1. Consider a square of side length 1 and erect equilateral triangles of side length 1 on all four sides of the square such that one triangle lies inside the square and the remaining three lie outside. Going clockwise around the square, let A, B, C, D be the circumcenters of the four equilateral triangles. Compute the area of $ABCD$.

   Answer: $\mathbf{\frac{3+\sqrt{3}}{6}}$

   Solution: Let A denote the circumcenter of the triangle which lies inside the square. We first note that in an equilateral triangle of side length 1, its altitudes goes through the circumcenter and the circumcenter divides the altitude into two segments of length $\frac{1}{\sqrt{3}}$ and $\frac{1}{2\sqrt{3}}$. Thus, $ABCD$ is convex and its area is the the sum of the area of triangle ABD and the area of triangle CBD. The length of BD is $\frac{1}{2\sqrt{3}} + 1 + \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}}$. In triangle ABD, the altitude corresponding to base BD is $\frac{1}{2} - \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}-1}{2\sqrt{3}}$. Thus, the area of ABD is $\frac{1}{2} \cdot \frac{\sqrt{3}+1}{\sqrt{3}} \cdot \frac{\sqrt{3}-1}{2\sqrt{3}} = \frac{1}{6}$. Next, in triangle CBD, the altitude corresponding to base BD is $\frac{1}{2} + \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}+1}{2\sqrt{3}}$. Thus, the area of CBD is $\frac{1}{2} \cdot \frac{\sqrt{3}+1}{\sqrt{3}} \cdot \frac{\sqrt{3}+1}{2\sqrt{3}} = \frac{2+\sqrt{3}}{6}$. Therefore, the area of $ABCD$ is $\frac{1}{6} + \frac{2+\sqrt{3}}{6} = \boxed{\frac{3+\sqrt{3}}{6}}$.

2. Let ABC be a triangle with sides $AB = 19$, $BC = 21$ and $AC = 20$. Let ω be the incircle of ABC with center I. Extend BI so that it intersects AC at E. If ω is tangent to AC at the point D, then compute the length of DE.

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: Since I is the incenter, we know that BE is the angle bisector of $\angle ABC$. By the angle bisector theorem, $\frac{AE}{CE} = \frac{AB}{BC} = \frac{19}{21}$. Also, we know that $AE + CE = AC = 20$ so $CE = 20 - AE$. Thus, $\frac{19}{21} = \frac{AE}{20-AE}$ so $AE = \frac{19}{2}$.

   Because D is the point of tangency, we also know that $AD = s - BC$, where s is the semiperimeter $\frac{AB+BC+AC}{2} = 30$. This implies that $AD = 9$. Finally, $DE = AE - AD = \frac{19}{2} - 9 = \boxed{\frac{1}{2}}$.

3. Compute the perimeter of the triangle that has area $3 - \sqrt{3}$ and angles $45°$, $60°$, and $75°$.

   Answer: $\mathbf{3\sqrt{2} + 2\sqrt{3} - \sqrt{6}}$

   Solution: Let the triangle be denoted ABC, with $\angle A = 45°$, $\angle B = 60°$, and $\angle C = 75°$. We first find side AB. Drop an altitude D onto side AB and note that ACD is a 45-45-90 triangle and BCD is a 30-60-90 triangle. Therefore, $AD = CD$ and $BD = \frac{CD}{\sqrt{3}}$. Thus, $AB = (1 + \frac{1}{\sqrt{3}})CD$. Since CD is an altitude, the area of ABC is $\frac{1}{2} \cdot AB \cdot CD = \frac{1}{2} \cdot \frac{\sqrt{3}}{1+\sqrt{3}} \cdot AB^2$. Since the area of ABC is $3 - \sqrt{3}$, it follows that $AB^2 = 2 \cdot \frac{\sqrt{3}+1}{\sqrt{3}} \cdot (3 - \sqrt{3})$ and hence $AB = 2$.

   Now, $CD = \frac{\sqrt{3}}{1+\sqrt{3}} AB = 3 - \sqrt{3}$. Since ACD is a 45-45-90 triangle, $AC = \sqrt{2} \cdot CD = 3\sqrt{2} - \sqrt{6}$. Since BCD is a 30-60-90 triangle, $BC = \frac{2}{\sqrt{3}} \cdot CD = 2\sqrt{3} - 2$. Thus, the perimeter of ABC is $2 + 3\sqrt{2} - \sqrt{6} + 2\sqrt{3} - 2 = \boxed{3\sqrt{2} + 2\sqrt{3} - \sqrt{6}}$.

