University of Toronto – Faculty of Arts & Science – MAT235Y1: Multivariable Calculus

Term Test 1 – Fall 2024/Winter 2025

Family Name (PRINT): Sample Solutions

Given Name(s) (PRINT): ________________

Student Number: ________________

U of T Email: ________________

This exam contains **8** pages (including this cover page) and **6** problems. Once the exam begins, check to see if any pages are missing. There are **50** possible points to be earned in this exam.

- Duration: **90 minutes**
- **No aids or calculators are permitted on the exam.**
- Power off all cell phones, smartphones, and other electronic devices, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
- **Do not tear any pages off this exam.**
- **One scrap page is provided at the end.** This page will not be graded unless specifically indicated. Please enter all of your answers in the space provided.
- Do not write in the page margins. Make sure that your writing is dark enough to be readable.
- **Unsupported answers to long answer questions will not receive full credit**. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
- **Organize your work** in a reasonably neat and coherent way.
- You must use the methods learned in this course to solve all of the problems.

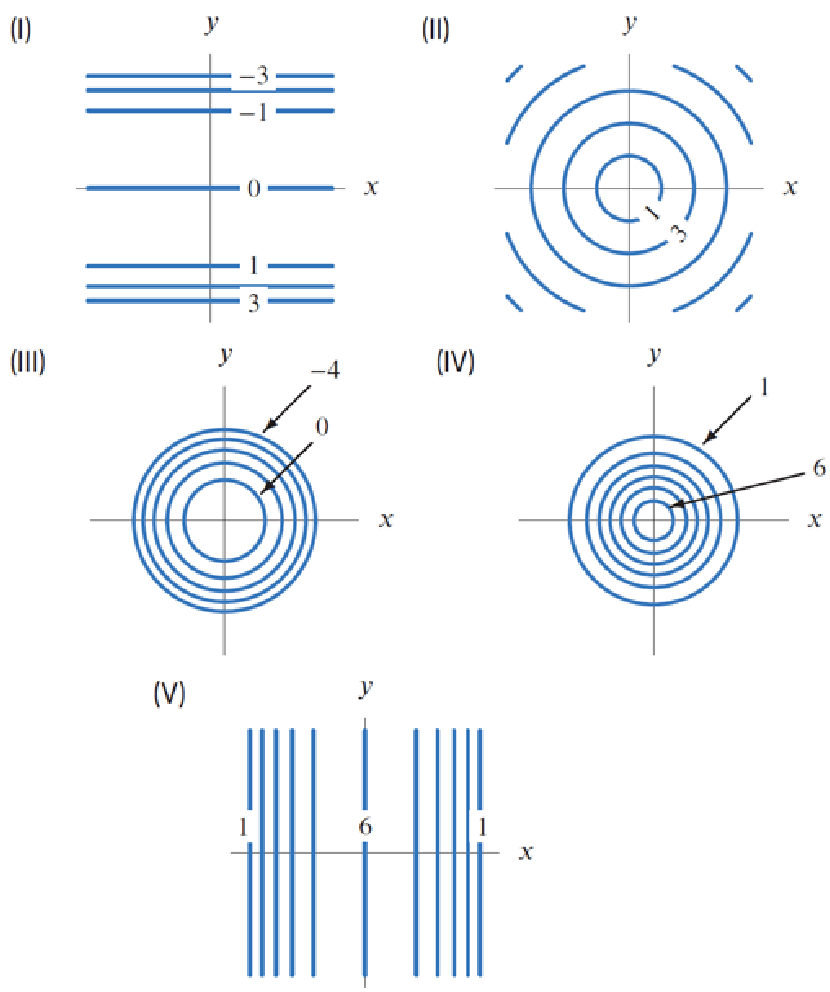
1. (6 points) Match the given equations with one of the following contour diagrams. Write **(I), (II), (III), (IV)** or **(V)** for each equation. Only your final answer will be graded. Each part is worth 1.5 marks.

