University of Toronto – Faculty of Arts & Science – MAT235Y1: Multivariable Calculus

Term Test 1 – Fall 2023/Winter 2024

Family Name (PRINT): ______________________________

Given Name(s) (PRINT): ______________________________

Student Number: ______________________________

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This exam contains **8** pages (including this cover page) and **6** problems. Once the exam begins, check to see if any pages are missing. There are **50** possible points to be earned in this exam.

- Duration: **80 minutes**
- **No aids or calculators are permitted on the exam.**
- **Do not tear any pages off this exam.**
- **One scrap page is provided at the end.** This page will not be graded unless specifically indicated. Please enter all of your answers in the space provided.
- Do not write in the page margins. Make sure that your writing is dark enough to be readable.
- **Unsupported answers to short answer questions will not receive full credit**. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
- **Organize your work** in a reasonably neat and coherent way.
- You must use the methods learned in this course to solve all of the problems.

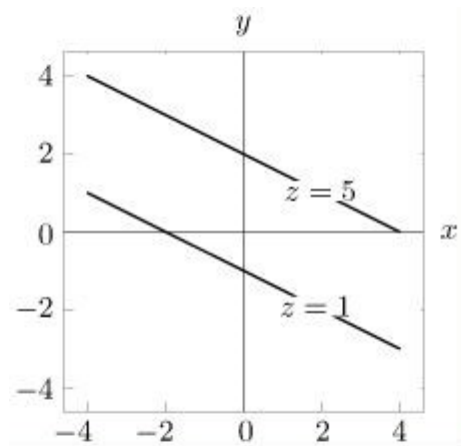
1. (12 points) (Multiple choice) For each part, write either **A, B, C, D** or **E** on the indicated line. Only your final answer will be graded for this question. Each part is worth 3 marks.

   (i) The points $(1, 3, 2)$, $(0, -5, 4)$, and $(-2, 2, 1)$ are the vertices of a triangle in space. Which of the vertices is closest to the origin?

   A $(1, 3, 2)$ **B** $(0, -5, 4)$ **C** $(-2, 2, 1)$
   D Both $(1, 3, 2)$ and $(0, -5, 4)$ **E** Both $(-2, 2, 1)$ and $(0, -5, 4)$

   Answer C $1^2+3^2+2^2=14$ $0^2+(-5)^2+4^2=41$

   (ii) Which linear function best matches the following partial contour diagram? $(-2)^2+2^2+1^2=9$



   A $z = 7/3 + 2/3x + 4/3y$ **B** $z = 7/3 - 2/3x + 4/3y$
   C $z = 7/3 + 2/3x^2 + 4/3y$ **D** $z = 7/3 + 2/3x - 4/3y$ **E** $z = 7/3 + 4/3y^2$

   Answer A (must have negative slope, linear function)

   (iii) If $\vec{w} = 2\vec{i} - 3\vec{j} + 2\vec{k}$, find a unit vector which is parallel to $\vec{w}$.
   A $\langle \frac{2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{2}{\sqrt{17}} \rangle$ **B** $\langle \frac{2}{7}, \frac{-3}{7}, \frac{2}{7} \rangle$ **C** $\langle \frac{2}{17}, \frac{-3}{17}, \frac{2}{17} \rangle$ **D** $\langle \frac{1}{2}, \frac{-3}{4}, \frac{1}{2} \rangle$ **E** None of the above

   Answer A

   (iv) Suppose we pull a rope in the xy-plane with a force $\vec{F}$ of 400 pounds southeast. What is the $\vec{i}$-component of $\vec{F}$?

