Nonlinear Midterm

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1 Nonlinear Dynamical Systems

$$\dot{x} = \lambda - x - \frac{1}{x^2} \tag{1}$$

$$\dot{x} = \frac{1}{e^{-\lambda x}} - x \tag{2}$$

$$\dot{x} = -\lambda \sin x - 2 \sin 2x \tag{3}$$

The above systems are partial differential equations, $\dot{x}$, derivatives of functions expressing changes in an original system. System one is a second order equation while systems two and three are first order. The following analysis was performed for all three functions whose original state was arbitrarily initiated with state vector X as 2000-D array for $\{x|x \in R, -6 < x < 6\}$ (see the following notebook for code reference (Click me)).

1. Find the fixed points and their stability
2. Find the potential functions and plot them together with the functions in phase space
3. What are 'interesting' values of λ, i.e. values where bifurcations occur;
4. Draw a bifurcation diagram for each system
5. Classify the bifurcation in terms of the basic types

1.1 Fixed Points and Stability

Fixed points, $\tilde{x}$, are defined as points where $\dot{x} = 0$; where the system is not expressing change. The initial state of the system at such points will stay the throughout the system dynamics. These points can be described as stable, unstable, and half-stable. Stable fixed points will attract the system at the current state while unstable fixed points will repel the system. Half-stable fixed points both attract and repel, creating a saddle (see sec. phase space plots). The following fixed points were found for each of the three systems.

System One:

$$\dot{x} = \lambda - x - \frac{1}{x^2}$$

No fixed points were found for $\lambda < 0$.

A stable fixed point was found for $\lambda = 0$ for the initial state vector at approximately x position 833, where $x \approx -0.99$ (see Figure 1).

An unstable fixed point for the initial state vector and $\lambda > 0$ was found at approximately x position 1053, where $x \approx 0.32$ (see Figure 2).

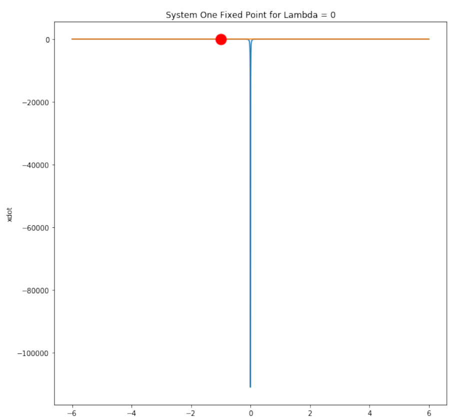


Figure 1: System one, stable fixed point for $\lambda = 0$.

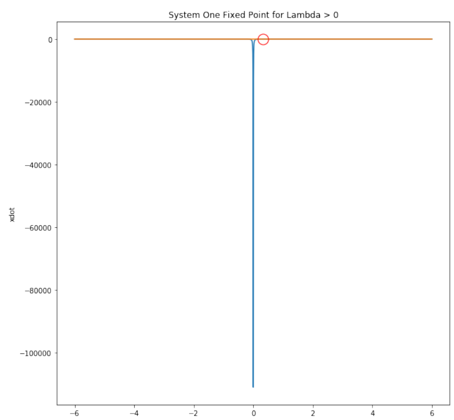


Figure 2: System one, unstable fixed point for $\lambda > 0$.

System Two:

$$\dot{x} = \frac{1}{e^{-\lambda x}} - x$$

No fixed points were found for $\lambda < 0$.

A stable fixed point was found for $\lambda = 0$ for the initial state vector at approximately x position 1166, where $x \approx 0.99$ (see Figure 3).

Another stable fixed point for the initial state vector and $\lambda > 0$ was found at approximately x position 1029, where $x \approx 0.17$ (see Figure 4).

