Title: Mathematical Modeling of Acceleration Profiles for Mountain Bike Takeoff Ramps: A Comparative Study of Logarithmic, Circular, and Clothoid Curves

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Abstract:

This investigation explores the application of mathematical modeling to design optimal mountain bike jumps. By analyzing the geometry and acceleration profiles of various jump curves, the study seeks to create safer and more predictable takeoff structures that enhance rider control and experience. The research evaluates three types of curves—logarithmic, circular, and clothoid—to determine their suitability under different conditions. Key factors considered include smooth acceleration transitions, consistency of peak acceleration across velocity ranges, and the impact of curve geometry on rider stability.

During the modeling process mathematical tools such as spline interpolation, exponential functions, and parametric equations were utilized to refine jump profiles. For the logarithmic curve, exponential functions were employed to ensure a gradually increasing slope, minimizing abrupt changes in acceleration. Circular curves was modeled with an adaptation of the circle equation, but the transition from zero curvature to a constant curvature became a problem. clothoid curves, commonly used roller coaster and railways, were examined to resolve this. This was modeled using parametric equations. Differentiating this curve required use of basic differential geometry, such as differentiating tangential angle over curvature.

Through the principle of energy conservation and vector analysis of tangential and centripetal accelerations, the study compares acceleration profiles under different initial velocities. Results indicate that logarithmic curves balance peak acceleration and consistency, while clothoid curves excel in maintaining a uniform trend but may produce exponentially higher acceleration at higher velocities.

Introduction:

This investigation aims to model the total acceleration experienced by a mountain bike rider as they ride up a mountain bike jump. I will explore different mathematical curves to determine an ideal jump profile that offers smooth acceleration, helping to manage the forces experienced by the rider. The geometry of the takeoff can vary from linear to curved, mellow to steep, and small to large. It is important to perfect the design of a jump to guarantee entertainment and safety at the same time.

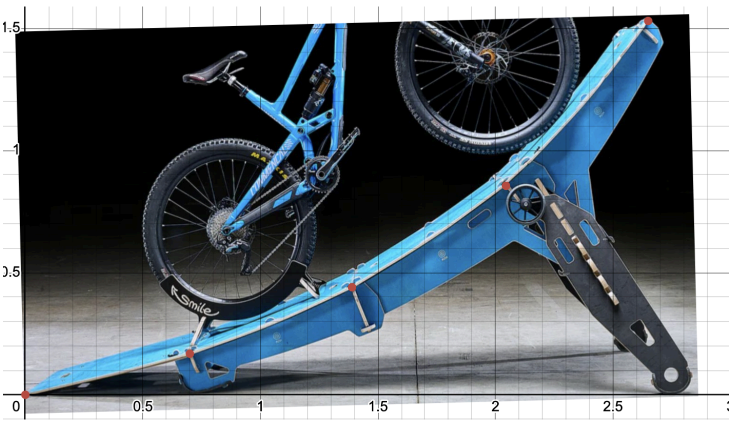
An important distinction is that acceleration is a vector component, and therefore gravity is not the only acceleration experienced. Riders will also experience centripetal acceleration, similar to those experienced on a roller coaster. A suitable jump shape should ideally provide a gradual transition that increases smoothly with minimal sudden changes in acceleration. This helps ensure that the rider maintains control and experiences safe levels of acceleration throughout the jump. A jump should also be consistent thorough velocities, so riders know what to expect when they approach the jump with different speeds.

When analyzing the shape and acceleration profile of a takeoff, I will consider the following factors.
1. Length, height, and takeoff angle.
2. Smooth delivery of acceleration through the starting point and ending point.
3. Tendency of acceleration should remain consistent across a velocity interval.
4. Peak acceleration should be as insensitive as possible to velocity changes.
5. The position of peak acceleration during the takeoff should remain consistent across the velocity interval.

In this study, I will use the following structure:
Section 1: Logarithmic - this is the simplest curve and helps me refine the method
Section 2: Circular - I will apply the method from section 1 in a circular curve
Section 3: Clothoitd - I will apply and expand on the method to work parametrically
From the three types of curves, I will determine the suitability of curves to different situations:

Section 1 - Logarithmic Curves
I have chosen an existing jump to start my modeling process.:

x_1	y_1
0.000	0.000
0.700	0.168
1.390	0.440
2.045	0.855
2.650	1.530



The picture is a portable takeoff ramp made by mtb-hopper. The recommended speed is between 7-10 ms^{-1}, the flight trajectory peaks at a height of 2.8m, and ends at a length of 7m. The image resolution is 1434x782px - Scaling the image proportionally with a factor of 2 increases its length to 2.868mm, which closely represents the real dimension of 2875mm. Each unit in demos representings 1 meter in real life. The image is tilted by 0.025 radians align the base at the x axis and takeoff point at (0,0).

Points are plotted on the joints which model the ramp’s construction We can visualize the construction through 4 linear equations.

This can be easily found, since $\frac{y_2 - y_1}{x_2 - x_1} = m$. Rewriting the slope formula gives:

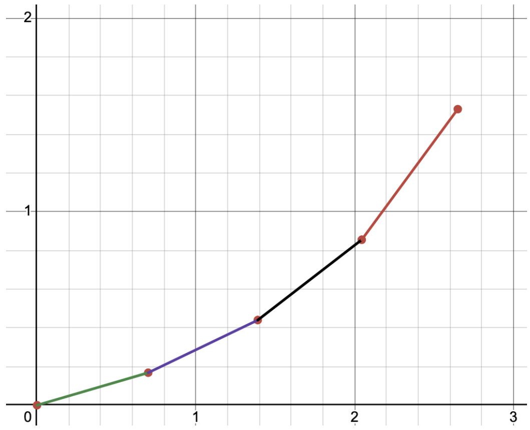
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

4 $y = \frac{0.168}{0.7}(x)\ \{0 \le x \le 0.7\}$

5 $y - 0.168 = \frac{(0.44 - 0.168)}{(1.39 - 0.7)}(x - 0.7)\ \{0.7 \le x \le 1.39\}$

6 $y - 0.44 = \frac{(0.855 - 0.44)}{(2.045 - 1.39)}(x - 1.39)\ \{1.39 \le x \le 2.045\}$

7 $y - 0.855 = \frac{(1.53 - 0.855)}{(2.65 - 2.045)}(x - 2.045)\ \{2.045 \le x \le 2.650\}$



The four planks could be bent in a certain way so they form a smooth line.

Let the x coordinates of the 5 points starting from (0,0) be x_0, x_1, x_2, x_3, x_4 respectively, and y coordinates be y_0, y_1, y_2, y_3, y_4. We can use cubic spline interpolation to find an accurate equation for each interval. For example function $S_0(x)$ covers interval $\{x_0 \le x \le x_1\}$, and function $S_n(x)$ covers interval $\{x_n \le x \le x_{n+1}\}$.
To make them connect smoothly, following conditions should be met:
Let $S_j(x_j) = y_j\ for\ each\ j = 0, 1, 2, 3$

1. $S_0(x_1) = S_1(x_1)$ This makes sure the functions connect
2. $S_0'(x_1) = S_1'(x_1)$ This makes sure the functions connect smoothly
3. $S_0''(x_1) = S_1''(x_1)$ This makes sure the functions have similar tendencies (concave/convex)
4. $S_0''(x_0) = S_3''(x_4) = 0$ Natural boundary indicates the tendency of the slope cease to change

However, although spline interpolation produces a more accurate and realistic model compared to singular equations, the difficulty in calculating curvature exceeds the scope of this investigation.

For simplicity, I estimated using an exponential function. This is a superior choice over cubic functions, as it can guarantee slope constantly increases, implying that there will be no undulations on the ramp. Desmos approximation shows high accuracy with the coefficient of determination being 0.9998.

$y_1 \sim a^{(x_1 - b)} + c$

STATISTICS

$R^2 = 0.9998$

RESIDUALS

e_2 plot

PARAMETERS

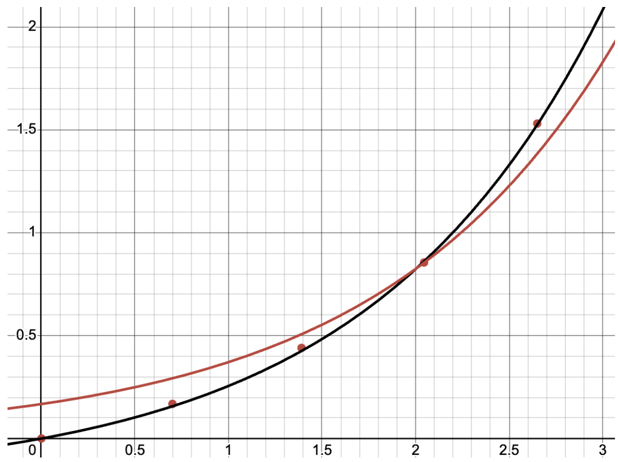
$a = 2.2119$
$b = 1.958$
$c = -0.205474$

Hence, the approximation could be rewritten as the function of height(h) over displacement(x).

