ELEC 2100 Lecture Note

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1 Math Background

2 Signals & Systems

Loosely speaking, signals represent information or data about some phenomenon of interest. For the purposes of this course, a system is an abstract object that accepts input signals and produces output signals in response. From a mathematical perspective, signals can be regarded as functions of one or more independent variables (e.g video signals as a function of x-coordinate, y-coordinate and time).

2.1 Continuous-Time and Discrete-Time Signals

- A **continuous-time signal (CT signals)**, represented by $f(t)$ where t are all real numbers
- A **discrete-time signal (DT signals)**, represented by $f[n]$ where n is a discrete set of values (e.g.,$n \in \mathbb{Z}$) One way to obtain discrete-time signals is by **sampling** continuous-time signals (i.e., by selecting only the values of the continuous-time signal at certain time intervals).

2.2 Transformation of Signals

2.2.1 Time shift, reversal and scaling

Throughout the course, we will be interested in manipulating and transforming signals into other forms. Here, we start by considering some very simple transformations on independent time variable.

- **Time shift:** 2 signals are identical in shape, but that are displaced or shifted relative to each other. $x(t - t_0)$ is a delayed or shift right of $x(t)$ if t_0 is positive(same thing will happen after t_0).
- **Time reversal:** $x(-t)$ is obtained by reflecting $x(t)$ about $t = 0$.
- **Time scaling:** $x(2t)$ is 2 times "thinner" than $x(t)$.

The general form of transformation of the independent variable is $x(\alpha t + \beta)$, where α and β are real numbers. The principle is to do time shift first, and then the other 2 operations.[1]

[1] We can define $h(t) = f(t + \beta)$, and $g(t) = h(\alpha t)$. Thus, we have $g(t) = f(\alpha t + \beta)$ as required. If we applied the operations in the other order, we would first get the signal $h(t) = f(\alpha t)$, and then $g(t) = h(t + \beta) = f(\alpha(t + \beta)) = f(\alpha t + \alpha\beta)$. In other words, the shift would be by $\alpha\beta$ rather than β.

2.2.2 Periodic Signal

A periodic continuous time signal has the following property that there is a time T where

$$x(t+T) = x(t)$$

Similarly, a periodic discrete time signal will be unchanged in a period shift of N, where

$$x[n+N] = x[n]$$

One application of periodic is to express the signal as Poisson sum to construct periodic signals, where the periodic extension is defined as:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-kT)$$

2.2.3 Odd and Even

- **Odd signals:** One that is negated under time reversal.
- **Even signals:** One that is unchanged under time reversal.

An important property is that any signal can be broken into a sum of an even signal and an odd signal.

$$\textbf{even}(t) = \frac{(f(t)+f(-t))}{2}, \textbf{odd}(t) = \frac{(f(t)-f(-t))}{2}$$

2.3 Exponential and Sinusoidal Signals

These signals serve as basic building blocks from which many other signals are constructed.

2.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

The CT complex exponential signal is of the form

$$x(t) = Ce^{at}$$

where a and C are complex numbers. Depending on the values of parameters, the complex exponential can exhibit several different characteristics.

Real Exponential

When C is a real number. The natural response value of RC circuits are represented by certain decaying exponential. The time shift in real exponential is a multiplication of some constants.

Periodic Complex Exponential and Sinusoidal Signals

Periodic complex exponential can be obtained by constraining a to imaginary. Considering the base case where

$$x(t) = e^{j\omega_0 t}$$

This signal is periodic since

$$x(t+T) = e^{j\omega_0(t+T)} = e^{j\omega_0 t}e^{j\omega_0 T}$$

Obviously the signal is periodic when $j\omega_0 T = 0 \Rightarrow T_0 = \frac{2\pi}{|\omega_0|}$. Periodic complex exponential can be written in terms as sinusoidal signals with the same fundamental period by using Euler's relation:

$$e^{j\omega_0 t} = cos\omega_0 t + jsin\omega_0 t$$

Periodic complex exponential will play a central role in much of our treatment of signals and systems. For example, the sets of **harmonically related** complex exponential are sets of periodic exponential, all being periodic with period T_0, T_0 not necessary the fundamental period.

$$\phi_k(t) = e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2$$

for any $k \neq 0$, $\phi_k(t)$ is periodic with fundamental period $\frac{T_0}{k}$. The k-th harmonic is still periodic with period T_0 as it goes through exactly $|k|$ of its fundamental periods during any time interval of T_0.

General Complex Exponential Signals

The general case can be expressed in terms of real exponential and periodic complex exponential.

$$x(t) = Ce^{at} = |C|e^{j\theta}e^{(j\omega_0+r)t} = |C|e^{rt}cos(\omega_0 t + \theta) + j|C|e^{rt}sin(\omega_0 t + \theta)$$

2.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals

Similar to CT signal cases, the complex exponential signal in DT signal is defined by

$$x[n] = C\alpha^n$$

where C and α are complex numbers. This could expressed in the form similar to CT signals by replacing α by e^{β}.

Real Exponential Signals

The function is then similar to exponential function, with special notice to the case then $\alpha < 0$, $x[n]$ alternate between C and -C.

Sinusoidal Signals

When the β is purely imaginary so that $|a| = 1$, complex exponential is in its special case sinusoidal signal. Consider the base case:

$$x[n] = e^{j\omega_0 n}$$

With similar process in CT signals,

$$x[n] = cos\omega_0 n + jsin\omega_0 n$$

General Complex Exponential Signals

$$C\alpha^n = |C|e^{j\theta}|\alpha|e^{j\omega_0 n} = |C||\alpha|^n cos(\omega_0 n + \theta) + j|C||\alpha|^n sin(\omega_0 n + \theta)$$

2.3.3 Periodicity Properties of DT Complex Exponential

For CT signals, we can identify that for $e^{j\omega_0 t}$, it is periodic for any value of ω_0, and the larger the magnitude of ω_0, the higher the rate of oscillation in the signal. This is not the case in DT signal.