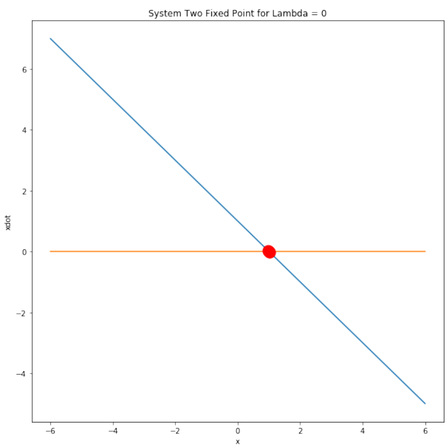


Figure 3: System two, stable fixed point for $\lambda = 0$.

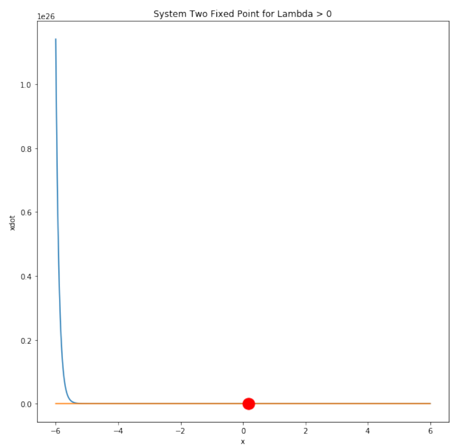


Figure 4: System two, stable fixed point for $\lambda > 0$.

System Three:

$$\dot{x} = -\lambda \sin x - 2 \sin 2x$$

Two stable fixed points were found for $\lambda < 0$ for the initial state vector at approximately x positions 476 and 1523, where $x \approx -3.14, 3.14$, respectively. One unstable fixed point was also found at positions 999, where $x \approx -0.003$. (see Figure 5).

Two stable fixed points were found for $\lambda = 0$ for the initial state vector at approximately x positions 476 and 1523, where $x \approx -3.14, 3.14$, respectively. Two unstable fixed points were also found at positions 738 and 1261, where $x \approx -1.56, 1.56$, respectively. (see Figure 6).

Two unstable fixed points for the initial state vector and $\lambda > 0$ were found at approximately x positions 476 and 1523, where $x \approx -3.14, 3.14$, respectively (see Figure 7). Note, the stable fixed points when $\lambda \leq 0$ have now become unstable as λ has become positive.

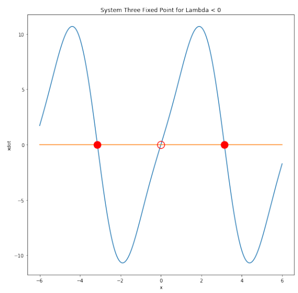


Figure 5: System three, two stable and one unstable fixed points for $\lambda < 0$.

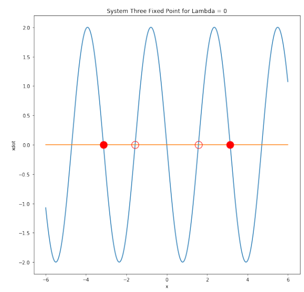


Figure 6: System three, two stable and two unstable fixed points for $\lambda = 0$.

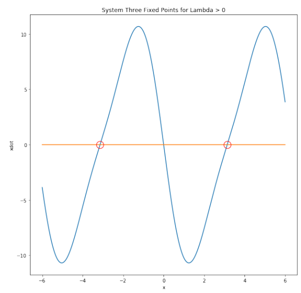


Figure 7: System three, two unstable fixed points for $\lambda > 0$.

1.2 Potential Functions in Phase Space

The potential function, V(x), is a possible original function the differential equation is modelling. If possible, the potential function can be found by taking the negative integral of the differential equation, such as $V(x) = -\int \dot{x}dx$. In other words, the potential function implies the magnitude of change occurring in the system. While potential functions for most first order differential equations are definable, it is not always possible, especially for higher ordered systems. Furthermore, the derivative of the potential function defines the behavior of V(x), which will only decrease towards zero or a local/global minimum.

Phase space plots show the relationship trajectory of two functions. Thus, the above figures are phase plots of $\dot{x}$ and $f(x)$. Such plots show the relationship between the state vector and the derivatives at such points in the state vector. Therefor, we can see for any current state, how the dynamical system will change. For each of the three systems, the following phase space plots were generated to show the potential function, V(x), with the original function, f(x), for the various conditions of λ.