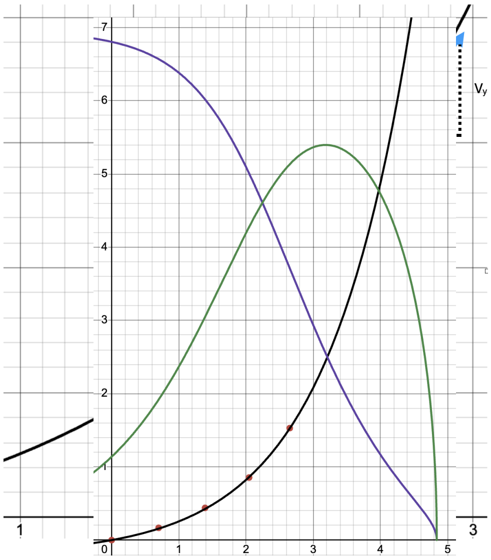
$$h(x) = 2.22^{(x-1.96)} - 0.21$$

Deriving this function gives the slope

$$h'(x) = \frac{d}{dx}(2.22^{(x-1.96)}) - \frac{d}{dx}(0.21)$$
$$= \frac{d}{dx}(2.22^{(x-1.96)})$$
$$= ln(2.22) \cdot 2.22^{(x-1.96)} \cdot 1$$
$$= 0.798 \cdot 2.22^{(x-1.96)}$$



Slope is defined as $\frac{rise}{run}$. We can visualize rise as the height of a right triangle and run as the length. Since this ratio is the same as tangent, one can use the Arctan function to find the tangential angle at a certain point, expressed as: $arctan(h'(x))$



The slope at the end point is h'(2.65)=1.38. The angle of takeoff is arctan(1.38) = 53.8 degrees.
Since we know the angle, we can express tangential velocity at any point as the sum of horizontal and vertical vectors

$$\widehat{V}_x = |V| \cdot cos(\theta) \qquad \widehat{V}_y = |V| \cdot sin(\theta)$$

$$\widehat{V}_x = |V| \cdot cos(arctan(h'(x))) \qquad \widehat{V}_y = |V| \cdot sin(arctan(h'(x)))$$

Splitting velocity into components allows us to regard it as a vector quantity. This is important because when we regard velocity as a scalar, we disregard the change in direction. Therefore, the resulting acceleration will only be gravity. However, the ramp does change the direction of the velocity through acceleration in the y component and deceleration of the x component, creating net acceleration

Another factor is the acceleration of gravity on velocity. Ignoring friction, we don't have to account for rotational momentum. Therefore, energy is conserved through kinetic energy and gravitational potential energy, and we can write tangential velocity v1 as an expression of initial velocity v0:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + mgh$$

$$v_0^2 = v_1^2 + 2gh$$

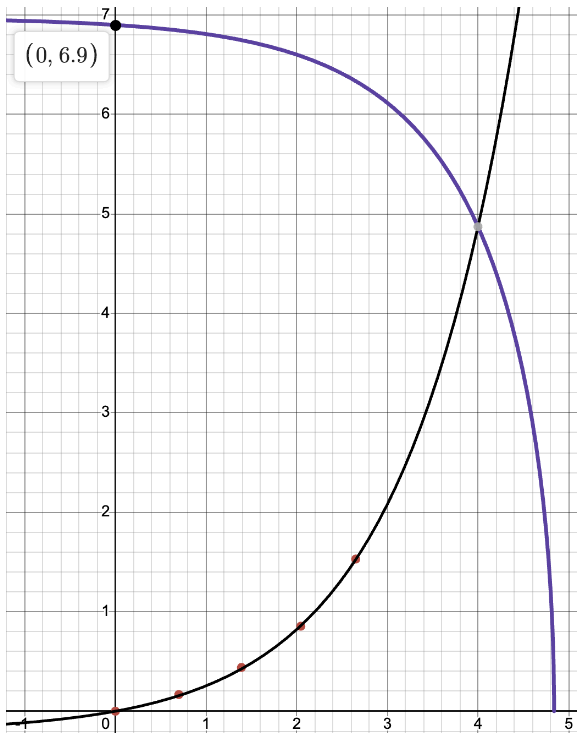
Here, m denotes mass, v_0 denotes initial velocity, v_1 denotes velocity at a certain point, g denotes gravitational acceleration, and h denotes height.

$$v_1 = \sqrt{v_0^2 - 2gh}$$

Then we can substitute function h(x) for height.

$$v_1 = \sqrt{v_0^2 - 19.6(2.22^{(x-1.96)} - 0.21)}$$

We can set the initial velocity v_0 between the recommended 25-35 kph, which is 6.9 - 9.7 ms^{-1}, respectively. This produces a graph of the tangential velocity.

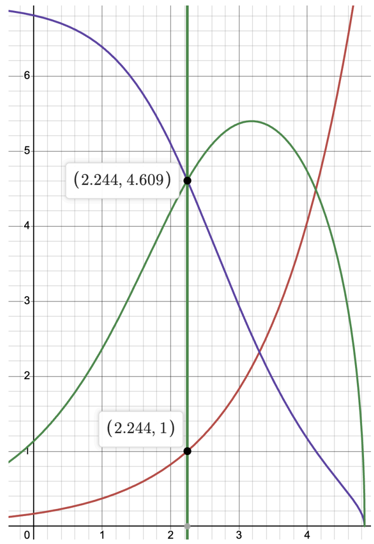


We can substitute the velocity equation back into the horizontal and vertical velocity equations.

$$\widehat{V_x} = \sqrt{v_0^2 - 19.6(2.22^{(x-1.96)} - 0.21)} \cdot cos(arctan(h'(x)))$$

$$\widehat{V_y} = \sqrt{v_0^2 - 19.6(2.22^{(x-1.96)} - 0.21)} \cdot sin(arctan(h'(x)))$$

The violet and green graphs intersect at (2.244, 4.61), meaning at 2.244 meters from the start, horizontal and vertical velocities are both 4.61 ms^{-2}. Hence, the ratio of $\frac{rise}{run}$ should be 1, implying that the slope should be 1. Plugging 2.244 into h'(x) verifies this:



$$h'(2.244) = 0.798 \cdot 2.22^{(2.244-1.96)}$$

$$= 0.798 \cdot 2.22^{(0.284)}$$

$$= 1$$

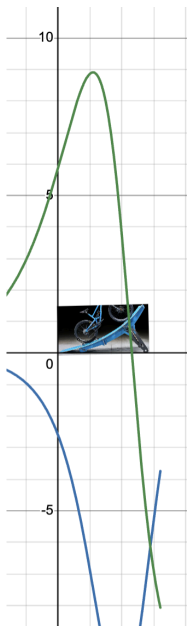
Through the graphs, we see that horizontal velocity decreases while vertical velocity increases. This translates to a shit from forward momentum to upwards momentum in real life.

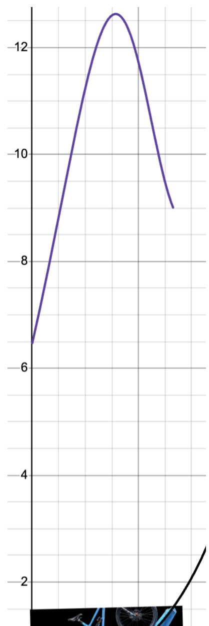
We can find horizontal and vertical acceleration through deriving velocities. Acceleration is defined as $\frac{dv}{dt}$, but deriving the current equations gives us $\frac{dv}{dx}$. To find horizontal acceleration, we can derive horizontal velocity over displacement, and multiply by displacement over time, which is horizontal velocity.

$$a_x = \frac{dv_x}{dt} = \frac{d}{dx}(v_x) \cdot \frac{dx}{dt} = \frac{d}{dx}(v_x) \cdot v_x$$

Similarly, we can find the vertical acceleration. Notice that the y acceleration is plotted against horizontal distance, rather than conventional against time. This requires the following transformation:

$$a_y = \frac{dv_y}{dt} = \frac{dv_y}{dx} \cdot \frac{dx}{dt} = \frac{dv_y}{dx} \cdot v_x$$





The two accelerations are plotted in the left picture

The total acceleration is indicated as a sum of the two vectors

$$\bar{a}(x) = a_x(x)\,\hat{i} + a_y(x)\,\hat{j}$$

To find the magnitude of this vector, we use the Pythagorean theorem:

$$|a| = \sqrt{a_x(x)^2 + a_y(x)^2}, \{0 \le x \le 2.65\}$$

This page involves a manual derivative of a_y:

Expanding a_y we have

$$a_y = (\frac{d}{dx}(|V| \cdot sin(arctan(h'(x)))) \cdot v_x$$

Since we have to use the product rule, I will start with the left hand side.

