Lecture 17: Fri Oct 10th

How to take partial derivatives: $f(x,y) = x^2y^3$

$f_x = 2xy^3, \quad f_y = x^2(3y^2)$

Treat one variable as a constant and diff.

Recap: Find the partial derivatives.

1) $f(x,y) = xe^{\sqrt{xy}}$

A: $f_x = u\frac{dv}{dx} + v\frac{du}{dx}$

$$x\frac{d}{dx}\left(e^{\sqrt{xy}}\right) + e^{\sqrt{xy}}\frac{dx}{dx}$$

$$= x\frac{d}{dx}(\sqrt{xy})e^{\sqrt{xy}} + e^{\sqrt{xy}}$$

$$= \frac{x\sqrt{y}}{2\sqrt{x}}e^{\sqrt{xy}} + e^{\sqrt{xy}}$$

$$= e^{\sqrt{xy}}\left(\frac{\sqrt{xy}}{2} + 1\right)$$

$f_y = \frac{df}{du}\frac{du}{dy}$

$* \; u = \sqrt{xy}$

$\frac{du}{dy} = \frac{\sqrt{x}}{2\sqrt{y}}$

$$= xe^{u}\frac{\sqrt{x}}{2\sqrt{y}}$$

$$= \frac{x\sqrt{x}\,e^{\sqrt{xy}}}{2\sqrt{y}}$$

More examples (if you need it!)

2) $f(x,y) = z\ln(y\cos x)$

A: $f_x = \frac{z}{y\cos x}(-y\sin x) = -z\tan x$

$f_y = \frac{z}{y\cos x}(\cos x) = \frac{z}{y}$

$f_z = \ln(y\cos x)$

3) $f(x,y) = x^7 + 2^y + x^y$

A: Recall: $a^b = e^{\ln(a^b)} = e^{b\ln(a)}$

$f(x,y) = x^7 + e^{y\ln(2)} + e^{y\ln(x)}$

$f_x = 7x^6 + \frac{y}{x}e^{y\ln(x)} = 7x^6 + yx^{y-1}$

$f_y = \ln(2)e^{y\ln(2)} + \ln(x)e^{y\ln(x)}$

$= \ln(2)\,2^y + \ln(x)\,x^y$

Review:

§12

12.1

12.2 Graphs and cross-sections in $x+y$ (definition, drawing)

- common surfaces (spheres, paraboloid, cones, planes)
- general equations of circles, ellipses, hyperbolas, trig, polynomial, exp, log

completing the □

12.3 Contours (definition, drawing, matching)

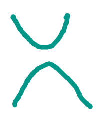
ellipses: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

12.4 Planes

- linear functions
- linear equation from table
- linear equation from contour diagram
- properties of tables of linear functions

hyperbolas: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$



	12.5	Level surface (definition, drawing, how do they vary) - matching level surfaces - interpretation
	12.6	Limits - DNE: show 2 paths with 2 different limits - Exist: Squeeze Thm (helpful inequalities) Continuity - Find where f is ctns - Find c that makes f ctns
§13	13.1, 2	Vectors - displacement vectors - $\lVert\cdot\rVert$, unit vectors - components of 2d vectors, simple physics situation
	13.3	Dot product $\vec{v}\cdot\vec{w}$ (Geometric, algebraic) - find θ - find orthogonal/parallel vectors - Work $W=\vec{F}\cdot\vec{d}$
	13.4	Cross product $\vec{v}\times\vec{w}$ (Geometric, algebraic) - 3 equations of plane - Find plane from 3 pts - Find area of parallelogram - Find plane parallel to a plane + 1pt - Find plane $\perp$ to a plane + 2 pts - Find plane containing a line + 1pt - When are planes $\perp$ or // or neither

Review:

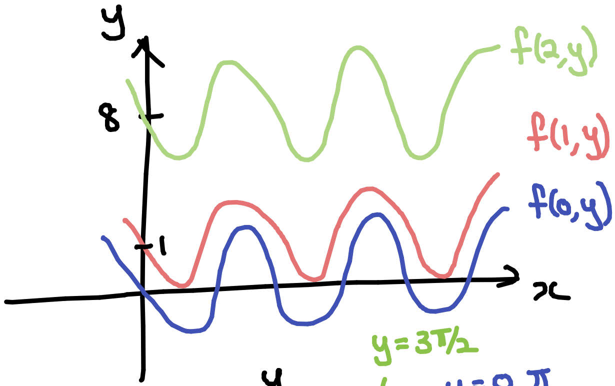
1) Draw at least 3 cross-sections of $z = x^3 - \sin(y)$.