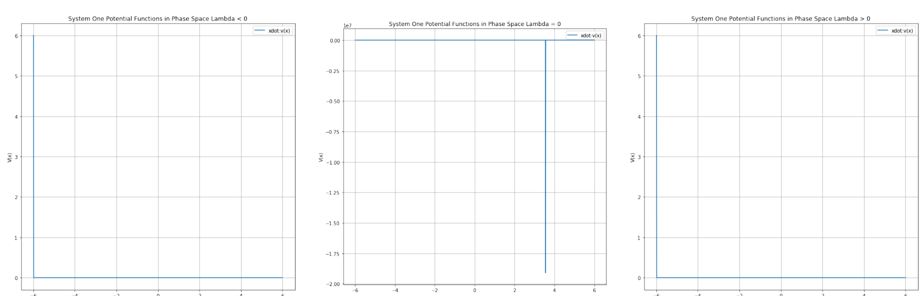


Figure 8: System one, phase space plot of potential function, V(x) and state vector, x.

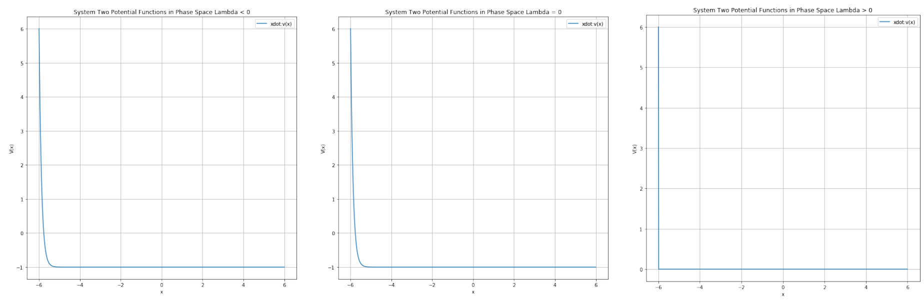


Figure 9: System two, phase space plot of potential function, V(x) and state vector, x.

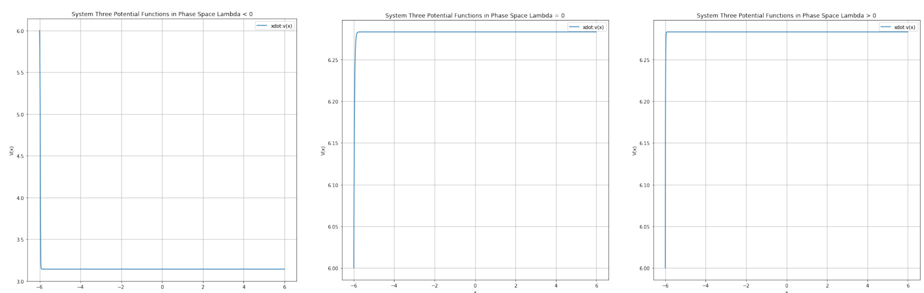


Figure 10: System three, phase space plot of potential function, V(x) and state vector, x.

1.3 Bifurcation Diagrams

Similar to phase space plots, we can generate bifurcation diagrams by plotting the fixed point locations for different values of λ. By doing so, we can see the influence of parameter, λ, on the change in the dynamical system for the given state. Such diagrams show how the system dynamics progress as the parameter changes. There are four basic types of bifurcations: saddle-node, transcritical, and super- and subcritical pitchfork bifurcation, as well as systems with hysteresis. Below, bifurcation diagrams were constructed for $\lambda|\lambda \in R, -10 < \lambda < 10$ for each of the three systems.

System One:

$$\dot{x} = \lambda - x - \frac{1}{x^2}$$

Bifurcation seemingly occurs for $\lambda > 2$ as these are the states in the system that begin to alternate in stability and begin to converge towards a central attractor. The bifurcation diagram is represented by Figure 12. This type of bifurcation can almost be classified as transcritical, however it is missing key stable fixed points for negative λ values.