The 2 major differences are

1. periodic frequency with the change of ω_0.

2. non-periodic signal under some condition.

2.4 Unit Impulse and Unit Step function

2.4.1 DT Unit Impulse and Unit Step

Unit Impulse:

$$\delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$

Unit Step:

$$u[n] = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}$$

Unit step function can be generated by unit impulse function:

$$\delta[n] = u[n] - u[n-1]$$

Unit impulse function can also be generated by unit step function:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

or equivalently,

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

An interpretation of the last equation is as a superposition of delayed impulses, i.e., we can view the equation as the sum of a unit impulse $\delta[n]$ at $n = 0$, a unit impulse $\delta[n-1]$ at $n = 1$, ect.

The unit impulse sequence can be used as sampling. $\delta[n - n_0]$ can sample the values at $n = n_0$:

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$$

2.4.2 CT Unit Impulse and Unit Step

Unit Step:

$$u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases}$$

Also unit step:

$$u(t) = \int_{-\infty}^{t} \delta(t)dt$$

Unit impulse is the derivative of unit step at $t = 0$:

$$\delta(t) = \frac{du(t)}{dt} \text{where t at 0}$$

Since the unit step is discontinuous at $t = 0$, this derivative is an approximation of an infinite value at within a very narrow area near $t = 0$ where the area under the curve is 1.

Sampling property:

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$
$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = \int_{-\infty}^{\infty} x(t_0)\delta(t - t_0)dt = x(t_0)$$

2.5 CT and DT systems

Many different applications have very similar mathematical descriptions. Here are some examples of systems.

A continuous-time system is a system in which continuous-time input signals are applied and result in continuous-time output. Similarly, a discrete-time system is a system that transforms discrete-time inputs into discrete -time outputs.

2.5.1 Interconnection

Many complex real systems are build from interconnections of several simple subsystems. **Serial** interconnection, **parallel** interconnection, and **feedback** interconnection are commonly used interconnection methods.

2.5.2 Fundamental Properties

1. **memoryless:** A system is said to be memoryless if its output for each value at a given time is dependent only on the input at that same time.

   Here are some examples of systems with memory:

   - delay $y[n] = x[n - n_0]$
   - accumulator or summer $y[n] = \sum_{k=-\infty}^{n} x[k] = y[n-1] + x[n]$

   Here is a tricky example of a system without memory, i.e., memoryless: though include $t+1$, $cos(t+1)$ is not an input, but a value changing with time, which is known.

   $$y(t) = x(t)cos(t+1)$$

2. **causality:** A system is causal if the output at any time depends only on values of the input at the present time and in the past. Every memoryless system is causal.

3. **invertibility:** A system is said to be invertible if distinct inputs lead to distinct outputs. The function of the system should be monotone for the system to be invertible.

4. **stability:** A stable system is one in which small inputs lead to responses that do not diverge. In other words, a bounded input lead to bounded output.

5. **time invariant:** A system is time invariant if the behavior and characteristics of the system are fixed overtime. In other words, a system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.

   Here are some examples of time-variant system:

   - system with time-varying gain $y(t) = (t+1)x(t)$, note that this is a memoryless system.
   - imparted-compressed time system: $y(t) = x(2t)$

6. **linearity:** A linear system is a system that possesses the important property of superposition: if an input consists of the weighted sum of several signals, then the output is the superposition of the responses of the system to each of those signals. It should be additivity and homogeneity.

   One way to check is that 0 input will always have 0 output for linear system.

3 Linear Time-Invariant System: LTI system

3.1 DT LTI Systems: the Convolutional Sum

DT signal can be view as a sequence of impulses by the shifting property of unit impulse.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

For a linear system, not necessary time-invariant, we can denote the output through input $x[n]$ and the response $h_k[n]$ of the system to the shifted unit impulse $\delta[n-k]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

If the linear system is also time-invariant, then $h_k[n]$ is a time-shifted version of $h[n]$, i.e.,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The result is referred to as the convolution sum or superposition sum, denoted by $*$. So $y[n] = x[n] * h[n]$. The following Fig.1 shows the idea of convolution.

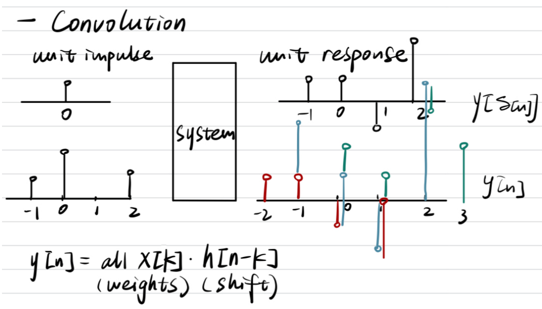


Figure 1: The Idea of Convolution

Here is another way to view and check the convolution: $g[k] = x[k]h[n-k]$ at time k which represent the contribution of $x[k]$ to the output at time n. In calculation, it's convenient to flip the convolutional kernel, which is $h[n]$, to $h[n-k]$.

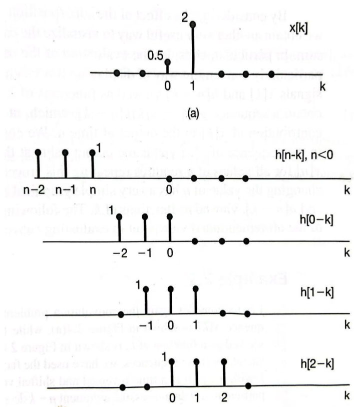


Figure 2: Convolution

3.2 CT LTI Systems: the Convolution Integral

Similarly, DT signals can also be viewed as integral of shifted unit impulse. To make analogy with DT signal, we use $\delta_\Delta(t)\Delta$, which is the square area of the narrow pulse with width Δ and total area 1. Approximate a CT signal $x(t)$ using shifted $\delta_\Delta(t)\Delta$ signals by **staircase approximation**.

$$x(t) = \lim_{\Delta\to 0}\sum x(k\Delta)\delta_\Delta(t-k\Delta)\Delta = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Similarly, for linear system, we have

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau$$

Moreover, for time-invariant system, $h_\tau(t)$ is exactly time shift of $h(t)$, which is $h(t-\tau)$.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

The procedure for evaluating the convolutional integral is similar to that for its DT counterpart. First obtain the signal $h(t-\tau)$, regarded as a function of τ with t fixed, from $h(\tau)$ by a reflection about the original and shift to left or right by t.

