University of Toronto – Faculty of Arts & Science – MAT235Y1: Multivariable Calculus

Term Test 1 (Practice Test) – Fall 2024/Winter 2025

Family Name (PRINT): Sample Solutions

Given Name(s) (PRINT): ______________________

Student Number: ______________________

U of T Email: ______________________

This exam contains **8** pages (including this cover page) and **6** problems. Once the exam begins, check to see if any pages are missing. There are **50** possible points to be earned in this exam.

- Duration: **90 minutes**
- **No aids or calculators are permitted on the exam.**
- Power off all cell phones, smartphones, and other electronic devices, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
- **Do not tear any pages off this exam.**
- **One scrap page is provided at the end.** This page will not be graded unless specifically indicated. Please enter all of your answers in the space provided.
- Do not write in the page margins. Make sure that your writing is dark enough to be readable.
- **Unsupported answers to long answer questions will not receive full credit**. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
- **Organize your work** in a reasonably neat and coherent way.
- You must use the methods learned in this course to solve all of the problems.

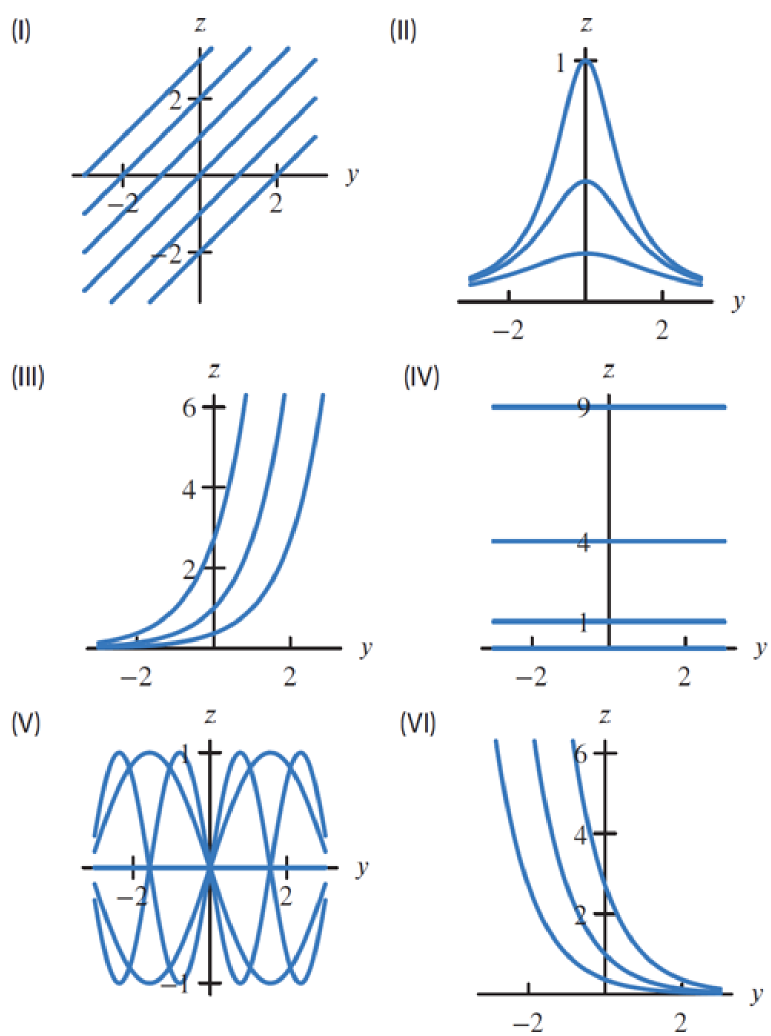
1. (6 points) Match each equation with one of the following cross-sections with x fixed. Write **(I)**, **(II)**, **(III)**, **(IV)**, **(V)** or **(VI)** for each equation. Only your final answer will be graded. Each part is worth 1.5 marks.

   1. $z = \sin(xy)$ Answer **(V)**

   2. $z = e^{x-y}$ Answer **(VI)**

   3. $z = 1 + x + y$ Answer **(I)**

   4. $z = x^2$ Answer **(IV)**



2. (8 points) Verify whether the following limits exist and, if they do, calculate their values. If they do not exist, provide a full justification for your conclusions.