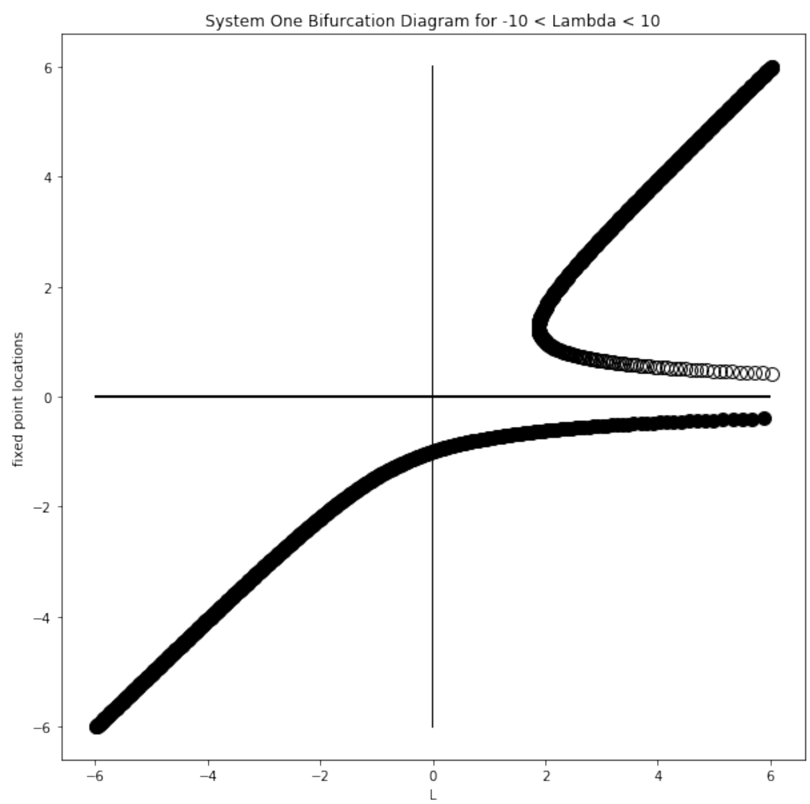


Figure 11: Fixed point locations for various values of λ in system one.

System Two:

$$\dot{x} = \frac{1}{e^{-\lambda x}} - x$$

Bifurcation does not seem to occur, although for $\lambda < 0$, states in the system that begin to are repelled towards stability and begin to converge towards a central attractor as λ increases. The bifurcation diagram is represented by Figure 12. This type of bifurcation cannot be classified as it is not truly bifurcating.

