Lecture 12: Vectors

Last Time: (1) Vectors (in $\mathbb{R}^2$ & $\mathbb{R}^3$)

(2) Properties of Vectors ($+$, $-$, scaling)

(3) Distinguished Vectors ($\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{0}$)

$\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$, $\vec{0} = (0,0,0)$

Today: (1) Brief Recap

(2) Dot Product

Properties of Vectors: (1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(2) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(3) $\vec{a} + \vec{0} = \vec{a}$ where $\vec{0} = \langle 0, 0, 0 \rangle$

(4) $\vec{a} + (-1)\vec{a} = \vec{0}$

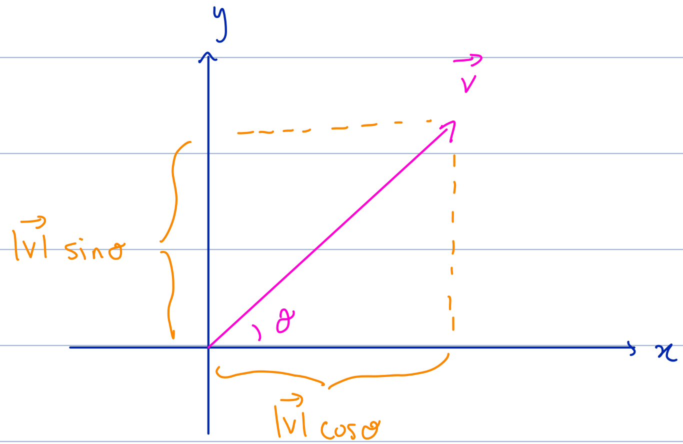
(5) $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$

(6) $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$

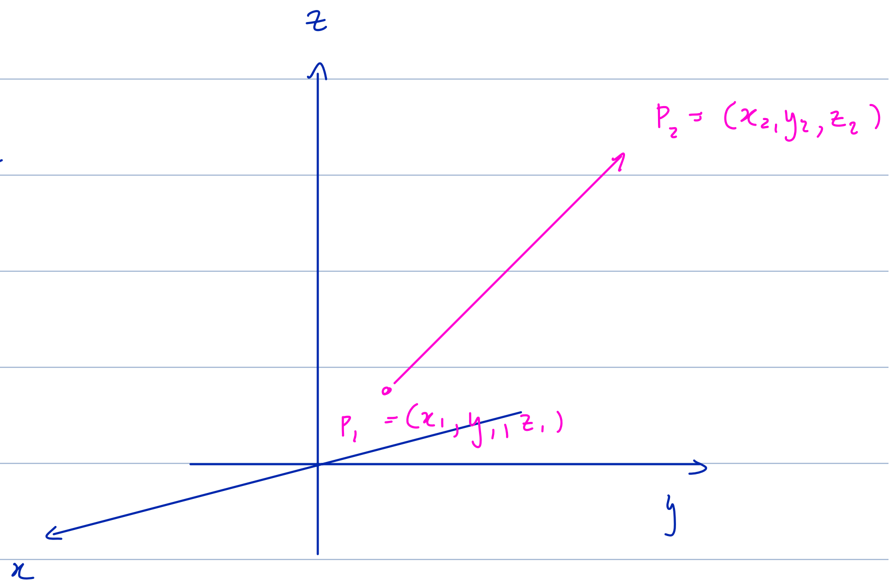
(7) $0 \cdot \vec{a} = \vec{0}$

(8) $\vec{a} \cdot 1 = \vec{a}$

Vector Decomposition:



Components of a displacement vector



$$\overrightarrow{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

$$\|\overrightarrow{P_1P_2}\| = \left((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right)^{1/2}$$

If we want to normalize a vector $\overrightarrow{P_1P_2}$ we divide by its length

$\vec{u} = \dfrac{\overrightarrow{P_1P_2}}{\|\overrightarrow{P_1P_2}\|}$ this is a unit vector in the director of $\overrightarrow{P_1P_2}$ starting at P_1.

