1. Compute the remainder when 2^{30} is divided by 1000.

   Answer: 824

   Solution: Note that $2^{30} \equiv 1024^3 \equiv 24^3 \pmod{1000}$. We will now consider $24^3 \pmod 8$ and $24^3 \pmod{125}$. Note that 24^3 is divisible by 8, but $24^3 \equiv 74 \pmod{125}$, so therefore $2^{30} \equiv \boxed{824} \pmod{1000}$.

2. Consider all right triangles with integer side lengths that form an arithmetic sequence. Compute the 2014th smallest perimeter of all such right triangles.

   Answer: 24168

   Solution: We seek triples (a, b, c) such that $a^2 + b^2 = c^2$. Without loss of generality, assume that $a < b < c$. If they form an arithmetic sequence, then we need $b = a + d$ and $c = a + 2d$ for some positive integer d. Thus $a^2 + (a + d)^2 = (a + 2d)^2$. The solution to this is $3d = a$. Thus, the triples all take the form $(3x, 4x, 5x)$. Thus the perimeter is $2014 \cdot (3 + 4 + 5) = \boxed{24168}$.

3. A segment of length 1 is drawn such that its endpoints lie on a unit circle, dividing the circle into two parts. Compute the area of the larger region.

   Answer: $\frac{5\pi}{6} + \frac{\sqrt{3}}{4}$

   Solution: Let O be the center of the circle, and X and Y be the endpoints of the line segment. Note that OXY is an equilateral triangle with area $\frac{\sqrt{3}}{4}$. Segments OX and OY divide the circle into two regions, where the bigger region is 5 times bigger than the smaller one. That bigger region has area $\frac{5\pi}{6}$, implying that the area of the larger region in the original problem is $\boxed{\frac{5\pi}{6} + \frac{\sqrt{3}}{4}}$.

4. A frog is hopping from $(0, 0)$ to $(8, 8)$. The frog can hop from (x, y) to either $(x + 1, y)$ or $(x, y + 1)$. The frog is only allowed to hop to point (x, y) if $|y - x| \leq 1$. Compute the number of distinct valid paths the frog can take.

   Answer: 256

   Solution: The frog will hop on (a, a) for $0 \leq a < 8$. At each of these 8 points, the frog has two choices for where it can hop, and then its next hop is uniquely determined. This means it has $2^8 = \boxed{256}$ different paths it can take.

5. Given a triangle ABC with integer side lengths, where BD is an angle bisector of $\angle ABC$, $AD = 4$, $DC = 6$, and D is on AC, compute the minimum possible perimeter of $\triangle ABC$.

   Answer: 25

   Solution: By the angle bisector theorem, we have that $\frac{AB}{4} = \frac{BC}{6}$. Note that $AB + BC > 10$ by the Triangle Inequality. Therefore, $AB = 6$ and $BC = 9$ gives us the minimum perimeter of $\boxed{25}$.

6. Compute the largest integer N such that one can select N different positive integers, none of which is larger than 17, and no two of which share a common divisor greater than 1.

   Answer: 8

   Solution: There are 7 primes less than or equal to 17. Therefore, we can have a maximum of $\boxed{8}$ positive numbers, each of the 7 primes and 1.

7. Eddy draws 6 cards from a standard 52-card deck. What is the probability that four of the cards that he draws have the same value?

Answer: $\mathbf{\frac{3}{4165}}$

Solution: There are $\binom{52}{6}$ different six-card hands. The number of valid hands is $13 \cdot \binom{48}{2}$. Therefore, the probability of drawing a four of a kind is

$$\frac{13 \cdot \binom{48}{2}}{\binom{52}{6}} = \boxed{\frac{3}{4165}}.$$

8. Equilateral triangle DEF is inscribed inside equilateral triangle ABC such that DE is perpendicular to BC. Let x be the area of triangle ABC and y be the area of triangle DEF. Compute $\frac{x}{y}$.

Answer: 3

Solution: For clarity, let D be on BC. We have that $2 \cdot CD = CE$, but $AE = CD$, so therefore $DC = \frac{BC}{3}$. Therefore, the area of triangle CDE is $\frac{\sqrt{3}BC^2}{9}$, so the area of triangle DEF is $\frac{\sqrt{3}BC^2}{12}$, giving a ratio of $\boxed{3}$.

9. Find the sum of all real numbers x such that $x^4 - 2x^3 + 3x^2 - 2x - 2014 = 0$.

Answer: 1

Solution: First, note that there must be a real root because the quartic has a positive leading coefficient but takes on negative values.

We can factor the expression as $\frac{1}{2}x^4 + \frac{1}{2}(1-x)^4 - 4029 = 0$. Note that if r is a real root, then so is $1 - r$. Thus, the real roots must sum up to $\boxed{1}$.