(a) $f(x, y) = 2 - x^2 - y^2$ Answer ___(III)___

(b) $f(x, y) = 6 - x^2$ Answer ___(V)___

(c) $f(x, y) = \sqrt{x^2 + y^2}$ Answer ___(II)___

(d) $f(x, y) = -y^3$ Answer ___(I)___



2. (7 points) An oceanographer is studying the temperature of a coastal region. They collect data on the ocean temperature (T) in °C at various depths (d) in metres and distances from the shore (x) in kilometres.

   (a) (3 points) The collected data is presented in the following table of values. However, two data points T_1 and T_2 are missing:

$x \setminus d$	5	10	20	30
10	18	17	15	13
20	15	14	T_1	10
30	12	11	9	7
60	T_2	2	0	-2

   The oceanographer suspects that temperature is a linear function of depth and distance. Find values for T_1 and T_2 which make the above table linear. Put your final answer in the indicated box. Only your final answers will be graded for this part.

   $T_1 = \boxed{12}$ $T_2 = \boxed{3}$

   (b) (4 points) The oceanographer decides to incorporate time into their model, as they noticed temperature variations throughout the day. They now consider temperature as a function of depth (d), distance from shore (x), and time (t) in hours since midnight:

$$T(d, x, t) = 20 - 0.2d - 0.1x + 2\sin(\pi t/12).$$

   For a fixed time of day t, describe the general shape of the level surfaces of this function when $T = 15$°C. How do they change throughout the day? Give a brief description.

   Fix a time of day $t = c$. When $T = 15$, we can rearrange the above equation to get

$$0.2d + 0.1x = 5 + 2\sin(\pi c/12).$$

   If we view this equation in the xd-plane then this is just a line, and so in $\mathbb{R}^3$ the surface becomes a (vertical) plane (since the equation does not depend on the third variable t, as t is fixed). Thus, all of these level surfaces are parallel planes for each fixed time t.

   As t varies throughout the day, this changes the vertical shift of the line $0.2d + 0.1x = 5 + 2\sin(\pi c/12)$ (for each fixed $t = c$) in the xd-plane. This causes the planes to oscillate throughout the day according to the $\sin(\pi t/12)$ term.

3. (10 points) (a) (6 points) Find the set of points of continuity of the function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0),(1,1) \\ 1, & (x,y) = (1,1) \\ 2, & (x,y) = (0,0) \end{cases}$$

Justify your answer.

When $(x,y) \neq (0,0),(1,1)$, the function f is clearly continuous at (x,y), as it is the quotient of polynomials, which are continuous functions, and the denominator is nonzero.

At $(1,1)$, we have

$$\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(x,y)\to(1,1)} \frac{x^2y}{x^4+y^2} = \frac{1^2\cdot 1}{1^4+1^2} = \frac{1}{2} \neq 1 = f(1,1).$$

It follows that f is not continuous at $(1,1)$.

At $(0,0)$, consider the curve $C_k := \{(x,y) \in \mathbb{R}^2 : y = kx^2\}$ for some real number k. Taking the limit of f as (x,y) approaches $(0,0)$ along C_k, we get

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y) \text{ in } C_k}} f(x,y) = \lim_{\substack{x\to 0 \\ y=kx^2}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2\cdot kx^2}{x^4+(kx^2)^2} = \lim_{x\to 0}\frac{kx^4}{(1+k^2)x^4} = \frac{k}{1+k^2},$$

which implies that the limit of f at $(0,0)$ does not exists, since the value of $\frac{k}{1+k^2}$ varies as k varies. Hence f is not continuous at $(0,0)$.

Therefore, the set of points of continuity of the function f is all of $\mathbb{R}^2$ minus the points $(0,0)$ and $(1,1)$.

(b) (4 points) Use the Squeeze Theorem to evaluate the limit

$$\lim_{(x,y)\to(0,0)} 2x^2\, y^4 \sin(\ln(x^2+y^2)).$$

Give a full justification.

