Lecture 10: Limits & Continuity

Last Time: (1) Fixed a few from last friday's lecture

↳ Domain

↳ Eq of a plane

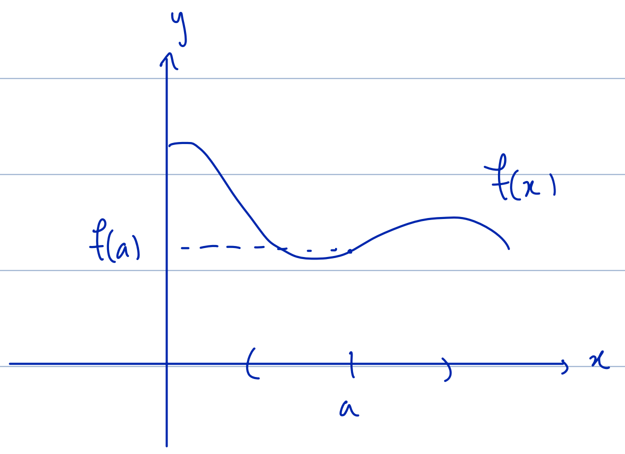
(2) Limits

↳ Non - Existence.

Today: (1) Finish Limits

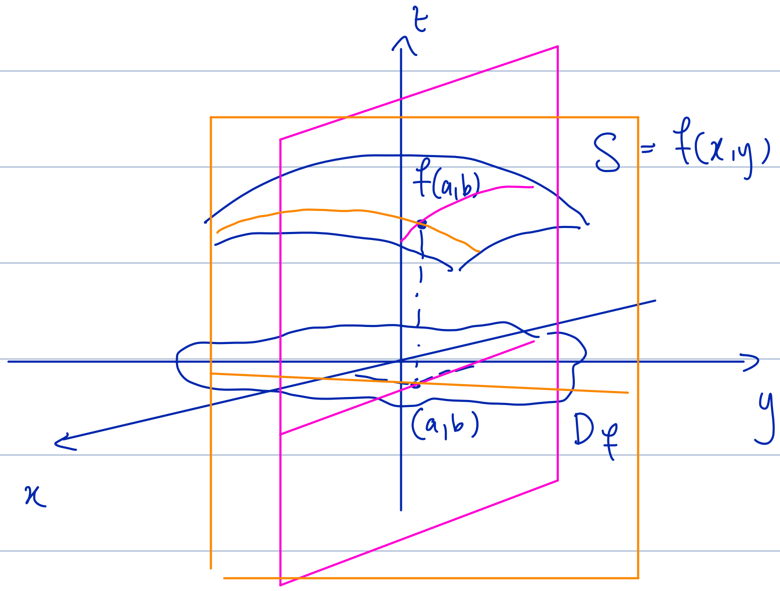
(2) Continuity.

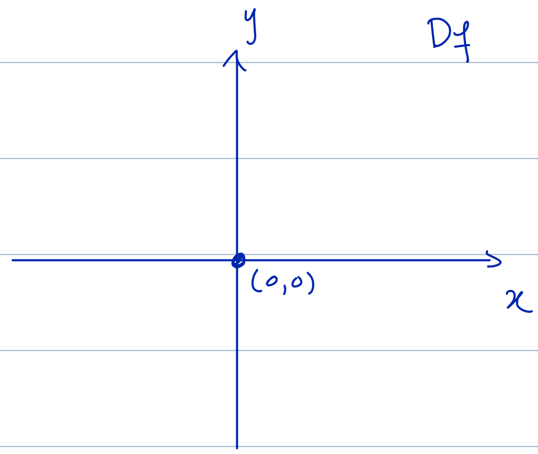
Intuition: In 1D calc for a limit to exist we needed the limit from the left and right to match



$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

However in 2D fcts we saw that to approach a pt in the domain there are infinitely many directions. And for this limit to exist we need the limit from all directions to match. But for non-existence we only two different directions to misalign.





Ex of Non-Existence: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

Consider the line $y=0$ i.e. approach along the x-axis then;

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Consider the line $x=0$. i.e. approach $(0,0)$ along the y-axis then;

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2}$$

$$= \lim_{y\to 0} \frac{-y^2}{y^2} = -1.$$

Since these give different limits upon approach to the same pt for two different curves the limit DNE.

If we want to do this quickly for this particular limiting pt $(0,0)$ we can approach along the general line $y=mx$. Then;

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,mx)\to(0,0)} \frac{x^2-(mx)^2}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}(1-m^2)}{\cancel{x^2}(1+m^2)}$$

$$= \frac{1-m^2}{1+m^2}$$

This depends on the choice of line i.e. "m" and $\therefore$

the limit DNE.

