Lecture 15: Cross Product & Partial Derivatives

Last Time: (1) Matrix Mult.

(2) Determinants

↳ Cross Product

↳ $\vec{a} \times \vec{b} = \det \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \overset{-}{a_1} & \overset{+}{a_2} & \overset{-}{a_3} \\ \overset{+}{b_1} & \overset{-}{b_2} & \overset{+}{b_3} \end{vmatrix}$

$= (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$

(3) $\vec{a} \times \vec{a} = 0$

↳ simple computation

(4) Recall: $V \times V \to V \Rightarrow \|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$

$V \cdot V \to \mathbb{R} \Rightarrow \vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

Need to know for T1.

$(\vec{a} \times \vec{b}) \perp$ to both $\vec{a}$ and $\vec{b}$

↳ compute $(\vec{a} \times \vec{b}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} = 0.$

Today: (1) Finish Cross Product

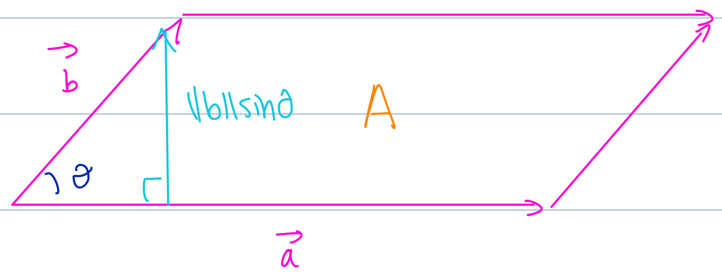
(2) Partial Derivatives.

Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$, then the length of the cross product $\vec{a} \times \vec{b}$ is given $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$.

Proof: Refer to textbook.

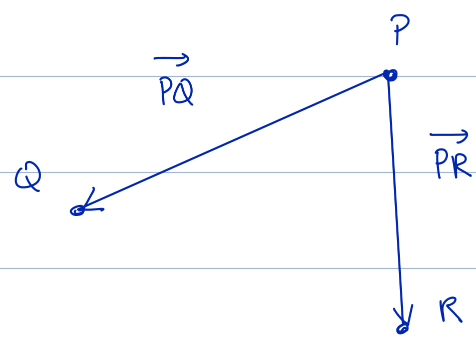
Corollary: Two non zero vectors $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a} \times \vec{b} = 0$.

Lemma: The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$



$$A = \|a\| \left(\|b\| \sin\theta \right)$$
$$= \|a\| \, \|b\| \sin\theta$$
$$= |\vec{a} \times \vec{b}|$$

Ex. Find a vector $\perp$ to the plane which contains the pts $P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$



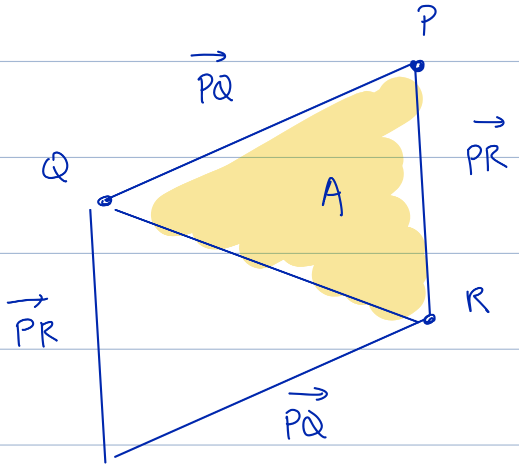
Form two vectors which lie in the plane using P, Q, and R.

$\vec{PR} = -5\vec{j} - 5\vec{k}$ $\qquad R - P$

$\vec{PQ} = -3\vec{i} + \vec{j} - 7\vec{k}$ $\qquad Q - P$

The vector $\vec{v} = \vec{PQ} \times \vec{PR}$ will be the vector $\perp$ to the plane containing P, Q, and R.

Ex. Find the area of the triangle w/ vertices P, Q, and R.



