PHYS 1114 Notes

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1 Electric Field and Electric Forces

1.1 Electric field in different model

1. **Electric field generated by point charge**
   By Coulomb's Rule, the electric field of a point charge is calculated as:

$$F = \frac{1}{4\pi\epsilon_0} \times \frac{|q_1 q_2|}{r^2}$$

2. **Electric field of a charged line segment** A line segment $2a$ with charge Q distributed uniformly throughout. We can divide it into an infinite number of small segment dy, each like a point charge with **linear charge density** being $\lambda = \frac{Q}{2a}$.

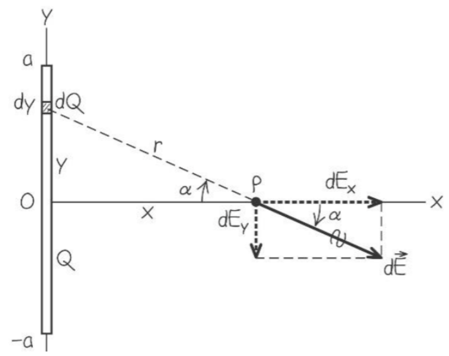
   for a small segment point at (0,y), $dE = k\frac{dq}{r^3}\vec{r} = k\frac{\lambda dy}{\sqrt{x^2+y^2}^3}(x,-y)$

   take the integration $E = \int dE = \int_{-a}^{a} k\frac{\lambda dy}{\sqrt{x^2+y^2}^3}(x,-y)$

   for y component $E_y = \int_{-a}^{a} k\frac{\lambda dy}{\sqrt{x^2+y^2}^3}(-y)\hat{y}$ which is an odd function and therefore 0.

   for x component $E_x = \int_{-a}^{a} k\frac{\lambda dy}{\sqrt{x^2+y^2}^3}x\hat{x} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x\sqrt{x^2+a^2}}$

   according to the conclusion, for infinite long stick $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{xa} = \frac{\lambda}{2\pi\epsilon_0 x}$



3. **Electric field of a ring** A ring with charge Q distributed uniformly around it. We can divide it into

infinie number of small segments ds, with each like a point charge and linear density being $\lambda = \frac{Q}{2\pi a}$.

$$\text{for a point charge } dE = k\frac{\lambda ds}{(x^2+a^2)}$$
$$\text{for y components } dE_y = dEsin\alpha\text{symmetric property cancel out all forces}$$
$$\text{for x components } dE_x = dEcos\alpha = k\frac{\lambda ds}{(x^2+a^2)}\frac{x}{\sqrt{x^2+a^2}}$$
$$\text{take the integration of x } E = \int dE_x = \int k\frac{\lambda ds}{(x^2+a^2)}\frac{x}{\sqrt{x^2+a^2}}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{\lambda x}{\sqrt{x^2+a^2}^3}\int ds = \frac{1}{4\pi\epsilon_0}\frac{\lambda x}{\sqrt{x^2+a^2}^3}2\pi a = \frac{\lambda ax}{2\epsilon_0\sqrt{x^2+a^2}^3}$$

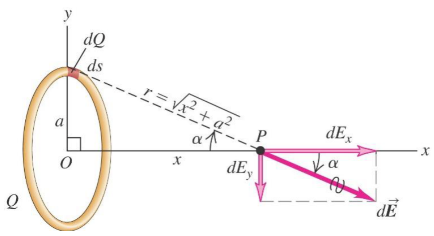


Figure 1: Electric Filed of a Ring

4. **Electric field of a disk** By the combination law $\vec{E} = \sum \vec{E}_n$, where all $\vec{E}_n$ always point to the right which makes integration easier.

$$\text{for a small segment of ring in the disk } \vec{E}_r = \hat{x}\int dE = \hat{x}\int \frac{1}{4\pi\epsilon_0}\frac{dQx}{\sqrt{x^2+r^2}^3} \text{ where } \sigma = \frac{Q}{\pi R^2}$$
$$= \hat{x}\int \frac{1}{4\pi\epsilon_0}\frac{\sigma 2\pi rx}{\sqrt{x^2+r^2}^3}dr = \hat{x}\frac{Qx}{2\pi R^2\epsilon_0}\int_0^R \frac{r}{\sqrt{x^2+r^2}^3}dr = \frac{\sigma}{2\epsilon_0}[1 - \frac{1}{\sqrt{\frac{R^2}{x^2}+1}}]$$

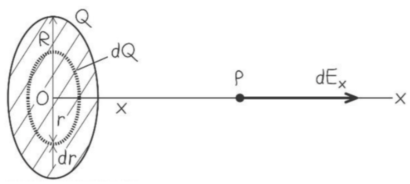


Figure 2: Electric Field of Disk

5. **Electric field of infinite sheet**
$$E = 2\frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

6. **Electric field of dipole** An electric dipole is a pair of equal and opposite charges +q and –q at a fixed distance d apart.
$$E = \frac{1}{2\pi\epsilon_0}\frac{p}{y^3}$$
Its electric dipole moment $\vec{p}$ is defined as $p = qd$ and points from –q to +q.

2 Gauss's Law

2.1 Electric flux

The **Electric flux** is defined to quantify the amount of electric field flowing in and out.

$$\Phi = \vec{E} \cdot \vec{A}$$

where the vector area A's direction is defined by its outward normal unit vector.

For electric flux through a sphere, since the surface is not flat, we would break up surface into infinitesimal flat patches $d\vec{A}$. We can then take integral of many patches of this kind.

$$\Phi E = \sum \vec{E} d\vec{A} = \oint \vec{E} d\vec{A}$$

We assume that the charge is at the center of the sphere (**important assumption!!!**), and can conclude that E and dA are in the same direction, we can simplify the formula into

$$\oint E dA = \oint k \frac{Q}{r^2} dA = \frac{kQ}{r^2} \oint dA = \frac{1}{4\pi\epsilon_0} \frac{Q 4\pi r^2}{r^2} = \frac{Q}{\epsilon_0}$$

In advanced courses, we can prove that even if the charge is not at the center of the sphere, we can still come to the same conclusion, which is the famous Gauss Law.

2.2 Gaussian surface and Gauss law

Gauss Law: the total electric flux through a **closed surface**, which is **Gaussian surface**, is equal to the net electric charge inside the surface divided by ϵ_0.

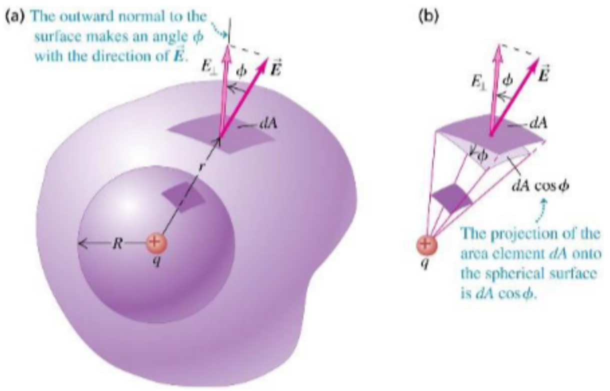


Figure 3: Gauss Law

There are two applications of Gauss Law:

- Finding the total charge inside a black box
- Calculating the electric field from the known charge inside

2.3 Distribution of excess charge in a conductor and Electrostatic Shield

Electrostatic condition means that there is no net flow of charge, which is current, inside a conductor. Under this condition, we know that the **electric field inside this conductor must be 0** (otherwise the charge would flow). We also know that the **electric field in the surface of the conductor must be perpendicular to the surface** due to the normal force on the surface (which is actually electric interaction). According to Gauss Law, we can connect the electric field to the distribution of charges, which means there are also **NO net charges** inside any electrostatic Gaussian surface.