4. Consider a square $ABCD$ with side length 4 and label the midpoint of the side BC as M. Let X be the point along AM obtained by dropping a perpendicular from D onto AM. Compute the product of the lengths XC and MD.

   Answer: $\mathbf{8\sqrt{5}}$

   Solution: First, by Pythagoras, $AM = DM = \sqrt{AB^2 + BM^2} = \sqrt{4^2 + 2^2} = 2\sqrt{5}$.

Next, we solve for XD by computing the area of triangle AMD in two different ways. Using AD as the base, we find the area to be $\frac{1}{2} \cdot 4 \cdot 4 = 8$. Using AM as the base, we find the area to be $\frac{1}{2} \cdot XD \cdot 2\sqrt{5} = \sqrt{5} \cdot XD$. Thus, $\sqrt{5} \cdot XD = 8$ and hence $XD = \frac{8}{\sqrt{5}}$.

Next, applying Pythagoras to triangle XMD, we get $XM = \sqrt{MD^2 - XD^2} = \sqrt{20 - \frac{64}{5}} = \frac{6}{\sqrt{5}}$. Finally, note that since $\angle DXM = \angle DCM = 90°$, the quadrilateral $DCMX$ is cyclic. Therefore, we may compute the product $XC \cdot MD$ via Ptolemy's theorem:

$$\begin{aligned} XC \cdot MD &= XM \cdot DC + XD \cdot MC \\ &= \frac{6}{\sqrt{5}} \cdot 4 + \frac{8}{\sqrt{5}} \cdot 2 \\ &= \boxed{8\sqrt{5}}. \end{aligned}$$

5. Consider a triangle ABC with $AB = 4$, $BC = 3$, and $AC = 2$. Let D be the midpoint of line BC. Find the length of AD.

   Answer: $\frac{\sqrt{31}}{2}$

   Solution: Let $\alpha = \angle ACB$. Then applying law of cosines to triangle ABC, we find that $\cos(\alpha) = \frac{AC^2+BC^2-AB^2}{2AC \cdot BC} = \frac{2^2+3^2-4^2}{2 \cdot 2 \cdot 3} = -\frac{1}{4}$. Next, applying law of cosines to triangle ADC, we find that $AD^2 = AC^2 + CD^2 - 2 \cdot \cos(\alpha) \cdot AC \cdot CD = 2^2 + \left(\frac{3}{2}\right)^2 + 2 \cdot \frac{1}{4} \cdot 2 \cdot \frac{3}{2} = \frac{31}{4}$. Thus, $AD = \sqrt{\frac{31}{4}} = \boxed{\frac{\sqrt{31}}{2}}$.

6. Consider a circle of radius 4 with center O_1, a circle of radius 2 with center O_2 that lies on the circumference of circle O_1, and a circle of radius 1 with center O_3 that lies on the circumference of circle O_2. The centers of the circle are collinear in the order O_1, O_2, O_3. Let A be a point of intersection of circles O_1 and O_2 and B be a point of intersection of circles O_2 and O_3 such that A and B lie on the same semicircle of O_2. Compute the length of AB.

   Answer: $\sqrt{6}$

   Solution: Let C be the intersection of O_1O_2 and circle O_2. First, note that $\angle AO_1O_2 = \angle BO_2O_3$. This is because angles are invariant under scaling and translation, and the setup of circles O_1 and O_2 is the same as the setup of circles of O_2 and O_3, only scaled by a factor of 2 and translated. Now, $O_1A = O_1O_2$ so triangle O_1AO_2 is isoceles, with $\angle O_1AO_2 = \angle O_1O_2A$. Thus,

$$\begin{aligned} \angle AO_2B &= 180° - \angle O_1O_2A - \angle BO_2O_3 \\ &= 180° - \angle O_1O_2A - \angle AO_1O_2 \\ &= \angle O_1AO_2 \\ &= \angle O_1O_2A \end{aligned}$$

