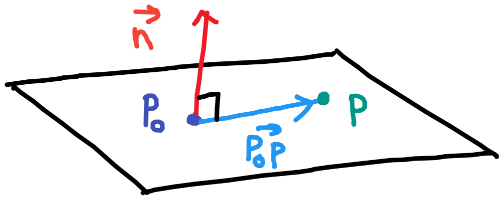
Lecture 13: Wed Oct 1st

Recap: Point-Normal form of a plane: $\vec{n} \cdot \overrightarrow{P_0P} = 0$

This equations describes all (x,y,z) that lies on the plane.

$\vec{n}$: normal vector (a,b,c); P_0: fixed point (x_0, y_0, z_0); P: (x,y,z)



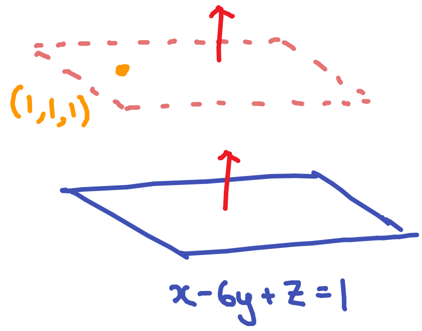
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

General equation: $ax+by+cz=d$

Point-slope form: $m(x-x_0)+n(y-y_0)+z_0=z$

Example: Find the equation of the plane parallel to $z=1-x+6y$ that contains the point $(1,1,1)$.

A: * Parallel planes have parallel normal vectors.



The plane $z=1-x+6y$ has $\vec{n}=(1,-6,1)$.

$x-6y+z=1$

The plane that contains $(1,1,1)$ has point-normal form

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

$$(1,-6,1)\cdot(x-1,y-1,z-1)=0$$

$$(x-1)-6(y-1)+(z-1)=0$$

$$x-1-6y+6+z-1=0$$

$$x-6y+z=-4.$$

Example: Which of the following planes are parallel to each other? Which are perpendicular?

(a) $4x+6y-2z=4$

(b) $f(x,y)=2x+3y$

(c) $2x+3y+z=4$

(d) $4x-5y+7z=2$

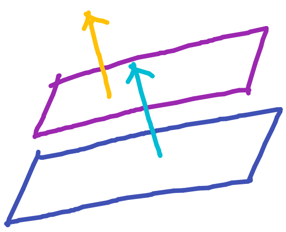
A: Note: 2 planes are parallel (resp. perpendicular) if their normal vectors are parallel (resp. perpendicular).

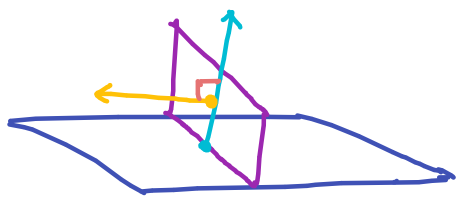
$\vec{n_a}=(4,6,-2)$

$\vec{n_b}=(2,3,-1)$

$\vec{n_c}=(2,3,1)$

$\vec{n_d}=(4,-5,7)$





$\vec{n}_a$ and $\vec{n}_b$ are scalar multiples. $\Rightarrow$ parallel $\vec{n}_a \parallel \vec{n}_b$.

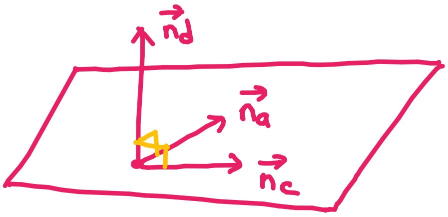
$\vec{n}_c \cdot \vec{n}_a = (2,3,1) \cdot (4,6,-2) = 8+9-2 = 15 \neq 0 \Rightarrow \vec{n}_c \not\perp \vec{n}_a, \vec{n}_b,\ \vec{n}_c \not\parallel \vec{n}_a, \vec{n}_b$

$\vec{n}_c \cdot \vec{n}_d = (2,3,1) \cdot (4,-5,7) = 8-15+7 = 0 \Rightarrow \vec{n}_c \perp \vec{n}_d$