The consequence of Gauss's law is that, under the electrostatic condition, all the charges inside the conductor would move to its surface. And by Gauss Law, we can calculate that the electric field on the surface should be

$$E = \frac{\sum Q}{\epsilon_0}$$

Figure 4: Hollow Conductor Under Electrostatic Condition

One important application of electrostatic condition of conductor is the Faraday's cage from his icepail experiment. Faraday's cage is a hollow metal box that shield its hollow inside from the external electric field. However, it cannot do the other way around since it will convert the charges inside onto its surface, as shown in Fig.5.

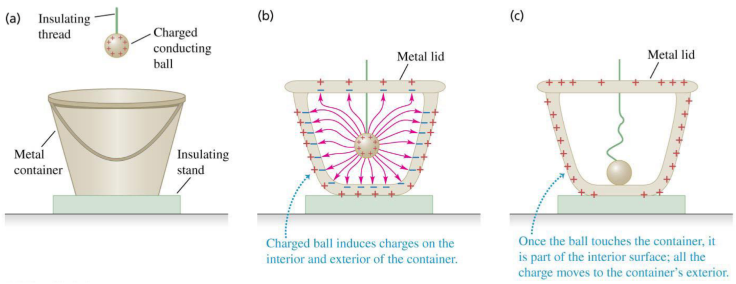


Figure 5: Faraday's Cage

2.4 Calculation of Electric Field

The other application of Gauss Law is to calculate the electric field through known charge distribution. It is much more difficult than calculate the charges from the given electric field, because we can easily do integration of electric field to get flux, but we cannot tell the applied area of electric field and thus is harder to directly know the electric field. However, we can consider the special case when the charge distributions are **symmetric**[1]. The trick here is to **choose a Gaussian surface** with the same symmetry as the charge distribution.[2] Under this special case, the Gauss Law can be rewrite as

$$\frac{\sum Q}{\epsilon_0} = \oint \vec{E} d\vec{A} = \oint E dA = E \oint dA = EA$$

Here we will focus on three models and see how electric field distribution under these conditions:

1. **Field of uniform distributed sphere:** Consider the two cases of calculation of electric field

$$\text{for electric field outside the charged sphere: } E = \frac{Q}{4\epsilon_0 \pi r^2}$$

$$\text{for electric field inside the charges sphere: } E = \frac{\rho dV}{4\epsilon_0 \pi r^2} = \frac{Qr}{4\epsilon_0 \pi R^3}$$

[1] Rotation symmetry and Flipping symmetry

[2] The reason why symmetry is required is because must make sure E and dA are on the same direction, and also make sure that E is uniformly distributed on dA to move the E out of integration

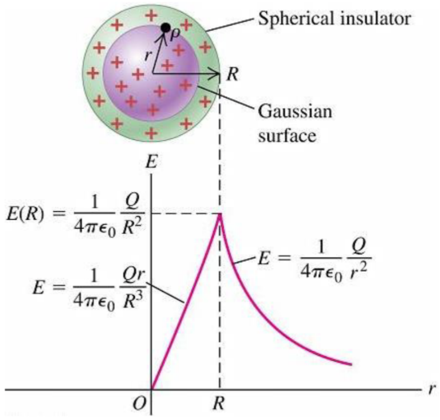


Figure 6: Electric Field of Symmetric Sphere

2. **Field of a long[3] uniform line charge:** By symmetry we can tell that $\vec{E}$ is cylindrical symmetry through the line, perpendicular to the wire and radially outward. We can make a **cylinder shaped** Gaussian surface and deal with integration, where

$$E = \frac{Q}{\epsilon_0 A} = \frac{Q/l}{2\pi R\epsilon_0} = \frac{\lambda}{2\pi R\epsilon_0}$$

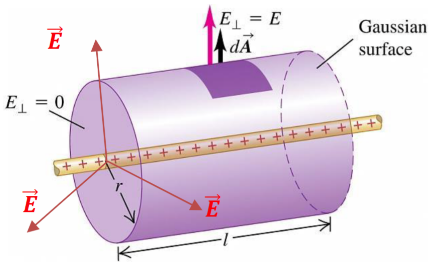


Figure 7: Electric Field of a long uniform line charge

3. **Field of an infinite sheet of charge:** By symmetry $\vec{E}$ must be perpendicular to the plain on both sides, so we can choose a cylinder (or a cube or random volume) as the Gaussian surface.Only the surface's flux need to be calculated, and they are of the same direction of the electric field.

$$2E(\text{both surface, with side's flux being } 0) = \frac{Q}{\epsilon_0 A} = \frac{Q/A}{\epsilon_0} = \frac{\sigma}{\epsilon_0} \text{ and thus } E = \frac{\sigma}{2\epsilon_0}$$

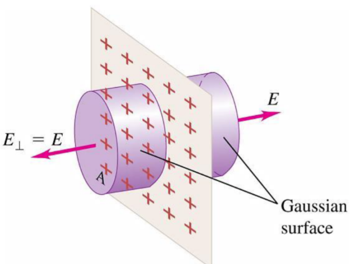


Figure 8: Electric Field of NonConducting Infinite Sheet

[3] Translational symmetry, considered as infinitely long

4. **Field between oppositely charged parallel conducting plates:** In this situation, the electric field inside the conductor are always 0 due to electrostatic condition, meaning that the flux is also 0, while the other four sides are perpendicular to the electric field. The difference between this situation and the sheet problem is that the charge are not uniformly distributed on 2 sides, with 1 side full of charge and other side no charge. To calculate $\oint \vec{E} d\vec{A}$, we only need to calculate the flux on one surface. Thus we can have

$$E = \frac{Q/A}{\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

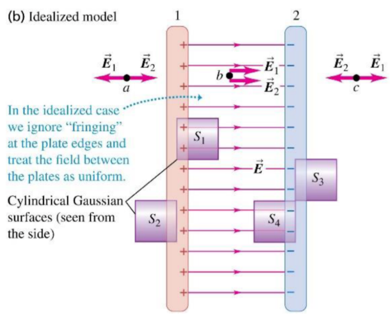


Figure 9: Electric Field between oppositely charged parallel conducting plates

5. **Field at the surface of a conductor of arbitrary shape:** At the close surface of the conductor, we can approximately assume that the surface is a plate and the part inside the conductor has no charge. We can borrow the conclusion from last one and get $E = \frac{\sigma}{\epsilon_0}$.

