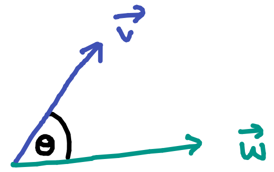
Lecture 12: Mon Sept 29th

§13.3 Dot product:



Defn: The dot product of $\vec{v}$ and $\vec{w}$ is given by

vector • vector = scalar

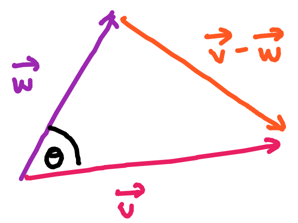
$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta \qquad (0 \le \theta \le \pi)$$ Geometric defn

$$= v_1w_1 + v_2w_2 + v_3w_3, \quad \text{where } \vec{v} = (v_1, v_2, v_3),\ \vec{w} = (w_1, w_2, w_3).$$ Algebraic defn

$$\vec{v}\cdot\vec{v} = \|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$ (Here, $\theta = 0$. So, $\cos(0) = 1$)

Why: Geometric defn = Algebraic defn?

$$\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$$ (Law of cosines)



$\|\vec{v}\|^2 = v_1^2+v_2^2+v_3^2$, $\|\vec{w}\|^2 = w_1^2+w_2^2+w_3^2$

$$\|\vec{v}-\vec{w}\|^2 = (v_1-w_1)^2 + (v_2-w_2)^2 + (v_3-w_3)^2$$
$$= v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2 + v_3^2 - 2v_3w_3 + w_3^2$$

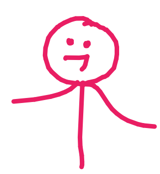
Cancel out $v_1^2, v_2^2, v_3^2, w_1^2, w_2^2, w_3^2$ on both sides. You are left with:

$$-2v_1w_1 - 2v_2w_2 - 2v_3w_3 = -2\|\vec{v}\|\|\vec{w}\|\cos\theta$$
$$v_1w_1 + v_2w_2 + v_3w_3 = \|\vec{v}\|\|\vec{w}\|\cos\theta$$

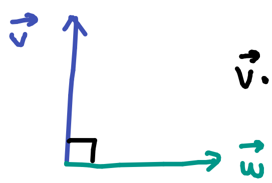
Properties of the dot product:

- $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$ (commutativity of •)
- $\vec{v}\cdot(\lambda\vec{w}) = \lambda(\vec{v}\cdot\vec{w}) = (\lambda\vec{v})\cdot\vec{w}$ (compatibility with scalar mult)
- $(\vec{v}+\vec{w})\cdot\vec{u} = \vec{v}\cdot\vec{u} + \vec{w}\cdot\vec{u}$ (distributivity)

How would you prove this?



Defn: Two non-zero vectors are orthogonal or perpendicular iff $\vec{v}\cdot\vec{w} = 0$



$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\pi/2) = 0$$

Example: Let $\vec{v} = 3\vec{i} + 3\vec{j} + 4\vec{k}$
$\vec{w} = 3\vec{i} + 4\vec{k}$

(a) Find angle θ between $\vec{v}$ and $\vec{w}$

(b) Find the value "a" so that $a\vec{i} + 2a\vec{j} + 3\vec{k}$ is perpendicular to $\vec{v}$.

A: a) $\vec{v}\cdot\vec{w} = (3,3,4)\cdot(3,0,4) = 9+0+16 = 25$

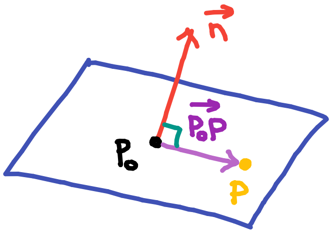
$\|\vec{v}\| = \sqrt{9+9+16} = \sqrt{34}$

$\|\vec{w}\| = \sqrt{9+16} = \sqrt{25}$

$$25 = \|\vec{v}\|\|\vec{w}\|\cos\theta$$
$$25 = \sqrt{34}\sqrt{25}\cos\theta$$
$$\frac{25}{\sqrt{34}\sqrt{25}} = \cos\theta, \quad \theta = \cos^{-1}\left(\sqrt{\frac{25}{34}}\right)$$

b) $(a, 2a, 3) \cdot (3, 3, 4) = 0$

$$3a + 6a + 12 = 0$$
$$a = -12/9 = -4/3$$



Defn: A normal vector to a plane is a vector perpendicular to it.

Fix $P_0 = (x_0, y_0, z_0)$ on the plane. Any point $P = (x, y, z)$ on the plane satisfies $\vec{n} \perp \vec{P_0P}$.

Point-Normal form of a plane

$$\vec{n} \cdot \vec{P_0P} = 0$$
$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (1)$$

Set $d = \vec{n} \cdot \vec{OP_0} = ax_0 + by_0 + cz_0$ in (1) gives:

General Equation of a plane

$$ax + by + cz = d$$

If you take (1) and divide by c: $\frac{a}{c}(x - x_0) + \frac{b}{c}(y - y_0) + (z - z_0) = 0$

$$z = -\frac{a}{c}(x - x_0) - \frac{b}{c}(y - y_0) + z_0 = 0$$

Set $m = -a/c$ and $n = -b/c$.

Point-slope form of a plane

$$z = z_0 + m(x - x_0) + n(y - y_0).$$

Example: What is the normal vector to the planes:

(a) $2x + 3y - 5z = 4$ A: $\vec{n} = (2, 3, -5)$

(b) $z = x - 2y$ A: $x - 2y - z = 0 \Rightarrow \vec{n} = (1, -2, -1)$

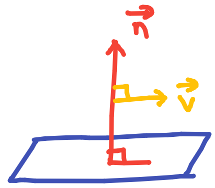
(c) $z = 1 - 2y$ A: $2y + z = 0 \Rightarrow \vec{n} = (0, 2, 1)$

Example: a) Find the plane perpendicular to $(1, 4, -7)$ and passing through $(2, -3, 5)$.

b) Find a vector parallel to this plane.

A: a) Use Point-Normal form: $\vec{n} \cdot \vec{P_0P} = 0$

$$(1, 4, -7) \cdot (x - 2, y + 3, z - 5) = 0$$
$$(x - 2) + 4(y + 3) - 7(z - 5) = 0$$
$$x - 2 + 4y + 12 - 7z + 35 = 0$$
$$x + 4y - 7z = -45$$



b) Such a vector $\vec{v}$ satisfies $\vec{n} \cdot \vec{v} = 0$

$$(1,-4,7) \cdot (v_1, v_2, v_3) = 0$$

$$v_1 - 4v_2 + 7v_3 = 0$$ (Any v_1, v_2, v_3 that satisfies this works!)

$\Rightarrow \vec{v} = (7,0,1)$ or $(-3,1,1)$ or $(4,1,0)$ etc.