1. Compute the smallest positive integer that is 3 more than a multiple of 5, and twice a multiple of 6.

   **Answer: 48**

   **Solution:** If the number is 3 more than a multiple of 5, then its last digit is either 3 or 8. If it's twice a multiple of 6 then it's a multiple of $2 \cdot 6 = 12$. No multiples of 12 end in 3, but the smallest multiple of 12 that ends in 8 is $\boxed{48}$.

2. Compute the number of integers between 1 and 100, inclusive, that have an odd number of factors. Note that 1 and 4 are the first two such numbers.

   **Answer: 10**

   **Solution:** We consider the first few odd numbers. Only one number has one factor: 1.

   Note that a number whose prime factorization is $p^i$ for some $i$ will have $i+1$ factors: $p^0, \dots, p^i$, and a number whose prime factorization is $p^i q^j$ will have $(i+1)(j+1)$ factors.

   Hence, any number with 3 factors must be of the form $p^2$ for some $p$. Of these only $2^2, 3^2, 5^2, 7^2$ are $\leq 100$. Similarly, any number with 5 factors must be of the form $p^4$, which leaves only $2^4$ and $3^4$, and any number with 7 factors must be of the form $p^6$, which yields $2^6$.

   Additionally, a number with 9 factors is either $p^8$ (which means it's $> 100$) or $p^2q^2$. Of these only $2^2 3^2$ and $2^2 5^2$ are $\leq 100$. It's clear that no number $\leq 100$ can have an odd number of factors greater than 9.

   Hence, there are a total of $\boxed{10}$ numbers.

3. A mouse is playing a game of mouse hopscotch. In mouse hopscotch there is a straight line of 11 squares, and starting on the first square the mouse must reach the last square by jumping forward 1, 2, or 3 squares at a time (so in particular the mouse's first jump can be to the second, third, or fourth square). The mouse cannot jump past the last square. Compute the number of ways there are to complete mouse hopscotch.

   **Answer: 274**

   **Solution:** It is easy to see that to get to the $n^{\text{th}}$ square, one must come from one of the three squares immediately before it, and so if $b_n$ is the number of ways to get to the $n^{\text{th}}$ square, we must have $b_n = b_{n-1} + b_{n-2} + b_{n-3}$. Using this recurrence relation and the initial value $b_1 = 1$ (since we start on the first square, there is only one way to get there), and setting $b_n = 0$ if $n \leq 0$, we see $b_2 = 1, b_3 = 2, b_4 = 4, b_5 = 7, b_6 = 13, b_7 = 24, b_8 = 44, b_9 = 81, b_{10} = 149, b_{11} = \boxed{274}$.

4. Cynthia and Lynnelle are collaborating on a problem set. Over a 24-hour period, Cynthia and Lynnelle each independently pick a random, contiguous 6-hour interval to work on the problem set. Compute the probability that Cynthia and Lynnelle work on the problem set during completely disjoint intervals of time.

   **Answer:** $\mathbf{\frac{4}{9}}$

   **Solution:** Consider the axis-aligned square with vertices at $(0,0)$ and $(18,18)$. Points within this square correspond to all possible combinations of start times, where $(x,y)$ corresponds to Cynthis starting at time $x$ and Lynnelle starting at time $y$. We want the area of the region $R$ which consists of all points $(x,y)$ where $|x-y| > 6$. This region forms two isosceles-right triangles with legs of length 12, which has area 144. The probability is therefore $\frac{144}{324} = \boxed{\frac{4}{9}}$.

5. Compute the smallest 9-digit number containing all the digits 1 to 9 that is divisible by 99.

   **Answer: 123475869**

   **Solution:** Note that $99 = 9 \cdot 11$ and since $\sum_{i=1}^{9} i = 45$ which is divisible by 9 it suffices to find the smallest such number that is divisible by 11. We begin with the smallest possible number 123456789. This is divisible by 11 if $(9+7+5+3+1) - (8+6+4+2) = 5$ is divisible by 11. This is clearly not the case. In order to get a difference of 11 we can do one of the following: $(1 \leftrightarrow 4), (3 \leftrightarrow 6), (5 \leftrightarrow 8)$. The first yields a number that cannot have 1 in highest place value and thus is excluded since it is not the smallest one. The second change yields $\{1,5,6,7,9\}$ for the first, third, fifth, seventh and ninth place values and $\{2,3,4,8\}$ for the second, fourth, sixth and eigth place values. The smallest such number is 125364789. By a similar logic the last change yields $\boxed{123475869}$ which is clearly smaller than the latter.