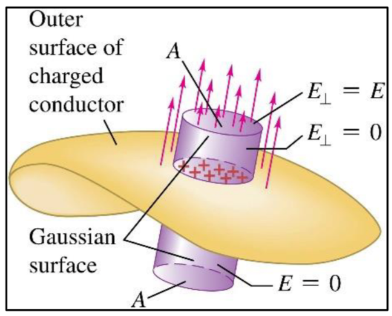
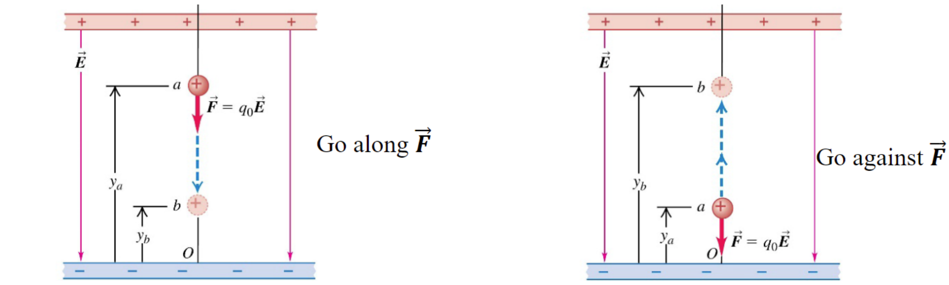


Figure 10: Field at the surface of a conductor of arbitrary shape

3 Electric Potential

3.1 Electric Force as Conservative Force and Electric Potential Energy

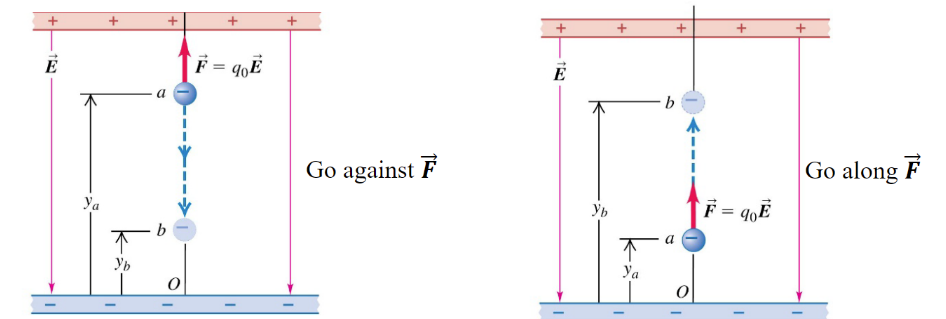
As learned in 1112, the gravitation is a conservative field, for any conservative field there are potential energy inside. By analogy, **electric field is also conservative**, which means path independent in the field. The electric field situation is a bit complicated than the gravitational field, due to the positive and negative charge of the object. For positive charge:



$W = \vec{F}\cdot\vec{s} > 0 \implies \Delta U = -W < 0, \therefore U \downarrow$ $\quad$ $W = \vec{F}\cdot\vec{s} < 0 \implies \Delta U = -W > 0, \therefore U \uparrow$

Figure 11: Positive

and for negative charge:



$W = \vec{F}\cdot\vec{s} < 0 \implies \Delta U = -W > 0, \therefore U \uparrow$ $\quad$ $W = \vec{F}\cdot\vec{s} > 0 \implies \Delta U = -W < 0, \therefore U \downarrow$

Figure 12: Negative

The thing that is same for both situation is that

$$W = \vec{F}\vec{s}$$
$$W = -\Delta U$$

In the real world, the electric field is not always ideally distributed. Here we will go through some situations and abstract them into different physical models.

1. **Electric potential of Point Charge:** In a field set up by a fixed charge q, test charge q_0 moves from a to b alone the electric field line.

$$W_{a\to b} = \int \vec{F}d\vec{s} = \int Fds = \int_a^b \frac{kqq_0}{r^2}dr = kqq_0\int_a^b \frac{1}{r^2}dr = kqq_0(-\frac{1}{r})|_a^b = \frac{q_q 0}{4\pi\epsilon_0}(\frac{1}{r_a} - \frac{1}{r_b})$$

For the general case where the test charge q_0 move randomly, the statement is also true because with dot product, any movement at any position can be mapped onto the radiation line.

since the total work done $W_{a\to b} = -\Delta U = (U_a - U_b)$, we can define the potential energy be

$$U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}$$

What we need to note is that the electric potential does not belong to only the source point charge. It actually belong to both the source and the test point charge q_0.

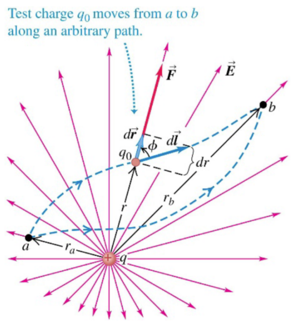


Figure 13: Electric Force

2. **Electric potential for pairs:** Similar to gravitational energy, the electric energy can also be calculated as a system, using the **conservation of electric potential energy and mechanical energy, with the conservation of momentum**.

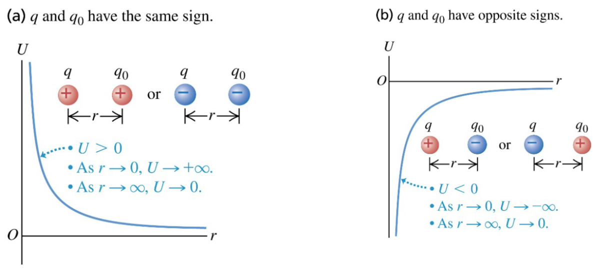


Figure 14: Electric Potential for pairs

3. **multi point charge problem:** If the field on q_0 due to several charges, each produces a field of $\vec{E}_i$. U can be viewed as the workdone by the total electric field to move q_0 from that position to ∞, which is actually $\int \sum_i \vec{E}_i q_0 ds = \sum \int E_i q_0 ds = \sum U_{i\to q_0}$. Based on this idea, we can calculate the total electric energy needed to construct a multi-point charges system. To add one more charge into a n element system, the potential energy is:

$$U_{n+1} = kq_{n+1}\sum_{i=0}^{n}\frac{q_i}{r_{i\to n+1}}$$

For the total amount of energy,

$$U = \sum U_i = k\sum_{i=0}^{n}\sum_{i<j}\frac{q_i q_j}{r_{ij}}$$

3.2 Electric Potential

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

which is similar to the idea of **electric field**, with a unit J/C which is Volta (V). Potential itself does not matter, what is important is the potential between the point charges.

$$V_{ab} = -\left(\frac{U_b}{q_0} - \frac{U_a}{q_0}\right) = \frac{W_{a\to b}}{q_0}$$

meaning the work needed to move 1 C of charge from b to a against the electric force. And from previous formula, we know the work can be calculated by electric field, we can connect electric field with the electric potential. On the other way,

$$V_{ab} = \int_a^\infty \vec{E}d\vec{s} = \int_a^\infty E ds = \int_a^\infty \frac{Q}{4\pi\epsilon_0 r^2}dr = \frac{1}{4\pi\epsilon_0 r_a}$$

Application of $\vec{E}d\vec{s}$:

1. **Corona Discharge** happen if E is too large, air molecules will be ionized, leading to an electric breakdown.
2. **Oppositely charged parallel plates:**

$$V_y - V_b = \int_y^b \vec{E}d\vec{s} = Ey$$

3. **Infinite Charged Conducting Cylinder:**

   Inside cylinder: V constant and equals to the value on the surface.

   Outside cylinder: by Gauss Law, $E = \frac{\lambda}{2\pi\epsilon_0 r}$, then for the calculation of potential,

$$V(r) = \int_r^\infty \vec{E}d\vec{l} = \int_r^\infty \frac{\lambda}{2\pi\epsilon_0 r}dr = \infty$$

   This is actually a wrong calculation because the log function. That is to say the infinite faraway is not a good choice as a reference. For this question, we can choose the conductor as potential 0.

$$V(r) = \int_r^R \vec{E}d\vec{l} = \int_r^\infty \frac{\lambda}{2\pi\epsilon_0 r}dr = \frac{\lambda}{2\pi\epsilon_0} ln\frac{R}{r} < 0$$

4. **A Ring of Charge:** in some problems we does not necessary need to calculate potential from electric field. We can calculate from the charge.

$$V = k\int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0 r}\int dq = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2+a^2}}$$

   Note that in this case the distribution of charge on the ring does not necessarily have to be uniform.