   Since $\angle AO_2B = \angle AO_2C$, it follows that triangles AO_2C and AO_2B are congruent. We will apply Ptolemy's theorem to quadrilateral ABO_3C in order to find the length AB. First, by Pythagoras, $BC = \sqrt{CO_3^2 - BO_3^2} = \sqrt{16-1} = \sqrt{15}$. Next, if we denote $x = AB$, then $AC = x$

and by Pythagoras, $AO_3 = \sqrt{CO_3^2 - AC^2} = \sqrt{16 - x^2}$. Therefore, by Ptolemy's:

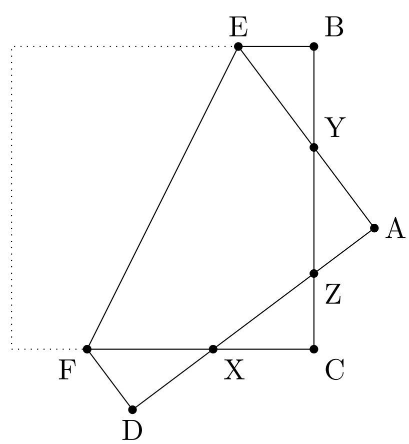
$$\begin{aligned}
AO_3 \cdot BC &= AB \cdot CO_3 + AC \cdot BO_3 \\
\sqrt{16 - x^2} \cdot \sqrt{15} &= 4x + x \\
\frac{\sqrt{3}}{\sqrt{5}} \cdot \sqrt{16 - x^2} &= x \\
\frac{3}{5} \cdot (16 - x^2) &= x^2 \\
\frac{8}{5}x^2 &= \frac{48}{5} \\
x^2 &= 6 \\
x &= \sqrt{6}
\end{aligned}$$

Thus, the length of AB is $\boxed{\sqrt{6}}$.

7. Let $ABCD$ be a square piece of paper with side length 4. Let E be a point on AB such that $AE = 3$ and let F be a point on CD such that $DF = 1$. Now, fold $AEFD$ over the line EF. Compute the area of the resulting shape.

Answer: $\mathbf{\frac{28}{3}}$

Solution: Here is a diagram of the shape:



Let X denote the intersection of AD and CF, Y denote the intersection of AE and BC, and Z denote the intersection of AD and BC. We will show that triangles BYE, AZY, CXZ, and FXD are all congruent.

First, all four triangles are similar because they are all right triangles and $\angle FXD = \angle ZXC$, $\angle XZC = \angle YZA$, and $\angle EYB = \angle ZYA$. Now, triangles EBY and FDX are congruent since $EB = FD = 1$. Next, let $x = XD$. Then by Pythagoras, $FX = \sqrt{1 + x^2}$ and hence $XC = FC - FX = 3 - \sqrt{1 + x^2}$. Now, $YA = EA - EY = 3 - \sqrt{1 + x^2} = XC$ so triangles ZXC and

AYZ are congruent. Now, since ZXC and FXD are similar,

$$\begin{aligned}\frac{ZC}{FD} &= \frac{XC}{XD}\\ \frac{ZC}{1} &= \frac{3-\sqrt{1+x^2}}{x}\\ ZC &= \frac{3-\sqrt{1+x^2}}{x}\end{aligned}$$

Applying Pythagoras,

$$\begin{aligned}YZ &= \sqrt{AY^2+AZ^2}\\ &= \sqrt{\left(3-\sqrt{1+x^2}\right)^2+\left(\frac{3-\sqrt{1+x^2}}{x}\right)^2}\\ &= \frac{3-\sqrt{1+x^2}}{x}\sqrt{1+x^2}\\ &= \frac{3\sqrt{1+x^2}-(1+x^2)}{x}\end{aligned}$$

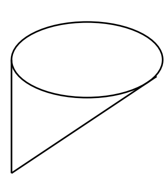
Next, since $BY+YZ+ZC=BC$, we have

$$\begin{aligned}x+\frac{3\sqrt{1+x^2}-(1+x^2)}{x}+\frac{3-\sqrt{1+x^2}}{x} &= 4\\ x^2+3\sqrt{1+x^2}-1-x^2+3-\sqrt{1+x^2} &= 4x\\ \sqrt{1+x^2} &= 2x-1\\ 1+x^2 &= 4x^2-4x+1\\ 3x^2-4x &= 0\\ x &= \frac{4}{3}\end{aligned}$$

Thus, $XD=\frac{4}{3}$ and by Pythagoras, $FX=\sqrt{1+\frac{16}{9}}=\frac{5}{3}$. We then compute $XC=3-\frac{5}{3}=\frac{4}{3}$ and hence FXD, ZXC, AYZ, and BYE are all congruent triangles with side lengths 1, $\frac{4}{3}$, and $\frac{5}{3}$.

