1. Alex gets 8 points on an exam, while his friend gets 3 times as many points as Alex. What is the average of their scores?

   Answer: 16

   Solution: Alex's friend gets $3 \times 8 = 24$ points. The average is $\frac{8+24}{2} = \boxed{16}$.

2. Sally rolls an 8-sided die with faces numbered 1 through 8. Compute the probability that she gets a power of 2.

   Answer: $\frac{1}{2}$

   Solution: There are a total of eight possibilities, and four of them $(1, 2, 4, 8)$ are powers of two. Thus, the answer is $\frac{4}{8} = \boxed{\frac{1}{2}}$.

3. Boris is driving on a remote highway. His car's odometer reads 24942 km, which Boris notices is a palindromic number, meaning it is not changed when it is reversed. "Hm," he thinks, "it should be a long time before I see that again." But it takes only 1 hour for the odometer to once again show a palindromic number! How fast is Boris driving in km/h?

   Answer: 110 km/h

   Solution: The next palindromic number after 24942 is 25052. So Boris travels $25052 - 24942 = 110$ km in one hour, at a speed of $\boxed{110}$ km/h.

4. If Bobby's age is increased by 6, it's a number with an integral (positive) square root. If his age is decreased by 6, it's that square root. How old is Bobby?

   Answer: 10

   Solution: $\boxed{10}$ is the unique solution to $a + 6 = (a - 6)^2$ where $a - 6 \geq 0$.

5. Screws are sold in packs of 10 and 12. Harry and Sam independently go to the hardware store, and by coincidence each of them buys exactly k screws. However, the number of packs of screws Harry buys is different than the number of packs Sam buys. What is the smallest possible value of k?

   Answer: 60

   Solution: The minimal way to obtain k screws with packs of a and b in two different ways is for $k = \text{lcm}(a, b)$, and then $k = xa = yb$ where $x = \frac{\text{lcm}(a,b)}{a}$ and $y = \frac{\text{lcm}(a,b)}{b}$. Clearly, $x \neq y$. For the minimal solution, neither Harry nor Sam can buy any more than $\text{lcm}(a, b)$. Therefore, the answer is $\text{lcm}(10, 12) = \boxed{60}$.

6. In triangle ABC, we have that $AB = AC$, $BC = 16$, and that the area of $\triangle ABC$ is 120. Compute the length of AB.

   Answer: 17

   Solution: If the area of $\triangle ABC$ is 120, then the length of the altitude from A to BC must be $\frac{120}{8} = 15$. Therefore, $AB = \sqrt{8^2 + 15^2} = \boxed{17}$.

7. Ben works quickly on his homework, but tires quickly. The first problem takes him 1 minute to solve, and the second problem takes him 2 minutes to solve. It takes him N minutes to solve problem N on his homework. If he works for an hour on his homework, compute the maximum number of problems he can solve.

   Answer: 10

Solution: We want the maximum N such that $\sum_{i=1}^{N} i \le 60$. The maximum such N is $N = \boxed{10}$.

8. George and two of his friends go to a famous jiaozi restaurant, which serves only two kinds of jiaozi: pork jiaozi, and vegetable jiaozi. Each person orders exactly 15 jiaozi. How many different ways could the three of them order? Two ways of ordering are different if one person orders a different number of pork jiaozi in both orders.

   Answer: 4096

   Solution: Each person can order 0–15 pork jiaozi, and all the others will be vegetable jiaozi. So each person has 16 possible orders. Since one person's order does not constrain anyone else's, the number of combinations of orders for all three people is therefore $16^3 = 2^{12} = \boxed{4096}$.

9. The operation $\odiv$, called "reciprocal sum," is useful in many areas of physics. If we say $x = a \odiv b$, this means that x is the solution to
$$\frac{1}{x} = \frac{1}{a} + \frac{1}{b}\ .$$
   Compute $4 \odiv 2 \odiv 4 \odiv 3 \odiv 4 \odiv 4 \odiv 2 \odiv 3 \odiv 2 \odiv 4 \odiv 4 \odiv 3$.