The Dot Product

The dot product of two vectors in $\mathbb{R}^n$ is a real number

The Dot Product
(Inner Product)

$\vec{a} \cdot \vec{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$

Theorem: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$

Important Products of Vectors

The cross product of two vectors in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$ and is $\perp$ to both vectors

Cross Product

$$\vec{a} \times \vec{b} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem: $|\vec{a} \times \vec{b}| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$

Def^n: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the dot product (inner product) is the real number $\vec{a} \cdot \vec{b}$ given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

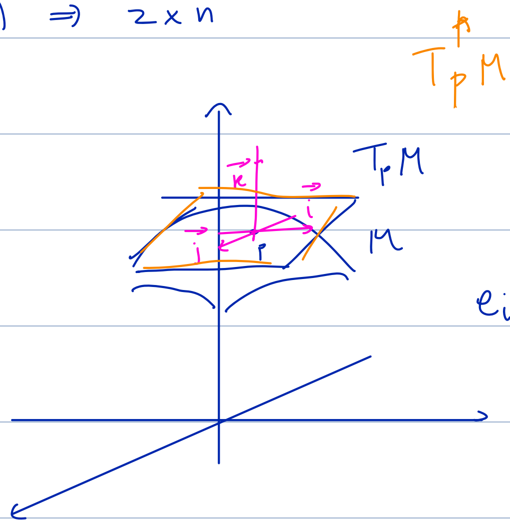
Aside: Vectors are matrices

Matrices can be in many forms. $\rightarrow$ 2×2 $\rightarrow$ $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$(2 \times 2) \times (2 \times n) \Rightarrow 2 \times n$$

$(1 \times n)$

$(n \times 1)$



$e_i d^j = \delta_i{}^j$

e_i $\quad i = 1 \quad e_1 = \langle 1, 0, 0 \rangle$

$i = 2 \quad e_2 = \langle 0, 1, 0 \rangle$

$i = 3 \quad e_3 = \langle 0, 0, 1 \rangle$

$$(1 \times 3) \times (1 \times 3) \rightarrow (1 \times 3) \times (3 \times 1) = 1 \times 1$$

Encoded in the dot product is conversion

$$\vec{a} \circ \vec{b} = (a_1, a_2, a_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Ex. (1) $\langle 2, 4 \rangle \circ \langle 3, -2 \rangle$

(2) $(2\vec{j} + 5\vec{k}) \circ (3\vec{i} - 4\vec{j} + 3\vec{k})$

(1) $\langle 2, 4 \rangle \circ \langle 3, -2 \rangle = (2)(3) + (4)(-2)$

$= 6 - 8$

$= -2.$

(2) $\langle 0, 2, 5 \rangle \circ \langle 3, -4, 3 \rangle = (0)(3) + (2)(-4) + (5)(3)$

$= -8 + 15$

$= 7$

Properties of the dot product; if $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors in $\mathbb{R}^3$ and $\lambda \in \mathbb{R}$ then

(1) $\vec{a} \circ \vec{a} = \|\vec{a}\|^2$

(2) $\vec{a} \circ (\vec{b} + \vec{c}) = \vec{a} \circ \vec{b} + \vec{a} \circ \vec{c}$

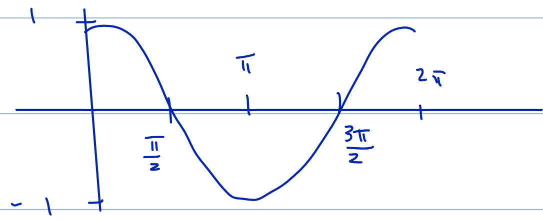
(3) $\vec{a} \circ \vec{b} = \vec{b} \circ \vec{a}$

(4) $\vec{0} \circ \vec{a} = 0$

(5) $(\lambda \vec{a}) \cdot \vec{b} = \lambda (\vec{a} \circ \vec{b}) = \vec{a} \cdot (\lambda \vec{b})$

Theorem: If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos\theta$$



Proof: Use the law of cosines

Q: What exactly does the dot product represent?

A: The dot product tells us what amount of one vector goes in the direction of another. We can think about it as the "sameness" of two vectors. For ex, if two vectors are $\perp$, they are not the same at all, and if they are $\parallel$ they are very much the same.

Minimized when $\perp$ and maximized when $\parallel$.