Expanding $|V|$ gives $\sqrt{v_0^2 - 19.6(h(x))}$

Then we apply chain rule to the overarching root function and the inner function

The derivative to the root function is $\frac{1}{2}x^{\frac{1}{2}}$ and the derivative to the inner function is -19.6h'(x), as v_0^2 is a constant, giving us

$$\frac{1}{2}(v_0^2 - 19.6(h(x))^{-\frac{1}{2}} \cdot - 19.6h'(x)$$

Now we derive the Right Hand Side, applying chain rule to $sin(arctan(h'(x)))$

The derivative to the sin function is cosx

The derivative to $arctan(h'(x))$ requires another chain rule, giving us $\frac{1}{h'(x)^2+1} \cdot h''(x)$

Now we use this back to the first chain rule, giving

$$cos(arctan(h'(x))) \cdot \frac{1}{h'(x)^2+1} \cdot h''(x)$$

Let's denote LHS as u and RHS as v. According to product rule

$$u \cdot v = u \cdot v' + v \cdot u'$$

Assembling the components and eliminating the negative exponent gives us

$$(v_0^2 - 19.6(h(x)))^{\frac{1}{2}} \cdot cos(arctan(h'(x))) \cdot \frac{h''(x)}{h'(x)^2+1} + sin(arctan(h'(x))) \cdot \frac{-19.6h'(x)}{2(v_0^2-19.6(h(x))^{\frac{1}{2}}}$$

Next we multiply by the expression $\sqrt{v_0^2 - 19.6h(x)} \cdot cos(arctan(h'(x)))$, simplifying the left hand term and right hand denominator. Giving us:

$$(v_0^2 - 4.9(h(x))) \cdot cos^2(arctan(h'(x))) \cdot \frac{h''(x)}{h'(x)^2+1} + sin(arctan(h'(x))) \cdot cos(arctan(h'(x))) \cdot - 9.8h'(x)$$

Next we know the properties of $cos(arctan(h'(x))) = \frac{1}{\sqrt{1+x^2}}$ and $sin(arctan(h'(x))) = \frac{x}{\sqrt{1+x^2}}$, which gives us a elegant simplification

$$(v_0^2 - 19.6(h(x))) \cdot \frac{1}{h'(x)^2+1} \cdot \frac{h''(x)}{h'(x)^2+1} + \frac{h'(x)}{h'(x)^2+1} \cdot - 9.8h'(x)$$

Simplifying further gives:

$$(v_0^2 - 19.6(h(x))) \cdot \frac{h''(x)}{(h'(x)^2+1)^2} - \frac{9.8h'(x)^2}{h'(x)^2+1}$$

Finally substituting in the equation h(x) and simplifying gives:

$$(v_0^2 - 19.6(2.22^{(x-1.96)} - 0.21)) \cdot \frac{0.64 \cdot 2.22^{x-1.96}}{((0.798 \cdot 2.22^{(x-1.96)})^2+1)^2} - \frac{9.8(2.22^{(x-1.96)}-0.21)^2}{(2.22^{(x-1.96)}-0.21)^2+1}$$

Deriving these equations manually is tedious, so I used another method with simpler derivation to verify my results. Instead of finding accelerations through the derivatives of velocity, we can split the total acceleration into centripetal acceleration and tangential acceleration.

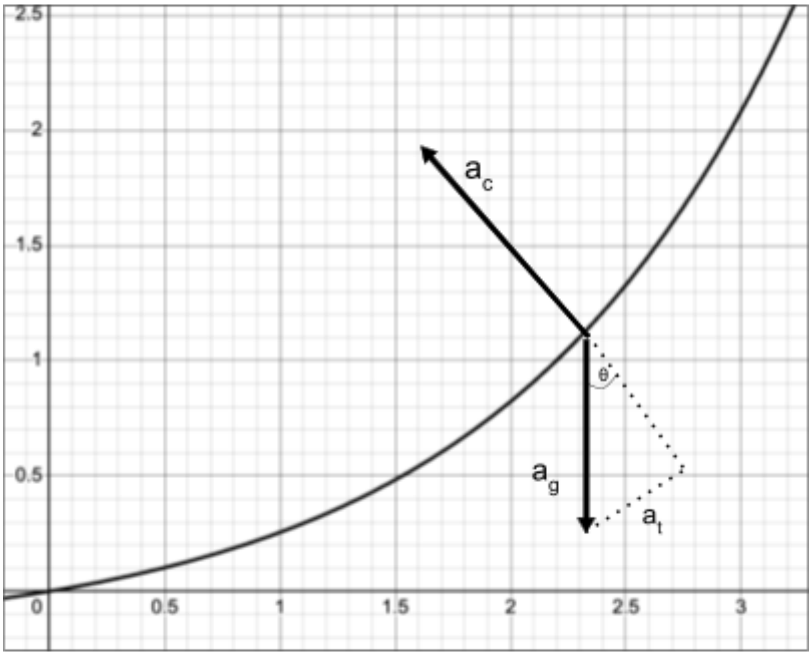
The equation for centripetal acceleration is:

$$a_c = \frac{v^2}{r}$$

Here, v is the tangential velocity, and r is the instant radius of the tangential circle. Knowing that curvature is defined as 1/r, we could also write the equation as:

$$a_c = v^2 \cdot k$$

We already have the velocity equation, so we would want to find curvature. curvature is also defined as the derivative of tangential angle over arclength, or $\frac{d\theta}{ds}$. Recall our equation for θ in terms of x from before is:



$$\theta(x) = arctan(h'(x))$$

Now, we can find arclength through integration. Imagine dividing the curve into many infinitesimally small diagonal lines, each composed of dx and dy, and we can add those together to find the total arclength. This is expressed as:

$$s = \int_0^x \sqrt{(\frac{dx}{dx})^2 + (\frac{dy}{dx})^2} dx$$

Here, $\frac{dx}{dx}$ is simply 1. Since we know that y is a function of x, we can write $\frac{dy}{dx}$ as $h'(x)$. Substituting these expression back into the main equation and adjusting the variables gives us:

$$s = \int_0^x \sqrt{1 + h'(u)^2} du$$

Knowing the equations for arclength(s) and angle(θ), we can find the curvature by deriving θ over x first, and then multiplying it by $\frac{dx}{ds}$.

$$\frac{d\theta}{ds} = \frac{d\theta}{dx} \cdot \frac{dx}{ds}$$

Now we derive θ over x first. Using the chain rule and performing the derivative of arctan gives

$$\frac{d\theta}{ds} = \frac{1}{(1+h'(x)^2)} \cdot \frac{dh'(x)}{dx} \cdot \frac{dx}{ds}$$

Let's denote the derivative of $h'(x)$ with $h''(x)$

$$\frac{d\theta}{ds} = \frac{h''(x)}{(1+h'(x)^2)} \cdot \frac{dx}{ds}$$

Now we derive the second term, $\frac{dx}{ds}$. Notice that $\frac{dx}{ds}$=$\frac{1}{(\frac{ds}{dx})}$, so we could derive s over xfirst, and then invert it.

Using the first fundamental theorem of calculus, we know that the derivative of an integral function is the function itself, so:

$$\frac{ds}{dx} = \frac{d}{dx}\int_{0}^{x} \sqrt{1 + h'(u)^2} du = \sqrt{1 + h'(x)^2}$$

Substituting this back into the original equation gives us

$$\frac{d\theta}{ds} = \frac{h''(x)}{(1+h'(x)^2)} \cdot \frac{1}{\sqrt{1+h'(x)^2}} = \frac{h''(x)}{(1+h'(x)^2)^{\frac{3}{2}}}$$

Finally, we can substitute our equation for curvature back to our equation for centripetal acceleration, yielding:

$$a_c = v^2 \cdot k = v^2 \cdot \frac{h''(x)}{(1+h'(x)^2)^{\frac{3}{2}}}$$

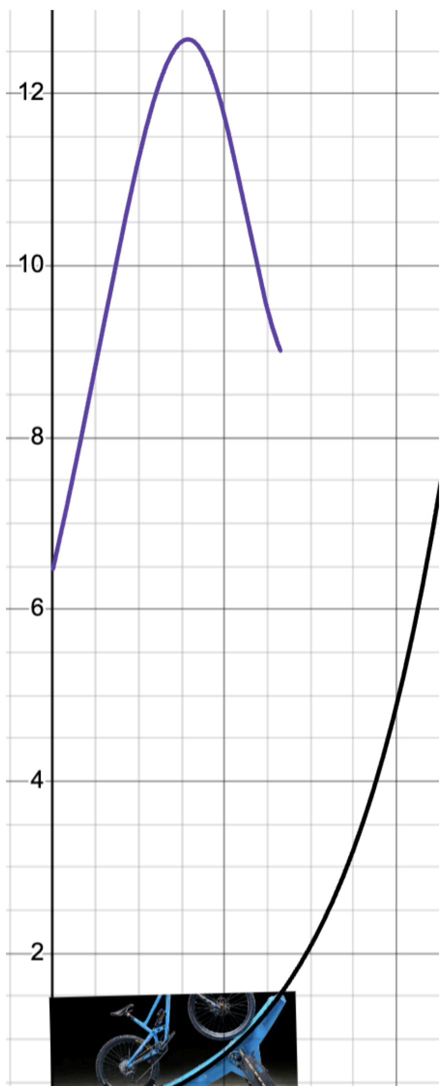
The diagram indicates that tangential acceleration is the “short leg” of gravity. We can express this with

$$a_t = g \cdot sin(\theta) = g \cdot sin(arctan(h'(x)))$$

The centripetal acceleration and tangential is perpendicular to each other, so we can use Pythagorean theorem to find the total acceleration.

$$a = \sqrt{a_c^2 + a_t^2}$$

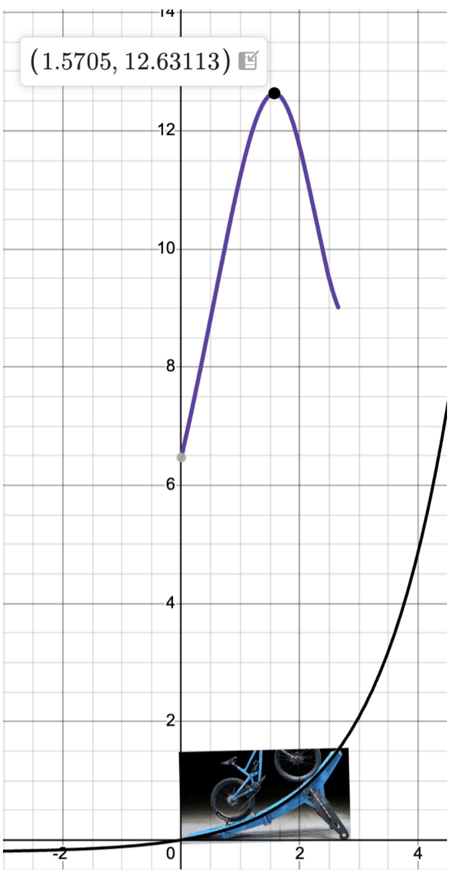
When a is plotted against x, we find that the shape of the graph completely overlaps with the graph obtained through velocity derivatives.