   Answer: $\mathbf{\frac{1}{4}}$

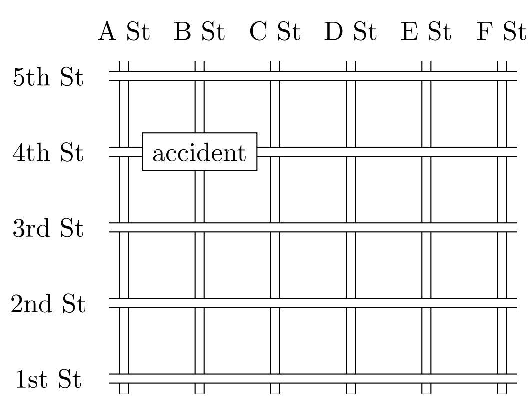
   Solution: $\odiv$ is associative and commutative, so the computation can be rewritten $(2\odiv 2\odiv 2)\odiv(4\odiv 4\odiv 4\odiv 4\odiv 4\odiv 4)\odiv(3\odiv 3\odiv 3\odiv 3)$. Furthermore, $1/(x_1\odiv x_2\odiv\cdots\odiv x_n) = \frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}$. So $2 \odiv 2 \odiv 2 = \frac{1}{3/2} = \frac{2}{3}$, $3 \odiv 3 \odiv 3 = 1$, and $4 \odiv 4 \odiv 4 \odiv 4 \odiv 4 \odiv 4 = \frac{1}{6/4} = \frac{2}{3}$. The answer is $1/(\frac{3}{2}+\frac{3}{2}+1) = \boxed{\frac{1}{4}}$.

10. Find the area of the smallest possible square that contains the points $(2, -1)$ and $(4, 4)$.

   Answer: $\mathbf{\frac{29}{2}}$

   Solution: The smallest possible square that contains the two given points is the square whose diagonal is the line segment connecting the points. A square with diagonal length d has side length $\frac{d}{\sqrt{2}}$ and area $\left(\frac{d}{\sqrt{2}}\right)^2 = \frac{d^2}{2}$. The line segment between $(2, -1)$ and $(4, 4)$ has length $\sqrt{(4-2)^2 + (4-(-1))^2} = \sqrt{4+25} = \sqrt{29}$, so the minimal square in this case has area $\frac{(\sqrt{29})^2}{2} = \boxed{\frac{29}{2}}$.

11. Mr. Ambulando is at the intersection of 5^{th} and A St, and needs to walk to the intersection of 1^{st} and F St. There's an accident at the intersection of 4^{th} and B St, which he'd like to avoid.



Given that Mr. Ambulando wants to walk the shortest distance possible, how many different routes through downtown can he take?

Answer: 56

Solution: The path must pass through either the intersection of 5th and C or the intersection of 3rd and A. From there, we just count the number of paths from corner to corner of a $w \times h$ rectangle, which is

$$\frac{(w+h)!}{w!h!},$$

which can be derived from enumerating permutations of southward and eastward one-block movements. If Ambulando goes south first, there are $\frac{7!}{2!5!} = 21$ possibilities, and if he goes east first, there are $\frac{7!}{3!4!} = 35$ possibilities, so the answer is $21 + 35 = \boxed{56}$.

An alternate solution is to write in the number of possible routes to a given intersection starting from intersections at the northwest, and proceeding by summing the numbers adjacent to the west and north (dynamic programming).

12. Consider a rectangular tiled room with dimensions $m \times n$, where the tiles are 1×1 in size. Compute all ordered pairs (m, n) with $m \leq n$ such that the number of tiles on the perimeter is equal to the number of tiles in the interior (i.e. not on the perimeter).

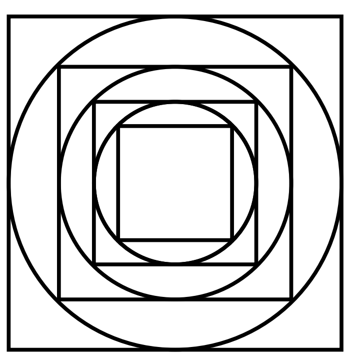
Answer: $\{(5, 12), (6, 8)\}$

Solution: The number of tiles on the perimeter is $2m + 2n - 4$ and thus to have this number equal to half the number of tiles in the rectangle we must have $\frac{mn}{2} = 2m + 2n - 4$. We thus have:

$$mn - 4n - 4m + 8 = 0 \implies mn - 4n - 4m + 16 = 8 \implies (m-4)(n-4) = 8$$

Since both m and n are integers we must have $(m-4, n-4) \in \{(1, 8), (2, 4)\}$ and thus $(m, n) \in \boxed{\{(5, 12), (6, 8)\}}$.

13. Square S_1 is inscribed inside circle C_1, which is inscribed inside square S_2, which is inscribed inside circle C_2, which is inscribed inside square S_3, which is inscribed inside circle C_3, which is inscribed inside square S_4.



Let a be the side length of S_4, and let b be the side length of S_1. What is $\frac{a}{b}$?

Answer: $2\sqrt{2}$

Solution: Rotate the squares so that they are 45 degrees offset from the next larger square surrounding them. Then it is clear that the area of $S_{n+1} = 2S_n$. So the area of $S_4 = 8S_1$, and the ratio of their side lengths must be $\boxed{\sqrt{8} = 2\sqrt{2}}$.