   A $-\frac{400}{\sqrt{2}}$ **B** $\frac{200}{\sqrt{2}}$ **C** $\frac{400}{\sqrt{2}}$ **D** $-\frac{200}{\sqrt{2}}$ **E** None of the above

   Answer C

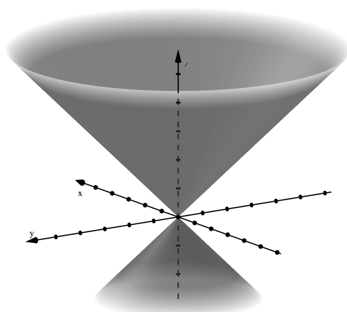
2. (4 points) Match each of the equations with one of the following surfaces, as shown below. Write **A, B, C** or **D** for each equation. Only your final answer will be graded. Each part is worth 1 mark.

(a) $x^2 + y^2 + z^2 = 1$ Answer C

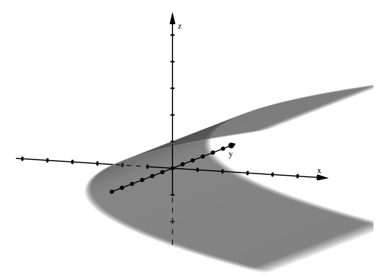
(b) $z^2 = x^2 + y^2$ Answer A

(c) $x = z^2$ Answer B

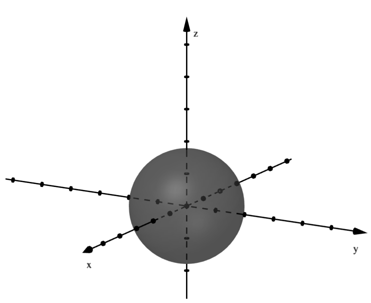
(d) $z + x + 2y = 0$ Answer D



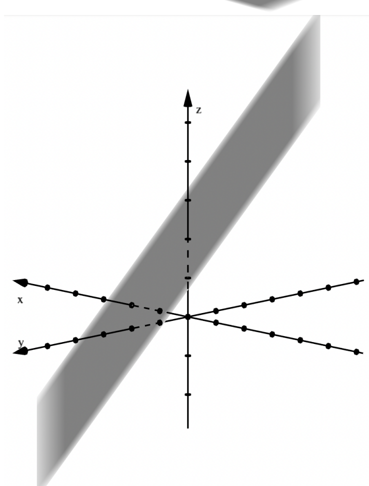
A



B



C



D

3. (9 points) (a) (5 points) What value(s) of c, if any, make the following function continuous at $(0,0)$? Justify your answer.

$$f(x,y) = \begin{cases} (x^{2024}+y^{2024})\cos\left(\frac{1}{\sqrt{x^{2023}+y^{2023}}}\right), & (x,y) \neq (0,0) \\ c, & (x,y) = (0,0) \end{cases}$$

$$\left|(x^{2024}+y^{2024})\cos\left(\frac{1}{\sqrt{x^{2023}+y^{2023}}}\right)\right| \leq x^{2024}+y^{2024} \xrightarrow[(x,y)\to(0,0)]{} 0$$

Let $\boxed{c=0}$. Then $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

(b) (4 points) Find the following limit if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}.$$

Along $y=0$, we have $\frac{x^3y}{x^6+y^2} = \frac{0}{x^6} = 0 \xrightarrow{x\to 0} 0$

Along $y=x^3$, we have $\frac{x^3y}{x^6+y^2} = \frac{x^6}{x^6+x^6} = \frac{1}{2} \xrightarrow{x\to 0} \frac{1}{2}$

$0 \neq \frac{1}{2}$ So $\boxed{\text{limit does not exist}}$

4. (9 points) Let $\vec{v_1} = 2\vec{i} - 4\vec{j} + \vec{k}$ and $\vec{v_2} = 5\vec{i} + \vec{j} + \vec{k}$.

(a) (4 points) Find a vector perpendicular to both $\vec{v_1}$ and $\vec{v_2}$.