3.3 Equipotential Surface

Potential is a scalar field, visualized as equipotential surfaces (on which every point has the same potential). The direction where the electric field line points to is the direction where the electric potential goes downward while perpendicular to the equipotential surface.

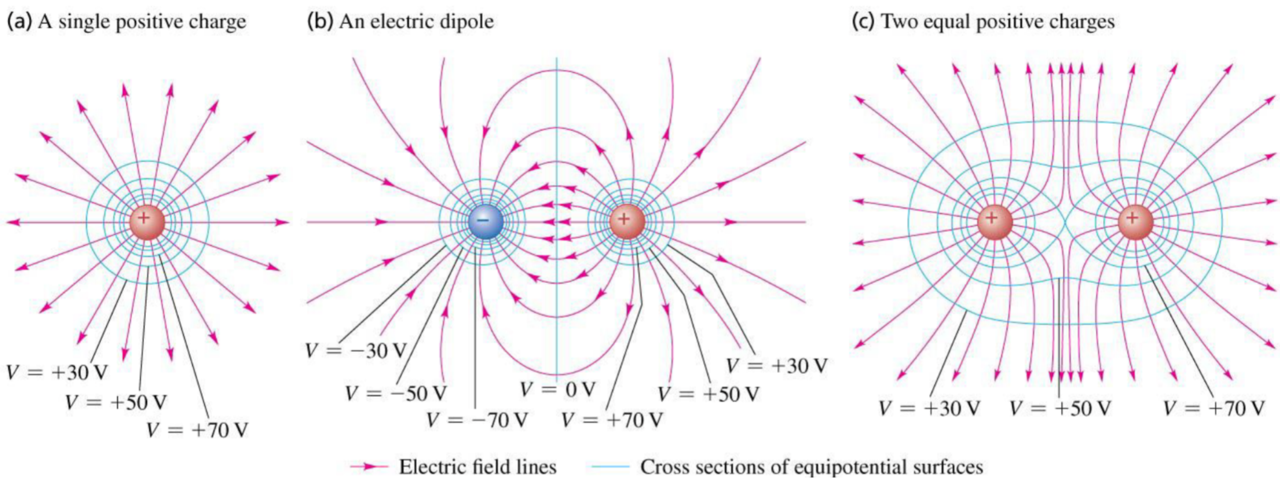


Figure 15: Equal Potential Surface

Consider the conductor in electrostatic situation again. Inside the conductor, the potential between any 2 points should be the same for the zero electric field inside, which means they are all **equipotential**. While its surface is also a equipotential surface, and there would be a potential difference between the conductor from inside to surface.

3.4 Gradient Operation

There is another variation of the famous formula$V_a - V_b = \int_a^b \vec{E} d\vec{l}$ by taking differentiation on both sides.

$$V_a - V_b = \int_a^b \vec{E} d\vec{l} = \int_b^a -\vec{E} d\vec{l}$$

$$dV = -\vec{E} d\vec{l} = -(E_x dx + E_y dy + E_z dz)$$

do partial differentiation: suppose dy=dz=0: $dV = \frac{\partial V}{\partial x} dx = -E_x dx$

$$E_x = -\frac{\partial V}{\partial x}, \text{ likewise, } E_y = -\frac{\partial V}{\partial y}, E_z = -\frac{\partial V}{\partial z}$$

$$\vec{E} = -(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}) = -\nabla V$$

Here is an example of how this method is powerful. Recall the electric field of a ring that was mentioned before.

Example 23.14 P. 797

We previous found that for P along the axis,

$$V = \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{x^2 + a^2}}$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{1}{4\pi\epsilon_0} \frac{Qx}{(x^2 + a^2)^{3/2}}$$

⚠ We have argued in Example 21.9, Lecture 2 notes P. 3, that $\vec{\boldsymbol{E}} = E_x \hat{\boldsymbol{\imath}}$, i.e., $E_y = E_z = 0$

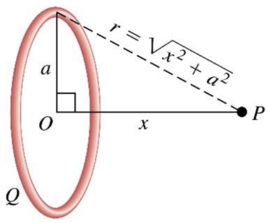


Figure 16: A Revisit to Ring Problem

4 Capacitance and Dielectrics

4.1 Capacitor

A **capacitor** is a device to store charge safely. Quantities to describe the ability of a capacitor:

- The **charge** stored in a capacitor: amount of $\pm Q$ contains in two conductors insulated in between.
- The potential difference, or **voltage**: the potential difference between 2 conductors.
- **Capacitance**: $C \equiv \frac{Q}{V_{ab}}$ depends only on the geometry of the conductors, not on charge or voltage.

1. **Parallel-Plate Capacitor:**

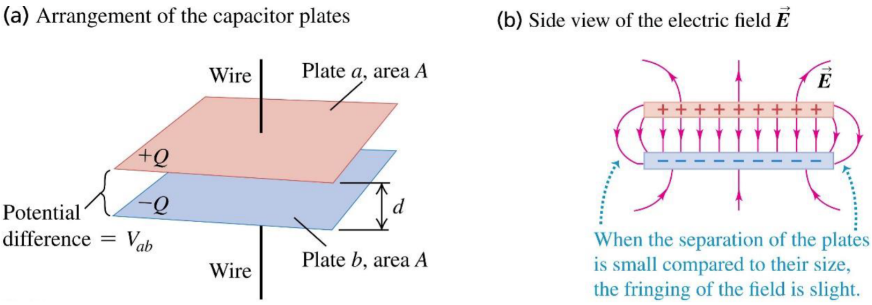


Figure 17: Parallel Plate Capacitor

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0}$$
$$V_{ab} = V_a - V_b = \int_a^b \vec{E} d\vec{l} = \int_a^b \frac{Q/A}{\epsilon_0} dL = \frac{Q/A}{\epsilon_0} d$$
$$C = \frac{Q}{V_{ab}} = \frac{A\epsilon_0}{d}$$

2. **Spherical Capacitor:**

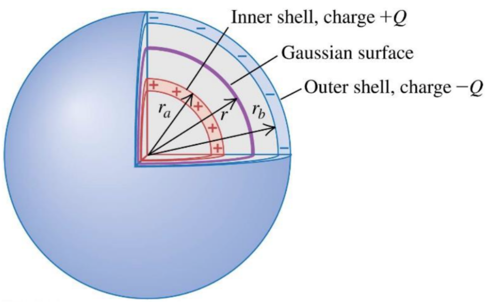


Figure 18: Spherical Capacitor

$$\oint \vec{E} d\vec{A} = \frac{Q}{\epsilon_0} \Rightarrow E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$
$$V_{ab} = V_a - V_b = \int_a^b E dr = \frac{Q}{4\pi\epsilon_0} \frac{(r_b - r_a)}{r_a r_b}$$
$$C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a} = \frac{\epsilon_0 \sqrt{A_a A_b}}{d}$$

