Lecture 5:

Last time : Quadric surfaces
: Cylinders

Today: 1) How to find the contours algebraically
2) Relate contour diagram to its algebraic and graphical representations

Recall: Contour lines / level curves are obtained from a surface by slicing it with horizontal planes.

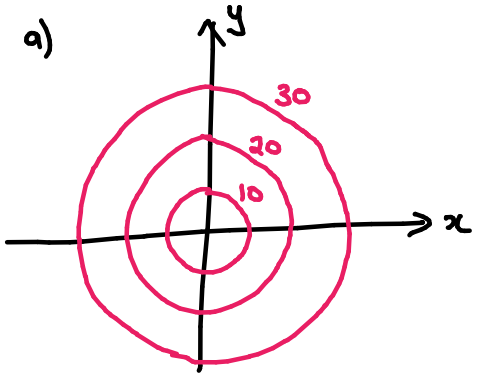
Contour diagram = collection of level curves labelled by function values $z = c$.
= drawn for equally spaced values of z.

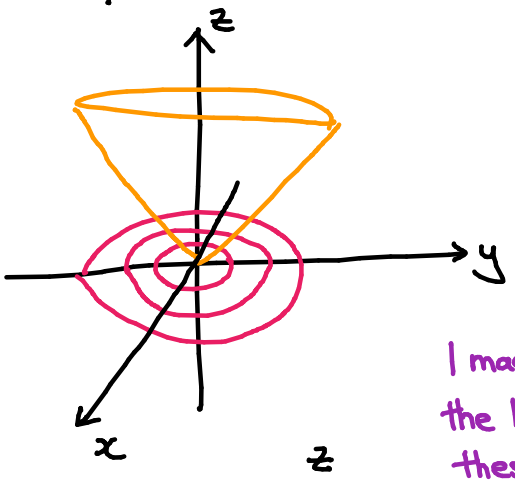
Q: How do you obtain contour lines algebraically?

A: The equation of a contour of f at height c is $f(x,y) = c$.

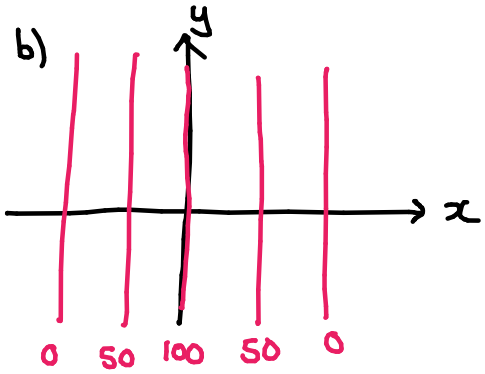
Intuition: What types of surfaces correspond to the contour maps?

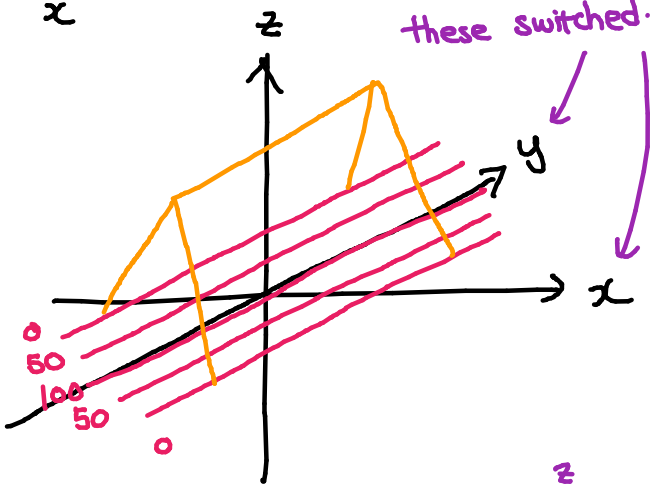
a)



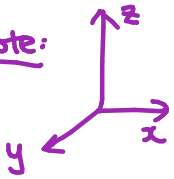


b)

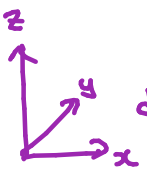




I made a mistake during the lecture and had these switched. This is correct.

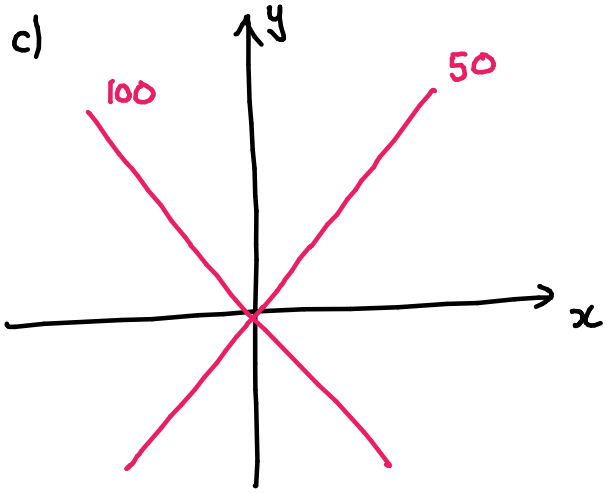
Note: 

does not follow right hand rule. ✗



does ✓

c)



Impossible b/c if it was, f maps (0,0) to 2 different z-values $z = 50 \& 100$.