A: For $x = a$, $f(a,y) = a^3 - \sin(y)$

$x = 0$, $f(0,y) = -\sin(y)$

$x = 1$, $f(1,y) = 1 - \sin(y)$

$x = 2$, $f(2,y) = 8 - \sin(y)$



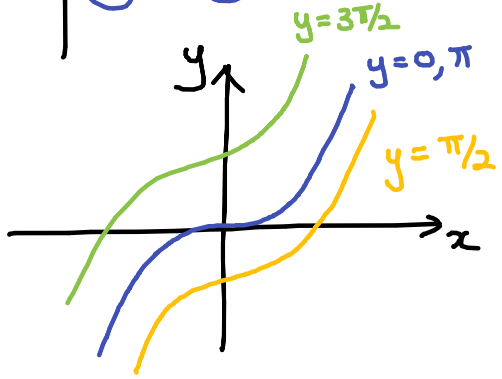
For $y = b$, $f(x,b) = x^3 - \sin(b)$

$y = 0$, $f(x,0) = x^3$

$y = \pi/2$, $f(x,\pi/2) = x^3 - 1$

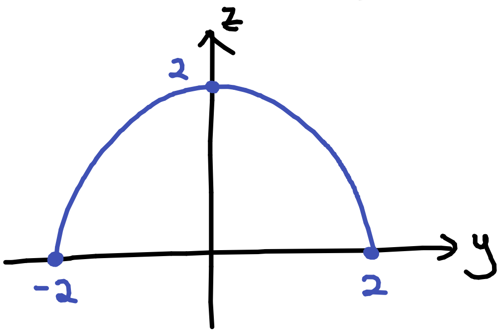
$y = \pi$, $f(x,\pi) = x^3$

$y = 3\pi/2$, $f(x, 3\pi/2) = x^3 + 1$

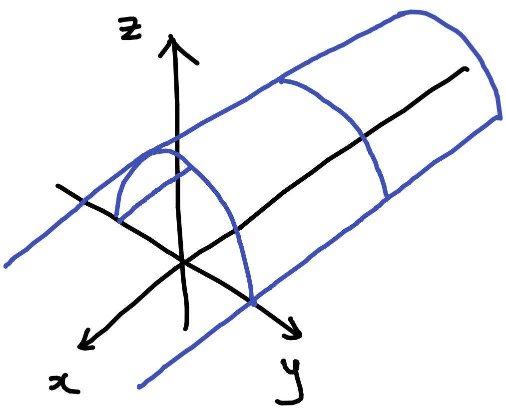


2a) An architect is designing a tunnel for cars.

If this is the cross-section for every fixed x, find the equation of the tunnel.



A: This is a half-cylinder of radius 2 along the x-axis.



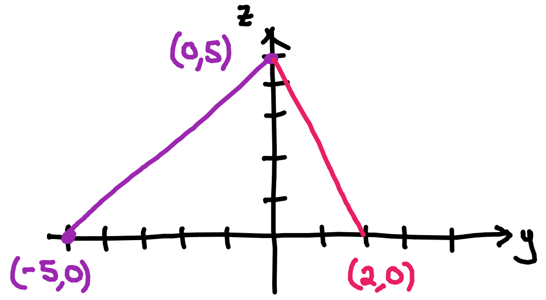
$y^2 + z^2 = 4$ with $z \geq 0$

OR

$z = \sqrt{4 - y^2}$

b) Here is another plan for a tunnel.

