\{1,2,3,4\}$.

2) $\mathcal{C}$ does NOT contain any 3-combinations.

Suppose $\mathcal{C}$ contains one of 3-combinations, say, $\{1,2,3\}$.

Then $\{1,2\}, \{1,3\}, \{2,3\} \notin \mathcal{C}$.

$\{1\}, \{2\}, \{3\} \notin \mathcal{C}$.

$\{4\} \notin \mathcal{C}$.

Five of $\{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\} \in \mathcal{C}$

3) $\mathcal{C}$ does NOT contain any 1-combinations.

Ex 4 Find # of circular permutations of $S = \{3\cdot a, 4\cdot b, 2\cdot c, 1\cdot d\}$.

where, for each type of letter, all letters of that type do NOT appear ~~consequ~~ consecutively.

Sol $A_1 = \{$circular permutations s.t. a's appear consecutively$\}$

$A_2 = \{$circular permutations s.t. b's appear consecutively$\}$

$A_3 = \{$circ. permutations s.t. c's appear consecutively$\}$

We count $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$. $X = \{$circular permutations of $S\}$.

$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |X| - |A_1 \cup A_2 \cup A_3|$.

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|$
$+ |A_1 \cap A_2 \cap A_3|$

$$|X| = \frac{(3+4+2+1)!}{3!\,4!\,2!\,1!} \cdot \frac{1}{3+4+2+1} = \frac{9!}{3!\,4!\,2!}$$

$S^* = \{\overset{1\cdot A}{\boxed{3\cdot a}}, \overset{1\cdot B}{\boxed{4\cdot b}}, \overset{1\cdot C}{\boxed{2\cdot c}}, 1\cdot d\}$

$$|A_1| = \frac{(1+4+2+1)!}{4!\,2!}\cdot\frac{1}{(1+4+2+1)} = \frac{7!}{4!\,2!}$$

$$|A_2| = \frac{6!}{3!\,2!}$$

$$|A_3| = \frac{8!}{3!\,4!}$$

$$|A_1 \cap A_2| = \frac{(1+1+2+1)!}{2!}\cdot\frac{1}{(1+1+2+1)} = \frac{4!}{2!}$$

$$|A_1 \cap A_3| = \frac{5!}{3!}$$

$$|A_2 \cap A_3| = \frac{6!}{4!}$$

$$\# = \frac{9!}{3!\,4!\,2!} - \left(\frac{7!}{4!\,2!} + \frac{6!}{3!\,2!} + \frac{8!}{3!\,4!}\right) + \left(\frac{4!}{2!} + \frac{6!}{4!} + \frac{5!}{3!}\right) - 3!$$

Ex 5

$$\#\text{ of sol. to } \begin{cases} x_1 + x_2 + x_3 = 10 \\ 0 \le x_1 \le 3 \\ 0 \le x_2 \le 5 \\ 0 \le x_3 \le 6 \end{cases}$$

Sol $S = \{\text{sol. to } x_1 + x_2 + x_3 = 10$ with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0\}$

$$|S| = \binom{10+3-1}{10}$$

$A_1 = \{\text{sol. to } x_1 + x_2 + x_3 = 10$ with $x_1 \ge 4,\ x_2 \ge 0,\ x_3 \ge 0\}$

$A_2 = \{\text{sol. to } x_1 + x_2 + x_3 = 10,\ \text{with } x_1 \ge 0,\ x_2 \ge 6,\ x_3 \ge 0\}$

$A_3 = \{\text{sol. to } x_1 + x_2 + x_3 = 10,\ \text{with } x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 7\}$

$\# = |S| - |A_1 \cup A_2 \cup A_3|$

M 4/13 Lecture

90-100 A

80-89 B

60-79 C

50-59 D

0-49 F

§7.1

$$\begin{cases} f_n = f_{n-1} + f_{n-2} \ ; \ n \geq 2 \\ f_0 = 0, \ f_1 = 1 \end{cases}$$

Thm 1 The Fibonacci number f_n satisfies

$$f_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n-m}{m-1}$$

where $m = \left\lfloor \frac{n+1}{2} \right\rfloor$.