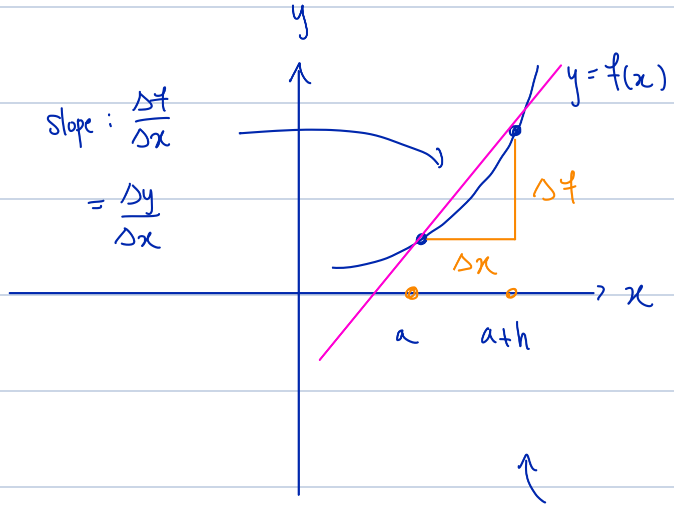
The area of the entire parallelogram is given by $\| \vec{PQ} \times \vec{PR} \|$ and so the area of the triangle is simply $\frac{1}{2} \| \vec{PQ} \times \vec{PR} \|$.

TT1

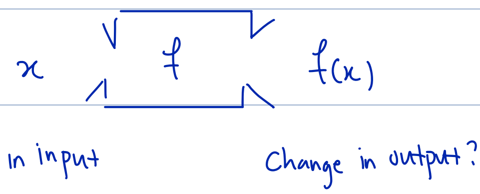
Partial Derivatives

Motivation: Let $f: \mathbb{R} \to \mathbb{R}$ be a fct $\bar{c}$ the following graph and let $a \in D_f$.

Slope : $\dfrac{\Delta f}{\Delta x} = \dfrac{\Delta y}{\Delta x}$

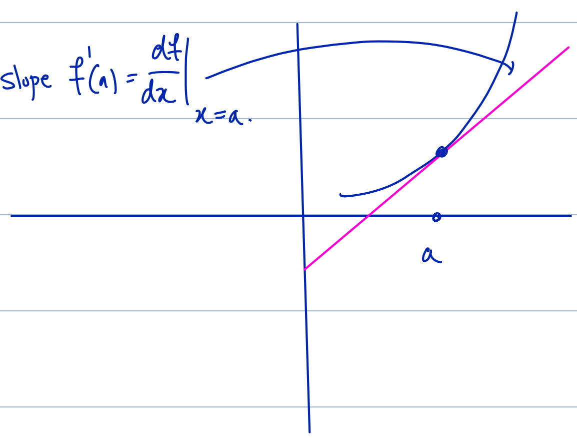


Input can only change along one axes.



$\Delta x \to 0$

the slope $f'(a) = \left.\dfrac{df}{dx}\right|_{x=a}$.