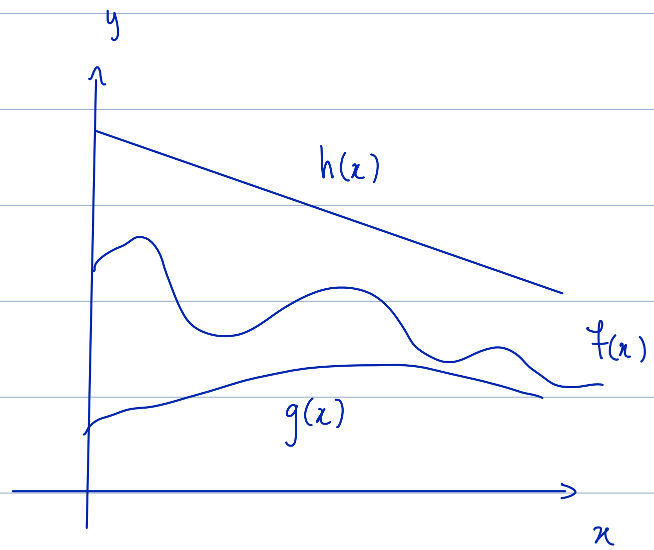
Showing a Limit Exists

To show a limit of $f(x,y)$ exists we need to show that the limit from all directions match. To do this we use the squeeze theorem.

Theorem: (Squeeze Theorem)

If $g(x,y) \le f(x,y) \le h(x,y)$ and the $\lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L$

then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.



Ex. $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} 3y \frac{x^2}{x^2+y^2}$

Notice that $0 \le x^2 \le x^2+y^2 \Rightarrow 0 \le \frac{x^2}{x^2+y^2} \le 1$

$-|y| \le y \le |y|$

From the inequality; $-3|y| \leq 3y \frac{x^2}{x^2+y^2} \leq 3|y|$

So $g(x,y) = -3|y|$ and $h(x,y) = 3|y|$

and $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} h(x,y) = 0.$

and since we have that $g(x,y) \leq f(x,y) = 3y \frac{x^2}{x^2+y^2} \leq h(x,y)$

we have that

$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$ by the squeeze theorem.

Aside: Polar Coordinates

↳ $x = r\cos\theta$

$y = r\sin\theta$

$r = (x^2+y^2)^{1/2} \Rightarrow r^2 = x^2+y^2.$

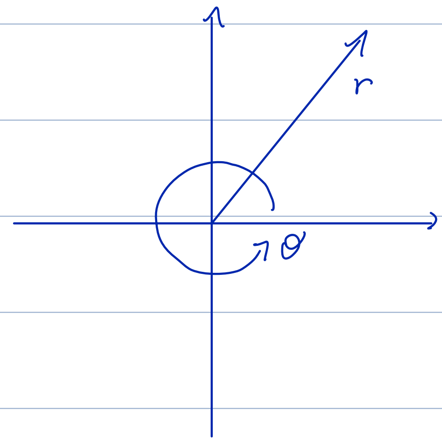
We can use this to refraim what we did previously as follows;

Ex. $\lim_{(x,y)\to(0,0)} \frac{e^{-(x^2+y^2)} - 1}{x^2+y^2}$

Convert into polar coords then the limit becomes

$$\lim_{r\to 0} \frac{e^{-r^2}-1}{r^2} \rightarrow \text{L'Hopitals}$$

$$\lim_{r\to 0} \frac{-\cancel{2r}e^{-r^2}}{\cancel{2r}} = \lim_{r\to 0} \frac{-e^{-r^2}}{1} = -1$$



If θ appears in the expression of the limit then the limit depends on the angle of approach and $\therefore$ DNE.

Remark: To show existence we always use squeeze theorem.

Continuity

Def^n: A fct $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Remark: f is continuous on a set D if f is continuous at every pt in the domain D.

Ex. of Continuous Functions:
- Polynomials, Rational Fcts
- Exponentials, Logarithms
- Trig Functions.

Rules of Limits & Cont - Add, Subtract, Mult., Divide, Composition.

Ex. What value of c makes the following fct continuous at $(0,0)$?

$$f(x,y) = \begin{cases} x^2+y^2+1 & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0). \end{cases}$$

We know that x^2+y^2+1 is cont on its domain $(x,y) \neq (0,0)$ since its a polynomial.

We need that $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Squeeze theorem

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} x^2+y^2+1 = 1$$

In order for $f(x,y)$ to be continuous at $(0,0)$ we require that $c = 1$.