Lecture 11: Vectors

Last Time: (1) Non-Existence of Limits

↳ Showing the limit along different directions has different values.

(2) Existence of Limits

↳ Squeeze Theorem.

(3) Continuity

Recall: Squeeze Theorem

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} 3y \cdot \frac{x^2}{x^2+y^2}$$

We noted that $0 \le x^2 \le x^2+y^2 \implies 0 \le \frac{x^2}{x^2+y^2} \le 1$

In addition $-|y| \le y \le |y| \implies -3|y| \le 3y \le 3|y|$

So combining these $\implies$ $-3|y| \frac{x^2}{x^2+y^2} \le 3y \frac{x^2}{x^2+y^2} \le 3|y| \frac{x^2}{x^2+y^2}$

$$0 \le -3|y| \frac{x^2}{x^2+y^2} \le 3y \frac{x^2}{x^2+y^2} \le 3|y| \frac{x^2}{x^2+y^2} \le 3|y|$$

We then have the following $0 \le 3y \frac{x^2}{x^2+y^2} \le 3|y|$

Take $g(x,y) = 0$ and $h(x,y) = 3|y|$

Then by the squeeze theorem we have that $\lim_{(x,y) \to (0,0)} f(x,y) = 0$.

Recall: Continuity

Ex. $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Q: Where is f continuous.

↳ f is cont. at all pts $(x,y) \neq (0,0)$ since it is a rational fct

↳ Check continuity at $(0,0)$ i.e. $\lim_{(x,y) \to (0,0)} \dfrac{x^2-y^2}{x^2+y^2} \overset{?}{=} 0$

↳ By the previous ex we saw that this limit DNE

So $f(x,y)$ cont only on $\mathbb{R}^2 / \{(0,0)\}$.

Ex: Non-Existence of Limits in P.C. $\rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ r^2 = x^2+y^2 \end{cases}$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$$

$$= \lim_{r\to 0} \frac{2(r\cos\theta)(r\sin\theta)}{(r\cos\theta)^2+3(r\sin\theta)^2}$$

$$= \lim_{r\to 0} \frac{2\cancel{r^2}\cos\theta\sin\theta}{\cancel{r^2}(\cos^2\theta+3\sin^2\theta)}$$

$$= \frac{2\cos\theta\sin\theta}{\cos^2\theta+3\sin^2\theta}$$

The lim. depends on θ $\therefore$ it does not exist.

When Computing Limits; If we're told to prove a limit exists

↳ Squeeze theorem

If we're told to prove a limit DNE

↳ Approach the pt along diff curves

If we're told to show whether a limit exists or does not

↳ (1) Choose different paths

- If the limit depends on the path then the lim. DNE
- If the lim. does not depend on the path then it most likely exists
  ↳ Pass to squeeze theorem to prove its existence.

Today: (1) Vectors. (Chapter 13)

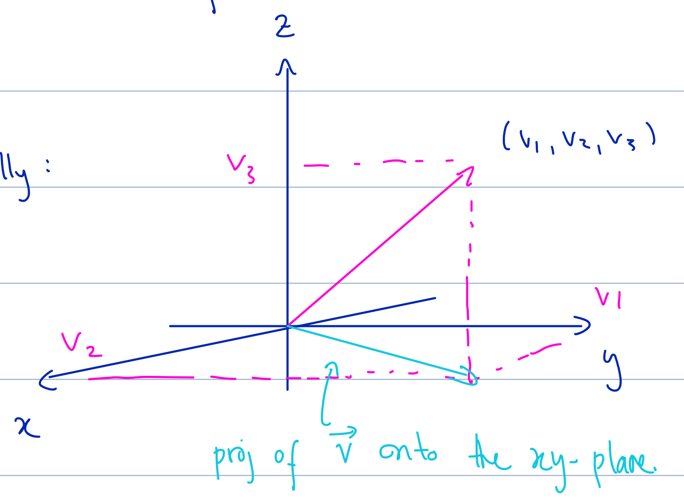
Def^n: The displacement vector from a pt A to a pt B is an arrow with its tail at A and tip at B

Def^n: The magnitude of the d.v. is the distance between A and B and is rep. the length of the arrow

Def^n: The direction of the d.v. is the direction of the arrow

Algebraically: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Graphically:



Addition: The sum $\vec{v} + \vec{w}$ of two vectors is the combined displacement resulting from first applying $\vec{v}$ and then $\vec{w}$ and is computed component wise.

