1. A college math class has N teaching assistants. It takes the teaching assistants 5 hours to grade homework assignments. One day, another teaching assistant joins them in grading and all homework assignments take only 4 hours to grade. Assuming everyone did the same amount of work, compute the number of hours it would take for 1 teaching assistant to grade all the homework assignments.

   Answer: 20

   Solution: Let W be the amount of work it takes to grade all the homework assignments. We have that $\frac{W}{N} = 5$ and $\frac{W}{N+1} = 4$. Therefore, $5N = 4N + 4$, so $N = 4$ and therefore there are 4 teaching assistants in the class. If 4 teaching assistants take 5 hours to grade homework assignments, it must take one teaching assistant $4 \cdot 5 = \boxed{20}$ hours to grade all the homework assignments.

2. Let a and b be positive integers such that $a > b$ and the difference between $a^2 + b$ and $a + b^2$ is prime. Compute all possible pairs (a, b).

   Answer: $(2, 1)$

   Solution: Let p be a prime number such that $p = a^2 + b - a - b^2 = (a - b)(a + b - 1)$. Since p is a prime and $(a - b) < (a + b - 1)$ then we must have $(a - b) = 1$ and $(a + b - 1) = p$. This implies that $2a - 1 = p + 1 \implies a = \frac{p}{2} + 1 \implies p$ is even $\implies p = 2$. Hence $(a, b) = \boxed{(2, 1)}$.

3. Compute all prime numbers p such that $8p + 1$ is a perfect square.

   Answer: 3

   Solution: Let $8p + 1 = k^2$. We see that k must be odd since $8p + 1$ is odd and thus take $k = 2m + 1$ for some m. We thus have $8p + 1 = 4m^2 + 4m + 1 \implies 2p = m^2 + m \implies 2p = m(m + 1) \implies p = \frac{m(m+1)}{2}$. Consider 2 cases:

   (a) m is even. Then $\frac{m}{2} = 1 \implies p = 3$.

   (b) $m + 1$ is even. Then if $m = 1 \implies p = 1$ which is a contradiction. Thus we must have $\frac{m+1}{2} = 1 \implies m = 1$ which leads to the same contradiction.

   Therefore the only possible p is $\boxed{p = 3}$.

4. Let $f(x) = \sum_{i=1}^{2014} |x - i|$. Compute the length of the longest interval $[a, b]$ such that $f(x)$ is constant on that interval.

   Answer: 1

   Solution: $f(x)$ is constant on $[1007, 1008]$, is decreasing for $x < 1007$, and is increasing for $x > 1008$. The answer is therefore $\boxed{1}$.

5. A positive integer k is 2014-ambiguous if the quadratics $x^2 + kx + 2014$ and $x^2 + kx - 2014$ both have two integer roots. Compute the number of integers which are 2014-ambiguous.

   Answer: 0

   Solution: Note that $2014 = 2 \cdot 19 \cdot 53$ - the eight integers k such that $x^2 + kx + 2014$ has integer roots do not generate a polynomial with integer roots for the polynomial $x^2 + kx - 2014$, thus giving an answer of $\boxed{0}$.

6. Compute $\cos\left(\frac{\pi}{9}\right) - \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{3\pi}{9}\right) - \cos\left(\frac{4\pi}{9}\right)$.

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: Note that $\cos\frac{\pi}{9} - \cos\frac{2\pi}{9} - \cos\frac{4\pi}{9} = 0$ because $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = 2\cos\frac{\pi}{9}\cos\frac{3\pi}{9}$. Therefore, $\cos\frac{\pi}{9} - \cos\frac{2\pi}{9} + \cos\frac{3\pi}{9} - \cos\frac{4\pi}{9} = \boxed{\frac{1}{2}}$.

7. $f(x)$ is a quartic polynomial with a leading coefficient of 1 where $f(2) = 4$, $f(3) = 9$, $f(4) = 16$, and $f(5) = 25$. Compute $f(8)$.

   Answer: 424

   Solution: Note that $f(x) - x^2 = (x-2)(x-3)(x-4)(x-5)$. Therefore, $f(8) = 8^2 + 6\cdot 5\cdot 4\cdot 3 = \boxed{424}$.

8. Consider the recurrence relation
$$a_{n+3} = \frac{a_{n+2}a_{n+1} - 2}{a_n}$$
with initial condition $(a_0, a_1, a_2) = (1, 2, 5)$. Let $b_n = a_{2n}$ for nonnegative integral n. It turns out that $b_{n+2} + xb_{n+1} + yb_n = 0$ for some pair of real numbers (x, y). Compute (x, y).