Proof Let $g_n = \binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{n-m}{m-1}$. Here $m = \left\lfloor \frac{n+1}{2} \right\rfloor$

$$= \sum_{k=0}^{m-1} \binom{n-k-1}{k}$$

Claim 1 $g_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$

Pf of Claim 1 $\sum_{k=0}^{n-1} \binom{n-1-k}{k} = \sum_{k=0}^{m-1} \binom{n-1-k}{k} + \sum_{k=m}^{n-1} \binom{n-1-k}{k}$

Notice that $m+1 \leq k \leq n-1 \Rightarrow k > n-1-k \Rightarrow \binom{n-1-k}{k} = 0$ $\square$

$g_0 = \binom{-1}{0} + \cdots = 0$

$g_1 = \sum_{k=0}^{0} \binom{n-1+k}{k} = \binom{n-1}{0} = \binom{1-1}{0} = \binom{0}{0} = 1$

Claim 2 $g_n = g_{n-1} + g_{n-2}$

Pf $g_{n-1} = \sum_{k=0}^{n-2} \binom{n-2-k}{k}$, $g_{n-2} = \sum_{k=0}^{n-3} \binom{n-3-k}{k} = \sum_{k=1}^{n-2} \binom{n-2-k}{k-1}$

$$g_{n-1}+g_{n-2} = \sum_{k=0}^{n-2}\binom{n-2-k}{k} + \sum_{k=1}^{n-2}\binom{n-2-k}{k-1}$$

$$= \binom{n-2}{0} + \sum_{k=1}^{n-2}\left[\binom{n-2-k}{k}+\binom{n-2-k}{k-1}\right] = \cancel{\sum_{k=1}^{n-1}\binom{n-1-k}{k}}$$

$$= \binom{n-1}{0} + \sum_{k=1}^{n-2}\left[\binom{n-2-k}{k}+\binom{n-2-k}{k-1}\right] = \sum_{k=0}^{n-2}\binom{n-1-k}{k}$$

$$= \sum_{k=0}^{n-1}\binom{n-1-k}{k} = g_n \quad \text{(By Claim 1)}$$

§7.2 Linear Homogeneous Recurence Relations

Let $(h_n)_{n=0}^{\infty}$ be a seq of real numbers.

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n$$

$$= \boxed{\sum_{j=1}^{k} a_j h_{n-j}} + b_n$$

Here $a_1, \ldots, a_k, b_n$ are real numbers.

$$(*)\ h_n = \sum_{j=1}^{k} a_j h_{n-j}$$

$(*)$ is called homogeneous of order k.

<u>Ex 1</u> $f_n = f_{n-1} + f_{n-2}$

<u>Ex 2</u> D_n – derangement #

Here $D_0 = 1$

$$\begin{cases} D_n = (n-1)(D_{n-2}+D_{n-1}) \\ D_0 = 1,\ D_1 = 0 \end{cases}$$

Replace h_n by q^n (Note: $h_n \neq q^n$); <u>$q \neq 0$</u> (non-trivial soln)

Then $(*)$ becomes $q^n = \sum_{j=1}^{k} a_j q^{n-j}$

$q^n - a_1 q^{n-1} - a_2 q^{n-2} - \dots - a_k q^{n-k} = 0$

$q^{n-k}(q^k - a_1 q^{k-1} - a_2 q^{k-2} - \dots - a_k) = 0$

$q^k - a_1 q^{k-1} - a_2 q^{k-2} - \dots - a_k = 0.$

$$\boxed{q^k - \sum_{j=1}^{k} a_j q^{k-j} = 0}$$

$\boxed{x^k - \sum_{j=1}^{k} a_j x^{k-j} = 0}$ is called char. eqn. of $(*)$.

<u>Thm</u> Let $(h_n)_n$ be a seq satisfying

$$(*) \quad h_n = \sum_{j=1}^{k} a_j h_{n-j}$$

Here $a_1, \dots, a_k \in \mathbb{R}$

Suppose that the char. eqn. $x^k = \sum_{j=1}^{k} a_j x^{k-j} = 0$

has $\boxed{\text{distinct}}$ roots $q_1, q_2, \dots, q_k$.

Then $h_n = c_1 q_1^n + c_2 q_2^n + \dots + c_k q_k^n$

$$= \sum_{j=1}^{k} c_j q_j^n$$

Here $c_1, \dots, c_k$ are constant (independent of n).

M 4/10 Lecture

Ex 2 Determine the generating function h_n = # of sol. of

$$\begin{cases} e_1 + e_2 + \cdots + e_k = n \\ e_1, e_2, \cdots, e_k \text{ are non-negative integers.} \\ \boxed{e_1, e_2, \ldots, e_k \text{ odd}} \end{cases}$$