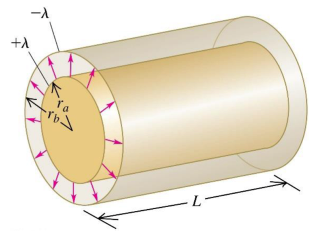


Figure 19: Cylindrical Capacitor

3. **Cylindrical Capacitor:**

$$\oint \vec{E}d\vec{A} = E2\pi rL = \frac{Q}{\epsilon_0} = \frac{\lambda L}{\epsilon_0} \Rightarrow E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$$
$$V_{ab} = \int_{r_a}^{r_b} Edr = \frac{\lambda}{2\pi\epsilon_0} ln\frac{r_b}{r_a}$$
$$C = 2\pi\epsilon_0 \frac{L}{ln(r_b/r_a)}$$

4.2 Capacitors in Series and Parallel

- **In Series:** Act as parallel resistor.
- **In Parallel:** Act as series resistor

4.3 Energy Stored in Capacitors

The energy stored in a capacitor is defined by the total work done to build charge difference between two conductors. We can calculate it by (consider a being the negative side and b being the positive side)

$$\Delta U = -W$$
$$\text{for dq: } dW = \int FdL = \int dq\vec{E}d\vec{l} = dq\int_a^b FdL = dq(v_a - V_b)$$
$$dW = dq(-\frac{q}{C})$$
$$\Delta U = -W = -\int dW = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} = \frac{1}{2}QV$$

There is another way to view this intuitively, which is from the view of **electric field energy**.

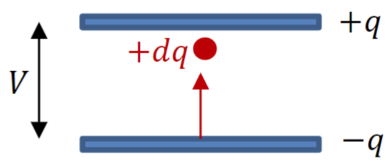


Figure 20: Energy in Capacitor

4.4 Effect of Dielectric

A **dielectric** is usually an insulator to prevent shot. One reason for creating dielectric is to prevent energy loss due to a current produced by high voltage. Another important reason for dielectric is that dielectric will increase the total capacitance.

It was discovered that with Q remain constant, voltage V decreases upon insertion of dielectric. The **dielectric constant** $K = \frac{C}{C_0} = \frac{V_0}{V} > 1$ where C_0 refer to the capacitance before the insertion of dielectric.

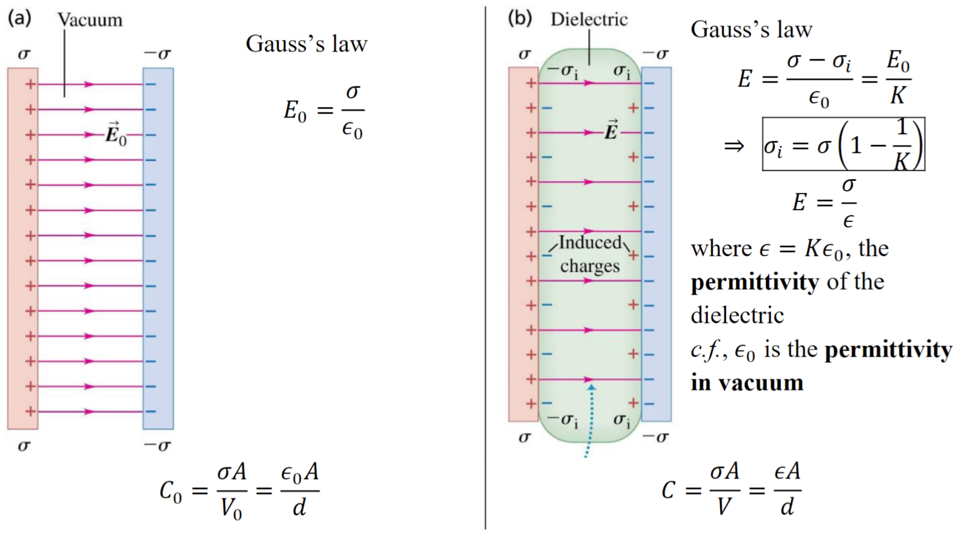


Figure 21: Calculations in Dielectric

How does energy changes when plugging in a dielectric into a capacitor? There are 2 different situations where the capacitor is connected to the power supply or not, i.e. they maintain the same voltage or charge.

- **Connected to the power supply:** voltage/ potential difference remain the same, $E = \frac{1}{2}CV^2$ increases.
- **Not connected to the power supply:** charge remains the same, $E = \frac{Q^2}{2C}$ decreases. (No external energies, internal energy are partially consumed by polarize the plugged in dielectric)

Another aspect to understand the dielectric is by considering their different type of charges with conductor. Conductors include **free charges** where charges can move freely, while dielectric contains **bounded charges** cannot move and only a limited electric field can be created. This provide a new perspective of the permittivity with application of Gauss's Law.

5 Circuit: Current, Resistance and EMF

5.1 Current

A **current** is a net drift of charge from one place to another, which means that it is not electrostatic condition. In a conductor, without external electric field, electrons move randomly without net drift. With external electric field, electrons move with a certain drift. In quantitative version, current is the charge flow through a cross section of the conductor per unit time.

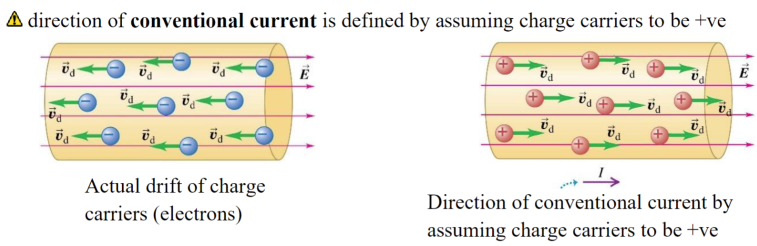


Figure 22: Current

$$I = \frac{dQ}{dt}$$
$$dQ = qdN = q(nAv_d dt)$$
$$I = \frac{dQ}{dt} = v_d qnA = venA$$
$$\text{current density } J = \frac{I}{A} = nqv_d$$
$$v_d = \frac{I}{nqA}$$

An interesting question here is that the speed is proportional to electric field, which is not the same case as electric static condition. This is because there are more than 1 electrons where their interaction will slow the **drift velocity**.

5.2 Ohm's Law

The initial form of Ohm's Law is $\vec{J} \propto \vec{E}$, which, surprisingly, can be derived into our familiar formula.

$$\vec{J} \propto \vec{E}$$
$$A\vec{J} = \frac{A\vec{E}}{\rho}$$
$$I = \frac{A}{\rho}\frac{V}{L}$$
$$I = \frac{V}{\rho L/A}$$
$$I = \frac{V}{R}$$

For a small enough temperature change, ρ can be approximated by a linear relation.

$$\rho(T) = \rho_0(1 + \alpha(T - T0))$$

Ohm's law is not a general law for all materials. For semiconductors and superconductors, they do NOT follow Ohm's law.