3.3 Properties of LTI system

The most important property of the previous representation is that the characteristics of an LTI system are completely determined by its impulse response. Take DT system as example.

1. commutative:
$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n] * x[n]$$
2. distributive:
$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$
this property tells that 2 systems in parallel is the same as adding them directly together first and then do convolution
3. associative:
$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * h_1[n] * h_2[n]$$
Impulse response of the cascade of 2 LTI system is the convolution of individual impulse responses, which means the order of cascade does not matter.
4. memoryless: memoryless only when $h[n] = K\delta[n]$.
5. causality: causal only when $h[n] = 0 for n < 0$
6. invertibility:
7. stability: stable only if impulse response is absolute integrable for CT or absolute summable for DT.

4 Fourier Series Representation of Periodic Signals

The starting point of discussion is the development of a representation of signals as linear combinations of a set of basic signals. Fourier series is another way to represent signals using **complex exponential**. The resulting representation are known as **CTFS**(Continuous Time Fourier Series) and **DTFS**(Discrete Time Fourier Series).

4.1 Response of LTI Systems to Complex Exponential

The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude, i.e., all complex exponential are **eigenfunctions** of LTI system.

$$e^{st} \to H(s)e^{st}$$
$$z^n \to H(z)z^n$$

where the complex **eigenvalue** $H(s)$ and $H(z)$ are **system functions**.

First show that complex exponential are indeed eigenfunctions of LTI systems.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{s(t-\tau)}h(t-\tau)d\tau \\ &= e^{st}\int_{-\infty}^{\infty} e^{s(-\tau)}h(t-\tau)d\tau \\ &= H(s)e^{st} \end{aligned}$$

For both CT and DT cases, if the input to an LTI system is represented as a linear combination of complex exponential, then the output can also be represented as a linear combination of the same complex exponential signals, each coefficient is obtained as a product of the corresponding coefficient of the input and the system's eigenvalue associated with the eigenfunction input.

As a special case of complex exponential, complex sinusoids are also eigenfunctions with physical meaning representing oscillations. In this case, $H(j\omega)$ and $H(e^{j\omega})$ are **frequency response**. The frequency response is the **CTFT** and **DTFT** of impulse response.

4.2 Fourier Series Representation of CT Periodic Signals

Fourier claimed that any T-periodic CT signal can be decomposed into, or synthesized as, a superposition of discrete set of T-periodic complex sinusoids with the **synthesis equation**:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where the complex sinusoid at frequency $k\omega_0$ is the **k-th harmonic** component. The $e^{jk\frac{2\pi}{T}t}$ is $\frac{T}{k}$ periodic and hence T periodic, the weighted (FS coefficients a_k) sum $x(t)$ are also T-periodic.

The k-th harmonic are determined as follows:

- the 0-th harmonic (the DC term) is the constant 1: $e^{j0\frac{2\pi}{T}t} = 1$. Its integration over duration T is T.
- all other harmonics $e^{jk\frac{2\pi}{T}t} = cos\left(k\frac{2\pi}{T}t\right) + jsin\left(k\frac{2\pi}{T}t\right)$ integrate to 0 over k period of duration $\frac{T}{k}$, which is in duration T.
- all harmonics have a constant power of 1.

Note that each k-th harmonic are orthogonal to each other because $< e^{jk\omega t}, e^{jm\omega t} >= \int_T e^{j(k-m)\omega t}dt = 0$ when $m \neq k$. This property is surprisingly useful, meaning that we can project any signal to its harmonic system.

FS coefficients a_k are determined by analysis equation(integration on arbitrary duration T).

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

which is a normalized inner product that computes a projection coefficient that specifies the component of $x(t)$ on $e^{-jk\omega_0 t}$. The following shows a proof to get analysis equation from synthesis equation.

$$\text{Multiplying both sides by } e^{-n\omega_0 t} : x(t)e^{-n\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{k\omega_0 t} e^{-n\omega_0 t}$$

$$\text{Integrating both sides from 0 to T: } \int_0^T x(t)e^{-n\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{(k-n)\omega_0 t}dt$$

$$\int_0^T x(t)e^{-n\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{(k-n)\omega_0 t}dt$$

Only when $k = n$ will the harmonic integrate to T, otherwise 0, so for each term a_k

$$\int_0^T x(t)e^{-k\omega_0 t}dt = Ta_k$$

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-k\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Some FS coefficients can be obtained by inspection.

$$sin\omega_0 t = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

This means that a_1 for $sin\omega_0 t$ is $\frac{1}{2j}$ and a_{-1} for $sin\omega_0 t$ is $-\frac{1}{2j}$, other FS coefficients are 0.

For other FS coefficients we must apply analysis equation. Here are some examples.

- Example 3.5 for periodic square wave. one period is defined as

$$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, T_1 < |t| < T/2 \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{-T/2}^{T/2} dt = \frac{2T1}{T} \\ a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\frac{-1}{jk\omega_0}e^{-jk\omega_0 t}|_{-T_1}^{T_1} \\ &= \frac{1}{jk\omega_0 T}(e^{jk\omega_0(T_1)} - e^{jk\omega_0(-T_1)}) = \frac{2}{k\omega_0 T}\frac{(e^{jk\omega_0(T_1)} - e^{jk\omega_0(-T_1)})}{2j} \\ &= \frac{2}{k\omega_0 T}sin(k\omega_0 T_1) = \frac{sin(k\omega_0 T_1)}{k\pi} \end{aligned}$$

- some observations:

  1. a_0 is the average value of $x(t)$ over one period.
  2. FS coefficients of the periodic square wave are the sampled values of a **sinc function**, i.e. in the form of $\frac{sin\pi x}{\pi x}$.

$$a_k = \frac{sin(k\omega_0 T_1)}{k\pi} = \frac{2T_1}{T}\frac{sin(k(2\pi/T)T_1)}{k\pi(2T_1/T)} = \alpha\frac{sin(\pi\alpha k)}{\pi\alpha k} = \alpha sinc(\alpha k)$$

where $\alpha = \frac{2T_1}{T} = a_0$ is the duty cycle of the wave.

