Lecture 10: Wed Sept 24th

Recap: Show that $\lim_{(x,y)\to(0,0)} \frac{2x-y^2}{x+y^2}$ does not exist.

A: Path $(0,y)\to(0,0)$: $x=0$ $\quad \lim_{(0,y)\to(0,0)} \frac{0-y^2}{0+y^2} = -1$

Path $(x,0)\to(0,0)$: $y=0$ $\quad \lim_{(x,0)\to(0,0)} \frac{2x-0}{x+0} = 2$

$-1 \neq 2$. $\therefore$ Limit does not exist.

* Path $x=y^2$ works as well.

$$\lim_{(y^2,y)\to(0,0)} \frac{2y^2-y^2}{y^2+y^2} = \frac{1}{2}$$

* Since path $y=x$ gives

$$\lim_{(x,x)\to(0,0)} \frac{2x-x^2}{x+x^2} = 2,$$

paths $y=x$ and $y=0$ do not show DNE.

:> Proving existence

Recall: Squeeze theorem: If $g(x,y) \le f(x,y) \le h(x,y)$ and $\underbrace{\lim_{(x,y)\to(a,b)} g(x,y)}_{=L} \le \lim_{(x,y)\to(a,b)} f(x,y) \le \underbrace{\lim_{(x,y)\to(a,b)} h(x,y)}_{=L}$,

then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Helpful inequalities for Squeeze Thrm:

- $-1 \le \sin(x) \le 1$
  $-1 \le \cos(x) \le 1$
  $-\pi/2 \le \tan^{-1}(x) \le \pi/2$
- $-|x| \le x \le |x|$
- $x \le |x| = \sqrt{x^2} \le \sqrt{x^2+y^2}$
- $|xy| \le \frac{x^2+y^2}{2}$

Note: $x \nleq x^2$

E.g. $x = 1/2 \nleq x^2 = 1/4$

How can I tell if a limit exists or not?

Tip: Here is a recommended approach.

1. If f is ctns at (a,b), just plug in (a,b) into f to get the limit.

   E.g. $\lim_{(x,y)\to(0,0)} e^{x^2+y^2} = e^{0+0} = 1$

2. If you get $\frac{0}{0}$ or $0\times\infty$, try easy paths to show DNE.

3. If all of the easy paths give the same limit, try squeeze theorem.

Example: $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2}$

A: $x^2 \le 2x^2 \le 2x^2+y^2$

(1) $\frac{x^2}{2x^2+y^2} \le 1 \Rightarrow \frac{-x^2}{2x^2+y^2} \ge -1$ (2)

Since $-|y^3| \le y^3 \le |y^3|$,

$$-|y^3| \overset{(2)}{\le} \frac{-x^2|y^3|}{2x^2+y^2} \le \frac{x^2y^3}{2x^2+y^2} \le \frac{x^2|y^3|}{2x^2+y^2} \overset{(1)}{\le} |y^3|$$

$$\underbrace{\lim_{(x,y)\to(0,0)} -|y^3|}_{=0} \le \lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} \le \underbrace{\lim_{(x,y)\to(0,0)} |y^3|}_{=0}$$

By Squeeze Thrm, the limit is 0.

Example: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|xy|^3}{x^2+y^2}$

A: Express $\dfrac{|xy|^3}{x^2+y^2} = \dfrac{|xy|\,|xy|^2}{x^2+y^2}$

Claim: $|xy| \leq \dfrac{x^2+y^2}{2}$.

Indeed, $0 \leq (|x|-|y|)^2 = x^2 - 2|xy| + y^2$

$2|xy| \leq x^2+y^2$

$|xy| \leq \dfrac{x^2+y^2}{2}$

$0 \leq \dfrac{|xy|}{x^2+y^2} \leq \dfrac{1}{2}$

So, $0 \leq \dfrac{|xy|\,|xy|^2}{x^2+y^2} \leq \dfrac{|xy|^2}{2}$

$$\underbrace{\lim_{(x,y)\to(0,0)} 0}_{=0} \leq \lim_{(x,y)\to(0,0)} \frac{|xy|^3}{x^2+y^2} \leq \underbrace{\lim_{(x,y)\to(0,0)} \frac{|xy|^2}{2}}_{=0}$$

$\therefore$ by Squeeze, the limit is 0.

Example: Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

A: Firstly, $x \leq |x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$,

① $\dfrac{|x|}{\sqrt{x^2+y^2}} \leq 1 \Rightarrow$ ② $\dfrac{-|x|}{\sqrt{x^2+y^2}} \geq -1$

Since $-|xy| \leq xy \leq |xy|$,

$$-|y| \overset{②}{\leq} \frac{-|x||y|}{\sqrt{x^2+y^2}} = \frac{-|xy|}{\sqrt{x^2+y^2}} \leq \frac{xy}{\sqrt{x^2+y^2}} \leq \frac{|xy|}{\sqrt{x^2+y^2}} = \frac{|x|\,|y|}{\sqrt{x^2+y^2}} \overset{①}{\leq} |y|$$

Take limits: $$\underbrace{\lim_{(x,y)\to(0,0)} -|y|}_{=0} \leq \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \leq \underbrace{\lim_{(x,y)\to(0,0)} |y|}_{=0}$$

By Squeeze thm, the limit is 0.

Example: Are the following functions ctns at $(0,0)$? What is the limit at $(0,0)$?

a) $\sin(x+y)$

A: Yes, it is ctns because sin is ctns and a composition of ctns functions is ctns.
The limit is $\sin(0+0) = 0$.

b) $\dfrac{x+y+1}{\sin(y)+2}$

A: Yes. Since $-1 \leq \sin(y) \leq 1$, the denominator never vanishes for all (x,y).
Since the numerator and denominator is ctns, the fraction $\dfrac{x+y+1}{\sin(y)+2}$ is ctns.
The limit is $\dfrac{0+0+1}{0+2} = \dfrac{1}{2}$.

c) $f(x,y) = \dfrac{x}{x^2+1}$.

A: Yes, the denominator never vanishes for all (x,y). Since the numerator and denominator are ctns, the fraction $\dfrac{x}{x^2+1}$ is.

The limit is $\dfrac{0}{0+1} = 0$.

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Example: Choose a value c to make the following function ctns.

$$f(x,y) = \begin{cases} x\tan^{-1}(x/y), & (x,y) \neq (0,0) \\ c, & (x,y) = (0,0) \end{cases}$$

A: $f(x,y) = x\tan^{-1}(x/y)$ is ctns at all $(a,b) \neq (0,0)$. Now, find $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

Use $-\pi/2 \leq \tan^{-1}(x/y) \leq \pi/2$.

If $x \geq 0$,

$$-\frac{\pi x}{2} \leq x\tan^{-1}(x/y) \leq \frac{\pi}{2}x$$

$$-\frac{\pi x}{2} \to 0, \quad \frac{\pi}{2}x \to 0 \quad \text{as } (x,y) \to (0,0).$$

If $x < 0$,

$$-\frac{\pi}{2}x \geq x\tan^{-1}(x/y) \geq \frac{\pi}{2}x \quad \text{(Inequalities flip)}$$

$$-\frac{\pi}{2}x \to 0, \quad \frac{\pi}{2}x \to 0 \quad \text{as } (x,y) \to (0,0).$$

In both cases, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

For f to be ctns, $f(0,0) = c = 0$.