Example: Find the equations for the contours of $f(x,y) = x^2+y^2$ and draw a contour diagram. Relate this to the graph of f.

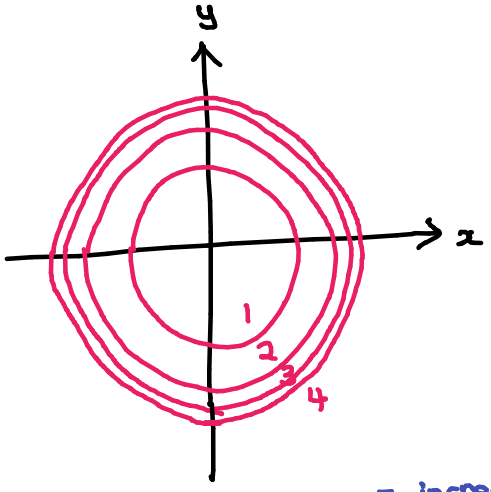
A: Let $f(x,y) = k = x^2+y^2$

$k=1$: $1 = x^2+y^2$ circle of radius 1

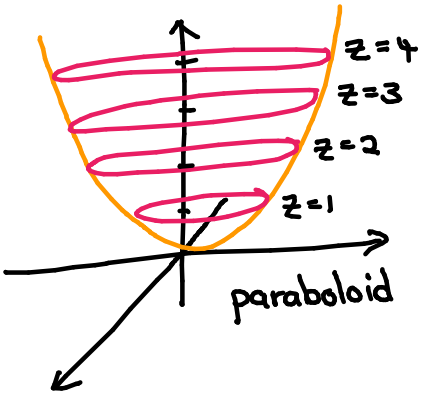
$k=2$: $2 = x^2+y^2$ circle of radius $\sqrt{2}$

$k=3$: $3 = x^2+y^2$ circle of radius $\sqrt{3}$

$k=4$: $4 = x^2+y^2$ circle of radius 2



Contours growing closer as z increases
⇒ surface gets steeper as z increases.



Example: Find the equations for the contours of $f(x,y) = \sqrt{x^2+y^2}$ and draw a contour diagram. Relate this to the graph of f.

A: Let $f(x,y) = k = \sqrt{x^2+y^2}$.

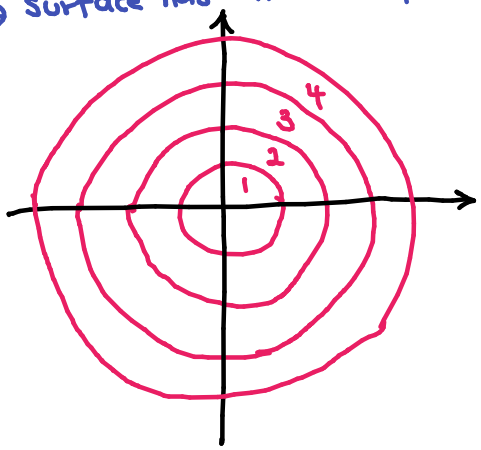
$k=1$: $1 = \sqrt{x^2+y^2} \Rightarrow 1 = x^2+y^2$ circle of radius 1

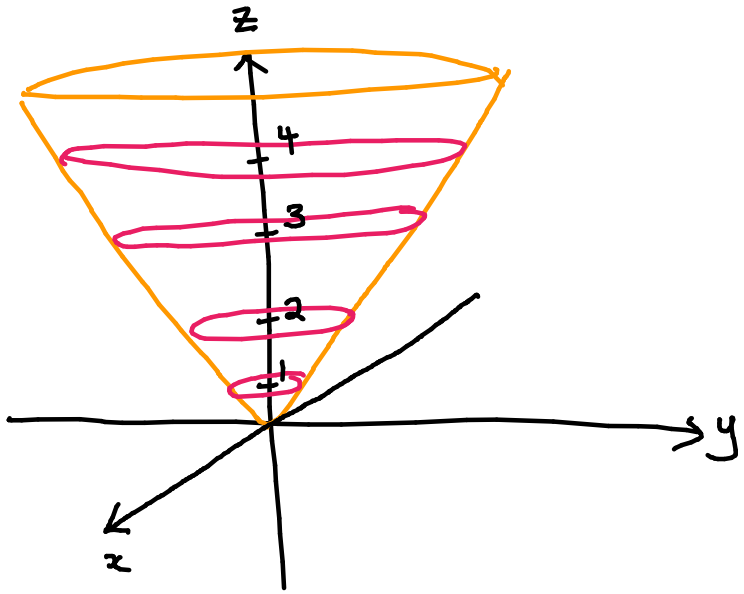
$k=2$: $2 = \sqrt{x^2+y^2} \Rightarrow 4 = x^2+y^2$ circle of radius 2

$k=3$: $3 = \sqrt{x^2+y^2} \Rightarrow 9 = x^2+y^2$ circle of radius 3

$k=4$: $4 = \sqrt{x^2+y^2} \Rightarrow 16 = x^2+y^2$ circle of radius 4

Contours are evenly spaced
⇒ surface has constant slope.