Sol ~~g(x)~~

w/o odd restriction: $\left(\sum_{e_1=0}^{\infty} x^{e_1}\right)\left(\sum_{e_2=0}^{\infty} x^{e_2}\right)\cdots\left(\sum_{e_k=0}^{\infty} x^{e_k}\right)$

$$\therefore \underset{(w/odd)}{g(x)} = \left(\sum_{e_1 \text{ odd}} x^{e_1}\right)\left(\sum_{e_2 \text{ odd}} x^{e_2}\right)\cdots\left(\sum_{e_k \text{ odd}} x^{e_k}\right)$$

$$= \underbrace{(x + x^3 + x^5 + x^7 + \cdots)(\cdots)\cdots(\cdots)}_{k}$$

$$= \frac{x}{1-x^2} \cdots \frac{x^k}{1-x^2} = \frac{x^k}{(1-x^2)^k} = \sum_{n=0}^{\infty} h_n x^n$$

by norm result $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ for $|z|<1$

Taylor series $g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$

$$h_n = \frac{g^{(n)}(0)}{n!}$$

Ex3 Find # of sol. of

$$(*) \begin{cases} e_1 + e_2 + e_3 = n \\ 0 \le e_1 \le 3, \quad 0 \le e_2 \le 2, \quad 0 \le e_3 \le 2 \end{cases}$$

Sol h_n = # of sol to $(*)$

The generating function $(h_n)_{n=0}^{\infty}$ is

$$g(x) = \left(\sum_{e_1=0}^{3} x^{e_1}\right)\left(\sum_{e_2=0}^{2} x^{e_2}\right)\left(\sum_{e_3=0}^{2} x^{e_3}\right)$$

$= (1+x+x^2+x^3)(1+x+x^2)(1+x+x^2)$ ~~$= 1+3x+6x^2+8x^3+8x^4+$~~

$= 1+3x+6x^2+8x^3+8x^4+6x^5+3x^6+x^7$

$h_0 = 1,\ h_1 = 3,\ h_2 = 6,\ h_4 = 8,\ \ldots,\ h_7 = 1,\ h_8 = 0,\ h_9 = 0, \ldots$

Ex 4 Find h_n = # of bags of n pieces of fruits that can be made out of apples, bananas, oranges, and pears, where, in each bag, # of a's is even

# of b's is a multiple of 5

# of O's is at most 4

# of p's is 0 or 1.

Sol. The generating function of $(h_n)_{n=0}^{\infty}$ is

$$g(x) = \Big(\sum_{\substack{\text{even}\\ e_1}} x^{e_1}\Big)\Big(\sum_{e_2 = 5 k} x^{e_2}\Big)\Big(\sum_{e_3=0}^{4} x^{e_3}\Big)\Big(\sum_{e_4=0}^{1} x^{e_4}\Big)$$

$$\underbrace{}_{1+x^2+x^4+x^6+\cdots} \quad \underbrace{}_{1+x^5+x^{10}+\cdots} \quad \underbrace{}_{(1+x+x^2+x^3+x^4)} \quad \underbrace{}_{(1+x)}$$

$|x|<1$: $g(x) = \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\cancel{(1+x+x^2+x^3+x^4)(1+x)}$

$$= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \binom{n+1}{n} x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

$\boxed{h_n = n+1}$

HW: §7.7: 13a) b) c), 17, 18, 19

§7.5 Recurrences and Generating Functions

Ex 1 $h_n = \boxed{5h_{n-1} - 6h_{n-2}}$ $(n \geq 2)$

with $h_0 = 1,\ h_1 = -2.$

Sol Let g be the generating function of $(h_n)_{n=0}^{\infty}$, that is,

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

$h_0 h_1 h_2 \cdots = (h_n)$

$$a_n = \begin{cases} h_{n-1} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

$$xg(x) = x\sum_{n=0}^{\infty} h_n x^n = \sum_{n=0}^{\infty} h_{\underset{n-1}{n}}\, x^{\underset{n}{n+1}}$$

$$= \sum_{n=1}^{\infty} h_{n-1} x^n$$

$$5xg(x) = \sum_{n=1}^{\infty} 5h_{n-1} x^n \qquad (1)$$

$$-6x^2 g(x) = \sum_{n=0}^{\infty} (-6h_{\underset{n-2}{n}}\, x^{\underset{n}{n+2}}) = \sum_{n=2}^{\infty} (-6h_{n-2})x^n \qquad (2)$$

(1)+(2)

$$5xg(x) - 6x^2 g(x) = h_0 - (h_1 - 5h_0)x + g(x)$$

$$(1 - 5x + 6x^2)\, g(x) = h_0 + (h_1 - 5h_0)\, x$$

$$= 1 - 7x$$

$$g(x) = \frac{1-7x}{1-5x+6x^2}$$

W 10/12 Lecture

Ex1 Solve $\begin{cases} h_n = 5h_n - 6h_{n-2} \\ h_0 = 1,\ h_1 = -2 \end{cases}$