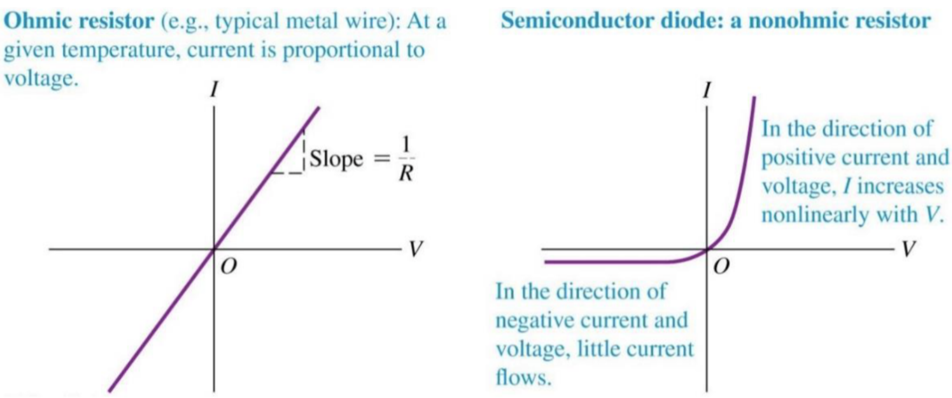


Figure 23: Ohmic Resistor

The electric field set up in an isolated conductor cannot drive a steady current(this is because $E = J\rho$, and if electric field is 0 due to counter electric field generated from inside, then the total current is 0). To maintain a **steady current**, we need a complete loop and a power supply to provide **electromotive force**, or **EMF** ε that bumps positive charge from lower to higher potential against electric field. Total energy consumed should be $W = \Delta V q = \varepsilon q$.

Ideally, $q\varepsilon = qV_{ab}$, while in reality, the device has internal resistance r, which means $V_{ab} = \varepsilon - Ir$.

5.3 Electric Power in Circuit

Electric power of a circuit can be easily calculated by $P = dW/dt = \Delta V q/dt = VI$.

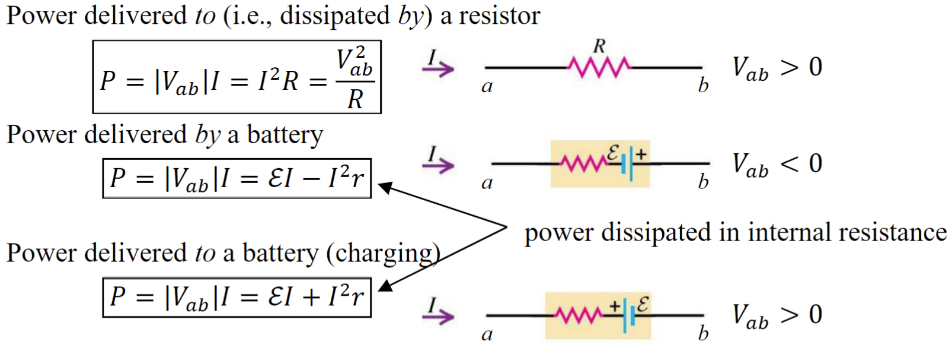


Figure 24: 3 Power Model

5.4 Complex Direct Current Circuit

For simple circuits, we can analyze its performance by parallel and series. For complex circuit, **Kirchhoff's rule** is more applicable. Kirchhoff's rule can be derived from the idea of conservation of **potential** and **charge**.

1. Kirchhoff junction rule: incoming current = outgoing current
2. Kirchhoff loop rule: total electric potential change in a close loop is 0.

One special situation is charging a capacitor.

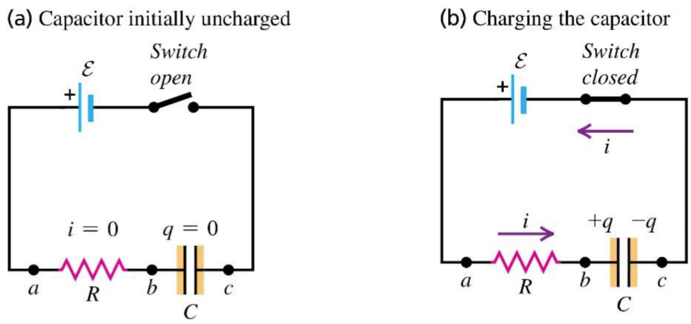


Figure 25: Capacitor in Circuit

When the switch is closed, the emf will continuously charge the capacitor until they reach the same potential difference, i.e. fully charged(ideally, it takes infinite times). The energy deposited to a capacitor is **non dissipative** and can be recovered when it discharges. On the other hand, the energy consumed by resistors are **dissipative**.

$$\varepsilon - iR - \frac{q}{C} = 0;\ \text{hence, } \frac{dq}{dt} = i \Rightarrow \varepsilon - \frac{dq}{dt}R - \frac{q}{C} = 0$$
$$dt = \frac{R}{\varepsilon - \frac{q}{C}}dq \Rightarrow \int dt = \int \frac{R}{\varepsilon - \frac{q}{C}}dq$$
$$t + \text{const} = -RC\ln(\varepsilon C - q)$$
$$\Rightarrow \varepsilon C - q = e^{-\frac{t}{RC}+\text{const}};\ \text{where the constant, when } t = 0 \text{ and } q = 0, \text{ is } -\ln \varepsilon C$$
$$\Rightarrow \varepsilon C - q = \varepsilon e^{-\frac{t}{RC}} \Rightarrow q = \varepsilon C\left(1 - e^{-\frac{t}{RC}}\right)$$

$$\Longrightarrow \boxed{q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right) = q(\infty)\left(1 - e^{-t/RC}\right)} \quad \text{and} \quad \boxed{i(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC} = i(0)e^{-t/RC}}$$

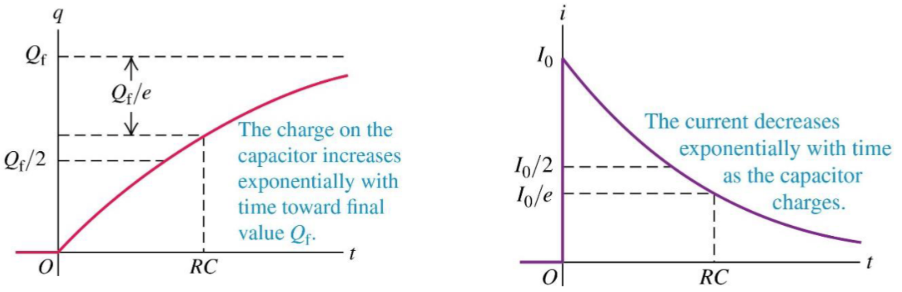


Figure 26: RC Circuit

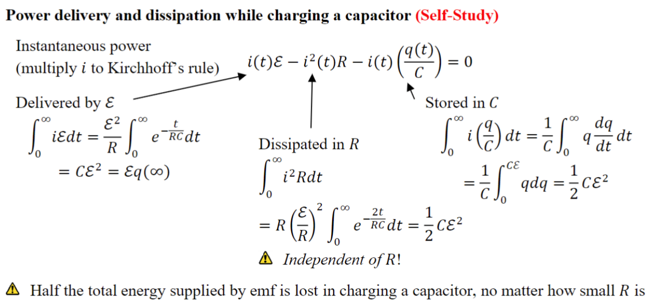


Figure 27: Energy Consumption

The above shows the charging procedure.
Another situation is discharging the capacitor.

$$\Rightarrow \boxed{q(t) = Q_0 e^{-t/RC}} \quad \text{and} \quad \boxed{i(t) = -\frac{Q_0}{RC}e^{-t/RC} = i(0)e^{-t/RC}}$$

⚠ *Decreasing in magnitude*

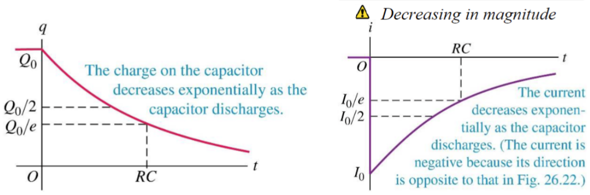


Figure 28: Discharging

6 Magnetic Field and Magnetic Forces

Define the **magnetic field** $\vec{B}$ through the magnetic force acting on a particle. In order to experience a magnetic force, a particle must be **charged** and **moving**.

$$\vec{F} = q\vec{v} \times \vec{B}$$

only the perpendicular part of velocity, with respect to the magnetic field, contribute to magnetic force.

