Lecture 13: Vectors, Work, Planes

Last Time: (1) Vectors

↳ $(+, -, \lambda)$

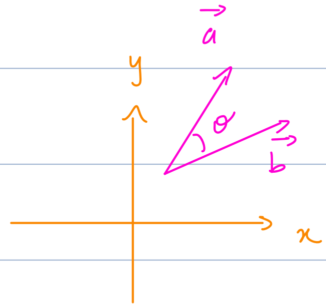
↳ $(\cdot, \times)$

dot product cross product

(2) Dot Product

↳ $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots$

↳ $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$ → for ex in $\mathbb{R}^2$



↳ "How aligned two vectors are"

↳ $\vec{a} \cdot \vec{b}$ is maximized when $\theta = 0^\circ, 180^\circ$

— " — minimized — " — $90^\circ, 270^\circ$

Today: (1) Projections

(2) Work

(3) Planes

(1) Projections

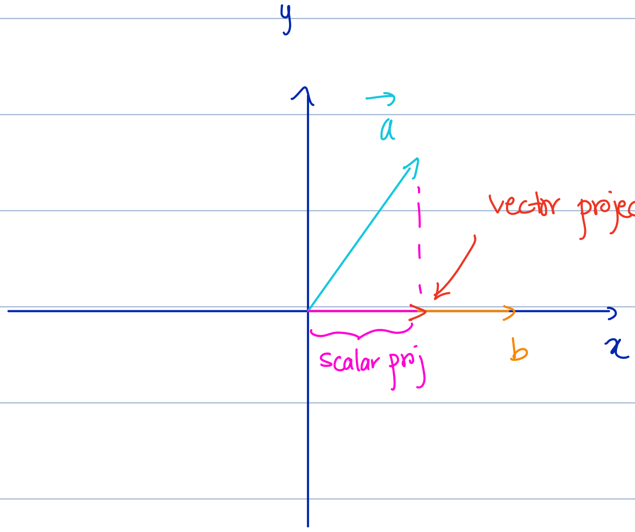
Def^n: Scalar Projection of $\vec{a}$ onto $\vec{b}$ $:= \dfrac{\vec{a} \cdot \vec{b}}{\|b\|} \in \mathbb{R}$

Def^n: Vector Projection of $\vec{a}$ onto $\vec{b}$:= $\text{proj}_{\vec{b}}\vec{a} = \underbrace{\left(\frac{\vec{a}\cdot\vec{b}}{\|b\|}\right)}_{\text{dot product}} \cdot \frac{\vec{b}}{\|b\|} \in \mathbb{R}^n$

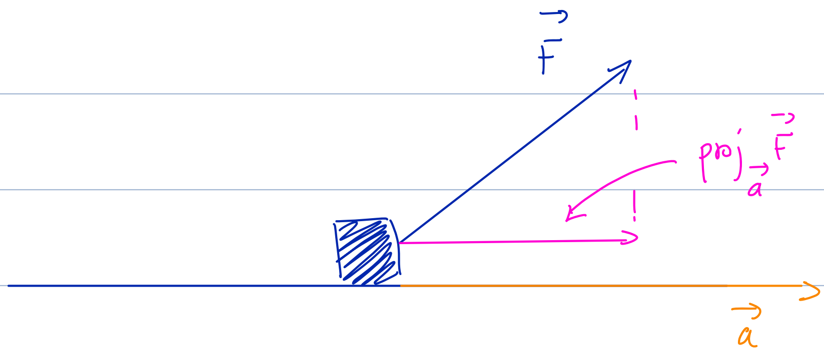
scalar multi.

$$= \left(\frac{\vec{a}\cdot\vec{b}}{\|b\|^2}\right)\cdot\vec{b}$$

$$= (\text{scalar proj})\cdot\frac{\vec{b}}{\|b\|}$$



Note: If $\vec{a}$ represents a force $(\vec{F} = m\vec{a})$ then $\text{proj}_{\vec{b}}\vec{a}$ represents the effective force in the direction of $\vec{b}$.



Corollary: If θ is the angle between two nonzero vectors $\vec{a}$ and $\vec{b}$ then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|a\|\|b\|} \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|a\|\|b\|}\right)$$

Corollary: Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal iff $\vec{a}\cdot\vec{b} = 0$

Ex. Find a unit vector that is $\perp$ to both $\vec{i}+\vec{j}$ and $\vec{i}+\vec{k}$

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ be a vector $\perp$ to $\vec{i}+\vec{j}$ & $\vec{i}+\vec{k}$.

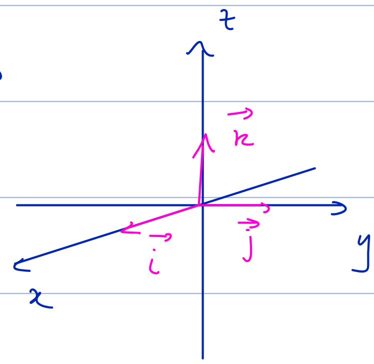
Then we will have that

(1) $(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot (\vec{i}+\vec{j}) = 0$

(2) $(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot (\vec{i}+\vec{k}) = 0$

(1) $\Rightarrow a_1 + a_2 = 0$ since $i \cdot j$ and $i \cdot k = 0$

(2) $\Rightarrow a_1 + a_3 = 0$



We also have $\|a\| = 1$ i.e. (3) $a_1^2 + a_2^2 + a_3^2 = 1$

Sub (1) & (2) into (3) $\Rightarrow a_1^2 + (-a_1)^2 + (-a_1)^2 = 1$

$\Rightarrow 3a_1^2 = 1$

$\Rightarrow a_1 = \pm \frac{1}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3}.$