Sol. $g(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{1-7x}{1-5x+6x^2} = \frac{1-7x}{(1-2x)(1-3x)}$

$= \frac{A}{1-2x} + \frac{B}{1-3x}$

$1-7x = A(1-3x) + B(1-2x) = (A+B) - (3A+2B)x$

$\begin{cases} A+B=1 \\ 3A+2B=7 \end{cases}$

$\begin{cases} A=5 \\ B=-4 \end{cases}$

$g(x) = \frac{5}{1-2x} + \frac{-4}{1-3x}$ $\qquad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,\ |z|<1$

$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$

$\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$

$g(x) = \sum_{n=0}^{\infty} [5 \cdot 2^n - 4 \cdot 3^n] x^n$

$h_n = 5 \cdot 2^n - 4 \cdot 3^n$

Ex2 Solve $\begin{cases} h_n = 4h_{n-1} - 4h_{n-2} + \boxed{3^n} \text{ for } n \geq 2 \\ h_0 = 0,\ h_1 = 1 \end{cases}$

Sol $g(x) = \sum_{n=0}^{\infty} h_n x^n$ $\qquad A \cdot 3^n + B \cdot n \cdot 3^n$

(*) $h_{n+2} = 4h_{n+1} - 4h_n + 3^{n+2}$ for $n \geq 0$

Multiply (*) by x^n.

$h_{n+2}x^n = 4h_{n+1}x^n - 4\boxed{h_n x^n} + 3^{n+2}x^n$ for $n \geq 0$.

$$\sum_{n=0}^{\infty} h_{n+2}x^n = \sum_{n=0}^{\infty} 4h_{n+1}x^n - \sum_{n=0}^{\infty} 4h_n x^n + \sum_{n=0}^{\infty} 3^{n+2}x^n$$

$4g(x)$

$$-4\sum_{n=0}^{\infty} h_n x^n = -4g(x)$$

$$\sum_{n=0}^{\infty} h_{n+2}x^{n+2-2} = \sum_{n=0}^{\infty} h_{n+2}x^{n+2}x^{-2} = \frac{1}{x^2}\sum_{n=0}^{\infty} h_{n+2}x^{n+2}$$

$n=2$, n, n

$$= \frac{1}{x^2}\left(g(x) - x\right)$$

$$4\sum_{n=0}^{\infty} h_{n+1}x^{n+1-1} = \frac{4}{x}\sum_{n=0}^{\infty} h_{n+1}x^{n+1} = \frac{4}{x}\left(g(x) - h_0\right) = \frac{4}{x}g(x)$$

$$\frac{1}{x^2}\left(g(x) - x\right) = \frac{4}{x}g(x) - 4g(x) + \frac{9}{1-3x}$$

$$\Longleftrightarrow g(x) = \frac{6x^2 + x}{(1-3x)(1-2x)^2} = \frac{A}{1-3x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

$$6x^2 + x = A(2x-1)^2 + B(1-3x)(1-2x) + C(1-3x)$$

Let $x = \frac{1}{2}$:

$$6\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} = 0 + 0 + C\left(1 - \tfrac{3}{2}\right) = -\tfrac{1}{2}C \Rightarrow C = -4$$

Let $x = \frac{1}{3}$: $\Rightarrow A = 9$

Let $x = 0$: $\Rightarrow B = -5$

$$g(x) = \frac{9}{1-3x} + \frac{-5}{1-2x} + \frac{-4}{(1-2x)^2}$$

$$= 9\sum_{n=0}^{\infty} 3^n x^n - 5\sum_{n=0}^{\infty} 2^n x^n - 4\left(\sum_{n=0}^{\infty} 2^n x^n\right)\left(\sum_{n=0}^{\infty} 2^n x^n\right)$$

$$= 9\sum_{n=0}^{\infty} 3^n x^n - 5\sum_{n=0}^{\infty} 2^n x^n - 4\sum_{n=0}^{\infty} (n+1)2^n x^n.$$

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^n$$

$$g(x) = \sum_{n=0}^{\infty}\underbrace{\left[9\cdot 3^n - 5\cdot 2^n - 4(n+1)2^n\right]}_{h_n} x^n$$

$$h_n = 9\cdot 3^n - 5\cdot 2^n - 4\cdot(n+1)\cdot 2^n$$

<u>Thm 1</u>

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}, \quad n \geq k, \quad a_k \neq 0$$

$$\left(= \sum_{j=1}^{k} a_j h_{n-j}\right)$$

Let $g(x) = \sum_{n=0}^{\infty} h_n x^n$

Then $g(x) = \frac{p(x)}{q(x)}$ where

$$p(x) = \sum_{n=0}^{k-1}\left(h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k}\right)x^n$$

where $h_j = 0$ if $j \leq -1$.

$$q(x) = 1 - a_1 x - a_2 x^2 - \cdots - a_k x^k$$

HW: <u>§7.7</u> 48 c) f), 51

F 4/14 Lecture

Thm 1 Let $h_n = \boxed{a_1 h_{n-1}} + a_2 h_{n-2} + \cdots + a_k h_{n-k}$ $(n \geq k)$ $a_k \neq 0$.