   Answer: $\mathbf{(-4, 1)}$

   Solution: Taking the difference of the two equations
$$\begin{aligned} a_{n+4}a_{n+1} &= a_{n+3}a_{n+2} - 2 \\ a_{n+3}a_n &= a_{n+2}a_{n+1} - 2 \end{aligned}$$
gives us
$$(a_{n+4} + a_{n+2})a_{n+1} = a_{n+3}(a_{n+2} + a_n), \quad \frac{a_{n+4} + a_{n+2}}{a_{n+3}} = \frac{a_{n+2} + a_n}{a_{n+1}}.$$
Therefore the value of $(a_{n+2} + a_n)/a_{n+1}$ is 2-periodic, so it depends only on the parity of n. For the case where $(a_0, a_1, a_2) = (1, 2, 5)$, we have $a_3 = 8$, so
$$\begin{aligned} \frac{a_{2k+2} + a_{2k}}{a_{2k+1}} &= \frac{a_2 + a_0}{a_1} = 3, \quad a_{2k+2} = 3a_{2k+1} - a_{2k}. \\ \frac{a_{2k+3} + a_{2k+1}}{a_{2k+2}} &= \frac{a_3 + a_1}{a_2} = 2, \quad a_{2k+3} = 2a_{2k+2} - a_{2k+1}. \end{aligned}$$
Substituting $3a_{2k+1} = a_{2k} + a_{2k+2}$ into the second equation gives
$$(a_{2k+4} + a_{2k+2}) = 6a_{2k+2} - (a_{2k+2} + a_{2k}), \quad a_{2k+4} - 4a_{2k+2} + a_{2k} = 0.$$
therefore we have $(x, y) = \boxed{(-4, 1)}$.

9. A sequence $\{a_n\}_{n\geq 0}$ obeys the recurrence $a_n = 1 + a_{n-1} + \alpha a_{n-2}$ for all $n \geq 2$ and for some $\alpha > 0$. Given that $a_0 = 1$ and $a_1 = 2$, compute the value of α for which
$$\sum_{n=0}^{\infty} \frac{a_n}{2^n} = 10$$

   Answer: $\mathbf{\frac{6}{5}}$

Solution: Let $S = \sum_{n=0}^{\infty} \frac{a_n}{2^n}$. Divide the given recurrence by 2^n then sum from $n = 2$ to ∞ to get

$$\frac{a_n}{2^n} = \frac{1}{2^n} + \frac{a_{n-1}}{2^n} + \frac{\alpha a_{n-2}}{2^n} \tag{1}$$

$$\sum_{n=2}^{\infty} \frac{a_n}{2^n} = \sum_{n=2}^{\infty} \frac{1}{2^n} + \sum_{n=2}^{\infty} \frac{a_{n-1}}{2^n} + \sum_{n=2}^{\infty} \frac{\alpha a_{n-2}}{2^n} \tag{2}$$

$$S - \frac{a_0}{2^0} - \frac{a_1}{2^1} = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{a_n}{2^n} + \frac{\alpha}{4}\sum_{n=0}^{\infty} \frac{a_n}{2^n} \tag{3}$$

$$S - 2 = \frac{1}{2} + \frac{1}{2}(S-1) + \frac{\alpha}{4}S \tag{4}$$

$$S = \frac{8}{2-\alpha} \tag{5}$$

Setting $S = 10$ we get $\alpha = \boxed{\frac{6}{5}}$.

10. Let $p(x) = c_1 + c_2 \cdot 2^x + c_3 \cdot 3^x + c_4 \cdot 5^x + c_5 \cdot 8^x$. Given that $p(k) = k$ for $k = 1, 2, 3, 4, 5$, compute $p(6)$.

Answer: $\mathbf{-50}$

Solution: Let $f(x) = (x-1)(x-2)(x-3)(x-5)(x-8) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$. We claim that the coefficients a_1 through a_5 of this polynomial satisfy $p(i) + a_1 \cdot p(i-1) + a_2 \cdot p(i-2) + a_3 \cdot p(i-3) + a_4 \cdot p(i-4) + a_5 \cdot p(i-5) = 0$.

To prove this claim, note that if we set $p(i) = 1^i$, then in our sequence, we get that $f(1) = 0$, which is correct. The same is true if we set $p(i) = 2^i$, $p(i) = 3^i$, $p(i) = 5^i$, or $p(i) = 8^i$. This then holds true for all $p(x)$ by linear combination.

Note therefore that $p(6) + a_1 \cdot p(5) + a_2 \cdot p(4) + a_3 \cdot p(3) + a_4 \cdot p(2) + a_5 \cdot p(1) = 0$. This means that $p(6) = \boxed{-50}$.