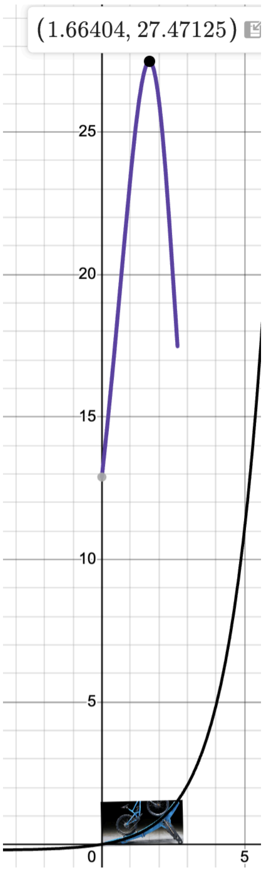
Graph Analysis

I will adjust the entry velocity v_0 to observe how acceleration changes at different speeds. Ideally, the trend of the graph should remain consistent.

Acceleration Profile at v_0= 7m/s



Acceleration Profile at v_0= 10m/s



At a speed of 7 ms^{-1}, acceleration peaks at 12.63 ms-2 and at 1.57 meters into the takeoff. In real life, this translates to the rider feeling a maximum g-force. At this point, the rider typically executes the jumping maneuver, because the larger normal force increases the input efficiency.

At a speed of 10 ms^{-1}, acceleration peaks at 27.5 ms^{-2}, and at 1.66 meters into the takeoff. The peak acceleration is equivalent to a gentle roller coaster. When we compare the profile at two speeds, we see that the acceleration more than doubles over a relatively small speed increase. This could be dangerous for less skilled riders because they are unable to accurately control speed, and the difference in compressive forces can set them off balance. The postive is that the vertex's x value remains similar. This translates to a similar point to jump, and therefor riders can apply the same technique.

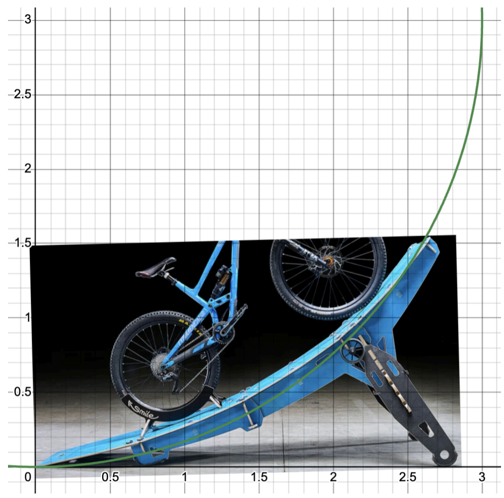
The y intercept of the acceleration curve is not zero because the slope of the curve starts at 0.17. When riders enter the curve, they feel a sudden jolt of acceleration when they transition from flat ground (slope zero) onto the takeoff. In most circumstances this wouldn't be a major problem, because the front suspension on mountain bikes will absorb the acceleration like a pump. However, if the rider is riding a trick bike without suspension, this can be detrimental.

Evaluation of two methods

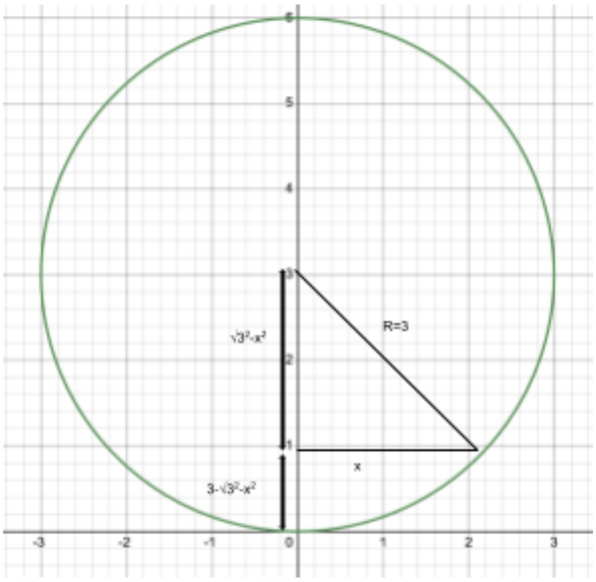
In this investigation, the first method involves finding the derivative of velocity vectors, while the second method involves centripetal and tangential acceleration. Although the two methods has reached the same conclusion, the first method is more flexible. This is immediately apparent in the next section, where I calculate the cetripetal acceleration of a circle.

Section 2: Circular Curves

I used the equation $(x)^2 + (y - 3)^2 = 3^2$ to model the takeoff ramp. This equation shows that the radius is 3, and the center of the circle is (0, 3). I shifted the circle upwards by three units so that flat bottom coincides with zero.

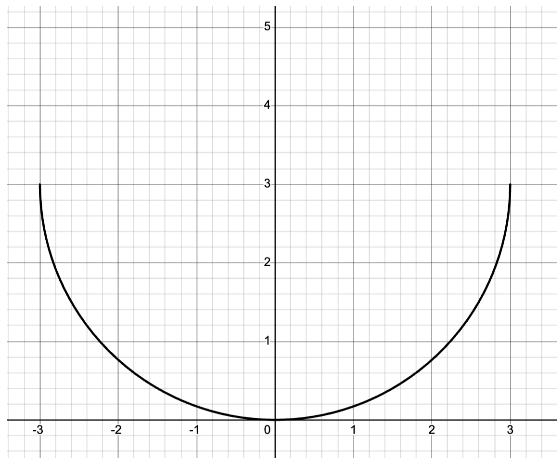


To find the slope of the circle, implicit differentiation would not be a good choice. This is because we cannot produce a graphable function. Instead, we can use reorganize the function with geometry and then derive.



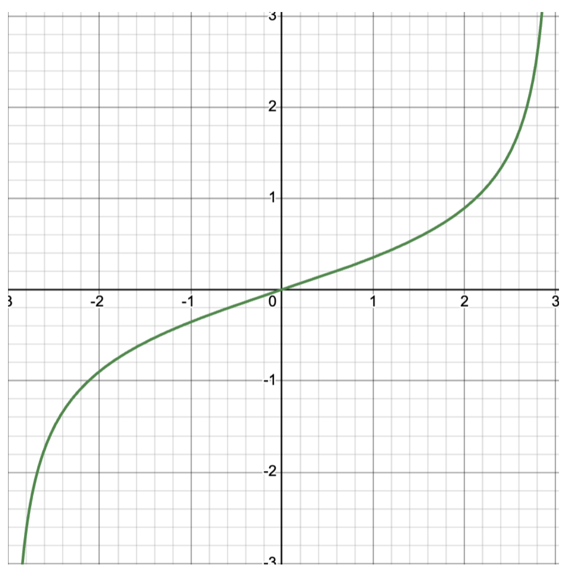
In this image, we could find the height of the circles bottom half, by subtracting the triangle leg from the radius. This gives us the function:

$$h(x) = 3 - \sqrt{3^2 - x^2}$$



$h(x)$ is plotted to the right, and we can see this is indeed a function because it passes the vertical line test
Then, we derive h(x) using the chain rule to find its slope:

$$h'(x) = \frac{d}{dx} 3 - \sqrt{3^2 - x^2}$$
$$= -\frac{d}{dx}\sqrt{9 - x^2}$$
$$= \frac{1}{2} \cdot \frac{1}{\sqrt{9-x^2}} \cdot - 2x$$
$$= -\frac{x}{\sqrt{9-x^2}}$$



This equation is graphed as the green curve to the right.
To find the tangential velocity on the circle, I used the same method of energy conservation as before:

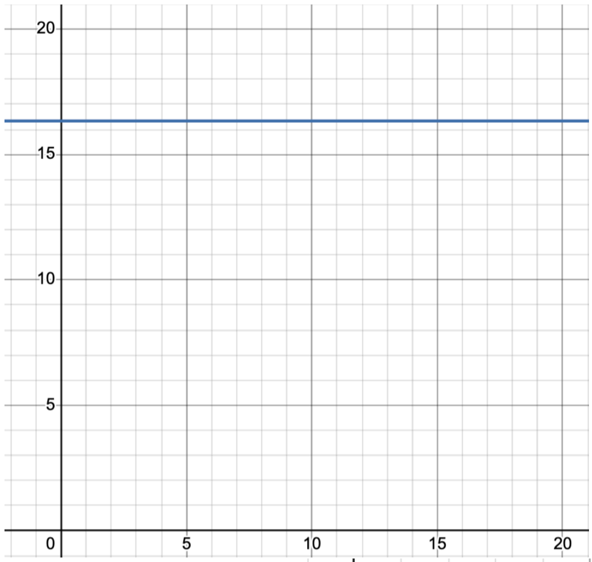
$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + mgh$$
$$v_0^2 = v_1^2 + 2gh$$
$$v_1 = \sqrt{v_0^2 - 2gh(x)}$$
$$v_1 = \sqrt{v_0^2 - 19.6(3 - \sqrt{9 - x^2})}$$

Moving forward, we can use two methods.