Let $g(x) = \sum_{n=0}^{\infty} h_n x^n$. Then $g(x) = \frac{p(x)}{q(x)}$,

where $p(x) = \sum_{n=0}^{k-1} (h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k}) x^n$

(here $h_j = 0$ if $j \leq -1$)

$q(x) = 1 - a_1 x - a_2 x^2 - \cdots - a_k x^k$.

$\frac{C}{(1-Bx)^k}$ $\frac{1}{(1-z)^k}$

$h_n = F(h_{n-1}, \ldots, h_{n-k})$

Pf $g(x) = \sum_{n=0}^{\infty} h_n x^n$. Let $h_j = 0$ if $j \leq -1$.

$$a_1 x g(x) = \sum_{n=0}^{\infty} (a_1 h_n) x^{n+1} = \sum_{n=1}^{\infty} (a_1 h_{n-1}) x^n = \sum_{n=0}^{\infty} (a_1 h_{n-1}) x^n$$

$$a_2 x^2 g(x) = \sum_{n=0}^{\infty} a_2 h_n x^{n+2} \overset{n \to n-2}{=} \sum_{n=2}^{\infty} a_2 h_{n-2} x^n$$

$$= \sum_{n=0}^{\infty} a_2 h_{n-2} x^n$$

$$a_k x^k g(x) = \sum_{n=0}^{\infty} a_k h_n x^{n+k} \overset{n \to n-k}{=} \sum_{n=k}^{\infty} a_k h_{n-k} x^n = \sum_{n=0}^{\infty} a_k h_{n-k} x^n$$

$$(a_1 x + a_2 x^2 + \cdots + a_k x^k) g(x) = \sum_{n=0}^{\infty} (a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}) x^n$$

$$= \sum_{n=k}^{\infty} h_n x^n + \sum_{n=0}^{k-1} (a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}) x^n$$

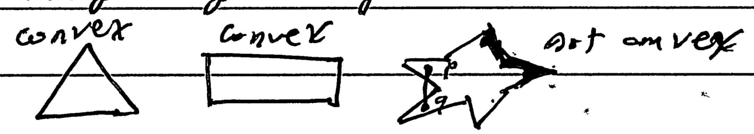
$$\underbrace{(1-a_1 x-a_2 x-\cdots-a_k x^k)}_{q(x)}\, g(x) = \underbrace{\sum_{n=0}^{k-1}(h_n - a_1 h_{n-1} - \cdots - a_k h_{n-k})x^n}_{p(x)}$$

$$g(x) = \frac{p(x)}{q(x)}$$

HW: §7.7: 48c)f), 51

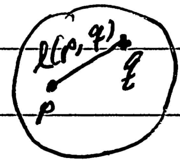
§7.6 A Geometry Problem

Def1 A polygon is a region whose boundary consists of finitely many line segments.



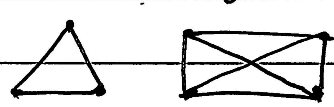
Def2 Let $K \subseteq \mathbb{R}^2$. K is called a "convex" set if any two $p, q \in K$,

the line segment connecting p, q $\ell(p;q) \subseteq K$



Let P_{n+1} be the convex polygon with $n+1$ sides.

Def3 A diagonal in a polygon is a line seg. joining two nonconsecutive vertices.



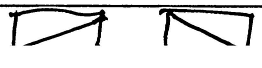
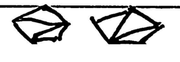
Ex1 Let P_n be a polygon with n sides. How many diagonals in P_n?

Sol P_n has n vertices.

$\binom{n}{2}$ line segments

$\#$ of diag. $= \binom{n}{2} - n$.

Let $h_n = \#$ of ways of dividing P_{n+1} into Δ-regions by inserting diagonals which do not intersect in the interior of P_{n+1}.

 $h_3 = 2$  Then $h_n = \frac{1}{n}\binom{2n-2}{n-1}$ for all $n \geq 1$.

M 4/17 Lecture

Let P_{n+1} be a convex polygon with $n+1$ sides.
Let h_n = # of ways of dividing P_{n+1} into Δ-regions by inserting diagonals which do NOT intersect in the interior of P_{n+1}.