Example: Find the equations for the contours of $f(x,y) = 4x^2+y^2+1$ and draw a contour diagram. Relate this to the graph of f.

A: Let $k = f(x,y) = 4x^2+y^2+1$.

$$k-1 = 4x^2+y^2$$

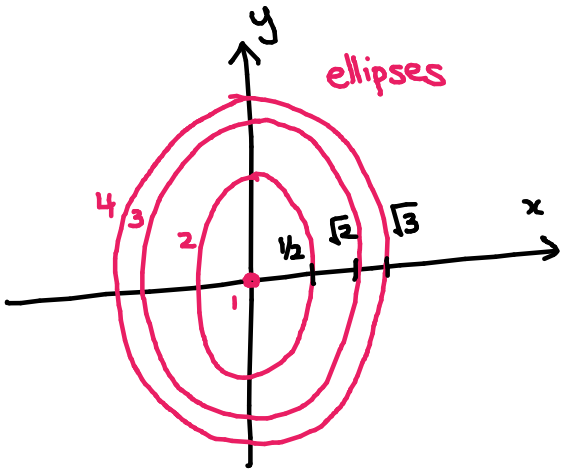
$$1 = \frac{x^2}{\frac{1}{4}(k-1)} + \frac{y^2}{(k-1)} \quad (\text{if } k \neq 1)$$

If $k=1$, $0 = 4x^2+y^2 \Rightarrow$ point 0

If $k=2$, $1 = \frac{x^2}{1/4} + \frac{y^2}{1}$ $\left(\begin{matrix} a=1/2 \\ b=1 \end{matrix}\right)$

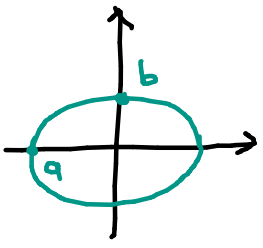
If $k=3$, $1 = \frac{x^2}{1/2} + \frac{y^2}{2}$ $\left(\begin{matrix} a=1/4 \\ b=\sqrt{2} \end{matrix}\right)$

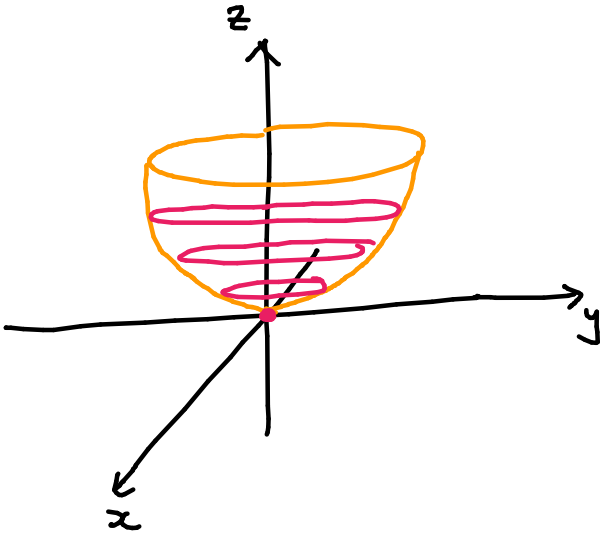
If $k=4$, $1 = \frac{x^2}{3/4} + \frac{y^2}{3}$ $\left(\begin{matrix} a=\sqrt{3}/2 \\ b=\sqrt{3} \end{matrix}\right)$



General equation for ellipses:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$





Example: Relate the table values of $f(x,y) = x^2 - y^2$ with its contour diagram.

This is Example 12.3.6 of textbook = how to relate table of values to contours.

If the (x,y)-cell of the table has value $f(x,y) = c$, then the point (x,y) on the contour diagram lies on the contour $z = c$.

(Did not manage to cover)

Example: Consider the production function $P = f(N,V) = cN^{\alpha}V^{\beta}$, where

N = # of workers

V = total value

P = work produced

c, α, β positive constants with $0<\alpha<1$, $0<\beta<1$.

What are the contours of this function?

A: Set $k = f(N,V) = cN^{\alpha}V^{\beta}$

$$\frac{k}{c}N^{-\alpha} = V^{\beta}$$

$$\left(\frac{k}{c}\right)^{1/\beta}N^{-\alpha/\beta} = V, \quad V \text{ is a power function of } N \text{ with negative exponents}$$