14. Patricia has a rectangular painting that she wishes to frame. The frame must also be rectangular and will extend 3 cm outward from each of the four sides of the painting. When the painting is framed, the area of the frame not covered by the painting is 108 cm^2. What is the perimeter of the painting alone (without the frame)?

    Answer: 24 cm

    Solution: Let the painting have width w and height h. The frame's area is $(w+6)(h+6) - wh = 108$. The perimeter we want to find is $2(w+h)$. We expand $(w+6)(h+6) - wh = wh + 6w + 6h + 36 - wh = 3(2(w+h)) + 36 = 108$, so $2(w+h) = \frac{108-36}{3} = \boxed{24 \text{ cm}}$.

15. Antoine, Benoît, Claude, Didier, Étienne, and Françoise go to the cinéma together to see a movie. The six of them want to sit in a single row of six seats. But Antoine, Benoît, and Claude are mortal enemies and refuse to sit next to either of the other two. How many different arrangements are possible?

    Answer: 144

    Solution: In order not to sit next to each other, A, B, and C must take seats $\{1,3,5\}$, $\{1,3,6\}$, $\{1,4,6\}$, or $\{2,4,6\}$ in any order. D, E, and F take the other three seats in any order. Within each grouping of three people, there are 3! ways for them to arrange themselves. Therefore, there are $4 \cdot 3! \cdot 3! = \boxed{144}$ permutations.

16. Compute the number of geometric sequences of length 3 where each number is a positive integer no larger than 10.

    Answer: 18

    Solution: The geometric sequences are $(1,1,1)$ through $(10,10,10)$, $(1,2,4)$, $(2,4,8)$, $(1,3,9)$, $(4,6,9)$, and the last four sequences reversed, for a total of $\boxed{18}$.

17. Given that the line $y = mx + k$ intersects the parabola $y = ax^2 + bx + c$ at two points, compute the product of the two x-coordinates of these points in terms of a, b, c, k, and m.

    Answer: $\boldsymbol{\frac{c-k}{a}}$

    Solution: The line $y = mx + k$ intersects the parabola at $x = \frac{m-b\pm\sqrt{(m-b)^2-4a(c-k)}}{2a}$. The product of both of these x-coordinates is $\frac{(m-b)^2-[(m-b)^2-4a(c-k)]}{4a^2} = \boxed{\dfrac{c-k}{a}}$.

18. A two-digit positive integer is *primeable* if one of its digits can be deleted to produce a prime number. A two-digit positive integer that is prime, yet not primeable, is *unripe*. Compute the total number of unripe integers.

    Answer: 5

    Solution: Note that 1 and 9 are the only odd digits which are not prime. Therefore, all unripe integers end in either 1 or 9.

    Furthermore, 1, 4, 6, 8, and 9 are the only positive digits which are themselves not prime numbers. Therefore, we have only 10 numbers which we need to check for primality, namely 11, 41, 61, 81, 91, 19, 49, 69, 89, and 99.

    The unripe integers are 11, 41, 61, 19, and 89. There are $\boxed{5}$ of them.

19. Given that $f(x) + 2f(4 - x) = x + 8$, compute $f(16)$.

Answer: $-\frac{32}{3}$

Solution: Plugging in $x = -12$ and $x = 16$, we get that $f(16) + 2f(-12) = 24$ and $f(-12) + 2f(16) = -4$. Therefore, $f(16) = \boxed{-\frac{32}{3}}$.

20. $ABCD$ is a parallelogram, and circle S (with radius 2) is inscribed inside $ABCD$ such that S is tangent to all four line segments AB, BC, CD, and DA. One of the internal angles of the parallelogram is $60°$. What is the maximum possible area of $ABCD$?

Answer: $\frac{32\sqrt{3}}{3}$

Solution: All tangential quadrilaterals that are parallelograms must also be rhombuses, so $ABCD$ is a rhombus, with two internal angles of $60°$ and two of $120°$. This fully constrains the rhombus (so no maximum needs to be found, just the only solution). The rhombus can be split into four 30-60-90 triangles with altitude 2, which will have legs of 4 and $\frac{4}{\sqrt{3}}$. So the total area is $4(\frac{1}{2} \times 4 \times \frac{4}{\sqrt{3}}) = \frac{32}{\sqrt{3}} = \boxed{\frac{32\sqrt{3}}{3}}$.

21. A bitstring of length ℓ is a sequence of ℓ 0's or 1's in a row. How many bitstrings of length 2014 have at least 2012 consecutive 0's or 1's?