(a) (4 points) $\lim_{(x,y)\to(0,0)} 2x^2\,y^3\sin(\cos(\ln(x^2+y^2)))$

Let $f(x,y) = 2x^2\,y^3\sin(\cos(\ln(x^2+y^2)))$. Note that

$$-1 \le \sin(\cos(\ln(x^2+y^2))) \le 1$$

for all (x,y) in the domain of $f(x,y)$, since $-1 \le \sin(t) \le 1$ for all real numbers t. Since $2x^2\,|y|^3$ is always non-negative, we can multiply through to get

$$-2x^2\,|y|^3 \le 2x^2\,|y|^3\sin(\cos(\ln(x^2+y^2))) \le 2x^2\,|y|^3.$$

By continuity, we know

$$\lim_{(x,y)\to(0,0)} -2x^2\,|y|^3 = 0 \text{ and } \lim_{(x,y)\to(0,0)} 2x^2\,|y|^3 = 0.$$

Thus, applying the Squeeze Theorem, we know the limit of $2x^2\,|y|^3\sin(\cos(\ln(x^2+y^2)))$ is 0. This means

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = 0.$$

Since $-|f(x,y)| \le f(x,y) \le |f(x,y)|$, using the Squeeze Theorem again tells us that

$$\lim_{(x,y)\to(0,0)} 2x^2\,y^3\sin(\cos(\ln(x^2+y^2))) = 0.$$

(b) (4 points) $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$

Consider the following two paths in the domain of the function:

1. Consider the direction along $y = 0$.

$$\lim_{(x,y)\to(0,0)} \left.\frac{2xy^2}{x^2+y^4}\right|_{y=0} = \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0.$$

   Note that $y = 0$ is a valid direction as it passes through $(0,0)$.

2. Consider the direction along $x = y^2$.

$$\lim_{(x,y)\to(0,0)} \left.\frac{2xy^2}{x^2+y^4}\right|_{x=y^2} = \lim_{(x,y)\to(0,0)} \frac{2y^4}{2y^4} = \lim_{(x,y)\to(0,0)} \frac{2}{2} = 1.$$

   Note that $x = y^2$ is a valid direction as it passes through $(0,0)$.

As the limit gives us two different values via two different valid directions, the limit cannot exist.

3. (9 points) (a) (5 points) Find the equation of the plane uniquely determined by the three points $P_1 = (1, 0, 0), P_2 = (0, 1, 0)$ and $P_3 = (0, 0, 1)$. Express your answer in the form $Ax + By + Cz = D$. Show all your work.

Let

$$\vec{v} = P_2 - P_1 = (0, 1, 0) - (1, 0, 0) = \langle -1, 1, 0\rangle,$$
$$\vec{u} = P_3 - P_1 = (0, 0, 1) - (1, 0, 0) = \langle -1, 0, 1\rangle.$$

The cross product between $\vec{u}$ and $\vec{v}$ gives a third vector $\vec{n}$ such that $\vec{n}\cdot\vec{u} = 0$ and $\vec{n}\cdot\vec{v} = 0$. In the other words, the cross product gives a vector $\vec{n}$ which is perpendicular to the both given vectors. Computing the cross product gives

$$\vec{n} = \langle -1, 1, 0\rangle \times \langle -1, 0, 1\rangle = \langle 1, 1, 1\rangle.$$

This means $\vec{n}$ is a normal vector of the plane. To find the equation of the plane, we also need any point on the plane. Pick any one points of P_1, P_2, P_3, e.g. $P_1 = (1, 0, 0)$. Then the equation of the plane is given by

$$1(x - 1) + 1(y - 0) + 1(z - 0) = 0,$$

which simplifies to $x + y + z = 1$.

(b) (4 points) Consider a different set of three provided points: $A = (1, 1, 0), B = (2, 2, 0)$, and $C = (4, 4, 0)$. Show that it is not possible to find a unique plane containing these three points. Give **two** justifications for this fact: One using the cross product, and the other using the dot product. (Hint: What does the dot product tell you about the angle between two vectors?)

By observation, we can see that the three points A, B, C are collinear (contained in a single line in three-dimensional space), and a line can't determine a unique plane. One way to justify this is by taking the cross product: If we form two vectors

$$\vec{v} = B - A = \langle 1, 1, 0\rangle \text{ and } \vec{u} = C - A = \langle 3, 3, 0\rangle$$

using the three points and take the cross product, then we get the zero vector. Thus, there is no uniquely determined normal vector and so there is no uniquely determined plane containing these three points.

Alternatively, we can use the dot product to compute the angle between two vectors $\vec{v}$ and $\vec{u}$ to verify that they are parallel. The angle between two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right).$$

The dot product of $\vec{u}$ and $\vec{v}$ gives a value of 6, while $\|\vec{u}\| = \sqrt{18}$ and $\|\vec{v}\| = \sqrt{2}$. Thus the expression for the angle simplifies to $\cos^{-1}\left(6/(\sqrt{18}\sqrt{2})\right)$, which is just $\cos^{-1}(1) = 0$ (where we only consider angles between 0 and 2π). Thus, the two vectors are parallel, and so the points A, B, C are indeed collinear.

4. (9 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x, y) = \ln(x^2 + y^2 - 4).$$

(a) (3 points) Find the domain and range of f.

