Lecture 8: Functions of 3 Variables

Last Time: (1) Finished linear fcts of 2 Variables

(2) Started fcts of 3 variables

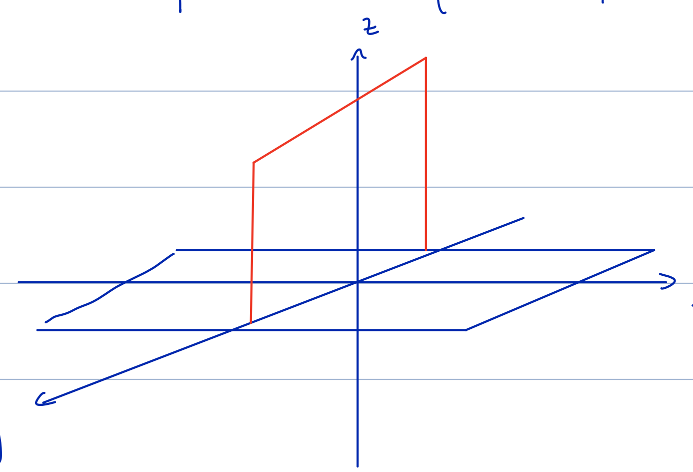
Today: (1) Finish fcts of 3 variables

Recall: A fct of 3 variables, f, is a rule that assigns to each ordered triple (x,y,z) in the $D_f \subseteq \mathbb{R}^3$ a unique (single) real number denoted $f(x,y,z) = w$.

Ex. Consider $f(x,y,z) = \ln(z-y) + xy\sin\left(\frac{z}{x}\right)$. What is the admissable domain?

The expression for $f(x,y,z)$ is defined as long as $z-y>0$ and $x \neq 0$.

So the domain of f is $D_f = \{(x,y,z) \mid z>y \text{ and } x \neq 0,\ x,y,z \in \mathbb{R}\}$.



The graph of f is $\{(x,y,z,w) \in \mathbb{R}^4 \mid w = f(x,y,z)\} \subseteq \mathbb{R}^4$ i.e. it is in a 4d space. This is difficult to sketch so instead we look at its level surfaces.

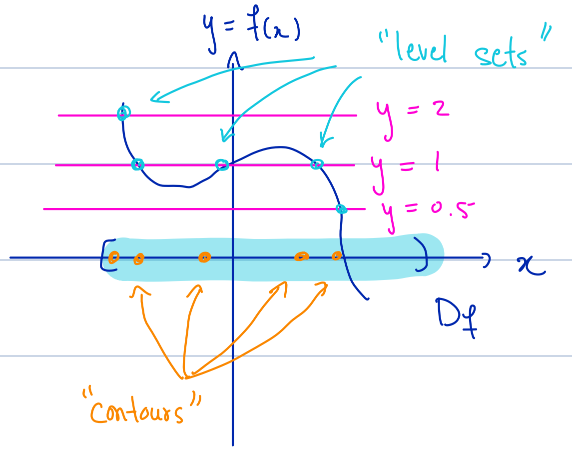
Recall: The set of pts $(x,y,z) \in \mathbb{R}^3$ where f has a constant value

$f(x,y,z) = c$ are level surfaces. In other words

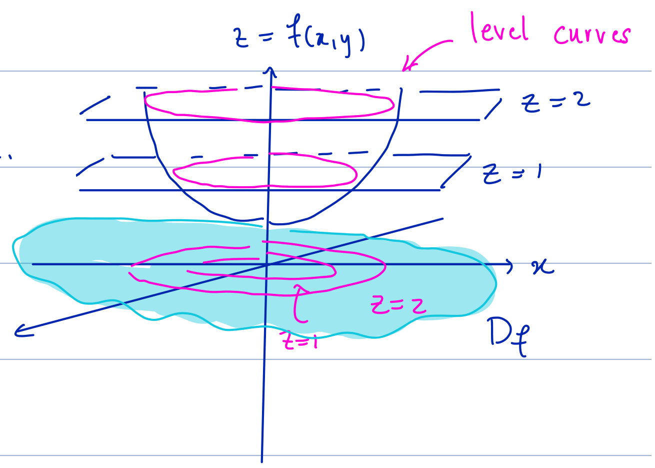
$f^{-1}(c) = \{(x,y,z) \in D_f \mid f(x,y,z) = c\}$.