For any real number z, we have $-1 \leq \sin(z) \leq 1$. From this it follows that

$$-2x^2y^4 \leq 2x^2y^4\sin(\ln(x^2+y^2)) \leq 2x^2y^4 \quad \text{for all } (x,y) \neq (0,0).$$

Since

$$\lim_{(x,y)\to(0,0)} -2x^2y^4 = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} 2x^2y^4 = 0,$$

the Squeeze Theorem implies that

$$\lim_{(x,y)\to(0,0)} 2x^2y^4\sin(\ln(x^2+y^2)) = 0.$$

4. (9 points) Alice pushes a heavy crate across a warehouse floor. The force applied to the crate is represented by the vector $\vec{F} = \langle 4, 3, -2 \rangle$, and the crate moves along the path represented by the displacement vector $\vec{d} = \langle 2, 1, 0 \rangle$.

   (a) (3 points) Calculate the work done by Alice in moving the crate. (Hint: Recall that work is given by the dot product of force with displacement.)

   Work done is

$$\begin{aligned} \vec{F} \cdot \vec{d} &= \langle 4, 3, -2 \rangle \cdot \langle 2, 1, 0 \rangle \\ &= (4)(2) + (3)(1) + (-2)(0) = 11 \end{aligned}$$

   (b) (3 points) Find an expression for the angle between $\vec{F}$ and $\vec{d}$. Your final expression should not involve the terms $\vec{F}$ or $\vec{d}$. (You do not need to fully simplify your answer.)

   Let $0 \leq \theta \leq \pi$ be the angle in radians between $\vec{F}$ and $\vec{d}$.

$$\begin{aligned} 11 = \vec{F} \cdot \vec{d} &= \|\vec{F}\| \, \|\vec{d}\| \cos\theta \\ &= \sqrt{(4^2 + 3^2 + 2^2)} \, \sqrt{(2^2 + 1 + 0)} \cos\theta \\ &= \sqrt{29}\sqrt{5} \cos\theta = \sqrt{145} \cos\theta \end{aligned}$$

   So $\cos\theta = 11/\sqrt{145}$ hence

$$\theta = \arccos\left(11/\sqrt{145}\right)$$

   (c) (3 points) Suppose Bob decides to apply a force of $\vec{G} = \langle 4, 3, c \rangle$, where c is an unknown constant. The crate moves along the same path as before, given by the displacement vector $\vec{d} = \langle 2, 1, 0 \rangle$. Bob wants to do the same amount of work as Alice but wants to minimize the magnitude of $\vec{G}$. Which value of c should Bob choose? Give a brief justification.

   We require the work done to be the same:

$$\begin{aligned} 11 = \vec{G} \cdot \vec{d} &= \langle 4, 3, c \rangle \cdot \langle 2, 1, 0 \rangle \\ &= 11 + 0c \, (= 11) \end{aligned}$$

   Notice that any choice of c results in the same work done. In order to minimize $\|\vec{G}\| = \sqrt{4^2 + 3^2 + c^2} = \sqrt{25 + c^2}$, Bob should choose $c = 0$.

5. (9 points) Consider a plane P in three-dimensional space. Assume that we have the following information about P:

- Two points on the plane P are given: $A(1,2,-1)$ and $B(3,0,2)$.
- There is another plane Q with equation $2x - y + 3z = 6$ which is perpendicular to P.

(a) (2 points) Find a vector $\vec{v}$ on the plane P using the two given points.

Since the given two points A and B are both on the plane P, the vector $\vec{v}$ we can find on the plane P is $\vec{v} = \vec{AB} = (3-1, 0-2, 2-(-1)) = (2,-2,3)$.

Remark: we can also write $\vec{v} = \vec{AB} = 2\vec{i} - 2\vec{j} + 3\vec{k}$. $\vec{BA} = -2\vec{i} + 2\vec{j} - 3\vec{k}$ is also on the plane P. So we can also have $\vec{v} = \vec{BA}$.