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= (-4-1)\hat{i} + (5-2)\hat{j} + (2+20)\hat{k}$$

$$= \boxed{-5\hat{i} + 3\hat{j} + 22\hat{k}}$$

(b) (5 points) Find the equation of the plane passing through the point $(-1, 1, -2)$ and with normal vector orthogonal (perpendicular) to both $\vec{v_1}$ and $\vec{v_2}$. Express your answer in the form $Ax + By + Cz = D$. Show all your work.

From above, we want $\langle -5, 3, 22\rangle \cdot \langle x+1, y-1, z+2\rangle = 0$

$\Rightarrow -5(x+1) + 3(y-1) + 22(z+2) = 0$

$\Rightarrow -5x - 5 + 3y - 3 + 22z + 44 = 0$

$\Rightarrow \boxed{-5x + 3y + 22z = -36}$

5. (9 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by $f(x, y) = 4 - x^2 - y^2$.

(a) (3 points) Find the domain and range of f.

Domain : $\mathbb{R}^2$

Range : $(-\infty, 4]$

(b) (2 points) Give a verbal description of the level curve $f(x, y) = c$ when $c = 2$.

$4 - x^2 - y^2 = 2 \Rightarrow x^2 + y^2 = 2$

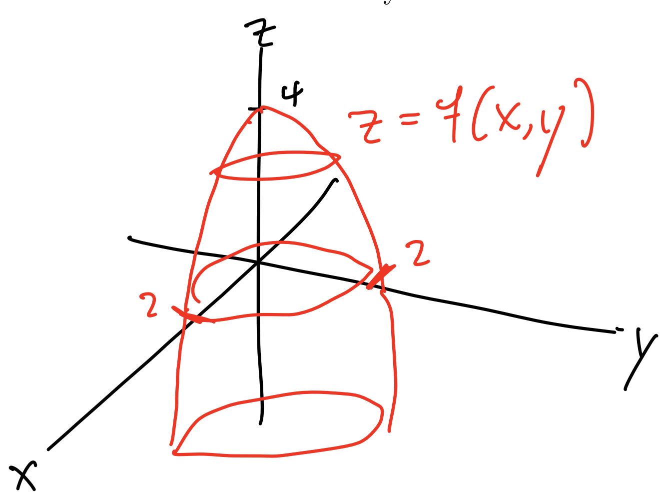
$\boxed{\text{Circle of radius } \sqrt{2}}$

(c) (2 points) Give a verbal description of the cross-sections of f with $x = \sqrt{2}$ and with $y = 1$.

$x = \sqrt{2} \Rightarrow f(\sqrt{2}, y) = 2 - y^2$ Parabola opening downward. (vertex at $z=2$)

$y = 1 \Rightarrow f(x, 1) = 3 - x^2$ Parabola opening downward. (vertex at $z=3$)

(d) (2 points) Sketch the graph of f. Label at least two points on the graph which intersect the coordinate axes. Make sure to label your axes.



6. (7 points) A person's basal metabolic rate (BMR) is the minimal number of daily calories needed to keep their body functioning at rest. The BMR (in kcal/day) of a person of mass m (in kg), height h (in cm) and age a (in years) can be approximated by

$$P = f(m, h, a) = 10m + 2h - 5a + 600.$$

(a) (4 points) Describe, in detail, the level surface $P = 2000$ and explain what the points on this level surface represent.

$2000 = 10m + 2h - 5a + 600 \Rightarrow 10m + 2h - 5a = 1400$

Equation of a plane

Points on this surface are the combinations of mass, height, and age that yield a BMR of 2000.

(b) (3 points) Suppose a 150 cm tall, 20-year-old person weighing 70 kg restricts their daily caloric intake to 1600 kcal. Should this person expect to lose weight?

$P = f(70, 150, 20) = 700 + 300 - 100 + 600$

$= 1500$

Assuming BMR is min. daily calories needed to function without losing weight, we see $1500 < 1600$, so this person should not expect to lose weight. (Unless they exercise enough.)

Do not tear this page off. This page is for additional work and will not be graded, unless you clearly indicate it on the original question page.