Answer: 16

Solution: We can count the strings with at least 2012 consecutive 0's, and the number of strings with at least 2012 consecutive 1's will be the same (just flip all the bits). There is one string with exactly 2014 consecutive 0's (000...000), two with exactly 2013 consecutive 0's (000...001, 100...000), and five with exactly 2012 consecutive 0's (000...010, 000...011, 100...001, 010...000, 110...000). Therefore, the answer is $(1 + 2 + 5) \cdot 2 = \boxed{16}$.

22. Apples cost 2 dollars. Bananas cost 3 dollars. Oranges cost 5 dollars. Compute the number of distinct baskets of fruit such that there are 100 pieces of fruit and the basket costs 300 dollars. Two baskets are distinct if and only if, for some type of fruit, the two baskets have differing amounts of that fruit.

Answer: 34

Solution: We want all nonnegative integer solutions to $x + y + z = 100$ and $2x + 3y + 5z = 300$. Note from these two equations, we can prove that $y + 3z = 100$, so therefore $x = 2z$. Coupled with the fact that $y + 3z = 100$, this means that $z \leq 33$. There are $\boxed{34}$ such choices for z.

23. Let triangle ABC have side lengths $AB = 11$, $BC = 7$, and $AC = 12$. Let D be a point on AC and E be a point on AB such that $\angle CDE = 90°$ and the area of triangle CDE is maximized. Find the area of triangle CDE.

Answer: $4\sqrt{10}$

Solution: Let BF be the altitude of AC. Then $CF = 3$, $FA = 9$, and $BF = 2\sqrt{10}$. Now, let $AD = x$. By similarity of BFA and EDA, it follows that $ED = \frac{x}{9} \cdot 2\sqrt{10}$ and hence the area of CDE is

$$\frac{1}{2} \cdot ED \cdot CD = \frac{\sqrt{10}}{9}x(12 - x)$$

Via your favorite method to maximize quadratics, we find that the area of CDE is maximized when $x = 6$. Since $6 < 9$, E does indeed lie on AB, and hence the maximum area of triangle CDE is $6^2 \cdot \frac{\sqrt{10}}{9} = \boxed{4\sqrt{10}}$.

24. It's pouring down rain, and the amount of rain hitting point (x, y) is given by

$$f(x,y) = |x^3 + 2x^2y - 5xy^2 - 6y^3|,$$

If you start at the origin $(0,0)$, find all the possibilities for m such that $y = mx$ is a straight line along which you could walk without any rain falling on you.

Answer: $\mathbf{-1, \frac{1}{2}, -\frac{1}{3}}$

Solution: Noticing that the alternating sum of the coefficients equals zero, $y = -x$ is a solution. Factoring it out yields $(x+y)(x^2+xy-6y^2) = 0$. The quadratic formula yields the two remaining roots $y = x/2$ and $y = -x/3$, leading to the complete solution $m = \boxed{-1, \frac{1}{2}, -\frac{1}{3}}$.

25. 300 couples (one man, one woman) are invited to a party. Everyone at the party either always tells the truth or always lies. Exactly 2/3 of the men say their partner always tells the truth and the remaining 1/3 say their partner always lies. Exactly 2/3 of the women say their partner is the same type as themselves and the remaining 1/3 say their partner is different. Find a, the maximum possible number of people who tell the truth, and b, the minimum possible number of people who tell the truth. Express your answer as (a, b).

Answer: $\mathbf{(500, 300)}$

Solution: By analyzing all four possible cases (men telling truth/lie, women telling truth/lie), we can see that any woman will say her partner is of the same type if and only if her partner always tells the truth, and any men will say his partner tells the truth if and only if his partner is of the same type as him. Thus we can see that exactly 2/3 of men are truthtellers, and 2/3 of couples have the same type. We can try to satisfy these constraints by choosing the numbers of types of women to assign to the men of each type.

The assignment of person types that yields the fewest truthtellers is to have the smallest number of truthtelling men assigned to women who tell the truth. Since there are 200 men and 100 women, we can assign truthtelling women to only to half of the truthtelling men. To fulfill the quota of 200 couples of opposing type, after we have 100 truthtelling couples and 100 couples with only truthtelling men, we need 100 more couples of the same type—so they must all lie. This is 300 truthtellers total. (If there are fewer than 100 men with truthtelling spouses, we can't fulfill the 200 same-type-couple constraint.)

The assignment of person types that yields the most truthtellers is to have the greatest number of truthtelling men assigned to women who tell the truth. In this case, every truthtelling man is assigned a truthtelling spouse. We still need the 100 couples with nonmatching types, so the 100 remaining lie-telling men must be assigned 100 truthtelling spouses. This results in 500 truthtellers total.

Therefore, the maximum number of truthtellers is $a = 500$, and the minimum number of truthtellers is $b = 300$. This means our answer is $\boxed{(500, 300)}$.