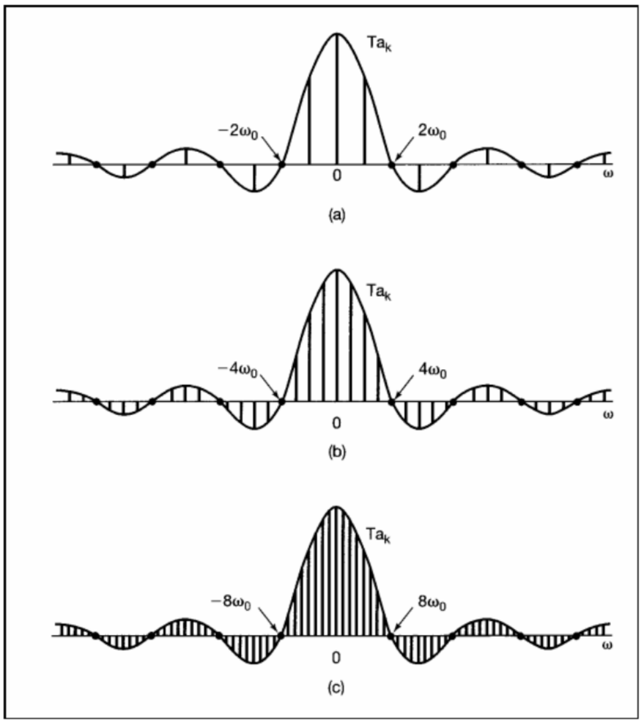


Figure 3: Sinc Function

4.3 Convergence of FS

The above example is not mathematically true, as we cannot produce a non-continuous function with a series of continuous function. However, it is valid that it can be approximate by the **truncated synthesis sum** as follows:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

and prove that for the above example, as N increase, $x(t)$ gets approximation closer. The overshoot, still remaining(Gibbs phenomenon), is very small as N goes larger, i.e., the FS is convergent. Generally, FS converges when the energy of error signal goes to 0.

4.4 Properties of CTFS

For a periodic signal $x(t)$ with period T, we denote the FS notation:

$$x(t) \leftrightarrow a_k$$

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting	$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Figure 4: CTFS Properties(Table 4.1)

4.4.1 Linearity

All FS decomposition and synthesis are linear.

$$z(t) = Ax(t) + By(t) \leftrightarrow c_k = Aa_k + Bb_k$$

4.4.2 Time Shifting

Time shift in time domain would lead to phase shift in frequency domain.

$$\begin{aligned} y(t) &= x(t - t_0) \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(t-t_0)} \\ &= \sum_{k=-\infty}^{\infty} e^{jk\omega_0(-t_0)} a_k e^{jk\omega_0 t} \\ b_k &= e^{jk\omega_0(-t_0)} a_k \\ x(t - t_0) \leftrightarrow b_k &= e^{jk\omega_0(-t_0)} a_k \end{aligned}$$

4.4.3 Time Reversal

Time reversal leads to frequency reversion.

$$y(t) = x(-t) \leftrightarrow b_k = a_{-k}$$

4.4.4 Time Scaling

Only change the fundamental frequency, a_k remains unchanged.

4.4.5 Multiplication

Multiplication in time domain leads to convolution of FS coefficients in frequency domain.

$$g(t) = x(t)y(t) \leftrightarrow c_k = \sum a_n b_{k-n}$$

$$\begin{aligned} g(t) &= x(t)y(t) = e^{jm\omega_0 t + jn\omega_0 t} \\ x(t) &= \sum a_k e^{jk\omega_0 t}, y(t) = \sum b_k e^{jk\omega_0 t} \\ g(t) &= \sum_n a_n e^{jn\omega_0 t} \sum_m b_m e^{jm\omega_0 t} = \sum_n \sum_m a_n b_m e^{j(m+n)\omega_0 t} \\ &= \sum_n \sum_k a_n b_{k-n} e^{jk\omega_0 t} = \sum_k \sum_n a_n b_{k-n} e^{jk\omega_0 t} = \sum_k e^{jk\omega_0 t}(a_k * b_k) \\ &= c_k \sum_k e^{jk\omega_0 t} \end{aligned}$$

4.4.6 Conjugation

Conjugation in time domain leads to conjugate in spectrum a_k and reverse in frequency. As a special case, for real exponential function.

4.4.7 Differentiation

Differentiation in time domain leads to multiplication by frequency with 90 degree phase shift.

4.4.8 Parseval's Relation

The average power in a periodic signal equals the sum of the power in all of its harmonies.
$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum |a_k|^2$

4.5 Fourier Series Representation of DT Periodic Signals

DTFS is to decompose an N-periodic DT signal into a weighted sum of the N distinct DT harmonic complex sinusoids with period N:

$$\phi_k[n] = e^{jk\frac{2\pi}{N}n}$$

where N is finite, i.e. the DTFS is N-periodic(Different from the infinite case in CT).

A DT signal, similarly to CT case, can be synthesized by N distinct DT harmonics:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk(\frac{2\pi}{N})n}$$

the analysis equation can be obtained in similar way in CT case:

$$a_k = \frac{1}{N}\sum_{k=0}^{N-1} e^{-jk(\frac{2\pi}{N})n}$$

In fact, due to the periodicity of DTFS coefficients, we can shift the summation window to sum over any N contiguous terms.

Similar to CTFS, some DTFS coefficients can be write directly while others should go through algebraic manipulation. Here is an example of rectangular DT signal.