Recall this concept in lower dimensions;

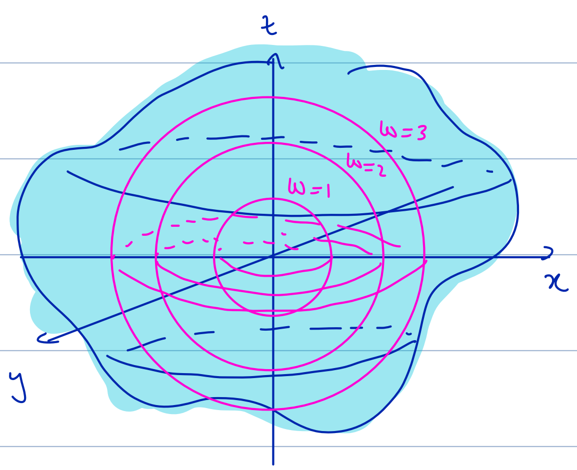
1d.



2d.



3d.



$w = f(x,y,z)$

Ex. Describe the level surfaces of the following fcts.

(1) $f(x,y,z) = x^2+y^2+z^2 \to$ a ball

To find level surfaces set $\omega = f(x,y,z) = \text{const.} = k$.

$k=0 \Rightarrow \{(0,0,0)\}$

$k=1 \Rightarrow \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}. \to$ sphere of radius 1

$k=2 \Rightarrow \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=2\} \to$ sphere of radius $\sqrt{2}$.

$\vdots$

$\omega=k \Rightarrow \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=k\} \to$ sphere of radius $\sqrt{k}$.

(2) $f(x,y,z) = 2y+z+x+1$

Set $k = 2y+z+x+1$

$k=0 \Rightarrow \{(x,y,z)\in\mathbb{R}^3 \mid 2y+z+x=-1\}. \to$ plane $n=\frac{\Delta z}{\Delta y}=2$, $m=\frac{\Delta z}{\Delta x}=1$

z-int $=-1$.

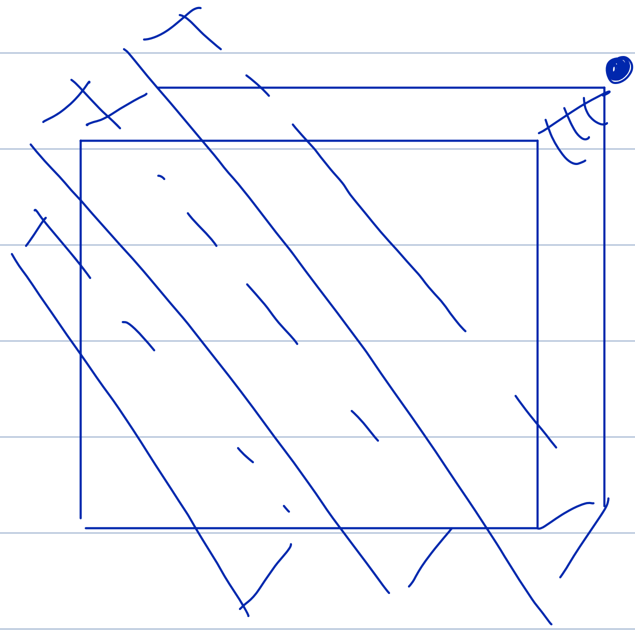
$k=1 \Rightarrow \{(x,y,z)\in\mathbb{R}^3 \mid 2y+z+x=0\} \to$ plane $n=\frac{\Delta z}{\Delta y}=2$, $m=\frac{\Delta z}{\Delta x}=1$

z-int $=0$.

$\vdots$

$\omega=k \Rightarrow \{(x,y,z)\in\mathbb{R}^3 \mid 2y+z+x=k-1\} \to$ plane $n=\frac{\Delta z}{\Delta y}=2$, $m=\frac{\Delta z}{\Delta x}=1$

z-int $= k-1$



Remark: The graph of a two variable fct $f(x,y)$ can be considered as a level surface of a fct of three variables. Namely, the graph of $f(x,y)$ is written as $z = f(x,y)$. This can rearranged as $z - f(x,y) = 0$. Set $g(x,y,z) = z - f(x,y)$ then $g(x,y,z)$ is a fct of 3 variables and the level surface $g(x,y,z) = 0$ is the graph of f.

Questin: Can a level surface of a fct of 3 variables be considered as a graph of a 2 variable fct?

Ex. Let $g(x,y,z) = x^2+y^2+z^2$

Consider the level surface $g(x,y,z) = 1$ i.e. $1 = x^2+y^2+z^2$.

We want to express $z = f(x,y)$

If we try to do this then $1 = x^2+y^2+z^2 \Rightarrow z^2 = 1-x^2-y^2$

and so $z = (1-x^2-y^2)^{1/2}$ or $z = -(1-x^2-y^2)^{1/2}$.

This tells us that z has two possible values for the same (x,y).

Takeaway: A level surface of a fct of 3 variables cannot be considered as a graph of a fct of 2 variables <u>unless</u> it is possible to solve $g(x,y,z) = c$ for z with a unique solution.