Lecture 13: Wed Oct 1st

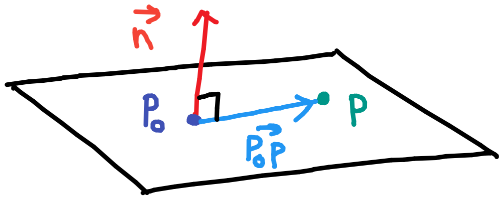
Recap: Point-Normal form of a plane: $\vec{n} \cdot \overrightarrow{P_0P} = 0$

This equations describes all (x,y,z) that lies on the plane.

(x,y,z)

(a,b,c) normal vector

fixed point (x_0, y_0, z_0)



$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

General equation: $ax+by+cz=d$

Point-slope form: $m(x-x_0)+n(y-y_0)+z_0=z$

Example: Find the equation of the plane parallel to $z=1-x+6y$ that contains the point $(1,1,1)$

A:

Example: Which of the following planes are parallel to each other? Which are perpendicular?

(a) $4x+6y-2z=4$

(b) $f(x,y)=2x+3y$

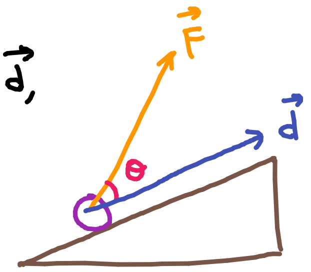
(c) $2x+3y+z=4$

(d) $4x-5y+7z=2$

A:

An application of the dot product:

When a force $\vec{F}$ acts on an object through displacement $\vec{d}$,



$$\begin{aligned} W = \text{work done by } \vec{F} &= \vec{F} \bullet \vec{d} \\ &= \vec{F} \cdot \vec{d} \\ &= \|\vec{F}\|\|\vec{d}\| \cos\theta \end{aligned}$$

Example: A force $\vec{F} = (1,3,-2)$ pushes a crate from point $P = (1,0,0)$ to $Q = (5,2,0)$.

a) What is the work done by $\vec{F}$?

b) Find another force $\vec{G} = (2,1,c)$ that gives the same work and has the smallest magnitude.

A:

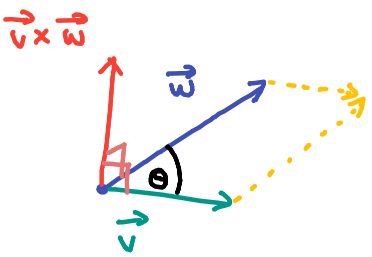
§13.4 Cross products:

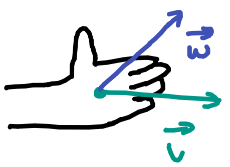
Given vectors $\vec{v}$ and $\vec{w}$,

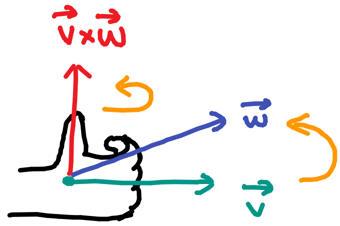
$\vec{v} \times \vec{w}$ = vector perpendicular to $\vec{v}$ and $\vec{w}$

$\|\vec{v} \times \vec{w}\|$ = area of parallelogram with edges/sides $\vec{v}$ and $\vec{w}$.

The direction of $\vec{v} \times \vec{w}$ is given right-hand rule.







Geometric definition

$$\begin{aligned} \vec{v} \times \vec{w} &= \|\vec{v} \times \vec{w}\| \vec{n} \longleftarrow \text{unit vector perpendicular to } \vec{v} \text{ and } \vec{w} \\ &= (\text{Area of parallelogram}) \vec{n} \\ &= (\|\vec{v}\| \|\vec{w}\| \sin\theta) \vec{n} \qquad (0 \le \theta \le \pi) \end{aligned}$$

Note: If $\vec{v} \parallel \vec{w}$, then $\theta = 0$ and $\vec{v} \times \vec{w} = \vec{0}$.

For the algebraic definition, we need determinants.

① Determinant of a 2x2 matrix

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

E.g. $\begin{vmatrix} 4 & -3 \\ 9 & 7 \end{vmatrix} = 4\cdot 7 - (-3)\cdot 9$
$= 28 - (-27)$
$= 55$

② Determinant of a 3x3 matrix

Note the sign!

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - b \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + c \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$\boxed{\text{Algebraic defn}}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - \vec{j} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \vec{k} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (v_2 w_3 - v_3 w_2)\vec{i} - (v_1 w_3 - v_3 w_1)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}.$$