4.6 Properties of DTFS

Most properties of DTFS are similar to those of CTFS, like linearity, time-shifting, time reversal, conjugate symmetry, etc.

There are some DTFS properties which manifest some differences because the signal is DT.

- **frequency shifting**: Multiply by the M-th harmonics lead to shift harmonics by M.
- **multiplication**: Multiply DT periodic signals lead to periodic convolution, i.e. convolution over the period of N, in DTFS coefficients.
- **periodic convolution**: The periodic convolution of 2 DT N-periodic signals, which is otherwise(for infinite convolution) meaningless, will lead to multiplication of DTFS coefficients.
- **first difference**: time shifting property

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Figure 5: DTFS Properties

4.7 LTI System as Filter

For LTI system, the output can be computed in 2 ways.

- time-domain: convolution
- frequency-domain: for all complex exponential, which are eigenfunction of LTI system, times frequency response(scale the input by a constant).
- **Conclusion:** LTI system is a **frequency-shaping filters** that scale the relative amplitude of different frequency components in signals.

Here are some idealized basic filter types:

1. **ideal low-pass filter(ILPF)**: frequency response $H(j\omega) = 1$ (pass) for low frequency components, and 0 (reject) for high frequency components.
2. **ideal high-pass filter(IHPF)**: frequency response $H(j\omega) = 1$ (pass) for high frequency components, and 0 (reject) for low frequency components.
3. **ideal band-pass filter**

5 Continuous-time Fourier Transform

Last chapter shows a representation of **periodic signals** as linear combination of complex exponential, more accurately, complex sinusoidal. The following part will extend these concepts to apply to signals that are not periodic by doing Fourier transforms.

5.1 Representation of aperiodic signals

One important idea is to see aperiodic signals as periodic in the limit of period T going to infinity. Consider a aperiodic signal with **finite duration(suppose first)** $x(t)$ with duration $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we can create periodic signals by repeating the signal on this interval.

$$\tilde{x}(t) = x(t - kT) \text{ if } kT - \frac{\pi}{2} \leq t < kT + \frac{\pi}{2}$$

In this case, $\tilde{x}(t)$ is periodic and thus we can apply CTFS to get a_k. Notice that within the special period of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tilde{x}(t) = x(t)$. So we have:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\frac{2\pi}{T}t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\frac{2\pi}{T}t}dt$$

Based on the finite duration consumption, $x(t)$ is 0 outside the first interval. So

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\frac{2\pi}{T}t}dt$$

Therefore **define** $X_T(j\omega)$ as the **inner product integral** of $x(t)$ with $e^{j\omega t}$.

$$X_T(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Then

$$a_k = \frac{1}{T}X_T(j\omega)$$

So the $x(t)$ within the first interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$ is :

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}X_T\left(j\frac{2\pi k}{T}\right)e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi}X_T(jk\omega_0)e^{jk\omega_0 t}$$

Now take limit of $T \to \infty$, then the Fourier transform integral is:

$$X(j\omega) = \lim_{T\to\infty} X_T(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

And the Interval Fourier Transform integral, obtained by the synthesis equation, is

$$x(t) = \lim_{T\to\infty}\sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi}X_T(jk\omega_0)e^{jk\omega_0 t} = \frac{1}{2\pi}\lim_{\omega_0\to 0}\sum_{k=-\infty}^{\infty} X_T(jk\omega_0)e^{jk\omega_0 t}\omega_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

5.2 CTFT Examples

CTFT for aperiodic signals:

- **One-sided causal exponential**. Signal $x(t) = e^{-at}u(t), a > 0$. Apply FT integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}|_0^\infty = \frac{1-0}{a+j\omega}$$

  when $Re\{a\} < 0$, $x(t)$ is unstable(goes to infinity) and its FT does not exist.

- **Unit impulse**. Signal $x(t) = \delta(t)$. Apply FT integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \delta(t)e^{-j\omega 0} = 1$$

- **Window signal**. Signal $x(t) = 1(t < |T_1|)$. Apply FT integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-T_1}^{T_1} e^{-j\omega t}dt = \frac{2sin\omega T_1}{\omega}$$

- **Window spectrum(frequency domain)**.Signal $X(j\omega) = 1(\omega < |W|)$. Apply IFT integral:

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{\sin(Wt)}{\pi t}$$

CTFT for periodic signals:

- **FT of complex sinusoid**. Consider a signal $x(t)$ with FT $X(j\omega) = 2\pi\delta(\omega - \omega_0)$. Apply IFT to obtain $x(t)$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

  That is to say, the FT of any complex sinusoid is a shifted impulse occurring at the harmonically related frequency, with area of $2\pi a_k$. Then any periodic signal can be view as a train of impulses.

- **Periodic impulse train**. Signal $x(t) = \sum_{k=-\infty}^{\infty}\delta(t - kT)$. Recall its FS coefficients:

$$a_k = \frac{1}{T}$$

  Then take linear combination

$$X(j\omega) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{T}\delta\left(\omega - \frac{k2\pi}{T}\right)$$

5.3 Properties of CTFT

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\}=\mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\}=-\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2d\omega$		

Figure 6: CTFT Properties

1. Linearity
2. Time shifting: time shift in time domain leads to phase shift in frequency domain.

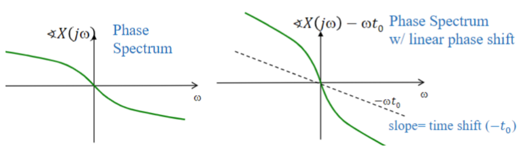


Figure 7: CTFT Shifting Property

3. Conjugation and conjugate symmetry: A conjugate of time signal leads to conjugation and frequency reversal of spectrum, which is

$$x^*(t) \Leftrightarrow X^*(-j\omega)$$

Similar to CTFS, if $x(t)$ is real, then $X(j\omega)$ has conjugate symmetry, that is:

$$X(j\omega) \Leftrightarrow x(t) = x^*(t) \Leftrightarrow X^*(-j\omega)$$

For real valued signal, we can conclude that:

In rectangular form: $\Re\{X(j\omega)\} = \Re\{X(-j\omega)\}$, and $\Im\{X(j\omega)\} = -\Im\{-j\omega\}$. That means the real part of FT is an even function of frequency, while the imaginary part is an odd function of frequency.