Thm1 $h_n = \frac{1}{n}\binom{2n-2}{n-1}$ ~~for all $n \geq 1$.~~ Let $h_1 = 1$. $h_2 = 1$.

Pf Claim1 $h_n = \sum_{k=1}^{n-1} h_k h_{n-k}$ ~~if $n \geq 2$.~~

Pf of Claim1 $h_2 = 1 = \sum_{k=1}^{1} h_k h_{2-k} = h_1 h_1 = 1$

Let $n \geq 3$. Consider P_{n+1}. Choose 1 side as a base.
After dividing P_{n+1} into Δ's. The base is one side of some Δ.
Divide P_{n+1} into $P_1 \cup T \cup P_2$ s.t. P_1 has $k+1$ sides, P_2 has $n-k+1$ sides. Here $1 \leq k \leq n-1$.

# of ways to divide P_1 into Δ's $= h_k$
# of ways to divide P_2 into Δ's $= h_{n-k}$
For fixed k, $h_k h_{n-k}$.

$$\boxed{h_n = \sum_{k=1}^{n-1} h_k h_{n-k}, \quad h_1 = 1, \quad n \geq 2} \quad \cancel{n \geq 3}$$

$$g(x) = \sum_{n=1}^{\infty} h_n x^n$$

$$(g(x))^2 = \left(\sum_{n=1}^{\infty} h_n x^n\right)\left(\sum_{n=1}^{\infty} h_n x^n\right) = \sum_{n=1}^{\infty} c_n x^n$$

$$= (x + h_2 x^2 + h_3 x^3 + \cdots)(x + h_2 x^2 + h_3 x^3 + \cdots)$$

$c_1 = 0$, $c_2 = 1$

$$C_n = \sum h_{n_1} h_{n_2} \qquad \begin{matrix} n_1, n_2 \geq 1 \\ n_1 + n_2 = n \end{matrix}$$

Let $k = n_1$, then $n_2 = n - k$.

$$C_n = \sum_{k=1}^{n-1} h_k h_{n-k}, \quad n \geq 2$$

$$= h_n$$

$$(g(x))^2 = \sum_{n=2}^{\infty} C_n x^n = \sum_{n=2}^{\infty} h_n x^n = g(x) - h_1 x = g(x) - x$$

$$\boxed{(g(x))^2 - g(x) + x = 0}$$

$Y = g(x)$

$Y^2 - Y + x = 0$

$$g(x) = \frac{1 \pm \sqrt{1-4x}}{2}$$

Notice

$g(0) = 0$, $\quad g(x) = \dfrac{1 - \sqrt{1-4x}}{2} = \dfrac{1}{2} - \dfrac{1}{2}\sqrt{1-4x}$

$$\sqrt{1-4x} = (1-4x)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n$$

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} = \frac{(-1)^{n-1}}{n \cdot 2^{2n-1}} \binom{2n-2}{n-1}$$

$$g(x) = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)(-4)^n \binom{\frac{1}{2}}{n} x^n$$

$$h_n = \left(-\frac{1}{2}\right)(-4)^n \binom{\frac{1}{2}}{n} = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1}$$

HW §7.7: 25, 40.

Ch 8

§8.1 Catalan Numbers

Catalan seq is $\{C_n\}_{n=0}^{\infty}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Let $n \in \mathbb{N}$.

Let $\{a_1, a_2, \ldots, a_{2n}\}$ be a seq of n $+1$'s and n (-1)'s

# of seq consisting of n $+1$'s and n (-1)'s $= \binom{2n}{n}$.

Let $S_k = a_1 + a_2 + \cdots + a_k$, $1 \le k \le 2n$.

Def If $S_k \ge 0$ for all $1 \le k \le 2n$, then $\{a_j\}_{j=1}^{2n}$ is called "acceptable". Otherwise unacceptable.

W 4/26

$S(p,k)$

$$n^p = \sum_{k=0}^{p} P(n,k)\,\boxed{S(p,k)} \qquad (*)$$

<u>Thm 1</u> If $1 \le k \le p-1$,

$$S(p,k) = \cancel{kS(p,k)}\ \cancel{k(S,}\ kS(p-1,k) + S(p-1,k-1)$$

<u>pf</u> By $(*)$,

$$n^{p-1} = \sum_{k=0}^{p-1} P(n,k)\,S(p-1,k)$$

$$n^p = n\,\boxed{n^{p-1}} = n\sum_{k=0}^{p-1} P(n,k)\,S(p-1,k)$$

$$= \sum_{k=0}^{p-1} n\,P(n,k)\,S(p-1,k)$$

$$= \sum_{k=0}^{p-1} ((n-k)+k)\,P(n,k)\,S(p-1,k)$$

$$= \boxed{\sum_{k=0}^{p-1} (n-k)P(n,k)\,S(p-1,k)} + \sum_{k=0}^{p-1} kP(n,k)\,S(p-1,k)$$

$$(n-k)P(n,k) = \underline{(n-k)}\frac{n!}{(n-k)!} = \frac{n!}{(n-k-1)!} = P(n,k+1)$$