Now, the area of the shape is the sum of the areas of trapezoid $EBCF$, triangle AYZ, and triangle FXD. Trapezoid $EBCF$ has area $\frac{1}{2}(EB+FC)\cdot BC=\frac{1}{2}(1+3)\cdot 4=8$. Triangle FXD has area $\frac{1}{2}\cdot FD\cdot XD=\frac{1}{2}\cdot 1\cdot\frac{4}{3}=\frac{2}{3}$. Since triangle AYZ is congruent to triangle FXD, it also has area $\frac{2}{3}$. Thus, the entire shape has area $8+\frac{2}{3}+\frac{2}{3}=\boxed{\frac{28}{3}}$.

8. Moor made a lopsided ice cream cone. It turned out to be an oblique circular cone with the vertex directly above the perimeter of the base (see diagram below). The height and base radius are both of length 1. Compute the radius of the largest spherical scoop of ice cream that it can hold such that at least 50% of the scoop's volume lies inside the cone.



Answer: $\frac{\sqrt{5}-1}{2}$

Solution: The problem is equivalent to finding the largest hemisphere fitting inside the cone, with its center lying on the base. This is because exactly 50% of the scoop should be contained inside the cone at the maximum radius, and the base plane should go through the center of the scoop.

Let the height of the cone lie along the z-axis with the vertex at $(0, 0, 1)$ and let the longest slant be between $(0, 0, 1)$ and $(2, 0, 0)$. Suppose that our hemisphere has radius $r < 1$ and point $(x, y, 0)$ as center. It lies inside the cone if and only if for all $0 \le h \le r$ its cross-section at $z = h$ is contained in the cross-section of the cone. As all cross-sections are circles symmetric about the line $x = 0$, we can push the hemisphere to $y = 0$ while not going outside of the cone. Thus, our hemisphere has center $(x, 0, 0)$.

Now, for the hemisphere to lie in the cone, it suffices to consider the cross-section at $y = 0$ and check that the semicircle lies in the right triangle. This is because the diameter of the cross-section of the hemisphere and cone at $z = h$ is the same as the length of the line in the cross section at $y = 0$ and $z = h$. Thus, we have transformed the original problem into a equivalent one: find the radius of the largest semicircle fitting in a right triangle with legs 1 and 2 such that the center lies on the leg of length 2. To solve this, consider a semicircle whose center lies on the leg of length 2, at a distance x from the right angle. The largest such semicircle must be either tangent to the hypotenuse or tangent to the leg of length 1. If the semicircle is tangent to the hypotenuse, then it has radius $\frac{2-x}{\sqrt{5}}$ by similar triangles. If the semicircle is tangent to the leg of length 1, then the semicircle has radius x. Thus, the largest semicircle with center at distance x has radius $\min(x, \frac{2-x}{\sqrt{5}})$. Since x is a monotonically increasing function of x and $\frac{2-x}{\sqrt{5}}$ is monotonically decreasing, it follows that the maximum is achieved when $x = \frac{2-x}{\sqrt{5}}$ and hence $x = \frac{2}{1+\sqrt{5}}$. Thus, the maximum radius is $x = \frac{2}{1+\sqrt{5}} = \boxed{\dfrac{\sqrt{5}-1}{2}}$.

9. We have squares $ABCD$ and $EFGH$. Square $ABCD$ has points with coordinates $A = (1, 1, -1)$, $B = (1, -1, -1)$, $C = (-1, -1, -1)$ and $D = (-1, 1, -1)$. Square $EFGH$ has points with coordinates $E = (\sqrt{2}, 0, 1)$, $F = (0, -\sqrt{2}, 1)$, $G = (-\sqrt{2}, 0, 1)$, and $H = (0, \sqrt{2}, 1)$. Consider the solid formed by joining point A to H and E, point B to E and F, point C to F and G, and point D to G and H. Compute the volume of this solid.