The function $\ln(x^2 + y^2 - 4)$ needs $x^2 + y^2 - 4 > 0$ in order to be defined. This means the domain of the $f(x, y)$ is

$$x^2 + y^2 > 4.$$

The logarithm's output gives all values in $\mathbb{R}$, therefore the range of f is $\mathbb{R}$.

(b) (2 points) Describe the level curve $f(x, y) = c$ when $c = 1$.

$$f(x, y) = 1 = \ln(x^2 + y^2 - 4)$$

This means

$$1 = \ln(x^2 + y^2 - 4).$$

This is the same as

$$e^1 = e^{\ln(x^2+y^2-4)},$$

or

$$e = x^2 + y^2 - 4,$$

or

$$e + 4 = x^2 + y^2.$$

The equation $e + 4 = x^2 + y^2$ determines the level curve when $c = 1$. This equation defines a circle centered at $(0, 0)$ with radius $r = \sqrt{e + 4}$.

(c) (1 point) Describe the cross-section of $f(x, y)$ when $y = 2$.

If we fix $y = 2$, then the function $f(x, y)$ becomes to

$$\ln(x^2 + 4 - 4) = \ln(x^2).$$

The cross-section is determined by the function $\ln(x^2), x \neq 0$.

(d) (3 points) Is the graph of f a level surface of some function $g(x, y, z)$? Justify your answer.

Yes, since if we write $z = \ln(x^2 + y^2 - 4)$ then we can group all variables together: $z - \ln(x^2 + y^2 - 4) = 0$. Now let $g(x, y, z) = z - \ln(x^2 + y^2 - 4)$. Then the graph of f corresponds to the level surface $g(x, y, z) = 0$.

5. (8 points) The pressure of gas in a storage container, in atmospheres, is given by

$$P = f(n, T, V) = \frac{82nT}{V}$$

where n is the amount of gas, in kilomoles, T is the temperature of the gas, in Kelvin, and V is the volume of the storage container, in liters.

(a) (4 points) Find a formula for the level surface of f containing the point $(n, T, V) = (1, 200, 41)$ and explain the significance of this surface in terms of pressure.

If $(n, T, V) = (1, 200, 41)$, then we have

$$P = \frac{82 \cdot 1 \cdot 200}{41} = 400,$$

so the formula for the level surface containing this point is given by

$$\frac{82nT}{V} = 400.$$

Points on this surface represent the possible ways of achieving a gas pressure of 400 atmospheres through appropriate combinations of gas amounts, temperatures, and container volumes.

(b) (4 points) Describe the level surfaces of $P = f(n, T, V)$ algebraically for $P > 0$. Using your formula and viewing V as a function of n and T, what is the general shape of the cross-sections of the form $n = c$, where c is a constant?

Suppose $P = P_0 > 0$ is fixed to be a constant. Algebraically, these level curves are simply described by

$$P_0 = \frac{82nT}{V}.$$

Rearranging for V as a function of n and T gives us

$$V = \frac{82}{P_0}nT.$$

If $n = c$ is set to be a constant, then this expression becomes

$$V = \frac{82c}{P_0}T,$$

and so the cross-section for $n = c$ is a line with slope $\frac{82c}{P_0}$.

6. (10 points) Suppose $T = f(x, y)$ is a function which gives the temperature at point (x, y) in a room. Suppose we have the following table of values representing data collected about this function:

$x \backslash y$	1	3	5	6
0	2	4	6	8
2	3	5	7	9
4	4	6	8	10
6	5	7	9	11

(a) (4 points) Based on the above data, could T be represented by a linear function? If so, find an expression for T. If not, give a detailed explanation.

No. For example, the row corresponding to $x = 0$ does not represent a linear function, since the slope is not constant. Note that the same observation is true for all other rows of the table.

(b) (6 points) Approximate the partial derivatives $f_x(0, 1)$ and $f_y(2, 3)$ using the above table of values. Then give a practical interpretation of these values.

Based on the given data, the best possible approximation of $f_x(0, 1)$ is given by:

$$f_x(0,1) \approx \frac{f(2,1) - f(0,1)}{2-0} = \frac{3-2}{2} = \frac{1}{2}.$$

One possible approximation of $f_y(2, 3)$ is given by:

$$f_y(2,3) \approx \frac{f(2,5) - f(2,3)}{5-3} = \frac{7-5}{2} = \frac{2}{2} = 1.$$

A practical interpretation of $f_x(0, 1) \approx \frac{1}{2}$ says that if $y = 1$ is held constant and the x-position in the room changes from $x = 0$ to $x = 1$, then the temperature in the room increases by approximately 0.5 units. Similarly, $f_y(2, 3) \approx 1$ says that if $x = 2$ is held constant and the y-position changes from $y = 2$ to $y = 3$, then the temperature in the room increases by approximately 1 unit.

Do not tear this page off. This page is for additional work and will not be graded unless you clearly indicate it on the original question page.