Method 1: Centripetal Acceleration + Tangential Acceleration

$$a_c = \frac{v^2}{r} = \frac{v_1^{\ 2}}{3}$$

If we ignore gravitational acceleration here, and assume the the velocity remains constant, then the only acceleration that the rider will experience is a_c. The graph will also be flat, since the result is a constant. The blue line shows the constant acceleration experienced at a speed of 7m/s. In real life, a constant acceleration translates to a consistent jump feel.



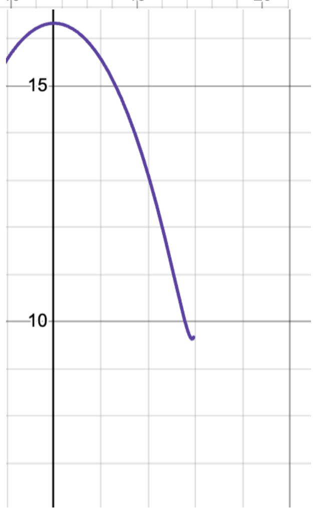
However, we live in a world with gravity, so a_c would depend on the equation v_1, and we also have to account for the tangential acceleration, $g \cdot sin(\theta)$

$$a_c = \frac{v_1^{\ 2}}{3}$$

$$a_c = \frac{v_0^{\ 2} - 19.6(3 - \sqrt{9-x^2})^2}{3}$$

$$a_t = g \cdot sin(arctan(h'(x)))$$

$$a_t = g \cdot sin(arctan(-\frac{x}{\sqrt{9-x^2}}))$$



Next, we perform a vector sum of the two accelerations like before, giving us:

$$a = \sqrt{a_c^{\ 2} + a_t^{\ 2}}$$

$$= \sqrt{(\frac{v_0^{\ 2} - 19.6(3 - \sqrt{9-x^2})}{3})^2 + g^2 \cdot sin^2(arctan(-\frac{x}{\sqrt{9-x^2}})}$$

Method 2: Velocity Derivatives:

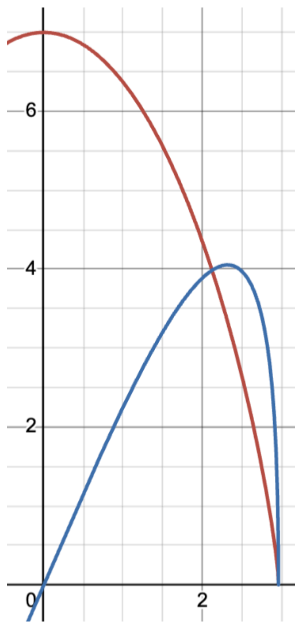
We can split velocity into x and y vectors like before

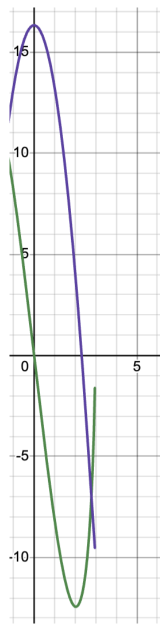
$$\widehat{V_x} = v_1 \cdot cos(arctan(h'(x)))$$

$$= \sqrt{v_0^{\ 2} - 19.6(3 - \sqrt{9 - x^2})} \cdot cos(arctan(-\frac{x}{\sqrt{9-x^2}}))$$

$$\widehat{V_y} = v_1 \cdot sin(arctan(h'(x)))$$

$$= \sqrt{v_0^{\ 2} - 19.6(3 - \sqrt{9 - x^2})} \cdot sin(arctan(-\frac{x}{\sqrt{9-x^2}}))$$





Next, we can derive the acceleration a_x and a_y Seperately:

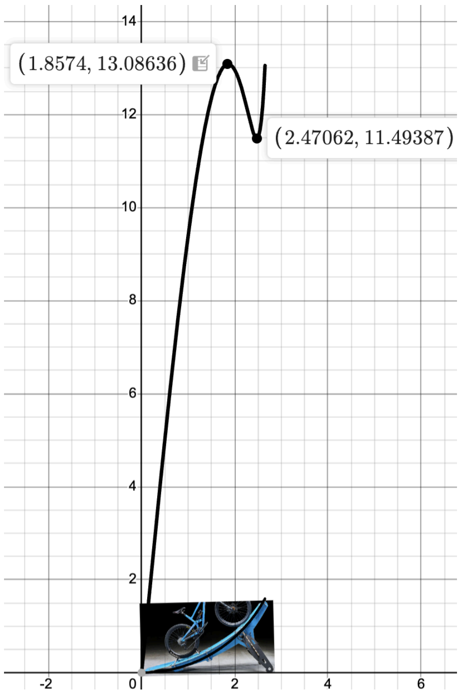
$$a_x = \frac{d}{dx}(v_x) \cdot v_x$$

$$a_y = \frac{d}{dx}(v_y) \cdot v_x$$

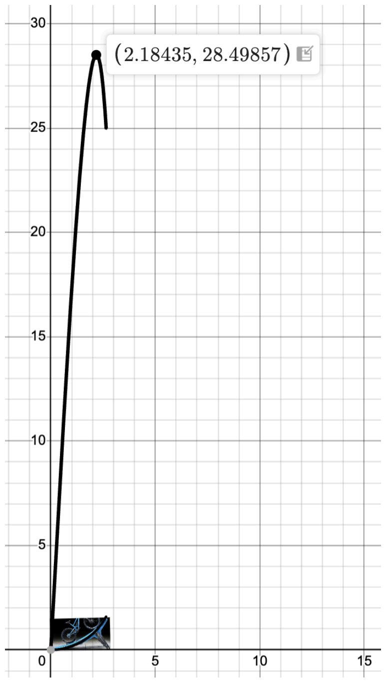
Finally, we can find the magnitude of the total acceleration by summing the two vectors together

- $|a| = \sqrt{a_x(x)^2 + a_y(x)^2}, \{0 \le x \le 2.65\}$

Acceleration Profile at v_0= 7m/s



Acceleration Profile at v_0= 10m/s



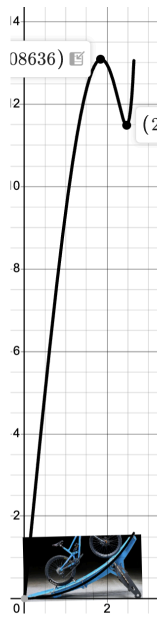
Graph Analysis

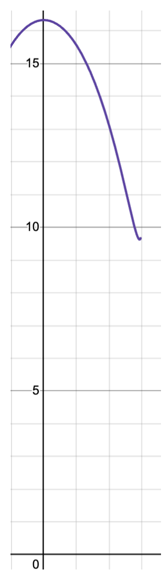
The left graph is the acceleration profile modeled at v_0=7ms^{-1} while the right graph corresponds to v_0 =10ms^{-1}. Across different speeds, we see that not only does the peak acceleration and its position change, the graphs tendency also changes. On the left graph, acceleration peaks, and then reduces to a local minimum, before rising again. This can translate to a wobble in real life, and create instability for the rider. On the right graph, the acceleration trend remains constant. One reason for this difference could be how the velocity decrease offsets the original acceleration of the curve. When velocity is high enough, then the acceleration will not be affected. This means that the circular takeoff is only suitable for higher speeds. However the logarithmic curve is still superior compared to the circular curve, as its acceleration peaks lower.

Method Evaluation

Compared to the second method(black), the first method(purpl) of centripetal + tangential acceleration seems unreasonable. The acceleration starts at a maximum and gradually decreases, when in reality, the initial portion of the takeoff is mellow, and the rider shouldn't feel excessive acceleration.

The issue lies in the method of summing centripetal and tangential acceleration. Centripetal acceleration assumes a object rolling on a path with curvature. In this case, the curvature is $\frac{1}{r}$, or $\frac{1}{3}$. However, when the rider is rolling on flat surface, the curvature is 0. There is an instant transition from 0 to $\frac{1}{3}$, causing the acceleration to start high on the y axis.

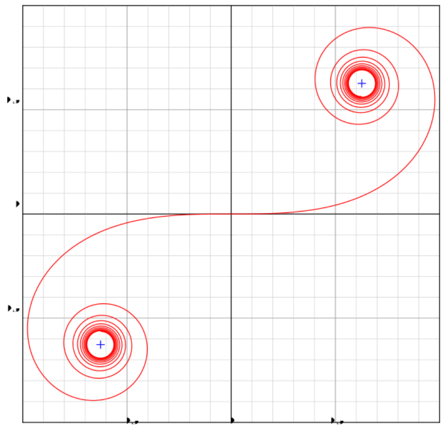




Section 3: Clothoid Curves

A problem faced with circular curves is that the curvature increases suddenly from flat ground, causing the two methods to deviate from each other. The clothoid curve should resolve this issue, as it is known as a "transition curve", often used in railways and roller coasters. The special property of the clothoid is that curvature (k) increases linearly with arclength (s), expressed in:

$$k = a \cdot s$$



This means when the arclength(s) is zero, the curvature(k) is zero, so the centripetal acceleration will also start at zero. Here, a is a constant. We can adjust this constant according to the specifications we want:

Say that I want a curve that has a circle of tangential radius R_l at arclength S_l. We can set the equation as:

$$\frac{1}{R_l} = a \cdot S_l$$

The constant a would therefore become:

$$a = \frac{1}{R_l \cdot S_l}$$

The clothoid curve is defined as a set of parametric equations, known as the fresnel integrals:

$$S(t) = \int_0^t cos(\frac{1}{2} \cdot a \cdot u^2)du$$

$$C(t) = \int_0^t sin(\frac{1}{2} \cdot a \cdot u^2)du$$

Here, t is used as the parameter. S(t) gives the x value of the clothoid curve, while C(t) gives the y value of the clothoid curve. The constant a is used to scale the curve.