10. Three real numbers x, y, and z are chosen independently and uniformly at random from the interval $[0, 1]$. Compute the probability that x, y, and z can be the side lengths of a triangle.

Answer: $\mathbf{\frac{1}{2}}$

Solution: We frame this as a geometric probability problem: the feasible region is the unit cube in three-dimensional Cartesian space, and we're looking for the volume of the region that corresponds to tuples (x, y, z) that form triangles.

Consider the case where $x, y \le z$. We look for (x, y, z) that do not form a triangle. Such a tuple cannot form a triangle if and only if $z > x + y$. So, we consider the region bounded by $z > x + y$ and $0 \le x, y, z \le 1$. This is a tetrahedron that is contained in the unit cube. It has vertices $(0, 0, 0)$,$(0, 0, 1)$,$(0, 1, 1)$, and $(1, 0, 1)$. This tetrahedron has volume $1/6$ because its height is 1 and its base has area $1/2$. By symmetry, we get congruent (and disjoint) tetrahedra for the cases $x, z \le y$ and $y, z \le x$. So, the total volume that does not work is $3 \cdot \frac{1}{6} = \frac{1}{2}$, which means the answer is $1 - \frac{1}{2} = \boxed{\frac{1}{2}}$.

11. In the following system of equations

$$|x + y| + |y| = |x - 1| + |y - 1| = 2,$$

find the sum of all possible x.

Answer: $\mathbf{\frac{4}{3}}$

Solution: The solutions are $(x, y) = (-\frac{2}{3}, \frac{4}{3}), (2, 0)$. The best way to see is to draw graphs for both equations. The equation $|x+y|+|y| = 2$ encloses a region $|x+y|+|y| \le 2$, which is equivalent to

$$(x+y)+y \le 2, (x+y)-y \le 2, -(x+y)+y \le 2, -(x+y)-y \le 2.$$

Those four lines bound a parallelogram with vertices at $(2, 0), (-2, 2), (-2, 0), (2, -2)$. Similarly $|x-1|+|y-1| = 2$ becomes a square with vertices at $(3, 1), (1, 3), (-1, 1), (1, -1)$. By drawing the figure, we can check that the square intersects the parallelogram at two points: one at vertex $(2, 0)$, the other at the intersection of the side of the square $y = x+2$ and the side of the rectangle $y = -1/2x + 1$. Solving this system gives $(x, y) = (-\frac{2}{3}, \frac{4}{3})$. This gives us an answer of $\boxed{\frac{4}{3}}$.

12. Find the last two digits of $\binom{200}{100}$. Express the answer as an integer between 0 and 99. (e.g. if the last two digits are 05, just write 5.)

    Answer: 20

    Solution: By computing the exponents of 2 and 5 in the factorization of $\binom{200}{100}$, using the standard formula

$$p^e \| n! \quad \Rightarrow \quad e = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor,$$

we see that the exponents of 2 and 5 in $\binom{200}{100}$ are 3 and 1 respectively. Since we are concerned with computing the remainder after dividing by $100 = 2^2 \cdot 5^2$, we need to consider $\binom{200}{100}/5 \bmod 5$. Consider

$$\begin{aligned}\frac{1}{5}\binom{200}{100} &= \frac{1}{5}\prod_{k=1}^{100}\frac{100+k}{k} \\ &= \frac{1}{5}\left(1+\frac{100}{25}\right)\left(1+\frac{100}{50}\right)\left(1+\frac{100}{75}\right)\left(1+\frac{100}{100}\right)\cdot\prod_{1\le k\le 100, 25\nmid k}\left(1+\frac{100}{k}\right) \\ &= 14\prod_{1\le k\le 100, 25\nmid k}\left(1+\frac{100}{k}\right).\end{aligned}$$

We can interpret the dividing by integers coprime to 5 as multiplying by their multiplicative inverses mod 5, and in this perspective all terms of the form $1 + 100/k$ becomes $\equiv 1 \bmod 5$. Thus we have

$$14\prod_{1\le k\le 100, 25\nmid k}\left(1+\frac{100}{k}\right) \equiv 14 \equiv 4 \pmod 5$$

so that $\binom{200}{100} \equiv 4 \cdot 5 = 20 \pmod{25}$. Combining this result with $\binom{200}{100} \equiv 0 \pmod 4$ shows that the last two digits of $\binom{200}{100}$ should be $\boxed{20}$.

13. Let α, β, γ be the three real roots of polynomial $x^3 - x^2 - 2x + 1 = 0$. Find all possible values of $\frac{\alpha}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha}$.

    Answer: $-4, 3$

    Solution: Vieta's theorem gives

$$\alpha+\beta+\gamma = 1, \quad \alpha\beta+\beta\gamma+\gamma\alpha = -2, \alpha\beta\gamma = -1.$$

Then from the identity

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)((\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha))$$

we can see that $\alpha^3+\beta^3+\gamma^3=4$.