$k \to k-1$:

$$n^p = \sum_{k=1}^{p} P(n,k)\,S(p-1,k-1) + \sum_{k=1} kP(n,k)\,S(p-1,k-1)$$

$$= P(n,p)\,S(p-1,p-1) + \sum_{k=1}^{p-1} P(n,k)\left[S(p-1,k-1) + kS(p-1,k)\right]$$

$S(p,p) = 1$

$$n^p = P(n,p)\,S(p,p) + \sum_{k=1}^{p-1} P(n,k)\,\boxed{\left[S(p-1,k-1) + kS(p-1,k)\right]}$$

$$= P(n,p)\,S(p,p) + \sum_{k=0}^{p-1} P(n,k)\,\boxed{S(p,k)}$$

For $1 \le k \le p-1$.

$S(p,k) = S(p-1,k-1) + kS(p-1,k)$

$S(p,k)$ is uniquely determined by

$$\begin{cases} S(p,k) = S(p-1,k-1) + kS(p-1,k) & 1 \le k \le p-1 \\ S(p,0) = 0 \text{ if } p \ge 1 \text{ and } S(p,p) = 1 \text{ for } p \ge 0 \end{cases}$$

Thm ~~$S(p,k) = $ # of partitions of a set of p~~

elements into k indistinguishable boxes in which no box is empty.

Pf. Let $S^*(p,k) = $ # of partitions of a set of p elements into exactly k ~~box~~ parts.

We show that

$$\begin{cases} S^*(p,k) = S^*(p-1,k-1) + kS^*(p-1,k) \\ S^*(p,0) = 0 \text{ if } p \ge 1 \\ S^*(p,p) = 1 \text{ for } p \ge 0 \end{cases}$$

$S^*(p,p) = 1$.

$S^*(p,0)$. Let $p \ge 1$. $S^*(p,0) = 0$.

Consider the set $\{1, 2, \ldots, p\}$ after we divide the set into k parts.

Case 1 There exists one part containing only p.

Case 2 There are NO parts containing only p.

Case 1 The set $\{1, 2, \ldots, p\}$ is divided into

$\{p\}, A_1, A_2, \ldots, A_{k-1}$, all $\ne \phi$

Then $A_1, \ldots, A_{k-1}$ form a partition of $\{1, 2, \ldots, p-1\}$.

There are $\boxed{S^*(p-1, k-1)}$ ways to partition $\{1, \ldots, p-1\}$ into $k-1$ parts.

<u>Case 2</u> Suppose $A_1, A_2, \ldots, A_k$ form a partition of $\{1, 2, \ldots, p\}$ s.t. $A_j \neq \{p\}$.

$A_j \neq \emptyset$, for ~~all~~, $1 \leq j \leq k$

To get such a partition, we partition $\{1, 2, \ldots, p-1\}$ into $B_1, B_2, \ldots, B_k$ (all nonempty) first.

$S^*(p-1, k)$

Add p into one of $B_1, \ldots, B_k$ to get a partition of $\{1, \ldots, p\}$.

k

$\boxed{kS^*(p-1, k)}$

$$S^*(p, k) = S^*(p-1, k-1) + kS^*(p-1, k)$$

<u>§ 8.6</u> 11, 2

F 4/28 Lecture

Ex3 Determine h_n = # of n-digit numbers with each digit odd, in which 1, 3 occur an even number of times.

Sol $g^{(e)}(x) = \left(\sum_{m_1 \text{ even}} \frac{x^{m_1}}{m_1!}\right)\left(\sum_{m_2 \text{ even}} \frac{x^{m_2}}{m_2!}\right) e^x e^x e^x$

$$= \frac{1}{2}(e^x + e^{-x}) \frac{1}{2}(e^x + e^{-x}) e^{3x}$$

$$= \frac{1}{4}\left(e^{5x} + 2e^{3x} + e^x\right)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$h_n = \frac{5^n + 2 \times 3^n + 1}{4}$$

M 5/1 Lecture

May 12, 1:30 – 4:30 PM

Ch 2, 3, 5, 6.1–6.5, 7, 8.1–8.2

Ex1 Suppose that a codeword consisting of
4 A's, 2 U's, 2 C's, 2 G's.
How many codewords are there if no two of A's occur consecutively?