I am scaling the curve to a dimension similar to previous curves. After adding a slider to adjust the constants R_l and S_l, I landed at the values:

R_l=0.6
S_l=12.1

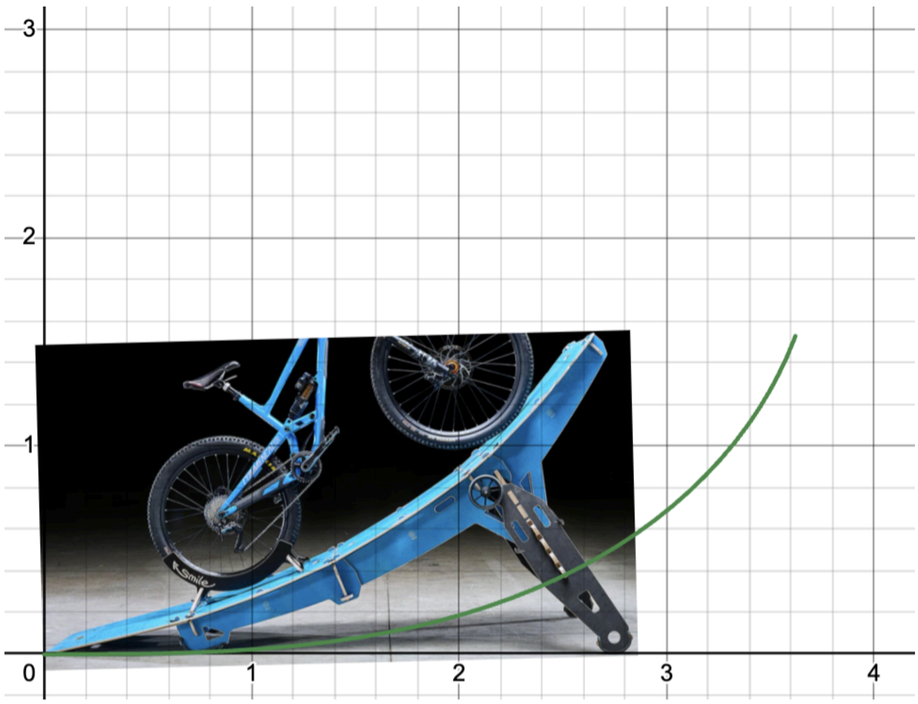
Substituting these into the parametric equations gives us:

$$S(t) = \int_0^t cos(\frac{1}{2 \cdot 0.6 \cdot 12.1} \cdot u^2)du$$

$$C(t) = \int_0^t sin(\frac{1}{2 \cdot 0.6 \cdot 12.1} \cdot u^2)du$$

I plotted this equation in Desmos using the expression

$$(S(t), C(t))\ \{0 \le t \le 4.2\}$$



We can try to transform parametric equations into a cartesian equation to use our previous methods.

To derive a cartesian equation, we want a function with the output C(t) and the input S(t).

For our x parameter function S(t), the current input is t and the output is S(t). The goal is to reverse the input and output makes t as a subject. This can be done through the inverse function, T=S^{-1}(x)

Then, we can plug S^{-1}(x) into the function C(t), giving
$y = C(S^{-1}(x))$

To find the inverse for S(t), we need to integrate the function first. However, $sin(u^2)$ is a non-elementary integral. A non-elementary integral is an integral that cannot be expressed in terms of elementary functions (polynomials, exponentials, logarithms, trigonometric functions, etc.). This happens because the antiderivative of the function involves special functions or infinite series that go beyond elementary expressions, such as e^{-x^2} (related to the error function).

Since we cannot rewrite this as a cartesian equation, we derive in parametric form.

Suppose that the derivatives for both parametrics, S'(t) and C'(t), exist. And we shall assume that $x'(t)$≠0. Then the derivative $\frac{dC}{dS}$ can derived as follows:
$\frac{dc}{ds} = \frac{\frac{dc}{dt}}{\frac{ds}{dt}} = \frac{C'(t)}{S'(t)}$

We can prove this equation by the chain rule. If the parameter t can be eliminated, yielding a function C = F(S), expanding S gives C=F(S(t)). Differentiating both sides gives us
C'(t)=F'(S(t)) · S'(t)

Isolating F'(S(t) gives
$F'(S(t)) = \frac{C'(t)}{S'(t)}$

Yet $F'(S(t))$ is $\frac{dc}{ds}$, therefore proving the theorem.

Now we can perform the derivation of the original function

$$\frac{dc}{ds} = \frac{\frac{d}{dt}\int_0^t sin(\frac{1}{2\cdot 0.6\cdot 12.1}\cdot u^2)du}{\frac{d}{dt}\int_0^t cos(\frac{1}{2\cdot 0.6\cdot 12.1}\cdot u^2)du}$$

Using the first fundamental theorem of calculus, we simplify both expressions.

Our expression is a definite integral of f(u), and it is defined as:
$F(t) = \int_a^t f(u)du$

We want to compute $\frac{d}{dt}F(t)$, so we can expand the equation using the first principle of derivatives
$\frac{d}{dt}F(t) = \lim_{h\to 0} \frac{F(t+h)-F(t)}{h}$

Now we expand $F(t + h) - F(t)$ using the definition of the function
$F(t + h) = \int_a^{t+h} f(u)du \qquad F(t) = \int_a^t f(u)du$

Thus,

$$F(t+h) - F(t) = \int_a^{t+h} f(u)du - \int_a^{t} f(u)du$$

We can simplify the difference between integrals using the additive property of definite integrals:

$$F(t+h) - F(t) = \int_t^{t+h} f(u)du$$

Substitue this equation into the derivative becomes:

$$\frac{d}{dt}F(t) = \lim_{h\to 0} \frac{\int_t^{t+h} f(u)du}{h}$$

Here, the integral $\int_t^{t+h} f(u)du$ represents the area of function F from t to t+h. When we divide it by h, we can interpret this as the average value of a function, just as how we divide a area by its base to find height.

$$\lim_{h\to 0} \frac{\int_t^{t+h} f(u)du}{h} = average\ valye\ of\ F(u)\ on\ \{t,\ t+h\}$$

As h→0, the interval $\{t,\ t+h\}$ shrinks to a single point t. Since the function is continuous, the average value converge at f(t). Finally, we have:

$$\frac{d}{dt}\int_a^{t} f(u)du = \lim_{h\to 0} \frac{\int_t^{t+h} f(u)du}{h} = f(t)$$

Therefor, our equations becomes:

$$\frac{dc}{ds} = \frac{sin(\frac{t^2}{2\cdot 0.6\cdot 12.1})}{cos(\frac{t^2}{2\cdot 0.6\cdot 12.1})} = tan(\frac{t^2}{2\cdot 0.6\cdot 12.1})$$

First Method of finding θ

Since this equation gives the slope of the graph, or $\frac{rise}{run}$, we can use the arctan function to find the tangential angle. Yet the arctan of tan canceles our, because the inverse function of that function is equal to the subject.

$$\theta = arctan(tan(\frac{t^2}{2\cdot 0.6\cdot 12.1})) = f(f^{-1}(\frac{t^2}{2\cdot 0.6\cdot 12.1})) = \frac{t^2}{2\cdot 0.6\cdot 12.1}$$

Second method of finding θ

we can verify this result of θ through the definition of curvature:

Curvature is defined as the derivative of tangential angle over the derivative of arclength, and for the clothoid curve it is also expressed as $k = a\cdot s$

$$k = \frac{d\theta}{ds} = a\cdot s$$

If we integrate curvature over arclength, we can eliminate the derivative and find the original equation for θ

$$\theta = \int_0^{s} k(u)du = \int_0^{s} a\cdot u\, du = \frac{1}{2}a\cdot s^2$$

Then we substitute the equation $a = \frac{1}{R_l\cdot S_l}$

$$\theta = \frac{s^2}{2\cdot R_l\cdot S_l}$$

Currently, the expression for constant (a), curvature (k) and tangential (θ) all involves the arclength (s). We would want to find its correlation with the parameter t gor graphing.

