1. Kevin is running 1000 meters. He wants to have an average speed of 10 meters a second. He runs the first 100 meters at a speed of 4 meters a second. Compute how quickly, in meters per second, he must run the last 900 meters to attain his desired average speed of 10 meters a second.

   Answer: 12

   Solution: He has 100 seconds to run 1000 meters. He spends 25 seconds running the first 100 meters, so he needs to run 900 meters in 75 seconds, for a speed of $\boxed{12}$ meters per second.

2. Let a and b be the roots of the quadratic $x^2 - 7x + c$. Given that $a^2 + b^2 = 17$, compute c.

   Answer: 16

   Solution: Note that $a + b = 7$ and $ab = c$, so that $a^2 + b^2 = (a + b)^2 - 2ab = 49 - 2c = 17$. Therefore, $c = \boxed{16}$.

3. Compute

$$\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right).$$

   Answer: $\frac{\sqrt{3}}{8}$

   Solution 1: Ideally, we would compute $\sin\left(\frac{\pi}{9}\right)$ immediately and then proceed with trigonometric formulas. Unfortuately 9 is not a product of distinct Fermat primes so $\sin\left(\frac{\pi}{9}\right)$ cannot be expressed in terms of sums, products, and finite root extractions on rational numbers. However we can use the triple angle identity (motivated because we know how to compute $\sin(\pi/3)$)

$$\sin(3\alpha) = 3\sin(\alpha) - 4\sin^3(\alpha)$$

   Thus $u = \sin\left(\frac{\pi}{9}\right)$ satisfies the cubic equation: $4u^3 - 3u + \sqrt{3}/2 = 0 = u^3 - \frac{3}{4}u + \frac{\sqrt{3}}{8}$. By Vieta's identity the product of the roots of this polynomial is $-\frac{\sqrt{3}}{8}$, the sum of pairs of roots equals $-3/4$, and the sum of the roots vanishes. If r_1, r_2 are the other roots then,

$$\begin{aligned}
-\frac{\sqrt{3}}{8} &= \sin\left(\frac{\pi}{9}\right) r_1 r_2 \\
-\frac{3}{4} &= \sin\left(\frac{\pi}{9}\right)(r_1 + r_2) + r_1 r_2 \\
0 &= \sin\left(\frac{\pi}{9}\right) + r_1 + r_2
\end{aligned}$$

   Now consider the other two factors in the expression, or goal is to write them in terms of $\sin^2\left(\frac{\pi}{9}\right)$ or the roots to its polynomial. Trigonometric identities can be used here but it is faster to write them with Euler's identiy and expand,

$$\begin{aligned}
\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right) &= \frac{1}{4}\left(e^{2\frac{\pi i}{9}} - e^{-2\frac{\pi i}{9}}\right)\left(e^{4\frac{\pi i}{9}} - e^{-4\frac{\pi i}{9}}\right) = \frac{1}{4}\left(1 + e^{2\frac{\pi i}{9}} - e^{7\frac{\pi i}{9}}\right) \\
&= \frac{1}{4}\left(1 + 2\cos\left(\frac{2\pi}{9}\right)\right) = \frac{1}{4}\left(3 - 4\sin^2\left(\frac{\pi}{9}\right)\right)
\end{aligned}$$

   But from the formulas above, $\frac{3}{4} = \sin^2\left(\frac{\pi}{9}\right) - r_1 r_2$. Thus $\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right) = -r_1 r_2$. And therefore,

$$\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right) = \boxed{\frac{\sqrt{3}}{8}}.$$

Solution 2: We square the equation, and multiply by $\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)$ to obtain

$$\begin{aligned}
&\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)\cdot\left(\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right)\right)^2 \\
&= \sin\left(\frac{3\pi}{9}\right)\sin\left(\frac{6\pi}{9}\right)\left(\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right)\right)\left(\sin\left(\frac{8\pi}{9}\right)\sin\left(\frac{7\pi}{9}\right)\sin\left(\frac{5\pi}{9}\right)\right) \\
&= \prod_{k=1} 8\sin\left(\frac{k\pi}{9}\right).
\end{aligned}$$

For $z = \exp(2\pi i/9)$ note that

$$\sin\left(\frac{k\pi}{9}\right) = \left|\frac{z^{k/2} - z^{-k/2}}{2i}\right| = \frac{|z^k - 1|}{2}.$$

We use the factorization

$$\frac{X^9 - 1}{X - 1} = X^8 + X^7 + \cdots + X + 1 = \prod_{k=1}^{8}(X - z^k)$$

to obtain

$$\prod_{k=1} 8\sin\left(\frac{k\pi}{9}\right) = \frac{1}{2^8}\prod_{k=1} 8|1 - z^k| = \frac{9}{2^8}.$$

Therefore our original equation has value

$$\sqrt{\frac{\frac{9}{2^8}}{\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)}} = \sqrt{\frac{3}{2^6}} = \boxed{\frac{\sqrt{3}}{8}}.$$