6. Consider 7 points on a circle. Compute the number of ways there are to draw chords between pairs of points such that two chords never intersect and one point can only belong to one chord. It is acceptable to draw no chords.

   **Answer: 127**

   **Solution:** This is the Motzkin number. Place $N$ points on the circle, and choose one. If it is connected to another point with a chord, then the chord bisects the circle. If there are $n$ points in one of the bisected sectors then there are $N - n - 2$ in the other sector. This construction can be continued in each of the bisected sectors separately; it makes no difference to identify all points along a selected chord, which produces two circles. That is, two smaller versions of the original problem with $n$ and $N - n - 2$ points respectively. If $T(n)$ is the number of ways to draw non-intersecting chords on a circle then the number of ways to do this such that the selected point is always from a chord is:

$$\sum_{n=0}^{N-2} T(n)T(N-n-2)$$

   The number of ways of drawing non-intersecting chords such that the selected point is never part of a chord is $T(N-1)$. To see this, note that this is the same as removing the point and considering the problem with $N-1$ points. Thus, including both possibilities the total number of ways there are to draw chords between pairs of points such that two chords never intersect is

$$T(N) = T(N-1) + \sum_{n=0}^{N-2} T(n)T(N-n-2)$$

   Now it is clear that, $T(0) = T(1) = 1$ because we can't draw any chords and $T(2) = 2$ because we can either draw a chord or not. So we can calculate the Motzkin numbers recursively, $T(7) = \boxed{127}$.

7. Two math students play a game with $k$ sticks. Alternating turns, each one chooses a number from the set $\{1,3,4\}$ and removes exactly that number of sticks from the pile (so if the pile only has 2 sticks remaining the next player *must* take 1). The winner is the player who takes the last stick. For $1 \leq k \leq 100$, determine the number of cases in which the first player can guarantee that he will win.

   **Answer: 71**

**Solution:** For the first player, the numbers $k = 1, 3, 4$ guarantee a win and $k = 2$ a loss. We denote these results by the boolean variables $l_1 = T$, $l_2 = F$, $l_3 = T$, and $l_4 = T$, where $T/F$ mean the first player won/lost. For any other number $k > 4$, $k = \neg(l_{k-1} \wedge l_{k-3} \wedge l_{k-4})$ where $\wedge$ and $\neg$ are the boolean AND and NOT functions. To see this, note that if one of $l_{k-1}, l_{k-3}, l_{k-4}$ is false then choosing to remove 1, 3, or 4 sticks respectively will guarantee you a win. If none of the three are false, the first player is effectively forced to start a new game for the second player with a winning number. In this manner, we can build up our table and find that the first few losing numbers are $2, 7, 9, 14, \ldots$. Notice that these are all numbers that are either 0 or 2 mod 7. Since $l_k$ depends only on $l_{k-1}$, $l_{k-3}$, and $l_{k-4}$, it's easy to see that $l_k = l_{k-7}$, so this pattern continues. Hence, the first player can guarantee a win with exactly 5 out of every 7 numbers, so 70 numbers from 1 to 98. Then 99 works and $100 \equiv 2 \bmod 7$ does not, so a total of $\boxed{71}$ numbers work.

8. Nick has a $3 \times 3$ grid and wants to color each square in the grid one of three colors such that no two squares that are adjacent horizontally or vertically are the same color. Compute the number of distinct grids that Nick can create.

   **Answer: 246**

   **Solution:** For clarity, we will use red, blue, and green as the colors.

   We will assume without loss of generality that the middle square is red. We then multiply the answer by 3. There are two cases that result:

   (a) The other squares in the middle row are the same color.

   (b) The other squares in the middle row are different colors.

   Note that independent of how those squares are colored, the top row and bottom row are now independent, so we will now uniquely determine how many ways the top row can be colored.

