1. Points A, B, C, and D lie in the plane with $AB = AD = 7$, $CB = CD = 4$, and $BD = 6$. Compute the sum of all possible values of AC.

   Answer: $4\sqrt{10}$

   Solution: Let M be the midpoint of BD. Since A and C are both on the perpendicular bisector of BD, they are collinear with M. Hence, the only two cases are when M is between A and C (so $ABCD$ is a kite) and when M is not between A and C (so $ABCD$ is a chevron). In the first case, $AC = AM + MC$, and in the second case, $AC = AM - MC$. The sum of these is $2AM$, which can be computed by the Pythagorean Theorem as

$$2 \times \sqrt{7^2 - 3^2} = 2\sqrt{40} = \boxed{4\sqrt{10}}.$$

2. Let $RICE$ be a quadrilateral with an inscribed circle O such that every side of $RICE$ is tangent to O. Given that $RI = 3$, $CE = 8$, and $ER = 7$, compute the length of IC.

   Answer: 4

   Solution: Let X, Y, Z, and W be points on RI, IC, CE, and ER, respectively, that are tangent to O. Because the two tangent line segments from any point outside the circle have the same length, $RX = RW$, $IY = IX$, $CZ = CY$, and $EW = EZ$. It follows that $RI + CE = IC + ER$ (also known as Pitot's Theorem) and hence $IC = 3 + 8 - 7 = \boxed{4}$.

3. Let ABC be a triangle and I its incenter. Suppose $AI = \sqrt{2}$, $BI = \sqrt{5}$, $CI = \sqrt{10}$ and the inradius is 1. Let A' be the reflection of I across BC, B' be the reflection across AC, and C' be the reflection across AB. Compute the area of triangle $A'B'C'$.

   Answer: $\frac{24}{5}$

   Solution: Let D, E, F be the points of tangency of the incircle such that D lies on side BC, E lies on AC, and F lies on AB. Note that triangle $A'B'C'$ is similar to triangle DEF where the ratio of corresponding sides is 2. Thus, the area of $A'B'C'$ is 4 times that of DEF. We now proceed to compute the area of triangle DEF.

   We will first compute the area of EFI. The area of DFI and DEI can be computed similarly. By Pythagoras, $AF = AE = \sqrt{AI^2 - FI^2} = \sqrt{2-1} = 1$. Now, let G denote the intersection of AI and EF. Then triangle FGI is similar to triangle AFI. Therefore,

$$\frac{FG}{AF} = \frac{FI}{AI}$$
$$\frac{FG}{1} = \frac{1}{\sqrt{2}}$$
$$FG = \frac{1}{\sqrt{2}}$$

   and

$$\frac{GI}{FI} = \frac{FI}{AI}$$
$$\frac{GI}{1} = \frac{1}{\sqrt{2}}$$
$$GI = \sqrt{2}$$

Therefore, it follows that triangle EFI has area $\frac{1}{2} \cdot EF \cdot GI = \frac{1}{2} \cdot \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$.

Similarly, we find the area of DFI to be $\frac{2}{5}$ and the area of DEI to be $\frac{3}{10}$. Therefore, the area of DEF is $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} = \frac{6}{5}$. Multiplying this by 4, we find that the area of $A'B'C' = \boxed{\frac{24}{5}}$.