In polar form: $X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$, meaning its magnitude is even and phase is odd.

4. Differentiation and Integration
5. Time and frequency scaling
6. **Duality**: as mentioned previously, the FT of window function is sinc function, while the FT of sinc function is window function. Generally, for any transform pair, there is a dual pair with the time and frequency variables interchanged. Consider the frequency shifting and time shifting example.
7. **Multiplication and convolution**: super important!!!

$$y(t) = h(t) * x(t) \Leftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

6 DTFT

Apply the similar technique in CTFT to obtain the synthesis equation and analysis equation of DTFT:

$$x[n] = \frac{1}{2\pi \int} \int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t),$ $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Figure 8: CTFT Pairs

7 Time and Frequency Characterization

This chapter introduces the relationship of time-domain and frequency domain characteristics and trade-offs in system design and analysis.

7.1 Magnitude Response and Phase Response

Recall the magnitude and phase response mentioned in the filter part. Magnitude response decide whether a frequency component will be passed or eliminated(in wireless communication, 1kHz is important information to be transmitted), while phase response will add an additional delay to each frequency components. In general, **changes in the phase function of X(jw) leads to changes in the time-domain characteristics of the signal x(t).**

For LTI system where $Y(j\omega) = H(j\omega)X(j\omega)$, the system scales the magnitude and shift the phase in time domain. Therefore, for $X(j\omega) = |X(j\omega)|\angle X(j\omega)$, $|X(j\omega)|$ is considered as **gain** and $\angle X(j\omega)$ is considered as **phase shift**. A phase response can be linear or non-linear.

The following linear phase system is a time delay system. Linear Phase response will not distort the input signal, i.e. will not change its shape.

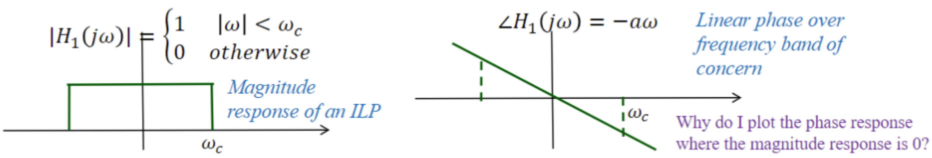


Figure 9: Linear Phase System

The following is a non-linear phase system. For input signal $e^{j\omega_1 t}$, the output signal is $e^{j\omega_1 t + \phi(\omega_1)} = e^{j\omega_1 (t - \frac{-\phi(\omega_1)}{\omega_1})}$, i.e, a signal with time delay by $\frac{-\phi(\omega_1)}{\omega_1}$. It means that each frequency component will have different time delay and thus result in a different signal shape(due to changes in relative phase) as the original one.

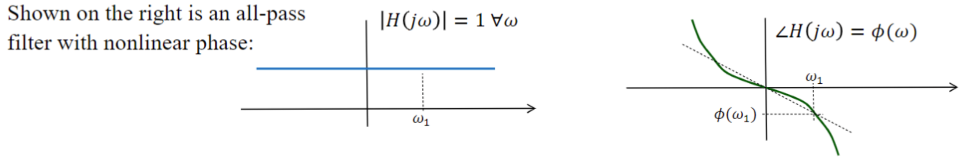


Figure 10: Non-linear Phase System

7.2 Signal Wave Burst

As discussed, systems with linear phase characteristics have the particularly simple interpretation as time shifts where the **phase slope tells the size of time shift**. That is, if $\angle H(j\omega) = -\omega t_0$ then the system imparts a time shift of $-t_0$(delay off t_0).

The concept of delay can be extended to nonlinear phase characteristics. For a narrow band signal $s(t)$, its FT is negligibly small outside a small band of frequencies centered at $\omega = \omega_0$

The basic wave burst is mathematically the product of a slow varying envelop signal $s(t)$ and a .

Group delay refers to the delay that a burst of oscillating signal at a given frequency ω_1 will suffer through an LTI system. For example, if the input signal is $x(t) = s(t)cos\omega_1 t$,

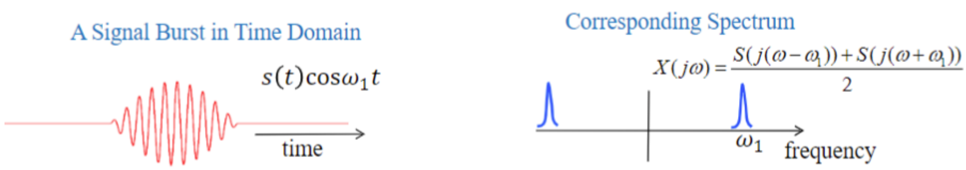


Figure 11: Burst

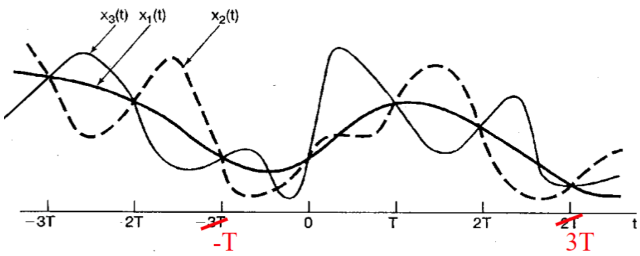
8 Sampling and Interpolation

Much of the importance of the sampling theorem lies in its role as a bridge between CT and DT signals. The fact that under certain conditions a CT signal can be completely recovered from a sequence of its samples provides a mechanism for representing a CT signal by a DT signal. The general process to deal with CT signals in real world by DT system(which is easier in designing) is:

1. Exploit sampling to convert a CT signal to a DT signal
2. Process the DT signal(which is easier for digital processing)
3. Interpolate the CT signal from DT signal

8.1 Sampling Theorem

In general, in the absence of any additional conditions or information, we would not expect that a signal could be uniquely specified by a sequence of equally spaced samples. For the following 3 CT signals, they can have the same sample values.