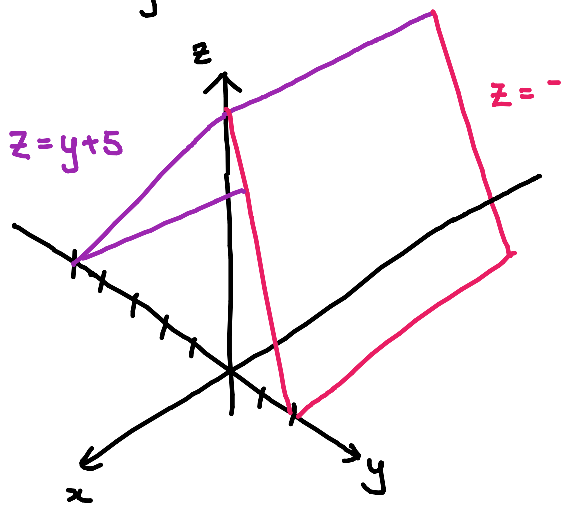
If this is the cross-section for every fixed x, find the equation of the tunnel.



A: Left: $m = \frac{5-0}{0--5} = 1$. Right: $m = \frac{0-5}{2-0} = \frac{-5}{2}$

$z = y+5$ with $0 \le z \le 5$ $\quad z = -\frac{5}{2}y + 5$ with $0 \le z \le 5$

The resulting surfaces are



$$z = \begin{cases} y+5, & -5 \le y \le 0, \\ -5/2\, y + 5, & 0 \le y \le 5. \end{cases}$$

c) Draw at least 4 contours of part (b).

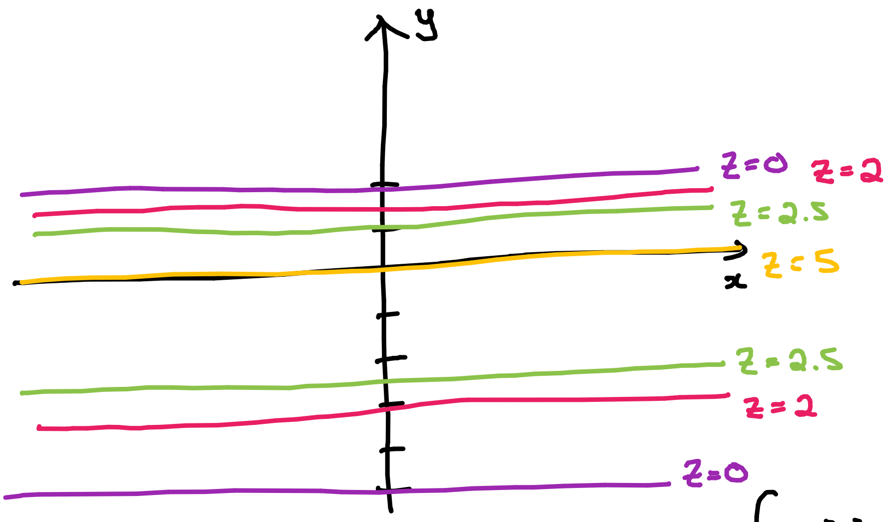
A: When $z=0$, Left: $0 = y+5$
$-5 = y$
Right: $0 = -\frac{5}{2}y + 5$
$-5 = -\frac{5}{2}y \Rightarrow y = 2$

When $z=5$, Left: $5 = y+5$
$0 = y$
Right: $5 = -\frac{5}{2}y + 5$
$0 = y$

When $z=1$, Left: $1 = y+5$
$-4 = y$
Right: $1 = -\frac{5}{2}y + 5$
$-4 = -\frac{5}{2}y \Rightarrow y = \frac{8}{5} = 1\,3/5$

When $z=2$, Left: $2 = y+5$
$-3 = y$
Right: $2 = -\frac{5}{2}y + 5$
$-3 = -\frac{5}{2}y \Rightarrow y = \frac{6}{5} = 1\,1/5$

When $z=2.5$, Left $2.5 = y+5$
$-2.5 = y$
Right $2.5 = -5/2\, y + 5$
$-5/2 = -5/2\, y$
$1 = y$



3) Find the set of points of continuity of $f(\ y) = \begin{cases} \dfrac{3x^2y^2}{x^2-5y^2}, & (x,y) \neq (0,0), (1,1) \\ 5 & (x,y) = (1,1) \\ 1 & (x,y) = (0,0) \end{cases}$

A: $f(x,y)$ is ctns on $\mathbb{R}^2 \setminus \{(0,0),(1,1)\}$ b/c the denom never vanishes & $f(x,y)$ is a quotient of ctns functions.