Through integrating the infnitessibmally small diagonals over the parameter t, we can find the total arclength between 0 to t.

$$s = \int_0^t \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

We can substitute $\frac{dx}{dt}$ and $\frac{dy}{dt}$ with our parametric equations, giving:

$$s = \int_0^t \sqrt{cos^2(\frac{1}{2} \cdot a \cdot u^2) + sin^2(\frac{1}{2} \cdot a \cdot u^2)} dt$$

Here, we know the trigonometric identity, $cos^2 + sin^2 = 1$, so the integral becomes

$$s = \int_0^t 1\, dt$$

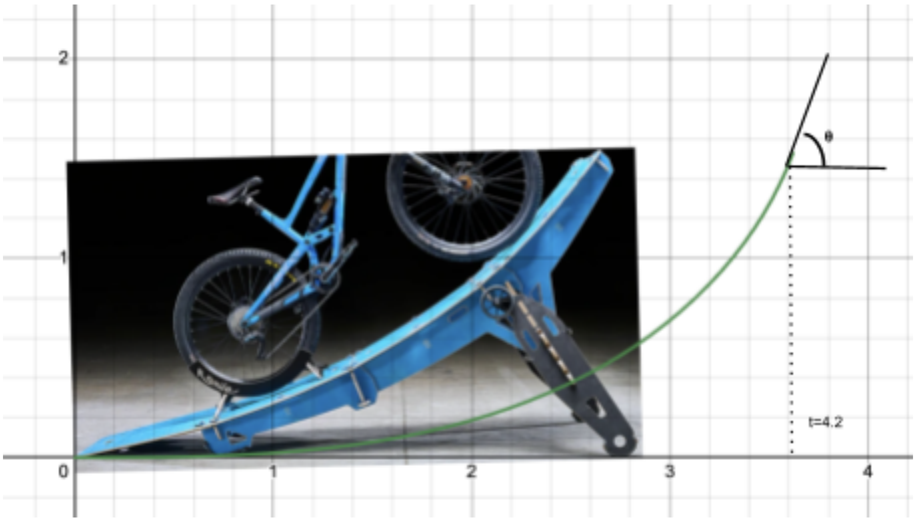
The integral of 1 is simply t, giving us the equation:

$s = t$

We can substitute this back into the expression for θ

$$\theta = \frac{s^2}{2 \cdot R_l \cdot S_l} = \frac{t^2}{2 \cdot R_l \cdot S_l}$$

Therefor, we can verify our results for θ because both methods yield the same results. Using this, we can approximate the tangential angle on takeoff, which is at point t=4.2, or $\int_0^{4.2} cos(\frac{1}{14.52} \cdot u^2) du = 3.6$ meters to the left of the start of the ramp.



$$\theta = \frac{t^2}{2 \cdot R_l \cdot S_l} = \theta = \frac{4.2^2}{2 \cdot 0.6 \cdot 12.1} = 1.21\, rad$$

This is equal to 69 degrees, which is steep takeoff that gives a high trajectory.

Moving forward, we can calculate acceleration using two methods:

Method 1: Centripetal Acceleration + Tangential Acceleration

The formula for centripetal acceleration is:

$$a_c = v^2 \cdot k$$

Yet we know in a clothoid curve, the curvature is proportional to the parameter:

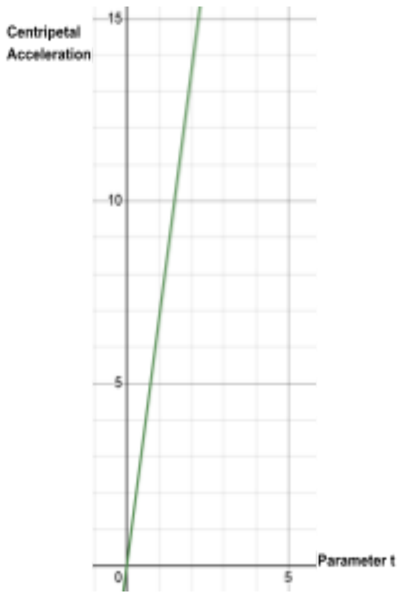
$$k = a \cdot s$$

We can substitute a with the a numerical value, and s with t:

$$k = \frac{1}{0.6 \cdot 12.1} \cdot t$$

Then substitute this equation back to a_c

$$a_c = \frac{v^2 \cdot t}{0.6 \cdot 12.1}$$

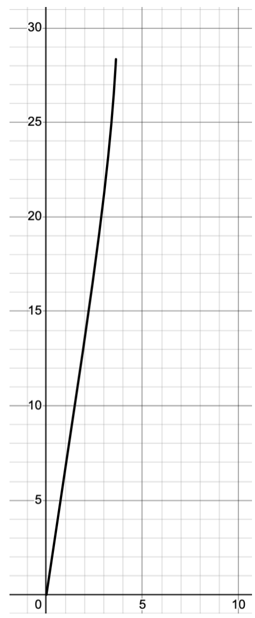


Here, if we ignore gravity and assume that velocity remains constant across the takeoff, then a_c will have a linear relationship with t, as indicated by the straight line of the graph.

Withouty gravity, the tangential acceleration will also disappear. Therefore, a_c will be the only source of acceleration.

However, the function a(t) is plotted against the parameter t. This doesn't translate to any value in real life. Instead, we want a(t) to be plotted against x.

Fortunately, when the velocity is constant, the acceleration is only dependent on the geometric property of the parameter (t), and this makes the relationship between a(t) and x(t) deterministic and static.



Therefore, we can transform a(t) into a(x) by plotting a(t) against x(t).
In this parametric equation:

$x(t) = \int_0^t \cos(\frac{1}{2 \cdot 0.6 \cdot 12.1} \cdot u^2)du$ and $y(t) = \frac{v_0^2 \cdot t}{0.6 \cdot 12.1}$

This is plotted in desmos as:

$$(S(t), a_c(t))$$

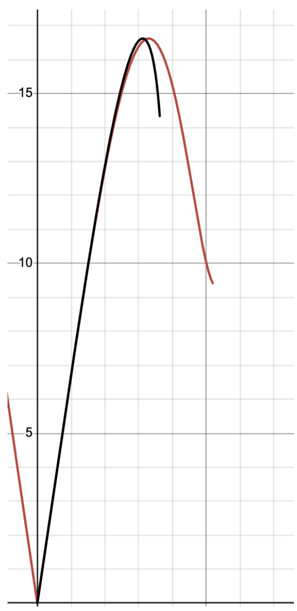
Now we can consider introducing variable velocity V, this can be derived from the work-energy principle:

$$\frac{1}{2}mv(t)^2 + mgh = \frac{1}{2}mv_0^2$$

We can substitute C(t) with h, since it gives the y value in terms of t

$$\frac{1}{2}mv(t)^2 + mgC(t) = \frac{1}{2}mv_0^2$$

$$v(t) = \sqrt{v_0^2 - 2gC(t)}$$



Acceleration a(t) now depends on **dynamic factors**, including:

Centripetal acceleration: $a_c(t) = v(t)^2 \cdot k$

Tangential acceleration: $a_t(t) = g \cdot \sin(\frac{v^2 \cdot t}{0.6 \cdot 12.1})$

Here, we notice the relationship of centripetal acceleration $a_c(t)$ with the curvature is constantly changing. therefor, its relationship with parameter t is also constantly changing. As a result, mapping a(t) onto S(t) might not be accurate.

When I introduce variable velocity v(t), the total acceleration a(t) becomes

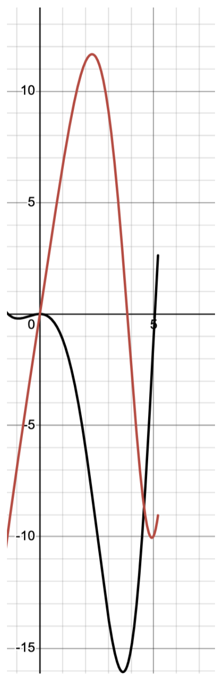
$$a(t) = \sqrt{a_c(t)^2 + a_t(t)^2} = \sqrt{(v(t)^2 \cdot k)^2 + (g \cdot \sin(\frac{v^2 \cdot t}{0.6 \cdot 12.1}))^2}$$

$$(S(t),\ a(t))$$

Here, the red graph is modeled correctly because it is plotted directly against parameter t. However, the black one is in accurate because of the **dynamic factors** mentioned before.

Method 2: Velocity Derivatives:

Recall that the tangential angle is expressed as $\theta = \frac{s^2}{2 \cdot R_l \cdot S_l}$



Therefor, we can write the velocities as:

$$\widehat{V_x} = |V| \cdot cos(\theta) \qquad \widehat{V_y} = |V| \cdot sin(\theta)$$

$$\widehat{V_x} = |V| \cdot cos(\frac{s^2}{2 \cdot R_l \cdot S_l}) \qquad \widehat{V_y} = |V| \cdot sin(\frac{s^2}{2 \cdot R_l \cdot S_l})$$

To find acceleration, we want the derivative of velocity over time $\frac{dv}{dt}$. Right now, can derive velocity over arclength (s), yielding $\frac{dv}{ds}$

We can perform a transformation to change this expression:

$$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt}$$

Yet, we notice that the derivative of arclength over time is tangential velocity. Substituting this in gives:

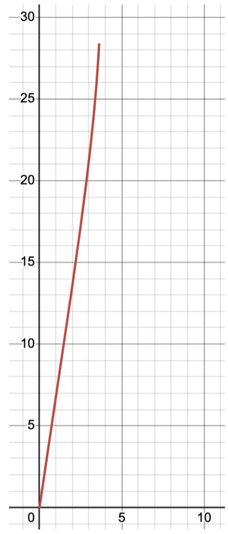
$$a = \frac{dv}{ds} \cdot v$$

Now we can write the acceleration for x and y separately. Lets first assume that velocity is constant.

$$a_x(t) = \frac{dv_x}{dt} \cdot v_0 = v_0(-v_0 \cdot sin(\frac{t^2}{2 \cdot R_l \cdot S_l}) \cdot \frac{t}{R_l \cdot S_l})$$

$$a_y(t) = \frac{dv_x}{dt} \cdot v_0 = v_0(v_0 \cdot cos(\frac{t^2}{2 \cdot R_l \cdot S_l}) \cdot \frac{t}{R_l \cdot S_l})$$

Here I used parameter t to substitute arclength s for graphing.