The **sampling theorem** states a condition that if a CT signal is smooth enough and the sampling interval T is small enough, then in theory the CT signal can be uniquely recovered from its sample.

- $x(t)$ is **band-limited** if its FT $X(j\omega) = 0 \ \forall \omega \ s.t. |\omega| > \omega_M$ and its bandwidth is ω_M
- A high enough sampling rate $\omega_s > 2\omega_M$ can reconstruct the signal(perfect reconstruction).
- **Nyquist's Rate** $= 2f_M$ which is the minimum sampling rate.

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T sin(\omega_c(t - nT))}{\pi(t - nT)}$$

The following is an explanation and proof of the sampling theorem.

One sampling method is **impulse train sampling** where the sampled signal $x_p(t) = x(t)p(t)$. The periodic impulse train $p(t)$ is referred to as the sampling function and its period T is the sampling period, while its fundamental frequency $\omega_s = \frac{2\pi}{T}$ is the sampling frequency.

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

$$X(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{T} X(j(\omega - k\omega_s))$$

Since $x(t)$ is band-limited, that means $X(j\omega)$ has limited bandwidth. Multiplication(sampling) in time domain leads to convolution in frequency domain, and the FT of impulse train is also an impulse train. Then the convolution result is also a Poisson sum of the single $X(j\omega)$.

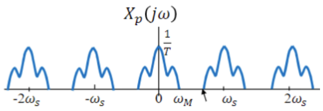


Figure 12: Non-overlapping

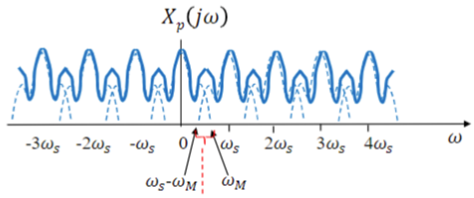


Figure 13: Aliasing

It is obvious that if there is no overlapping in frequency domain, we can apply low-pass filter, we can reconstruct the original signal. That is also where the sinc function in the formula comes from(IFT of windows function in time domain).

8.2 Sampling by Gating

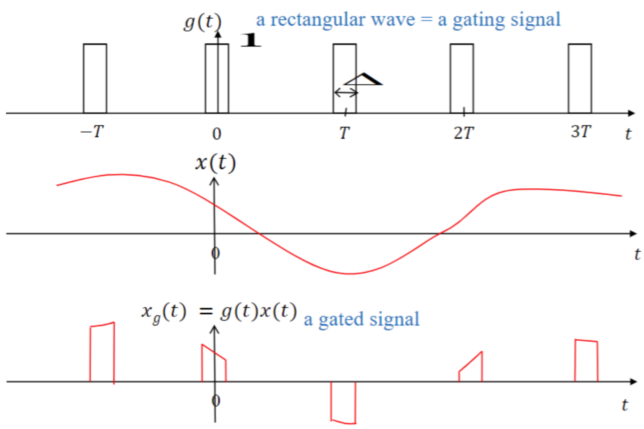


Figure 14: Sampling by Gating

8.3 DT Processing of CT Signals

For the condition that the sampling frequency is above the Nyquist rate, the sampled sequence $x_d[n]$ is the same as $x_c(t)$ in terms of spectrum.

The spectrum of $x_d[n]$ is given by its DTFT:

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n}$$

Recall that

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT}$$

We can relate them by a scaling in frequency:

$$x_d(e^{j\Omega}) = X_p(j\frac{\Omega}{T})$$

To sum up,

$\omega_M < \frac{\pi}{T}$ so that there is no aliasing, then:

Eq.7.21 of preceding slide

$$X_c(j\omega) = \begin{cases} TX_p(j\omega) = TX_d\left(e^{j\omega T}\right) & -\frac{\pi}{T} \le \omega \le \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases}$$

Figure 15: Relationship between Original Signal, Sampled Signal and DT signal

9 Modern Topics

9.1 Spectrum Analyzer

Spectrum analyzers (SA) is a piece of equipment for analyzing the spectrum of signals. The SA will analyze one **chunk** of signal $x_W(t)$ in duration W at a time and will compute the spectrum(FT or FS, in fact, FS is the sampled values of FT, and in finite duration case, both can synthesize $x_W(t)$) of a finite duration chunk.

It is obvious that with higher duration W, we can get the lower frequency components. To get higher resolutions, the duration of chunk must be longer.

Instead of doing integration, the SA, as a digital device, samples the integrand at interval Δ, and performs a numerical integration.

9.2 FFT

A faster computation method to compute DFT, which is similar to DTFS.

9.3 I/Q Channel

In radio/optical communication, we can actually use two carrier waves at the same frequency but 90-degrees out of phase (cosine and sine waves) to transmit two information signals at the same time. By convention, the carrier lagging in phase by 90-degree is called the I (In-Phase) channel, and the carrier leading in phase by 90-degree is called the Q (Quadrature Phase) channel.

10 Differential Equations as LTI Systems

As explained in previous parts, LCCDE(Linear Constant-Coefficient Differential Equation, equates a weighted sum of the 0 to N-th derivatives of the output to a weighted sum of the 0 to M-th derivatives of the input), is an LTI system.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

10.1 Analysis of LCCDE

The frequency response of an LCCDE as an LTI system can be determined by taking FT of both sides of the equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \Rightarrow \sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

The frequency response $H(j\omega)$ is obtained by $\frac{Y(j\omega)}{X(j\omega)}$.

$$H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} = \frac{Numerator(j\omega)}{Denominator(j\omega)}$$

Therefore we obtained the frequency response of **rational form**.