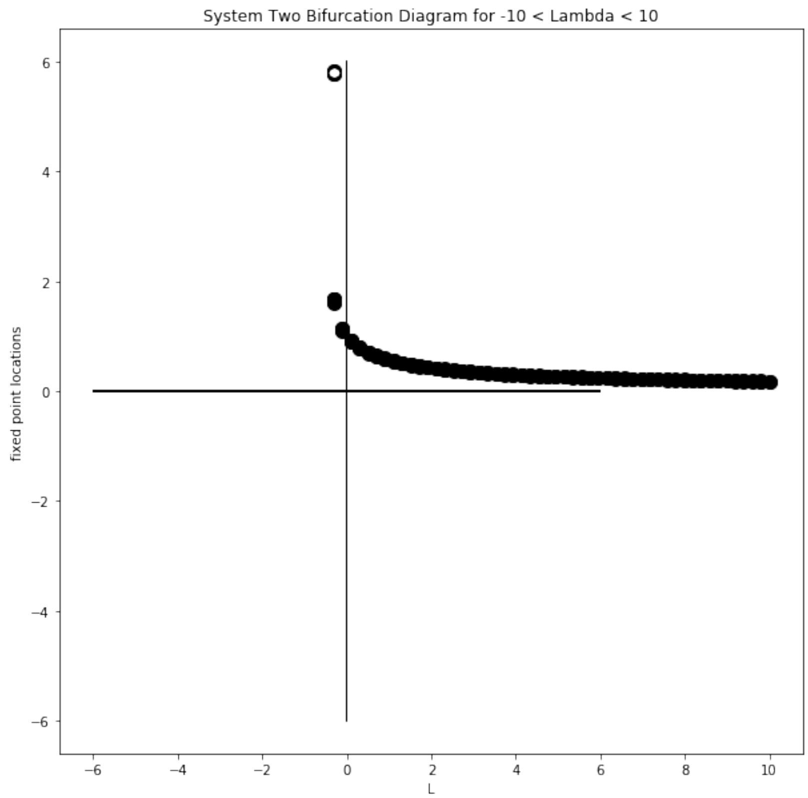


Figure 12: Fixed point locations for various values of λ in system two.

System Three:

$$\dot{x} = -\lambda \sin x - 2 \sin 2x$$

Bifurcation occurs most interestingly for $\lambda = \pm 3$. From these values, it appears the system has prefered states of attraction at locations -3, 0, and 3. The bifurcation diagram is represented by Figure 13. This type of bifurcation can be classified as subcritical, supercritical, then subcritical pitchfork bifurcation.

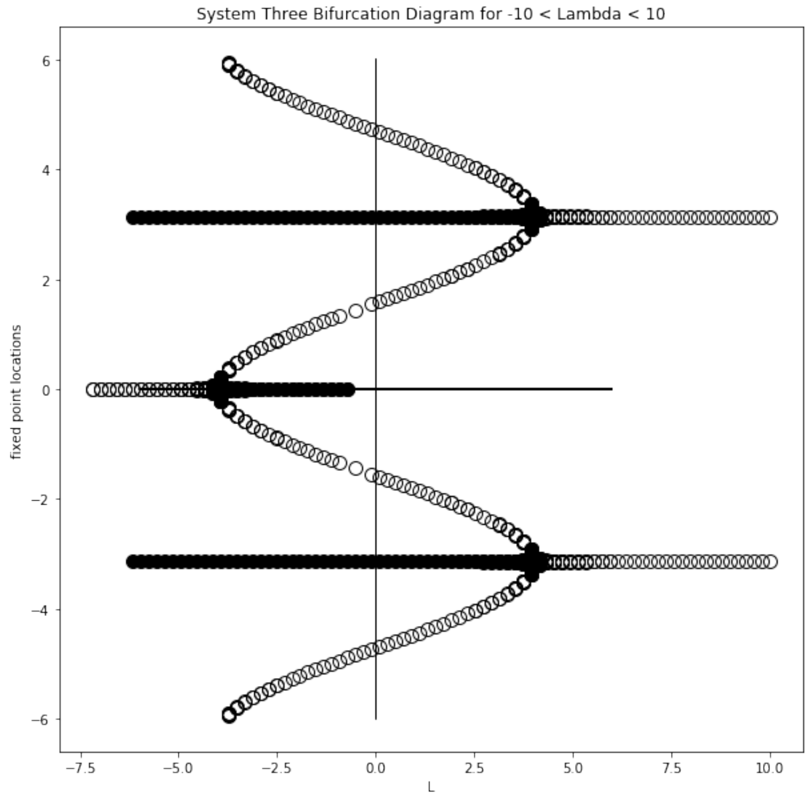


Figure 13: Fixed point locations for various values of λ in system three.

2 The Hodgkin–Huxley Equations

Read Reference: H-H Equations (Click me)
Translate HH python code to Insight Maker: Python H-H Equations (Click me)

2.1 Hodgkin-Huxley as Differential Equations

The Hodgkin-Huxley equation mathematically defines a neuronal cell spike. The equations can be modelled by coupling four differential equations for the change in membrane voltage (V(m)), electrochemical gradient of sodium (n), electrochemical gradient of potassium (k), and ion leakage (h). The change in V(m) is represented by a fourth order differential equation, while the other equations are first order and depend on two parameters each, alpha and beta respectively. Insight maker is a tool for simulating nonlinear dynamics and was used here to model the Hodgkin-Huxley equations (Click me). Figure 14 is the resultant Insight maker diagram.