$\vec{n}_d \cdot \vec{n}_a = (4,-5,7) \cdot (4,6,-2) = 16-30-14 = -28 \neq 0. \Rightarrow \vec{n}_d \not\perp \vec{n}_a, \vec{n}_b,\ \vec{n}_d \not\parallel \vec{n}_a, \vec{n}_b.$

<u>Note:</u> Even though $\vec{n}_c \not\parallel \vec{n}_a$ and $\vec{n}_c \perp \vec{n}_d$, it is possible for $\vec{n}_d \perp \vec{n}_a$.

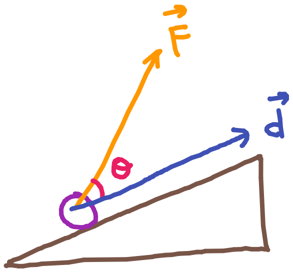
For example,



Application: Work

When a force $\vec{F}$ acts on an object through displacement $\vec{d}$,

$$W = \text{work done by } \vec{F} = \vec{F} \cdot \vec{d} = \|\vec{F}\|\|\vec{d}\| \cos\theta$$



<u>Example:</u> A force $\vec{F} = (1,3,-2)$ pushes a crate from point $P = (1,0,0)$ to $Q = (5,2,0)$.

a) What is the work done by $\vec{F}$?

b) Find another force $\vec{G} = (2,1,c)$ that gives the same work and has the smallest magnitude.

<u>A:</u> a) $\vec{d} = \vec{PQ} = (5,2,0) - (1,0,0) = (4,2,0)$

$W = \vec{F} \cdot \vec{d} = (1,3,-2) \cdot (4,2,0) = 4+6+0 = 10.$

b) $\vec{G} \cdot \vec{d} = (2,1,c) \cdot (4,2,0) = 8+2+0 = 10.$

No matter what c is, $\vec{G} = (2,1,c)$ gives the same work.

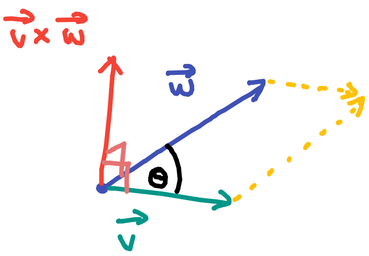
Since $\|\vec{G}\| = \sqrt{4+1+c^2}$, setting $c=0$ gives smallest magnitude.

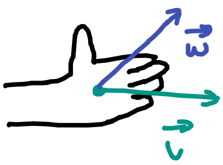
§13.4 Cross products:

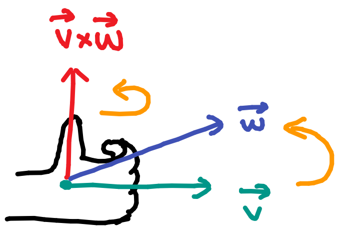
Given vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \times \vec{w}$ = vector perpendicular to $\vec{v}$ and $\vec{w}$

$\|\vec{v} \times \vec{w}\|$ = area of parallelogram with edges/sides $\vec{v}$ and $\vec{w}$.

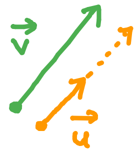
The direction of $\vec{v} \times \vec{w}$ is given right-hand rule.







In general, vector $= (\text{magnitude})\,\vec{u}$ ← unit direction vector



$\|\vec{v}\|\vec{u} = \vec{v}$

Geometric definition

$$\vec{v} \times \vec{w} = \|\vec{v} \times \vec{w}\|\,\vec{n}$$ ← unit vector perpencicular to $\vec{v}$ and $\vec{w}$

$$= (\text{Area of parallelogram})\,\vec{n}$$

$$= \left(\|\vec{v}\|\,\|\vec{w}\|\sin\theta\right)\vec{n} \qquad (0 \leq \theta \leq \pi)$$

Note: If $\vec{v} \parallel \vec{w}$, then $\theta = 0$ and $\vec{v} \times \vec{w} = \vec{0}$.