   In case 1, assume without loss of generality that the other squares were blue. If one of the top corners is green, then the top-middle square must be blue. Otherwise, it can be either green or blue. This gives us 5 choices, so there are 25 colorings for a fixed choice of the middle row, or 50 colorings in case 1 overall.

   In case 2, then for any choice of the middle square, there are two valid colorings that result. This gives us 4 choices, or 16 colorings that satisfy a fixed choice of the middle row, or 32 colorings that satisfy case 2 overall.

   This gives us an answer of $(50 + 32) \cdot 3 = \boxed{246}$.

9. Compute how many permutations of the numbers $1, 2, \ldots, 8$ have no adjacent numbers that sum to 9.

   **Answer: 13824**

   **Solution:** We can solve this using the Principle of Inclusion-Exclusion. Let $A_i$ be the number of permutations of $1, \ldots 2n$ that have at least $i$ adjacent pairs that sums to $2n+1$ (but overcounting permutations with more than $i$ pairs). Then each of these pairs will be of the form $(k, 2n+1-k)$ in some order and for some $1 \le k \le n$. Hence, to create a sequence with $i$ adjacent pairs there are $\binom{n}{i}$ ways to choose a $k$ for each pair, $2^i$ ways to order the numbers within all the pairs, and $(i + (2n - 2i))! = (2n - i)!$ ways to order the $i$ pairs and $2n - 2i$ remaining numbers. Hence, $A_i = \binom{n}{i} 2^i (2n - i)!$. Note that this clearly implies that $i \le n$.

Hence, using PIE and letting $n = 4$ (since the numbers above range from 1 to $2n$), the answer to the problem is

$$\begin{aligned} 8! - A_1 + A_2 - A_3 + A_4 &= 8! - \binom{4}{1}2^1(7!) + \binom{4}{2}2^2(6!) - \binom{4}{3}2^3(5!) + \binom{4}{4}2^4(4!) \\ &= 8! - 8! + 6 \cdot 4 \cdot 6! - 4 \cdot 8 \cdot 5! + 16 \cdot 4! \\ &= 4!(720 - 160 + 16) \\ &= 24 \cdot 576 \\ &= \boxed{13824}. \end{aligned}$$

10. Find the remainder when $(1^2+1)(2^2+1)(3^2+1)\cdots(42^2+1)$ is divided by 43. Your answer should be an integer between 0 and 42.

**Answer: 4**

**Solution:** Let $p = 43$. Consider the polynomial $f(x) = (x^2+1^2)(x^2+2^2)\cdots(x^2+(p-1)^2)$. For any $a$ coprime to $p$, the set $\{a, 2a, \cdots, (p-1)a\}$ is the same as $\{1, 2, \cdots, p-1\}$ modulo $p$. Thus we have

$$f(ax) \equiv \prod_{i=1}^{p-1}((ax)^2 + (ai)^2) \equiv a^{2(p-1)}f(x) \equiv f(x) \pmod{p}.$$

As the coefficient of $x^i$ is multiplied by $a^i$ in $f(ax)$, either $a^i \equiv 1$ or $x^i$-coefficient of $f$ should be zero. Thus $f$ has the following form $\equiv c_1x^{2(p-1)} + c_2x^{(p-1)} + c_3$. By expanding $f$ out and using Wilson's theorem, it is easy to see that $c_1 = c_3 = 1$.

The problem is computing $c_2$. We divide $f(x) \equiv x^{2(p-1)} + c_2x^{(p-1)} + 1$ by $x^2+1$, and it should be divisible mod $p$. But $x^{2(p-1)} + c_2x^{(p-1)} + 1$ divided by $x^2+1$ has remainder $1 - c_2 + 1$ (as $p \equiv 3 \pmod{4}$)), so $c_2$ should be 2. Therefore we have

$$f(x) \equiv x^{2(p-1)} + 2x^{(p-1)} + 1 \pmod{p}, \quad f(1) \equiv 1 + 2 + 1 = 4 \pmod{p}$$

and note that $f(1)$ represents our desired expression, giving us an answer of $\boxed{4}$.