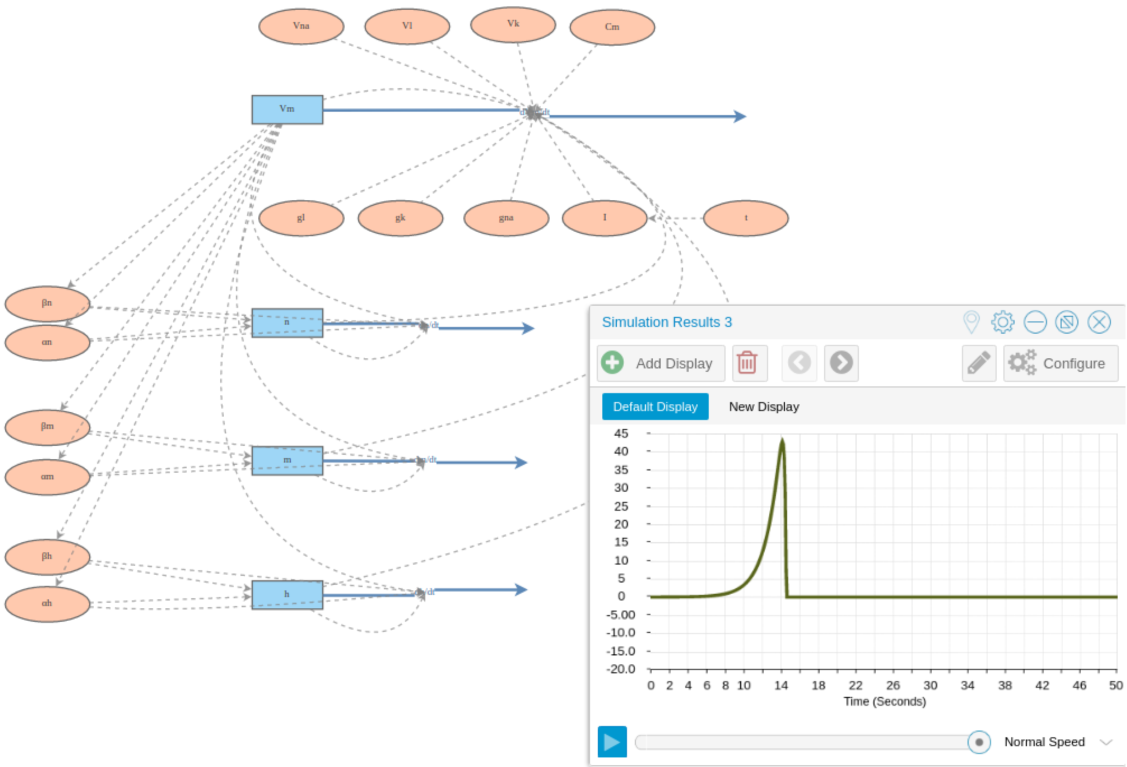
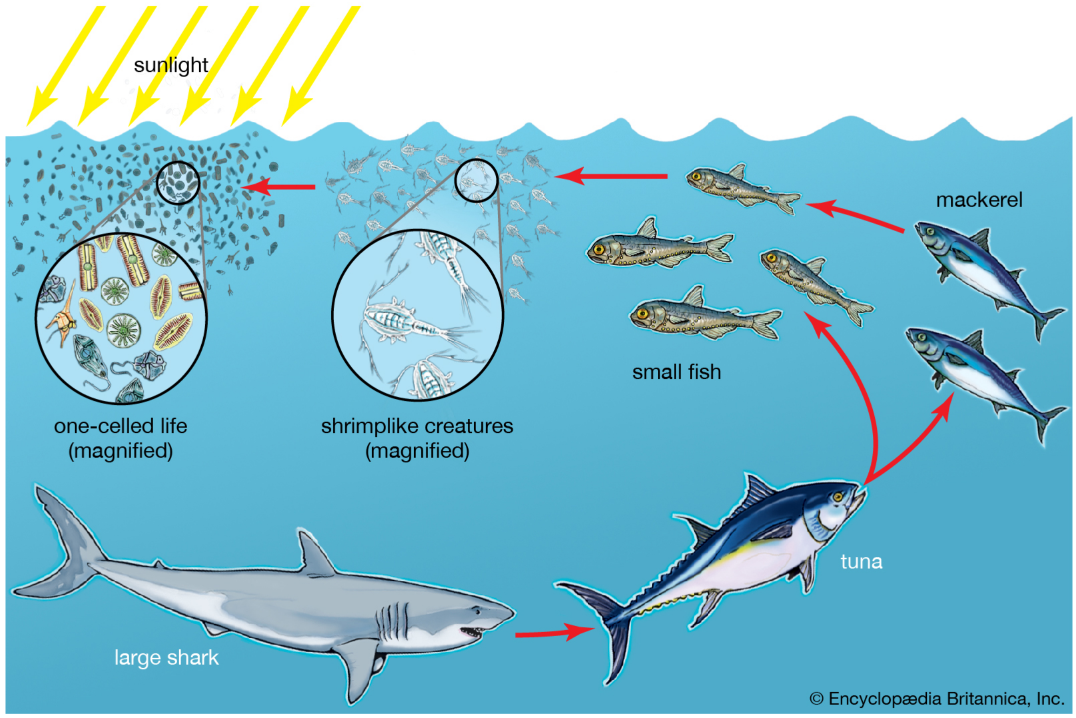