Sol Step1 Arrange 2 U's, 2 C's, and 2 G's.

$$\frac{(2+2+2)!}{2!\,2!\,2!} = \frac{6!}{2\cdot 2\cdot 2} = \frac{6!}{8}$$

Step2 Put 4 A's into O-positions

○ □ ○ □ ○ □ ○ □ ○ □ ○ □ ○

$$\binom{7}{4} = \binom{7}{3}$$

$$\boxed{\frac{6!}{8}\cdot\binom{7}{3}}$$

Ex2 Prove that a partially ordered set of $mn+1$ elements has a chain of size $m+1$ or an antichain of size $n+1$.

Pf Let $(X, \leq)$ be the partially ordered set of $mn+1$ elements.
Let r be the size of longest chain in X.

Case 1 $r \geq m+1$.
In Case 1, we get a chain of size $m+1$.

Case 2 $r < m+1$.
In this case, $r \leq m$.
X can be partitioned into r antichains.
By pigeonholing, there is an antichain of size $\geq n+1$.

Ex 3 Count the permutations of $\{1, 2, 3, 4\}$ where $i_1 \neq 3, 4$, $i_3 \neq 2, 3$, and $i_4 \neq 1, 2$.

Sol Count ways to put non-attacking rooks into 4×4 board with forbidden positions $(1,3), (1,4), (3,2), (3,3), (4,1), (4,2)$

		X	X
	X	X	
X	X		

$$\cancel{\# = 4! + \sum_{k=1}^{4} (-1)^k r_k (4-k)!}$$

$$= 4! - 3! r_1 + 2! r_2 - 1! r_3 + r_4$$

$r_1 = 6$

$r_2 = 10$

$r_3 = 4$

$r_4 = 0$

$\# = 4$

Ex 4 Let $S_n = \{2 \cdot a_1, 2 \cdot a_2, \ldots, 2 \cdot a_n\}$

Find # of permutations of S_n in which for each type of objects, the objects of the same type do NOT appear consecutively.

Sol For $1 \leq j \leq n$, let

$A_j = \{$all permutations of S_n in which a_j's appear consecutively$\}$

We count $A_1^c \cap A_2^c \cap \cdots \cap A_n^c$

$$|A_1^c \cap \cdots \cap A_n^c| = \left|\left(\bigcup_{j=1}^{n} A_j\right)^c\right| = |X| - \left|\bigcup_{j=1}^{n} A_j\right|.$$

Here X is the collection of all permutations of S_n.

$$|X| = \frac{(2n)!}{\underbrace{2!2!\cdots 2!}_{n}} = \frac{(2n)!}{2^n}$$

$$\left|\bigcup_{j=1}^{n} A_j\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1\le i_1 < i_2 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|$$

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} = \left\{ \begin{array}{l} \text{all permutations of } S_n \text{ in which} \\ a_{i_1}\text{'s appear consecutively} \\ a_{i_2}\text{'s } \text{———} \\ \vdots \\ a_{i_k}\text{'s } \text{———} \end{array} \right\}$$

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \frac{(\overbrace{1+\cdots+1}^{k} + \overbrace{2+\cdots+2}^{n-k})!}{\underbrace{2!\cdots 2!}_{n-k}} = \frac{(2n-k)!}{2^{n-k}}$$

$$\sum_{1\le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| = \frac{(2n-k)!}{2^{n-k}} \sum_{1\le i_1 < \cdots < i_k \le n} 1 = \frac{(2n-k)!}{2^{n-k}} \binom{n}{k}$$

$$\left|\bigcup_j A_j\right| = \sum_{k=1}^{n} (-1)^{k+1} \frac{(2n-k)!}{2^{n-k}} \binom{n}{k}$$

$$\# = \frac{(2n)!}{2^n} - \sum_{k=1}^{n} (-1)^{k+1} \frac{(2n-k)!}{2^{n-k}} \binom{n}{k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2n-k)!}{2^{n-k}}$$

Ex 5 Determine the number h_n of bags of fruit of apples, oranges, bananas and pears in which there are an even number of apples, at most two oranges, a multiple of three number of bananas, and at most one pear.

Sol $g(x) = \sum_{n=0}^{\infty} h_n x^n$

$$g(x) = \left(\sum_{m_1 \text{ even}} x^{m_1}\right)\left(\sum_{m_2=0}^{2} x^{m_2}\right)\left(\sum_{3 \mid m_3} x^{m_3}\right)\left(\sum_{m_4=0}^{1} x^{m_4}\right)$$