6.1 Magnetic flux

Magnetic Field lines always form a loop, which is different from electric field line. Define **magnetic flux** in exactly the same way as electric flux. Since magnetic field always in a loop, any closed surface will always get same amount in, same amount out.

$$\Phi_B = \int \vec{B} d\vec{A} = 0$$

Since the magnetic field never do works on charge, it is meaningless to discuss about potentials and energy. Magnetic field and magnetic force only provide the force to change the direction of velocity(to make circular motion), not the magnitude of velocity.

$$F = |q|vB = \frac{mv^2}{R}$$
$$\text{cyclotron frequency: } \omega = \frac{v}{R} = \frac{|q|B}{m}$$

If the velocity is not strictly perpendicular to $\vec{B}$, it will trace out a helical path.
For non-uniform magnetic field case:

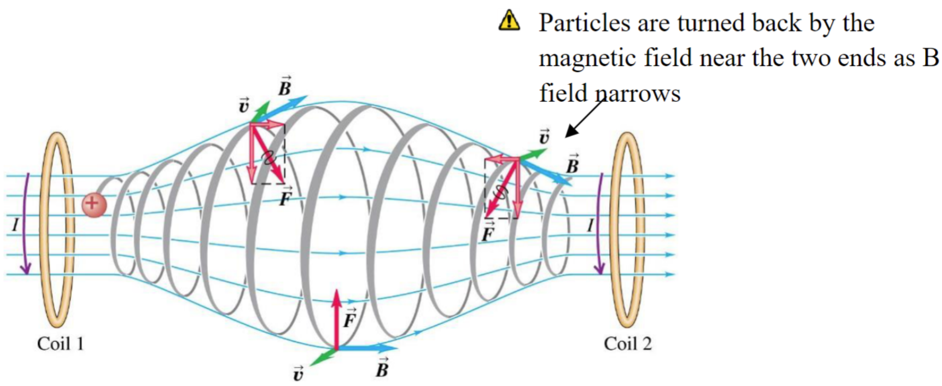


Figure 29: Magnetic confinement in a Thermonuclear Reactor

6.2 Charged particle in Electric and Magnetic Field

Lorentz force: a combination of electric and magnetic force.

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Many devices, overlapping electric field and magnetic field, are designed based on this formula.

6.2.1 Velocity Selector

$$qE = qvB$$
$$v = \frac{E}{B}$$

6.2.2 Mass Spectrometer

$$v = \frac{E}{B}$$
$$\frac{mv^2}{R} = qvB'$$
$$m = \frac{qB'R}{v}$$

6.3 Current in Electric and Magnetic Field

6.3.1 Magnetic Force on a Current-Carrying Conductor

Magnetic field does not do any work on free charged particles, it will do, however, some work on wires or currents, since particles are bounded by the wire and can only move through on direction. We can replace vector I by vector L because I must be the same direction of L, i.e bounded by wire.

$$\vec{F} = (qnLA)\vec{v}_d \times \vec{B} = I\vec{L} \times B$$

For the non-straight conductor case, just break them up into infinitesimal segments $d\vec{l}$, force on the infinitesimal segment is $d\vec{F} = Id\vec{l} \times \vec{B}$. It is obvious that under the same current in a conductor, the effect of other directions will be eliminate, leaving only the straight segment.

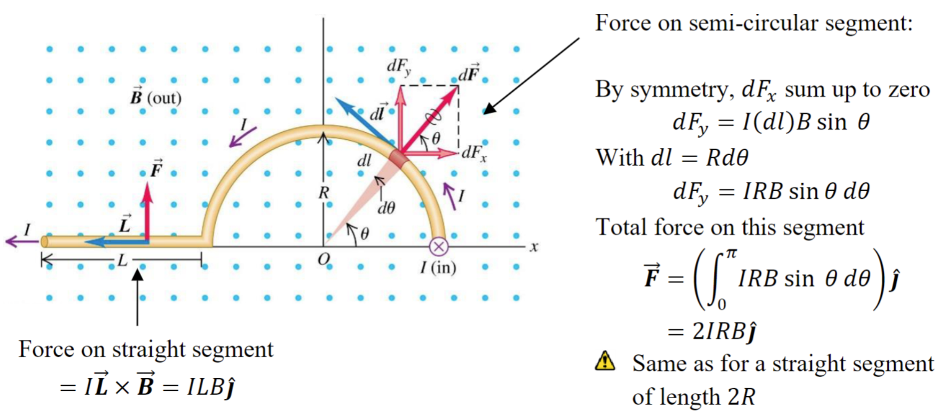


Figure 30: Magnetic Force Calculation

6.3.2 Turning Effect on a Circuit Loop

The net force on the system is 0 as 2 pairs of forces cancel each other(similar to electric dipole situation). However, the force on the a sides of the loop produce a torque on the loop, making it turning.

$$\vec{F} = I\vec{a} \times \vec{B} = IaB$$
$$\tau = 2\left(\frac{\vec{b}}{2} \times \vec{F}\right) = IabB\sin\phi = I\vec{A} \times \vec{B}$$
$$\vec{\tau} = \vec{\mu} \times \vec{B} \text{ where } \mu = I\vec{A}, \text{ direction according to current direction}$$

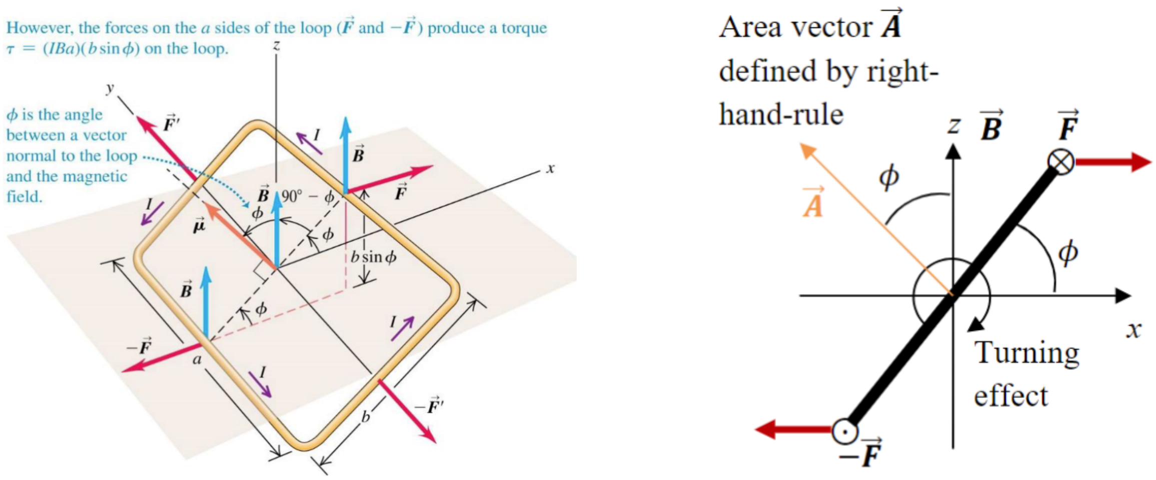


Figure 31: Turing Effect

Similar to electric dipole case, we also define the potential energy of an electric dipole as

