

Mechanics Wrap-Up

Static Equilibrium, Elasticity, Gravitation, Fluids, Oscillations

Physics Club

December 4, 2014

Static Equilibrium

Conditions for Equilibrium

The net external force and the net external torque about any point must remain zero for a rigid body to be in equilibrium.

$$\sum F = 0$$

$$\sum \tau = 0$$

The Center of Gravity

The center of gravity and the center of mass coincide for a uniform gravitational field. This can be seen more easily as follows. For an object of mass $M = \sum m_i$ in a uniform gravitational field g , the net torque

$$\tau_{\text{net}} = \sum_i r_i \times m_i g = r_{\text{cm}} \times Mg$$

Static Equilibrium

Couple

The torque produced by a couple, a pair of parallel forces of equal magnitudes, is the same about all points in space.

Static Equilibrium in an Accelerated Frame

An accelerated frame is a frame accelerating relative to an inertial reference frame. A static object in an accelerated reference frame has a center of mass acceleration equal to that of the reference frame. Also Newton's second law for rotation applies to rotation about the center of mass whether or not it is accelerating.

$$\sum F = ma_{cm}$$

$$\sum \tau_{cm} = 0$$

Elasticity

Elastic Modulus

A force may deform a solid object but when the force is removed, the object tends to return to its original shape and size. This is the case for forces up to a certain maximum, the **elastic limit**.

$$\text{Elastic modulus} \equiv \frac{\text{stress}}{\text{strain}}$$

Young's Modulus: Elasticity in Length

Measures the resistance of a solid to a change in its length

$$Y \equiv \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A}{\Delta L/L}$$

Elasticity

Shear Modulus: Elasticity of Shape

Measures the resistance to motion of the planes within a solid parallel to each other

$$S \equiv \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\Delta x/h} = \frac{F/A}{\tan \theta}$$

Bulk Modulus: Elasticity of Volume

Measures the resistance of solids or liquids to changes in their volume

$$B \equiv \frac{\text{volume stress}}{\text{volume strain}} = -\frac{\Delta F/A}{\Delta V/V_i} = -\frac{\Delta P}{\Delta V/V_i}$$

Basic Definitions of Gravity

Newton's Law of Universal Gravitation

$$F_g = G \frac{m_1 m_2}{r^2}$$

where $G = 6.674 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$.

Gravity Near the Surface of the Earth

$g \approx 9.81 \text{ m/s}^2$ near the surface of the Earth. By Newton's Law of Universal Gravitation

$$g = G \frac{M_E}{R_E^2}$$

where the mass of the Earth $M_E = 5.97 \times 10^{24} \text{ kg}$ and the average radius of the Earth $R_E = 3.39 \times 10^6 \text{ m}$.

Gravitational Fields

Gravity Above the Earth

$$g = \frac{GM_E}{(R_E + h)^2}.$$

$mg \rightarrow 0$ as $r \rightarrow \infty$.

Gravitational Fields

$$\vec{g} \equiv \frac{\vec{F}_g}{m_0}$$

Geosynchronous Orbits

Satellites can maintain a constant position above the Earth.

Gravitational Potential Energy

Gravitational Potential Energy

$U_g = mgh$ near the surface of the Earth. More generally

$$U = -\frac{Gm_1m_2}{r}.$$

Orbits

$$E_{\text{circle}} = -\frac{GMm}{2r},$$

$$E_{\text{ellipse}} = -\frac{GMm}{2a}$$

where $m \ll M$.

Escape Speed

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

Kepler's Laws

Kepler's First Law

All planets move in elliptical orbits with the Sun at one focus.

Kepler's Second Law

The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.

$$\frac{\Delta A}{\Delta t} = \frac{L}{2M_S}$$

where $M_S = 1.989 \times 10^{30}$ kg is the mass of the Sun.

Kepler's Third Law

$$T^2 = \left(\frac{4\pi^2}{GM_S} \right) a^3$$

where T is the orbital period of any planet and a is the semi-major axis of the elliptical orbit.

Basic Definitions of Fluid Mechanics

States of Matter

We have four basic states of matter: solids, liquids, gases, and plasma.

Density

$$\rho \equiv \frac{M}{V}.$$

The density of water is 1000 kg/m^3 .

Pressure

$$P \equiv \frac{F}{A},$$

which has the SI unit of Pascal ($\text{Pa} \equiv \text{N/m}^2$).

Variations of Pressure with Depth

When a fluid is at rest in a container, *all portions of the fluid must be in static equilibrium* - at rest with respect to the observer.

Changes in Pressure

All points at the same depth must be at the same pressure. The pressure isn't affected by the shape of the vessel.

$$P = P_0 + \rho gh$$

We can use this to measure pressure with open-tube manometers or mercury barometers.

Pascal's Principle

A change in pressure applied to an enclosed fluid is transmitted undiminished to every point of the fluid and to the walls of the container.

Buoyant Forces and Archimedes's Principle

Any object completely or partially submerged in a fluid is buoyed upward by a force with magnitude equal to the weight of the fluid displaced by the object.

$$F_B = \rho_{\text{fluid}} V_{\text{fluid}} g$$

- **Non-viscous:** no internal friction force between layers
- **Incompressible:** constant density
- **Steady motion:** no change in velocity, density, and pressure for each point in time
- **No turbulence:** no angular velocity about its center, which means no eddy currents in the moving fluid

Fluids in Motion

The Equation of Continuity

Constant flow rate through a tube

$$A_1 v_1 = A_2 v_2$$

Bernoulli's Equation

Energy conservation as applied to an ideal fluid

$$P + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}$$

Venturi Effect

When the speed of a fluid increases, the pressure drops

$$P + \frac{1}{2}\rho v^2$$

Simple Harmonic Motion

Hooke's Law

$$F_s = -kx$$

Acceleration

$$a = -\frac{k}{m}x$$

Position

$$x = A \cos(\omega t + \delta)$$

where $\omega = \sqrt{k/m}$ and A and δ can be uniquely determined by its **initial conditions** such as the initial displacement and velocity.

Energy

$$E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

Simple Harmonic Motion Definitions

- The maximum displacement from equilibrium is A , which is called the **amplitude**.
- The argument of the cosine function, $\omega t + \delta$, is called the **phase** of the motion.
- The **period** T is the shortest time satisfying the relation $x(t) = x(t + T)$.
- the **frequency** f is the number of cycles per second and is thus the reciprocal of the period. Its SI units are Hertz (Hz).
- ω is called the **angular frequency** because

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

The frequency and period of a simple harmonic motion are independent of the amplitude.

Simple Harmonic Motion and Circular Motion

When a particle moves with constant speed in a circle, its projection onto a diameter of the circle moves with S.H.M. Consider a particle moving with constant speed v in a circle of radius A . Its angular displacement relative to the x axis is

$$\theta = \omega t + \delta.$$

Recall that we can always consider the motion of a particle in a plane as the superposition of its motion along the x and y directions.

The x component of its displacement from the origin (the center of the circle), its velocity, and acceleration are

$$x = A \cos \theta = A \cos(\omega t + \delta)$$

$$v_x = -\omega A \sin \theta = -\omega A \sin(\omega t + \delta)$$

$$a_x = -\omega^2 A \cos \theta = -\omega^2 A \cos(\omega t + \delta) = -\omega^2 x$$

Thus the projection of the particle onto the x axis moves with simple harmonic motion.

Oscillating Systems

Many systems are analogous to simple harmonic motion

Object on a Vertical Spring

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Simple Pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Physical Pendulum

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$