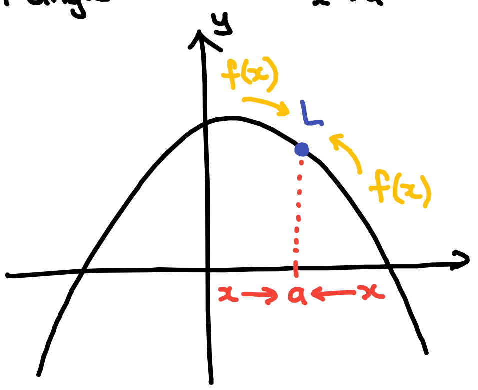
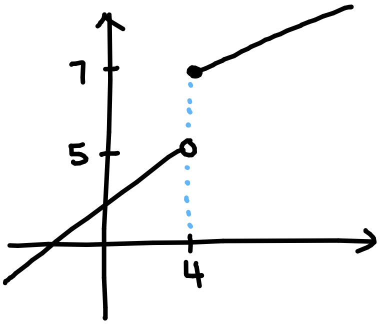
Lecture 9: Mon Sept 22nd

§12.6 Limits & Continuity

In single variables, $\lim_{x \to a} f(x) = L$ if $f(x)$ is close to L whenever x is close to a in any direction.



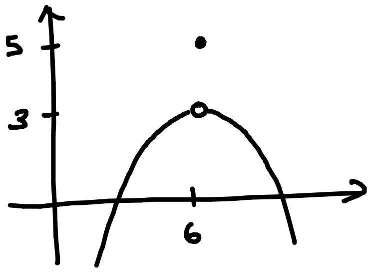


left hand limit:
$\lim_{x \to 4^-} f(x) = 5$

right hand limit:
$\lim_{x \to 4^+} f(x) = 7$

$4 \neq 7$. ∴ Limit does not exist.

$f(x)$ is continuous at $x=a$ if $\lim_{x \to a} f(x) = f(a)$.



$\lim_{x \to 6} f(x) = 3$ but $f(6) = 5$.

∴ f is not continuous.

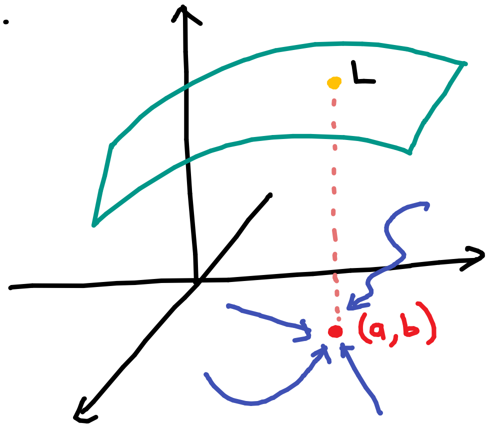
Two variables:

Defn: $\lim_{(x,y) \to (a,b)} f(x,y) = L$ if $f(x,y)$ is close to L whenever the distance between (x,y) and (a,b) is sufficiently close.

Defn: f is continuous at (a,b) if $\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$

f is ctns on a region R in xy-plane if it is ctns at each point of R.

In single variables, $x \to a$ from left or right. In two-variable calculus, $(x,y) \to (a,b)$ in many different ways.



If $\lim_{(x,y) \to (a,b)} f(x,y) = L$, then

$f(x,y) \to L$ for ALL paths $(x,y) \to (a,b)$.

Method to show non-existence of limits

- Find 2 different paths that lead to 2 different limits.

WARNING: If you find 2 paths that gives you the same limit L, that does not mean that L is the limit. This is because you need to check this for ALL paths.

Example: $g(x,y) = \begin{cases} \frac{x^2}{x^2+y^2}, & (x,y) \neq (0,0). \\ 0, & (x,y) = (0,0) \end{cases}$ Does the limit at $(0,0)$ exist?

A: Along $(0,y) \to (0,0)$: $\lim_{(0,y)\to(0,0)} g(x,y) = \lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0$
($x=0$ path)

Along $(x,0) \to (0,0)$: $\lim_{(x,0)\to(0,0)} g(x,y) = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$ $\therefore$ Limit does not exist.
($y=0$ path)

Example: Show that $f(x,y) = \frac{x+y}{x-y}$ $(x \neq y)$ does not have a limit. (Hint: use path $y = mx$).

A: $\lim_{(x,mx)\to(0,0)} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{x+mx}{x-mx} = \frac{1+m}{1-m}$.

Different m gives different limits.

E.g. $m=0$, $L=1$
$m=2$, $L=-3$ $\therefore$ Limit does not exist.

Example: Show that the following do not have limits.

a) $\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}}$

A: Path $y=0$: $\lim_{(x,0)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}} = 0$

Path $y=x$, $x>0$: $\lim_{(x,x)\to(0^+,0)} \frac{x}{\sqrt{2x^2}} = 1/\sqrt{2}$

$\therefore$ Limit does not exist.

b) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

A: Path $y=x^2$: $\lim_{(x,x^2)\to(0,0)} \frac{x^2 \cdot x^2}{x^4+x^4} = \frac{x^4}{2x^4} = \frac{1}{2}$

Path $y=0$: $\lim_{(x,0)\to(0,0)} \frac{x^2y}{x^4+y^2} = 0$.

$\therefore$ Limit does not exist.

Note: If you use path $x=0$, you need to say if $y>0$ or $y<0$ b/c they will give you different limits.

Path $x=0$, $y>0$: $\lim_{(0,y)\to(0,0^+)} \frac{y}{\sqrt{x^2+y^2}} = \lim_{(0,y)\to(0,0^+)} \frac{\overbrace{y}^{>0}}{\underbrace{\sqrt{0+y^2}}_{>0}} = 1.$

Path $x=0$, $y<0$: $\lim_{(0,y)\to(0,0^-)} \frac{y}{\sqrt{x^2+y^2}} = \lim_{(0,y)\to(0,0^-)} \frac{\overbrace{y}^{<0}}{\underbrace{\sqrt{0+y^2}}_{>0}} = -1.$

$\boxed{\text{Method to show existence:}}$

① Squeeze Theorem: If $g(x,y) \leq f(x,y) \leq h(x,y)$ and $\underbrace{\lim_{(x,y)\to(a,b)} g(x,y)}_{=L} \leq \lim_{(x,y)\to(a,b)} f(x,y) \leq \underbrace{\lim_{(x,y)\to(a,b)} h(x,y)}_{=L}$
then, $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

② Continuity of known functions (like polynomials, trig, exp)

- composition of ctns = ctns
- If f, g ctns and $g \neq 0$, f/g ctns.

<u>Example</u>: Find $\lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right)$

A: Since $-1 \leq \sin\left(\frac{1}{x^2+|y|}\right) \leq 1$,

$$-x^4 \leq x^4 \sin\left(\frac{1}{x^2+|y|}\right) \leq x^4$$

$$0 = \lim_{(x,y)\to(0,0)} -x^4 \leq \lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right) \leq \lim_{(x,y)\to(0,0)} x^4 = 0$$

$$\therefore \lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right) = 0.$$