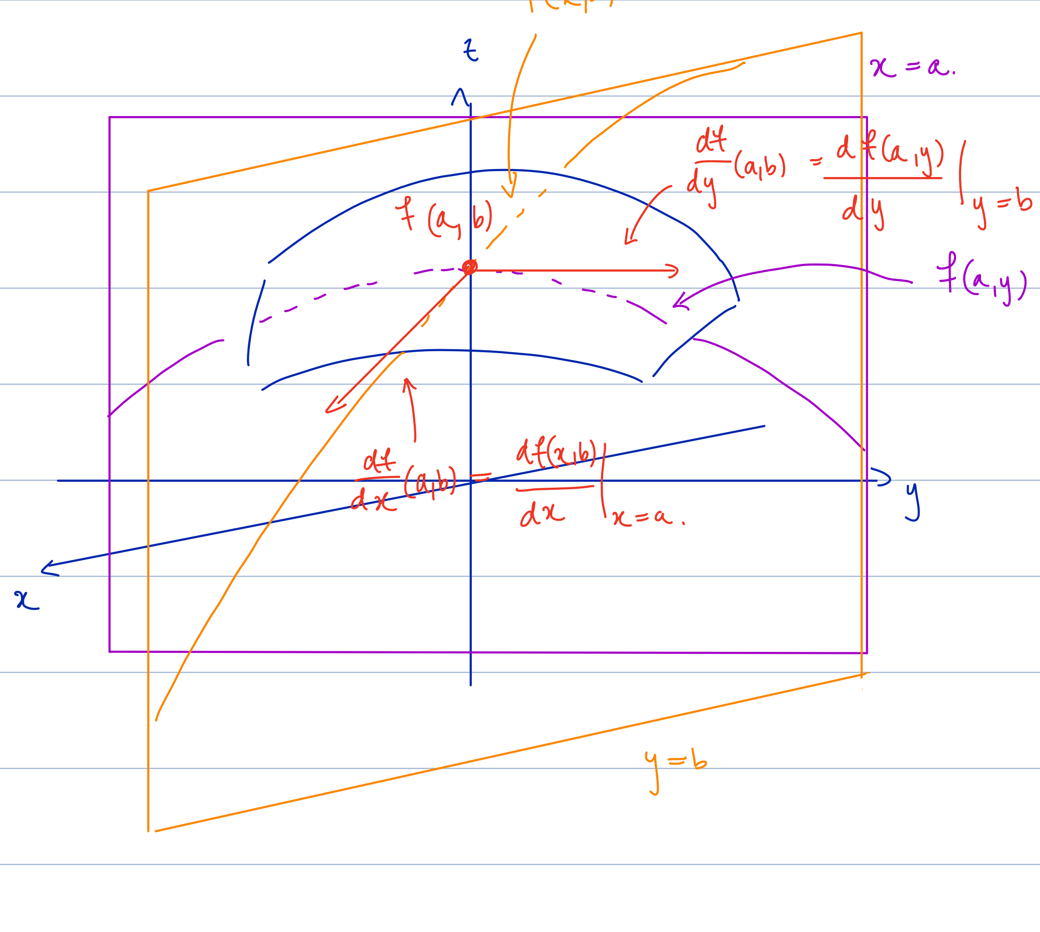


$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Q: How does it work for fcts of two variables?

$$(x,y) \to \boxed{f} \to f(x,y) \ ?$$

Answer: Similar, however we can change the input in many directions other than the x-direction.



Ex. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x^2 + 2y^2$.

Take $y=1$. Then we have $f(x,1) = x^2 + 2$.

Now set $g(x) = f(x,1)$. This is a fct only of x. Also this is a cross section of $f(x,y)$ to the plane $y=1$..

Consider the derivative of g w.r.t. x we have

$$g'(x) = \frac{d f(x,1)}{dx} = 2x \quad \text{so} \quad g'(1) = \left.\frac{d f(x,1)}{dx}\right|_{x=1} = 2.$$

This is in fact the slope of the tangent line to the cross section w.r.t. $y=1$ when $x=1$.

We call $g'(1)$ the partial derivative of f w.r.t. x at $(1,1)$ and denote it by $\frac{\partial f}{\partial x}(1,1)$ or $\partial_x f(1,1)$ or $f_x(1,1)$.

By the def of the derivative we have $g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$.

Now consider when $x=1$. Then we have $f(1,y) = 1 + 2y^2$

Set $h(y) = f(1,y)$

And consider $h'(y) = \frac{d f(1,y)}{dy} = 4y$

Similarly, the partial derivative of f w.r.t. y at $(1,1)$ is denoted $\frac{\partial f}{\partial y}(1,1)$ or $f_y(1,1)$.

$$f_y(1,1) = \lim_{h \to 0} \frac{f(1,1+h) - f(1,1)}{h}.$$

Def^n: Let $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and let $a \in D_f$.

1. The p.d. of f w.r.t. x at (a,b) denoted $f_x(a,b)$ is $f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$.

2. The p.d. of f w.r.t. y at (a,b) denoted $f_y(a,b)$ is $f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$.