Now we can perform a vector sum and find the magnitude of total acceleration, within the predefined interval:

$$a(t) = \sqrt{a_x(t)^2 + a_y(t)^2}, \{0 \le t \le 4.2\}$$

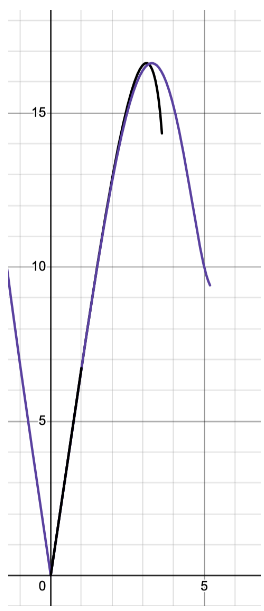
Since the velocity is constant, I could plot this against S(t) as the x axis to find a(x)

$(s(t), a(t))$

When we introduce variable velocity, vx and vy would become:

$$\widehat{V_x} = v(t) \cdot cos(\frac{s^2}{2 \cdot R_l \cdot S_l}) = \sqrt{v_0^2 - 2gC(t)} \cdot cos(\frac{t^2}{2 \cdot R_l \cdot S_l})$$

$$\widehat{V_y} = v(t) \cdot sin(\frac{s^2}{2 \cdot R_l \cdot S_l}) = \sqrt{v_0^2 - 2gC(t)} \cdot sin(\frac{t^2}{2 \cdot R_l \cdot S_l})$$



The acceleration of x and y components would become:

$$a_x(t) = \frac{dv_x(t)}{dt} \cdot v(t) = v(t)(v'(t) \cdot cos(\frac{t^2}{2 \cdot R_l \cdot S_l}) - v(t) \cdot sin(\frac{t^2}{2 \cdot R_l \cdot S_l}) \cdot \frac{t}{R_l \cdot S_l})$$

$$a_y(t) = \frac{dv_y(t)}{dt} \cdot v(t) = v(t)(v'(t) \cdot sin(\frac{t^2}{2 \cdot R_l \cdot S_l}) + v(t) \cdot cos(\frac{t^2}{2 \cdot R_l \cdot S_l}) \cdot \frac{t}{R_l \cdot S_l})$$

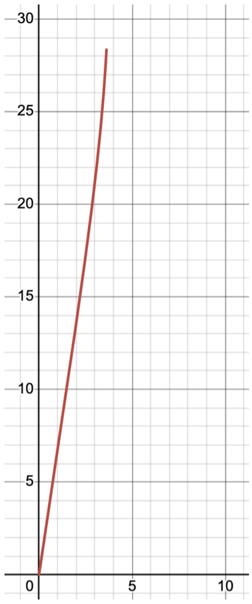
The expression for total acceleration in terms of t is:

$$a(t) = \sqrt{a_x(t)^2 + a_y(t)^2}$$

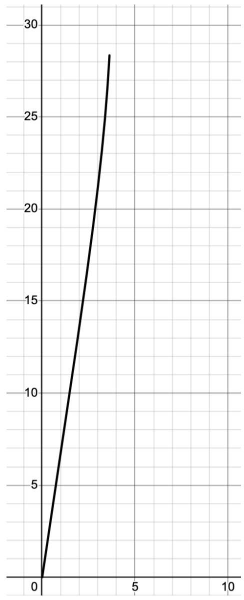
$(S(t), a(t))$

Comparison between two methods:
The two methods yielded the identical graphs, both with constant velocity and variable velocity. The graphs below compare the graphs produced through two methods:

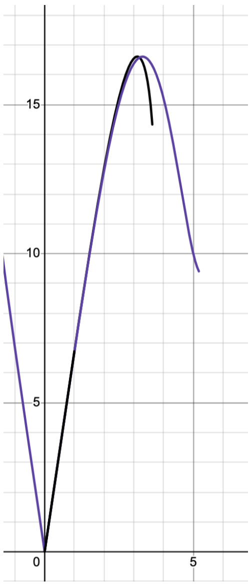
Acceleration profile at constant v_0=7
Using velocity derivatives



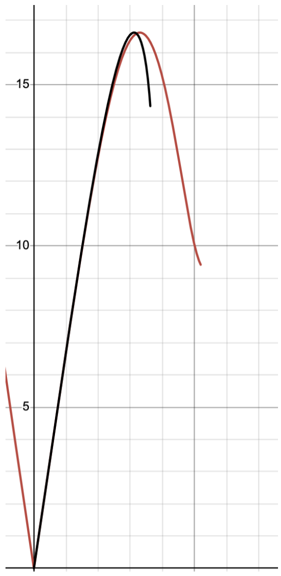
Acceleration profile at constant v_0=7
using centripetal acceleration



Acceleration profile with variable velocity
v(t) using velocity derivatives



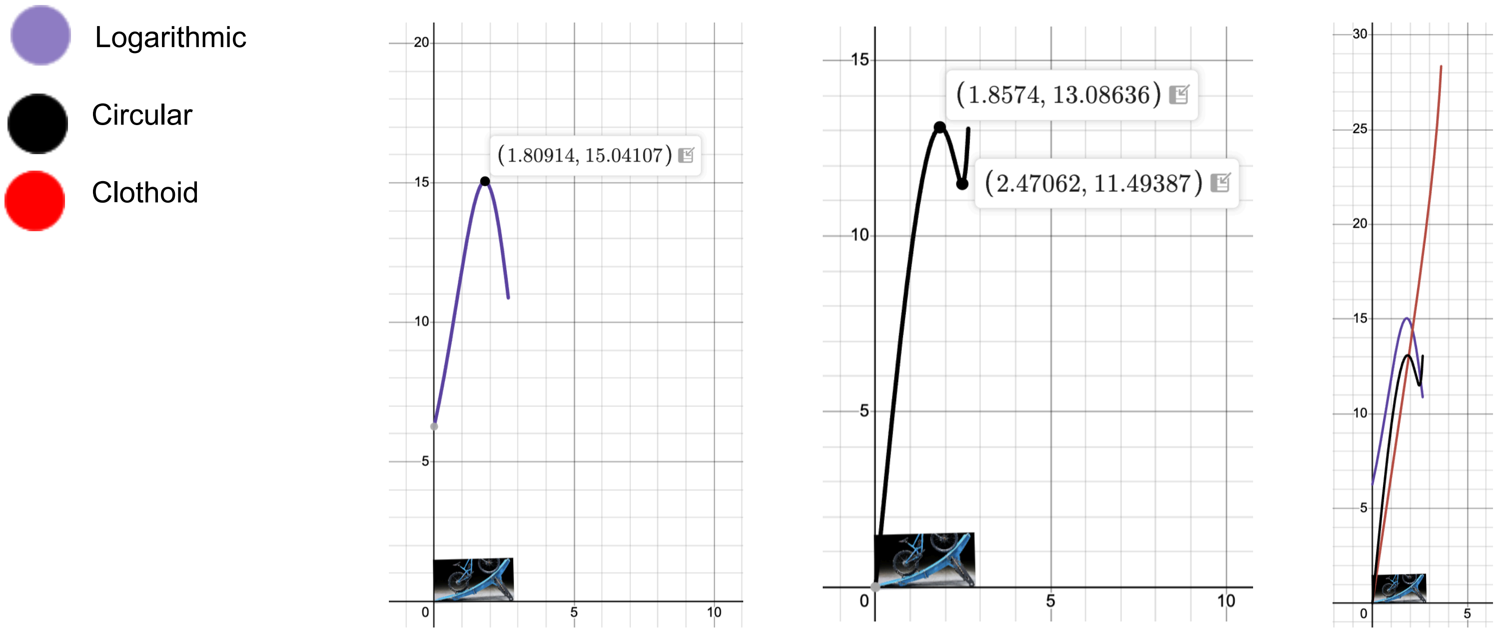
Acceleration profile with variable velocity
v(t) using centripetal acceleration



The acceleration profile in the first pair of graphs is accurate. This is because the non-linearities between x(t) and t do not affect the mapping. After all, v is constant, meaning a(t) depends only on curve geometry, which is tied to x(t). However, the acceleration profile (purple and red) in the second pair of graphs is inaccurate. This is because variable velocity introduces the expressions $v(t)$and $v'(t)$. As a result the relationship between a and t becomes dynamic, while t is tied stilled tied to x via a static relationship. We cannot use a static relationship to map a non-static relationship.

Conclusion:

For accuracy and consistency in comparing the three curves, I will assume that velocity remains constant.



From the three graphs, we can see the peak acceleration is lowest for the circular shape, but it has a change in tendency in the middle. The logarithmic curve reaches a good balance between consistent tendency and peak acceleration. The clothoid curve has the most consistent tendency. However, its acceleration reaches high labels approaching the end of the ramp. This makes it only suitable for mellower curves, where curvature doesn't decrease as much.

The correct method to find the acceleration profile of the clothoid curve against horizontal distance would be to invert the X parameter S(t) first. This would give us the parameter t in terms of x value, with the function t(x).

In the first method: we then use t(x) to find $a_c(t(x))$, $a_t(t(x))$, and $a(t(x))$. These functions are all plotted against the x axis.

In the second method, can substitute t(x) into $a_x(t)$ and $a_y(t)$. This will eliminate the parameter t within the equations, giving $a(x)$.

A potential extension to this IA is investigating the numerical inversion of the parameter x, which is $\int_0^t cos(\frac{1}{2} \cdot a \cdot u^2)du$. This includes methods of approximation that are used in graphing.