At $(1,1)$, $\displaystyle\lim_{(x,y)\to(1,1)} f(x,y) = \frac{3}{1+5} = \frac{3}{6} = \frac{1}{2} \neq f(1,1)$

To find $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$, use squeeze theorem.

$0 \leq x^2 \leq x^2+y^2 \leq x^2+5y^2$.

$0 \leq \dfrac{x^2}{x^2+5y^2} \leq 1$

$0 \leq \dfrac{x^2(3y^2)}{x^2+5y^2} \leq 3y^2$

$\downarrow 0$ $\quad \downarrow 0$ as $(x,y) \to (0,0)$. $\therefore$ Limit $= 0 \neq f(0,0)$.

Therefore, the set of pts of continuity is $\mathbb{R}^2 \setminus \{(0,0),(1,1)\}$.

4a) Which pt is furthest from $(1,2,3)$? $A = (1,5,1)$
$B = (0,0,0)$
$C = (2,1,2)$

A: $\|\vec{PA}\| = \|(1,5,1)-(1,2,3)\| = \|(0,3,-2)\| = \sqrt{9+4} = \sqrt{13}$

$\|\vec{PB}\| = \|(0,0,0)-(1,2,3)\| = \|(-1,-2,-3)\| = \sqrt{1+4+9} = \sqrt{14}$

$\|\vec{PC}\| = \|(2,1,2)-(1,2,3)\| = \|(1,-1,-1)\| = \sqrt{3}$

Distance from $\vec{B}$ is largest.

b) Find a vector in the direction of the longest displacement vector with length 4.

A: First, scale to 1. $\vec{u} = \frac{\vec{PB}}{\|\vec{PB}\|} = \frac{(-1,-2,-3)}{\sqrt{14}}$

Then, scale to 4. $4\vec{u} = \frac{4(-1,-2,-3)}{\sqrt{14}}$.

5) Consider two temperature functions $f(x,y,z) = -\ln(x^2+z^2)$ and $g(x,y,z) = x^2+z^2$

a) Describe the level surfaces of f and explain their significance.

A: $c = -\ln(x^2+z^2)$

$-c = x^2+z^2$

$e^{-c} = x^2+z^2$ Cylinder centered around the origin of radius $e^{-c/2}$ along the y-axis.

These are the points with temperature c.

b) Describe the level surfaces of g and explain their significance.

A: $c = x^2+z^2$. Cylinder centred around the origin of radius $\sqrt{c}$ along the x-axis.

These are the points with temperature c.

c) As $c \to \infty$, how do these level surfaces of f and g change?

A: As $c \to \infty$, the level surfaces of f shrink exponentially while the level surfaces of g grow as a square root function.

d) Can these level surfaces be written as a graph of a 2-variable function?

A: No because for f, $z^2 = e^{-c} - x^2$.

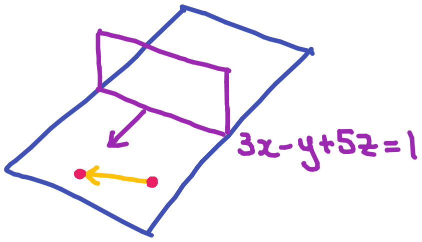
$z = \pm\sqrt{e^{-c} - x^2}$. We cannot express z as a single function.

For g, $z^2 = c - x^2$

$z = \pm\sqrt{c-x^2}$. We cannot express z as a single function.

6) Find the plane perpendicular to $3x - y + 5z = 1$ and containing $(1,0,-1)$, $(2,1,0)$.

A:



Let $\vec{v} = (3,1,5)$

$\vec{w} = (2,1,0) - (1,0,-1) = (1,1,1)$

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 5 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} -1 & 5 \\ 1 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= \vec{i}(-1-5) - \vec{j}(3-5) + \vec{k}(3+1)$$

$$= (-6, 2, 4).$$

Use $P_0 = (1,0,-1)$.

$\vec{n} \cdot \vec{P_0P} = 0$

$(-6,2,4) \cdot (x-1, y, z+1) = 0$

$-6(x-1) + 2y + 4(z+1) = 0$

$-6x + 2y + 4z = -10.$