Now, let

$$\sigma_1=\frac{\alpha}{\beta}+\frac{\beta}{\gamma}+\frac{\gamma}{\alpha},\quad \sigma_2=\frac{\beta}{\alpha}+\frac{\gamma}{\beta}+\frac{\alpha}{\gamma}.$$

Then

$$\sigma_1+\sigma_2=(\alpha+\beta+\gamma)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)-3=\frac{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma}-3=-1$$

and

$$\sigma_1\sigma_2=3+3\left(\frac{\alpha^3+\beta^3+\gamma^3}{\alpha\beta\gamma}\right)+\alpha\beta\gamma\left(\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}\right).$$

To evaluate the term $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}$, we consider the cubic equation having $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$ as roots, which is $x^3-2x^2-x+1=0$, the polynomial with its coefficients inverted.

Considering the similar identity,

$$\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}-\frac{3}{\alpha\beta\gamma}=\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)\left(\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^2-3\left(\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}\right)\right)$$

we can see that $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}=3$. Therefore, $\sigma_1\sigma_2=-12$.

From this we see that σ_1 and σ_2 are the two roots of quadratic equation $y^2+y-12=0$, which can be found to be $\boxed{3,-4}$.

14. Consider a round table on which 2014 people are seated. Suppose that the person at the head of the table receives a giant plate containing all the food for supper. He then serves himself and passes the plate either right or left with equal probability. Each person, upon receiving the plate, will serve himself if necessary and similarly pass the plate either left or right with equal probability. Compute the probability that you are served last if you are seated 2 seats away from the person at the head of the table.

Answer: $\frac{1}{2013}$

Solution: Assume for generality's sake that there are n people seated around the table. To be the last served we must have:

- the plate must make its way right to the person next to you (either on the left or right)
- the plate must turn around and make its way to the person on your other side and then get passed to you

The first condition must happen whether you are served last or not. Thus what differentiates being served last is whether the second condition happens in addition to the first or not. Given that the first condition has happened, there is some probability of the second condition happening too and this must be independent of the position along the table since it is a round table and different positions are equivalent. Since there are $n-1$ people who are not at the head of the table, the probability must be $\frac{1}{n-1}=\boxed{\frac{1}{2013}}$.

15. A point is "bouncing" inside a unit equilateral triangle with vertices $(0,0)$, $(1,0)$, and $(1/2, \sqrt{3}/2)$. The point moves in straight lines inside the triangle and bounces elastically off an edge at an angle equal to the angle of incidence. Suppose that the point starts at the origin and begins motion in the direction of $(1,1)$. After the ball has traveled a cumulative distance of $30\sqrt{2}$, compute its distance from the origin.

Answer: $\mathbf{30 - 17\sqrt{3}}$

Solution: It is useful to notice that equilateral triangles tessellate the plane. The basis vectors for this triangular lattice are $(1/2, \sqrt{3}/2)$ and $(1,0)$. We can think of the balls motion by drawing a line in the direction $(1,1)$ of distance $30\sqrt{2}$ through the triangular lattice formed by the tessellation. Each time the line intersects a grid-line represents a bounce int he original triangle. One can also see this representation by "unfolding" the trajectory after each bounce in the equilateral triangle. To get the distance from the origin we need to determine the location of the ball in the final triangle and which of its vertices corresponds to the origin.

The ball's location is $(30,30)$. The triangle that it ends up in has vertices v_1, v_2, v_3 such that $v_i = a(1/2, \sqrt{3}/2) + b(1,0)$. There are three points on the lattice that are closest to $(30,30)$, those are the vertices of the triangle.

$$(30,30) = a(1/2, \sqrt{3}/2) + b(1,0) \implies a \to 20\sqrt{3} \approx 34.641, b \to 30 - 10\sqrt{3} \approx 12.6795$$

The points are thus given by $a = 34, b = 13$ and $a = 35, b = 12$ and $a = 35, b = 13$.

To determine which of these is the origin we need to determine on what sub-lattice the origin lies. By drawing the lattice one can see that this sub-lattice is spanned by $(3/2, \sqrt{3}/2)$ and $(3/2, -\sqrt{3}/2)$. Thus the "origin" point in the final triangle is given by coordinate:

$$n(3/2, \sqrt{3}/2) + m(3/2, -\sqrt{3}/2) = a(1/2, \sqrt{3}/2) + b(1,0)$$

for a, b above and $n, m \in \mathbb{Z}$. The only pair that works is $a = 34b = 13$, then $n = 27, m = -7$. Thus the origin is at $(30, 17\sqrt{3})$. The distance is thus:

$$\sqrt{(30-30)^2 + \left(30 - 17\sqrt{3}\right)^2} = \boxed{30 - 17\sqrt{3}}$$

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