So $a_1 = \pm \frac{\sqrt{3}}{3}$ and hence subbing into (1) & (2) we get that $a_2 = a_3 = \mp \frac{\sqrt{3}}{3}.$

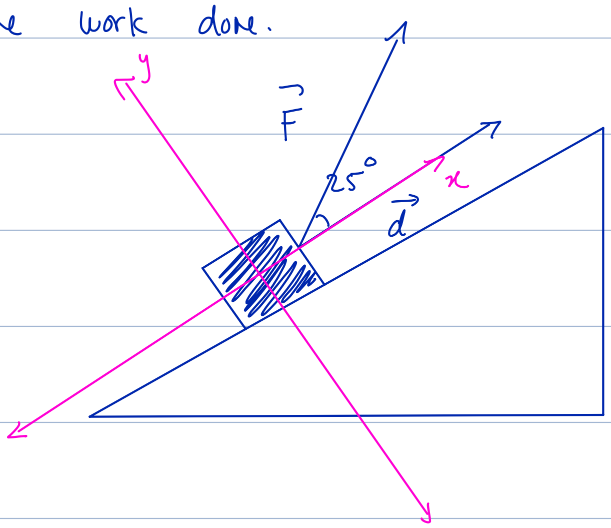
So $\vec{a} = \frac{\sqrt{3}}{3}\vec{i} - \frac{\sqrt{3}}{3}\vec{j} - \frac{\sqrt{3}}{3}\vec{k}$ or $\vec{a} = -\frac{\sqrt{3}}{3}\vec{i} + \frac{\sqrt{3}}{3}\vec{j} + \frac{\sqrt{3}}{3}\vec{k}.$

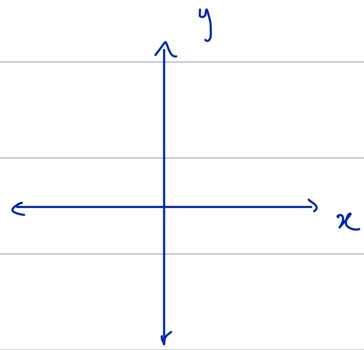
$\blacksquare$

Work

Def^n: The work W, done by a force $\vec{F}$ acting on an object through a displacement $\vec{d}$ is given by $W = \vec{F} \cdot \vec{d}$

Ex. A crate is hauled 6 meters up a ramp under a constant force of 300 N applied at an angle of 25° to the ramp. Use the def to calculate the work done.





Solution: Given $\|\vec{F}\| = 300\text{ N}$

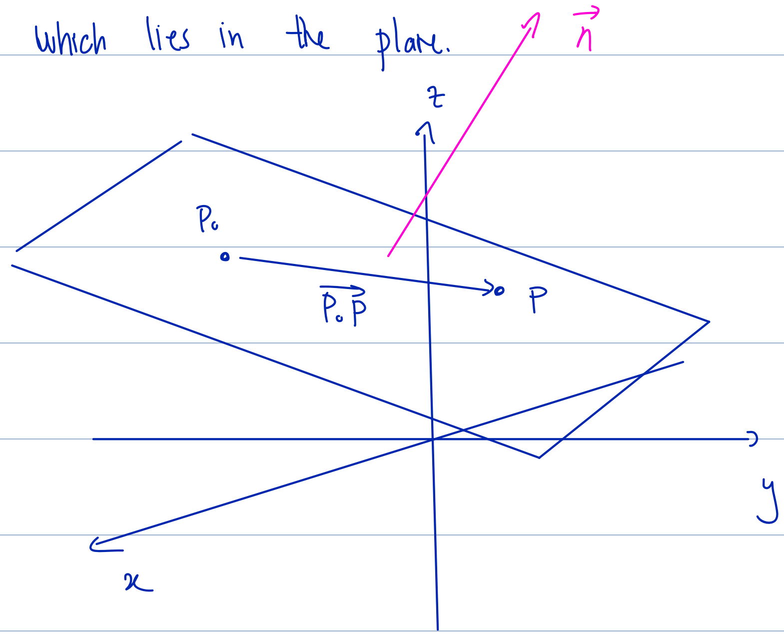
$\|\vec{d}\| = 6\text{ m}$

$\vec{F} \Rightarrow F_1\vec{i} + F_2\vec{j}$

$$
\begin{aligned}
W &= \vec{F} \cdot \vec{d} \\
&= \|F\|\,\|d\| \cos\theta \\
&= (300\text{ N})(6\text{ m})\cos(25°) \\
&\approx 1631 \text{ Joules.}
\end{aligned}
$$

Planes

Let $P_0 = (x_0, y_0, z_0)$ be a fixed pt and $P = (x, y, z)$ be an arbitrary pt which both lie in a plane. Then $\overrightarrow{P_0P} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$ is a vector which lies in the plane.



Consider a vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ which is normal to the plane.

Def^n: The equation of a plane in $\mathbb{R}^3$ that contains the pt (x_0, y_0, z_0) and has normal vector $\vec{n} = \langle a, b, c \rangle$ can be written as $\vec{n} \cdot \overrightarrow{P_0P} = 0$.

i.e. $\vec{n} \cdot \overrightarrow{P_0P} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Remark: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$$\Rightarrow \quad -c(z-z_0) = a(x-x_0)+b(y-y_0)$$

$$\Rightarrow \quad z = -\frac{a}{c}(x-x_0) - \frac{b}{c}(y-y_0) + z_0$$

$$\Rightarrow \quad z = m(x-x_0)+n(y-y_0)+z_0.$$

Note: To use the formula $\vec{n} \circ \overrightarrow{P_0P} = a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

we need a pt and a normal vector to the plane.

Ex. Find a normal vector to the plane $2x+3y-5z=4$.