From impulse response, we can obtain the frequency response by IFT. Here is an example:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$
$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{(j\omega)+2}{(j\omega)^2 + 4(j\omega) + 3}$$
$$\text{do partial fraction expansion and obtain: } H(j\omega) = \frac{1/2}{(j\omega+1)} + \frac{1/2}{(j\omega+3)}$$
$$\text{Corresponding impulse response is } e^{-at}u(t) \leftrightarrow \frac{1}{(j\omega+a)}$$
$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

From the following example, we can summarize the common steps of partial fraction expansion, since derivative forms will be transformed into a fraction form of polynomials. One trick that could follow is, for

$$H(j\omega) = \frac{N(j\omega)}{(j\omega-\alpha_1)(j\omega-\alpha_2)(j\omega-\alpha_3)} = \frac{c_1}{(j\omega-\alpha_1)} + \frac{c_2}{(j\omega-\alpha_2)} + \frac{c_3}{(j\omega-\alpha_3)}$$

To find the numerator constant c, called the **residue**, we can do the following calculation:

$$c_1 = H(j\omega)(j\omega-\alpha_1)|_{j\omega=\alpha_1}$$

10.2 Rational System

There are 3 equivalent ways of specifying the differential equation as LTI system:

1. By Differential Equation
2. By Frequency Response(3 standard forms)
   - Polynomial Form of fractions
   - Factored Form, easier to plot magnitude and phase
   - Partial Fraction Form: easier for IFT
3. By Impulse Response

For such system that

$$h(t) = \sum_{k=1}^{N} c_k e^{a_k t} u(t)$$

with residues c_k and roots a_k. For a real system, h(t) is real, then for any complex root a_k, there must be another root $a_{k'} = a_k^*$(because the calculation will form conjugate pair) and $c_{k'} = c^*$(calculated as result) to eliminate the complex part, which is

$$c_k e^{\alpha_k t} + c_k^* e^{\alpha_k^* t} = 2Re\{c_k e^{\alpha_k t}\} = 2|c_k| e^{Re\{a_k\}t} \cos(Im\{a_k\}t + \angle c_k)$$

This expression is a damped oscillation(if there are conjugate pairs of a_k), with frequency defined by imaginary part by a_k, and magnitude is damped by the real part of a_k.

The root $\alpha_k s$ are either real valued(correspond to real first order system), or conjugate pairs(correspond to real second order systems). So an n-th order real system can always be formulated by first order and second order system.

10.3 First Order System

An RC circuit gives a first order system. It act as a Non-ideal Low Pass Filter.

10.4 Second Order System

Consider a second order LCCDE:

$$y^{(2)}(t) + a_1 y^{(1)(t)} + a_0 y(t) = b_1 x^{(1)} + b_0 x(t)$$

Obtain the rational frequency response:

$$H(j\omega) = \frac{b_1(j\omega) + b_0}{(j\omega)^2 + a_1(j\omega) + a_0}$$

Write it in partial fraction form

$$H(j\omega) = \frac{c_1}{(j\omega - \alpha_1)} + \frac{c_1}{(j\omega - \alpha_2)}$$

and we can get the impulse response by IFT.

A second order system have the same oscillation performance as any rational system. It is stable if and only if $h(t)$ is integrable, i.e., α_1, α_2 are both negative in real parts. This means that a_0, a_1 must be **positive**.

Now we can define the following 2 parameters:

1. **Natural frequency**: $\omega_n = \sqrt{a_0}$ provides a scaling in frequency
2. **seta parameter**: $\zeta = \frac{a_1}{2\omega_n}$ shows if the roots are complex

$\alpha_1, \alpha_2 = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$

This lead to some conclusions:

1. If $\zeta \leq 0$: $a_1 \leq 0$, the system is unstable.
2. If $|\zeta| < 1$, α are complex conjugate pair and the system is **oscillatory**.
3. If $\zeta \gg 1$, then impulse response decays slowly, since one α is slightly less than 0. The system is **over-damped**.
4. If $\zeta \simeq 1$, both roots are near $-\omega_n$, the impulse response decays at the fastest rate possible. The system is **critically damped**.
5. If $\zeta \to 0^+$, the system is **under-damped** where $|H(j\omega)|$ is very large around ω_n.

11 Laplace Transform

Recall that the Laplace Transform of the impulse response $H(s)$ is the **system function**, which provides the eigenvalues of complex exponential signals e^{st}.

Generally, for a signal $x(t)$, its LT is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Fourier Transform is a cross section of Laplace Transforms where s are limited to purely imaginary number. While in Laplace Transform, we extend our attention to the entire s plane.

Let $s = \sigma + j\omega$, then its LT is:

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt = FT\{x(t)e^{-\sigma t}\}$$

11.1 LT Example for Causal and Anti-causal Exponential System

Here are some examples of LT.

1. Determine the LT of $x(t) = e^{-at}u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-(a+\sigma)t}e^{-j\omega t}u(t)dt$$

   which is the FT of $e^{-(a+\sigma)t}u(t)$

   According to the table, $FT\{e^{-(a+\sigma)t}u(t)\} = \frac{1}{j\omega+(a+\sigma)}$ for $Re\{a+\sigma\} > 0$

11.2 LT in Rational Forms

In general form, all LTs are in rational form

$$X(s) = \frac{N(s)}{D(s)}$$

- roots s where $D(s) = 0$ are poles.
- roots s where $N(s) = 0$ are zeros.

11.3 ROC

The ROC(Region Of Convergence) enables us to convey information concerning causality, by right-sidedness or left sidedness.

For general LTs, ROC is an area bounded by vertical lines parallel to the $j\omega$ axis on the s-plane where its strips are either:

1. the entire s plane if $x(t)$ is absolutely integrable and is of finite duration, or
2. the right-half plane if $x(t)$ is right-sided, or
3. the left-half plane if $x(t)$ is left-sided, or
4. a strip in the complex plane s that includes the line $Re(s) = \sigma_0$ if $x(t)$ is 2-sided.

For rational LT, its ROC is bounded by poles or extends to infinity. No poles are contained in the ROC(since at poles $X(s)$ always goes to infinity and is not absolute integrable). Its ROC also follow the above principles.

11.4 Inverse LT by Partial Fraction