Figure 14: Hodgkin-Huxley equations modelled in Insight Maker nonlinear dynamics simulation environment.

3 Nonlinear Population Growth Models

Read Reference: Logistic Growth with Harvesting (Click me)
Model the food chain in the figure below using coupled differential equations using: (1) Insight Maker and (2) Python/Colab



3.1 Population Modeling

A single population can be modelled by the logistic growth function $\frac{dP}{dt} = rP(K-P)$, where r is the rate of growth, K is the ultimate carrying capacity, and P is the population at time t. When we included multiple species of the food chain, there will be predation and thus our equation will change to include a harvesting parameter, H. Since harvesting will depend on the predation, the functions need to be coupled. We can accomplish this my expressing the harvesting rate as $H = QEP$, were Q is the catchability quotient, E is the predation effort, and P is the population. The catchability quotient can be thought of as the probability a predator will catch the prey while the predation effort can be expressed as how many predators there are. Each species of tuna, mackerel, small fish, shrimplike creatures, and one-celled life can be modelled by the following system $\frac{dP}{dt} = rP(K-P) - QEP$, where E is expressed as the population of predators (sharks for tunas, tunas for mackerels, and so forth). In this system, sharks are the apex predators and thus their equation will remain $\frac{dP}{dt} = rP(K-P)$. As for the one-celled life, they harvest photons from sunlight which can be very loosely approximated as $\frac{dP}{dt} = sin(P) - QEP$. Thus the total population dynamics for the ecosystem is given by $\frac{dP}{dt} = rP(K-P) - H|P = \sum(P), H = \sum(H)$. Before we can integrate the system, we need to set the parameter variables according to the logistic growth model of crab harvesting given by [1]. By setting the parameters r, k, and P to $.0004, 30000$, and 10000, respectively, we can initialize the dynamical system. However, there needs to be many more photons to continuously sustain the population so can initialize $P_{photons}(0) = 10000000$.

We should expect the changes in population of each animal to oscillate overtime without being driven to extinction. However, for this system to reach a homeostatic state of equilibrium, the parameters and initial populations of each species must be particularly calculated such that $0 < H < P$. In the following simulations, a sustainable state could not be obtained. This failure is likely because the number of possible parameter combinations for a seventh order differential equation system is extensive. Further considerations should concern finding the initial parameter values such that a steady state of oscillation occurs.

See python code here (Click me).

See Insight Maker here (Click me).