Answer: $\frac{16+8\sqrt{2}}{3}$

Solution: Consider a square $PQRS$ with vertices at the following locations

$$P = (\sqrt{2}, \sqrt{2}, 1), Q = (\sqrt{2}, -\sqrt{2}, 1), R = (-\sqrt{2}, -\sqrt{2}, 1), S = (-\sqrt{2}, \sqrt{2}, 1).$$

The solid described is obtained from removing the 4 pyramids $AEHP$, $BEFQ$, $CFGR$, and $DGHS$ from the frustrum $ABCD$-$PQRS$.

By symmetry, each of the 4 pyramids has the same volume so we compute the volume of $AEHP$ and then multiply that by 4. $AEHP$ has base PEH which is a 45-45-90 right triangle with legs $\sqrt{2}$ and hence area 1. The height of $AEHP$ is 2, so therefore $AEHP$ has area $\frac{1}{3} \cdot 1 \cdot 2 = \frac{2}{3}$. Thus, the total volume of the 4 pyramids is $\frac{8}{3}$.

Next, we compute the volume of the frustrum $ABCD$-$PQRS$. Extend the lines AP, BQ, CR, DS such that they meet at a point X. Then the volume is the difference between the volumes of the pyramids $XABCD$ and $XPQRS$. Now, let $Y = (1, 1, 1)$ and $Z = (0, 0, 1)$. Then triangles APY and PXZ are similar. We know that $|AY| = 2$, $|PZ| = 2$ and $|PY| = 2 - \sqrt{2}$. Therefore,

$$\frac{|XZ|}{|AY|} = \frac{|PZ|}{|PY|}$$
$$\frac{|XZ|}{2} = \frac{2}{2-\sqrt{2}}$$
$$|XZ| = 4 + 2\sqrt{2}$$

Thus, pyramid $XPQRS$ has base area $(2\sqrt{2})^2 = 8$ and height $|XZ| = 4 + 2\sqrt{2}$ and hence volume $\frac{1}{3} \cdot 8 \cdot (4 + 2\sqrt{2}) = \frac{32+16\sqrt{2}}{3}$. Pyramid $XABCD$ has base area $2^2 = 4$ and height $|XZ| - 2 = 2 + 2\sqrt{2}$ and hence volume $\frac{1}{3} \cdot 4 \cdot (2 + 2\sqrt{2}) = \frac{8+8\sqrt{2}}{3}$. Subtracting these gives us the volume of the frustrum $\frac{24+8\sqrt{2}}{3}$. Finally, subtracting the volume of the 4 pyramids $\frac{8}{3}$ gives the desired volume $\boxed{\dfrac{16+8\sqrt{2}}{3}}$.

10. In a convex quadrilateral $ABCD$ we are given that $\angle CAD = 10^\circ$, $\angle DBC = 20^\circ$, $\angle BAD = 40^\circ$, $\angle ABC = 50^\circ$. Compute angle BDC.

Answer: 40°

Solution: First, extend AD and BC to meet at a point E. Then triangle ABE is a 90-40-50 right triangle. We aim to show that CDE is a 60-30-90 right triangle, with $\angle EDC = 30^\circ$, which would then imply that $\angle BDC = 40^\circ$.

Via some simple angle chasing, we find that $\angle BDA = 110^\circ$ and $\angle ACE = 80^\circ$. Now, without loss of generality, we assume that $AB = 1$.

Since ABE is a right triangle, $BE = \sin(40^\circ)$ and $AE = \cos(40^\circ)$. BDE and ACE are also right triagles, so $CE = \cot(80^\circ) \cdot AE = \cot(80^\circ)\cos(40^\circ)$ and $DE = \tan(20^\circ) \cdot BE = \tan(20^\circ)\sin(40^\circ)$. Therefore,

$$\frac{DE}{CE} = \frac{\tan(20^\circ)\sin(40^\circ)}{\cot(80^\circ)\cos(40^\circ)} = \tan(20^\circ)\tan(40^\circ)\tan(80^\circ) = \tan(60^\circ)$$

Now, since CDE is a right triangle, it follows that $\angle DCE = \tan^{-1}(\frac{DE}{CE}) = 60^\circ$ and hence $\angle CDE = 30^\circ$. Therefore,

$$\angle BDC = 180^\circ - \angle BDA - \angle CDE = 180^\circ - 110^\circ - 30^\circ = \boxed{40^\circ}.$$