(b) (4 points) Find a normal vector $\vec{n}$ to the plane P.

Let $\vec{n}_1$ be the normal vector to the plane Q. Then $\vec{n}_1 = 2\vec{i} - \vec{j} + 3\vec{k}$. Since the plane Q is perpendicular to the plane P, we have the normal vector $\vec{n}_1$ to the plane Q is parallel to the plane P.

Thus, the normal vector $\vec{n}$ to the plane P is perpendicular to both $\vec{v}$ and $\vec{n}_1$. Therefore, we have

$$\vec{n} = \vec{v} \times \vec{n}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 3 \\ 2 & -1 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & 3 \\ -1 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -2 \\ 2 & -1 \end{vmatrix} = -3\vec{i} + 2\vec{k}$$

Remark: it's fine to have $\vec{n} = \vec{n}_1 \times \vec{v} = 3\vec{i} - 2\vec{k}$. Also, it is fine to solve the system $\vec{n} \cdot \vec{n}_1 = 0$ and $\vec{n} \cdot \vec{v} = 0$ to find $\vec{n}$.

(c) (3 points) Using the above information, find an equation of the plane P. Express your final answer in the form $ax + by + cz = d$ for some constants a, b, c and d.

Since we have found the normal vector $\vec{n} = -3\vec{i} + 2\vec{k}$ to the plane P, we can pick either the point $A(1,2,-1)$ or the point $B(3,0,2)$ to plug in the scalar equation of the plane. The equation of the plane P is

$$-3(x-1) + 0(y-2) + 2(z-(-1)) = 0 \implies -3x + 2z = -5$$

or

$$-3(x-3) + 0(y-0) + 2(z-2) = 0 \implies -3x + 2z = -5$$

Remark: $3x - 2z = 5$ is also correct.

6. (9 points) A meteorologist wants to model the relationship between altitude (h, in kilometres), latitude (θ, in degrees north of the equator), and average annual temperature (T, in °C) in a particular region. Assuming $T = f(h, \theta)$, the collected data is presented in the following table:

$h \setminus \theta$	30	35	40	45	50
0	25.0	22.5	20.0	17.5	15.0
1	19.0	16.5	15.0	11.5	9.0
2	12.0	10.5	8.0	5.5	3.0
3	6.0	4.5	2.0	-0.5	-3.0
4	1.0	-1.5	-4.0	-6.5	-9.5

(a) (2 points) Give the definition of the partial derivative $f_\theta(a, b)$ for a general point (a, b).

Since $T = f(h, \theta)$, we have

$$f_\theta(a, b) = \lim_{t \to 0} \frac{f(a, b+t) - f(a, b)}{t}.$$

(b) (4 points) Estimate the values of the partial derivatives $f_\theta(4, 40)$ and $f_h(4, 40)$ using the given data.

One possible approximation of $f_\theta(4, 40)$ is

$$f_\theta(4, 40) \approx \frac{f(4, 45) - f(4, 40)}{45 - 40} = \frac{-6.5 - (-4.0)}{5} = \frac{-2.5}{5} = -0.5.$$

Similarly, for $f_h(4, 40)$ we have

$$f_h(4, 40) \approx \frac{f(4, 40) - f(3, 40)}{4 - 3} = \frac{-4.0 - 2.0}{1} = -6.$$

(c) (3 points) Using your answer from part (b), give a practical interpretation of the value $f_h(4, 40)$. Include units in your interpretation.

Using the fact that

$$f_h(4, 40) \approx -6 \, \frac{°\mathrm{C}}{\mathrm{km}},$$

one possible interpretation is the following: When $\theta = 40$ is fixed and the altitude h increases from 4 km to 5 km, then the average annual temperature decreases by approximately 6° C. Alternatively, we can also say that if $\theta = 40$ and h decreases from 4 to 3 km, then the average annual temperature increases by approximately 6° C.

Do not tear this page off. This page is for additional work and will not be graded unless you clearly indicate it on the original question page.