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$$

(Similarly for subtraction) → The difference $\vec{v} - \vec{w}$ is the d.v. that when added to $\vec{w}$ gives $\vec{v}$ i.e.

$$\vec{v} = \vec{w} + (\vec{v} - \vec{w})$$

$$\vec{v} - \vec{w} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \\ v_3 - w_3 \end{bmatrix}$$

Scaling: If λ is a scalar and $\vec{v}$ is a d.v. the scalar mult. $\lambda\vec{v}$ is the d.v. $\overline{w}$ the following prop.

(1) The d.v. $\lambda\vec{v}$ is $\parallel$ to $\vec{v}$, pointing in the same direction if $\lambda > 0$ and in the opposite direction if $\lambda < 0$.

(2) the mag of $\lambda\vec{v}$ is $|\lambda|$ times the mag of $\vec{v}$, i.e.

$$\|\lambda \vec{v}\| = |\lambda| \, \|\vec{v}\|$$

$$\lambda \vec{v} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{bmatrix}$$

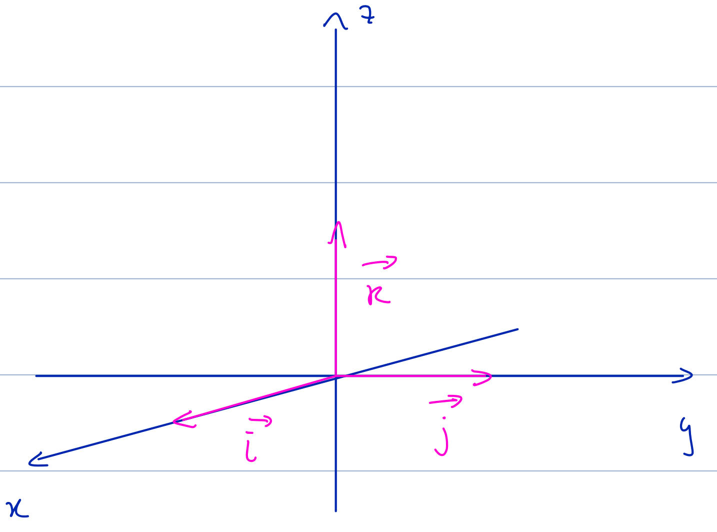
Q: Why do we have that $\vec{v} - \vec{w} = \vec{v} + (-1) \cdot \vec{w}$?

$$\vec{v} - \vec{w} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \\ v_3 - w_3 \end{bmatrix} = \begin{bmatrix} v_1 + (-1) w_1 \\ v_2 + (-1) w_2 \\ v_3 + (-1) w_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + (-1) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$= \vec{v} + (-1) \vec{w}$$

Unit Vectors: These are vectors of length 1 and are obtained by normalizing the given vector and they pt in the same direction as the vector.

$\vec{v}$ then $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$



There are 3 special vectors $\vec{i}, \vec{j}, \vec{k}$ defined as

$$\vec{i} = \langle 1, 0, 0 \rangle$$
$$\vec{j} = \langle 0, 1, 0 \rangle$$
$$\vec{k} = \langle 0, 0, 1 \rangle$$

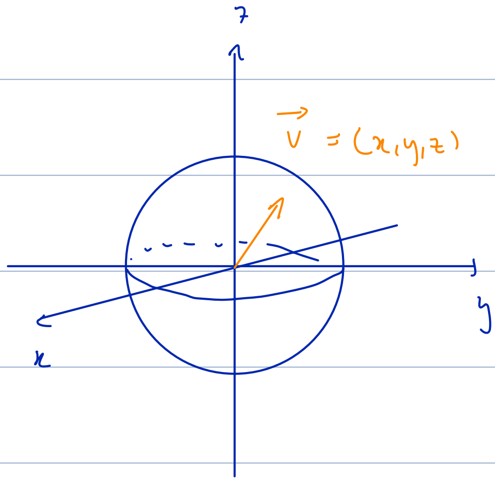
Remark: We can always decompose a vector into a linear combination of basis vectors.

Ex.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

Q: Unit vectors correspond to pts on which shape in 3-space?

A: The sphere i.e. $(x^2 + y^2 + z^2 = 1)$



then $\|\vec{v}\| = 1 \iff \sqrt{x^2 + y^2 + z^2} = 1$

$\iff x^2 + y^2 + z^2 = 1$