$$U = -\vec{\mu} \cdot \vec{B}$$

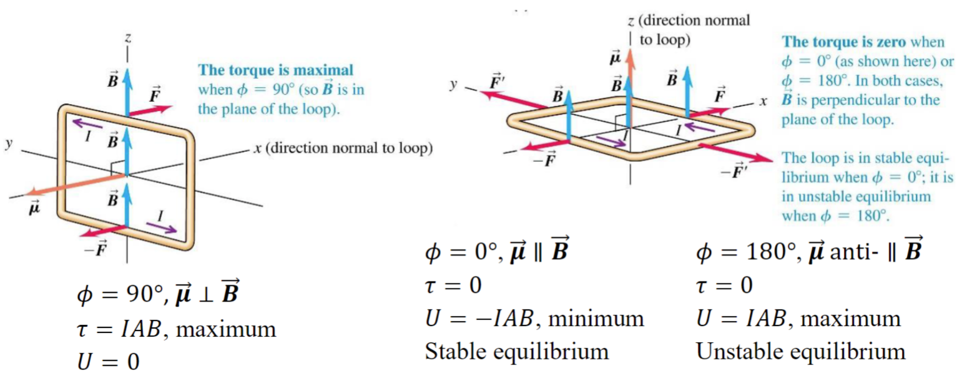


Figure 32: Magnetic potential Energy

7 Source of Magnetic Field

7.1 Biot–Savart law

1. Magnetic field due to moving charge

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$ where μ_0 is a proportionality constant called the vacuum permeability.

2. Magnetic field due to current

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{l}\times\hat{r}}{r^2}$$

From this formula we can calculate the magnetic field (at perpendicular distance x) due to straight current-carrying conductor: when the conductor is infinitely long, $B = \frac{\mu_0 I}{2\pi x}$.

- straight and parallel conductors

Force Between Parallel Conductors

Suppose the currents I and I' are in the same direction.
Magnetic field felt by upper wire (due to lower wire)

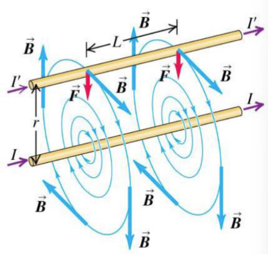
$$B = \frac{\mu_0 I}{2\pi r}$$

Magnetic force on upper wire is $F = I'LB$, therefore force per unit length

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r}$$

⚠ Lower wire experience the same force, but in opposite direction, due to the magnetic field of the upper wire, leading to **attraction** between the wires

⚠ If currents are in opposite direction, leads to repulsion



To conclude: **parallel currents attract, anti-parallel currents repel**.

Figure 33: Force from Magnetic Field generated by Straight Current

- circular conductor

Magnetic Field of a Circular Current Loop

By symmetry dB_y adds up to zero

$$dB_x = \frac{\mu_0 I}{4\pi}\frac{dl}{r^2}\cos\theta = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+a^2}\frac{a}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \quad B_x = \frac{\mu_0 I}{4\pi}\frac{a}{(x^2+a^2)^{3/2}}\underbrace{\int dl}_{2\pi a} = \frac{\mu_0 I}{2}\frac{a^2}{(x^2+a^2)^{3/2}}$$

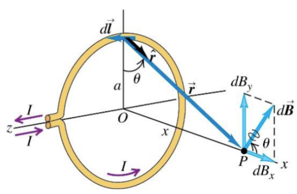


Figure 34: Force from Magnetic Field generated by Loop Current

⚠ Magnetic moment of current loop is $\mu = NIA$, magnetic field produced by a magnetic moment $\vec{\mu}$ along its direction is

$$B_x = \frac{\mu_0 \mu}{2\pi(x^2+a^2)^{3/2}}$$

If $x \gg a$,

$$B_x \cong \frac{\mu_0}{2\pi}\frac{\mu}{x^3}$$

c.f. for electric dipole of dipole moment p

$$E_x \cong \frac{1}{2\pi\epsilon_0}\frac{p}{x^3}$$

⚠ $\vec{\mu}$ can be viewed as a current-carrying loop with current defined by the right-hand-rule, or as a bar magnet where $\vec{\mu}$ points from S to N

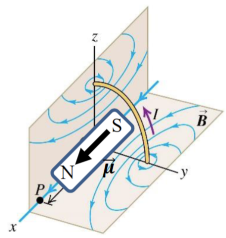


Figure 35: Similarity between Magnetic and Electric Field in moment view

7.2 Ampere's Law

For a close surface of a wire loop(Amperian loop)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

The direction is defined by the **right hand rule**, i.e, the positive direction is up when the magnetic field is calculated, or integrated counter clockwise. Similar to Gaussian's rule, we need to create a certain **symmetry structure** of wire to calculate out this equation. Here are some examples:

1. Magnetic field of a long straight conductor.

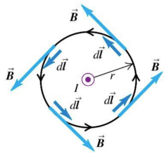


Figure 36: Magnetic field of a long straight conductor

$$\oint \vec{B} \cdot d\vec{L} = \oint BdL = 2\pi rB = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

2. Magnetic field of a long cylindrical conductor. Similar: electric field in/outside the spherical charge.

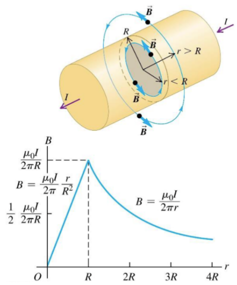


Figure 37: Magnetic field of a long cylindrical conductor

- Outside the conductor, the magnetic field is the same as previous one.
- Inside the conductor:

$$\oint \vec{B} \cdot d\vec{L} = \oint BdL = 2\pi rB = B\int dL = 2\pi rB = \mu_0 I$$

$$\mu_0 I = \mu_0 J\pi r^2 = \mu_0 \frac{I}{\pi R^2}\pi r^2 \Rightarrow B = \frac{\mu_0 I r}{2\pi R^2}$$

3. Magnetic field of a (long) solenoid.

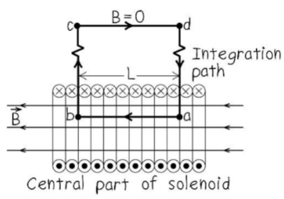


Figure 38: Cross Area of Solenoid

$$\oint \vec{B} \cdot d\vec{L} = BL = \mu_0 IN \Rightarrow B = \frac{\mu_0 IN}{L}$$

7.3 Magnetic Materials

An electron in an atom can be considered to create microscopic current loops, which create magnetic field.

8 Electromagnetic Induction

Faraday's Law for Electromagnetic induction: a time-varying magnetic flux induces an Emf, generating a effect to oppose the cause of the effect.

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

where Magnetic flux $\Phi_B = \int \vec{B} \cdot d\vec{A}$

Lenz's Law from this general law shows the trend that how a circuit will response to the flux change. For superconducting loop, the final flux will equal to initial flux, which means that it will completely resist all the changes.

Lots of devices are built from this law:

- A simple alternator
- A DC Generator
- A Slide ware Generator

In fact, the emf is induced as the form of electric field. An alternative form of Faraday's Law is:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

- Eddy Current Phenomenon
- Displacement Current