# Lecture 14: Matrices & The Cross Product

Last Time: (1) Finished the dot product

(2) Work $\longrightarrow$ $W = \vec{F} \cdot \vec{d}$

$= \|F\| \|d\| \cos\theta$

(3) Planes $\longrightarrow$ $\vec{n} \cdot \overrightarrow{P_0P} = 0$

Lets go back to (3): Def: The eq of the plane in $\mathbb{R}^3$ that contains the pt $(x_0, y_0, z_0)$ and has normal vector $\vec{n} = \langle a, b, c \rangle$ can be written as $\vec{n} \cdot \overrightarrow{P_0P} = 0$. i.e.

$$\vec{n} \cdot \overrightarrow{P_0P} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

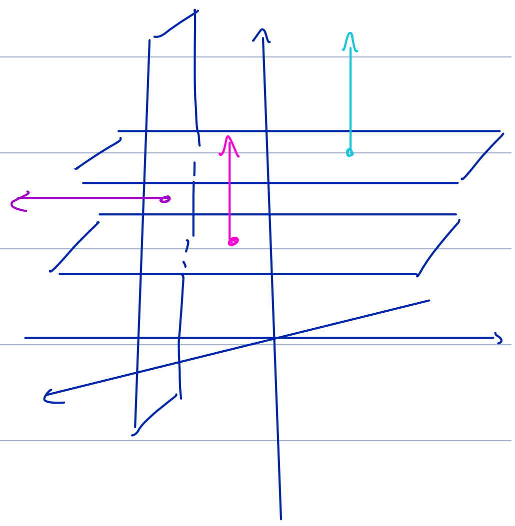
Ex. Find a normal vector to the plane

$2x + 3y - 5z = 4$

$\vec{n} = \langle 2, 3, -5 \rangle$.

$\vec{n}_\alpha = \alpha \langle 2, 3, -5 \rangle$ where $\alpha \in \mathbb{R} / \{0\}$.

Ex. Consider the following planes



(1) $4x + 6y - 2z = 4$ $\qquad \vec{n}_1 = \langle 4, 6, -2 \rangle$

(2) $f(x,y) = 2x + 3y$ $\qquad \vec{n}_2 = \langle 2, 3, -1 \rangle$

(3) $2x + 3y + z = 4$ $\qquad \vec{n}_3 = \langle 2, 3, 1 \rangle$

(4) $4x - 5y + 7z = 2$ $\qquad \vec{n}_4 = \langle 4, -5, 7 \rangle$

(a) Which of the planes are parallel to each other?

For 2 planes to be $\parallel$ their $\vec{n}$'s need to be scalar mult. of each other.

(1) & (2) are $\parallel$ since $\vec{n}_1 = 2\vec{n}_2$

(b) Which of the planes are perpendicular to each other?

For 2 planes to be $\perp$ their $\vec{n}$'s need to be s.t. $\vec{n}_1 \cdot \vec{n}_2 = 0$.

(3) & (4) are $\perp$ since $\vec{n}_3 \cdot \vec{n}_4 = 0$.

Today: (1) Matrices (Determinants)

(2) Cross Product

Def: An $m \times n$ matrix $A$ is a rectangular table of real numbers arranged in $m$ rows and $n$ columns.

Remark: If $m = n$ then the matrix is square.

Sometimes we use the notation $A = [a_{ij}]$

Properties: Let $A$ & $B$ be matrices then

(1) $C = A \pm B \Rightarrow c_{ij} = a_{ij} + b_{ij}$

(2) $C = \alpha A \Rightarrow c_{ij} = \alpha a_{ij}$

Only mat. of the order can be added/subtracted.

Def$^n$: The product $C = AB$ of a an $m \times n$ matrix $A$ and $n \times p$ matrix $B$ is the $m \times p$ matrix $C = [c_{ij}]$ where

$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$ where $i = 1, \dots, m$ and $j = 1, \dots, p$.

Ex. Consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$

$$C = AB = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (2)(3)+(1)(0) & (2)(2)+(1)(1) \\ (1)(3)+(0)(0) & (1)(2)+(0)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 5 \\ 3 & 2 \end{pmatrix}$$

Def$^n$: The determinant of a $2 \times 2$ matrix $A$ is the <u>real</u> number

$$\det(A) = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Def$^n$: The determinant of a $3 \times 3$ matrix $A$ is the real number defined by

$$\det(A) = \begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{vmatrix}$$

$$= +\, a_{11} \cdot \det \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \det \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \det \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Another Way:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$= (a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{33})$$

$$- (a_{13} a_{22} a_{31} + a_{11} a_{23} a_{32} + a_{12} a_{21} a_{33})$$

Def$^n$: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the cross product of $\vec{a}$ and $\vec{b}$ denoted $\vec{a} \times \vec{b}$ is the <u>vector</u>

$$\vec{a} \times \vec{b} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i}\,(a_2 b_3 - a_3 b_2) - \vec{j}\,(a_1 b_3 - a_3 b_1) + \vec{k}\,(a_1 b_2 - a_2 b_1)$$

Ex. Compute the cross product of $\vec{a}$ & $\vec{b}$ where $\vec{a} = \langle 1, 3, 4 \rangle$ & $\vec{b} = \langle 2, 7, -5 \rangle$.

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{pmatrix}$$

$$= \big( (3)(-5) - (4)(7) \big) \vec{i} - \big( (1)(-5) - (4)(2) \big) \vec{j} + \big( (1)(7) - (3)(2) \big) \vec{k}$$

$$= -43 \vec{i} + 13 \vec{j} + \vec{k}$$

Ex. Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$. Show that $\vec{a} \times \vec{a} = \vec{0}$.

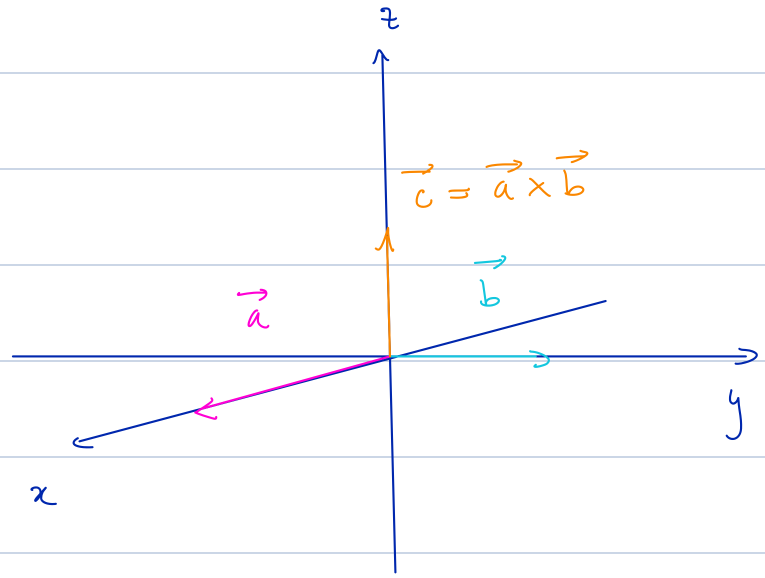
Compute

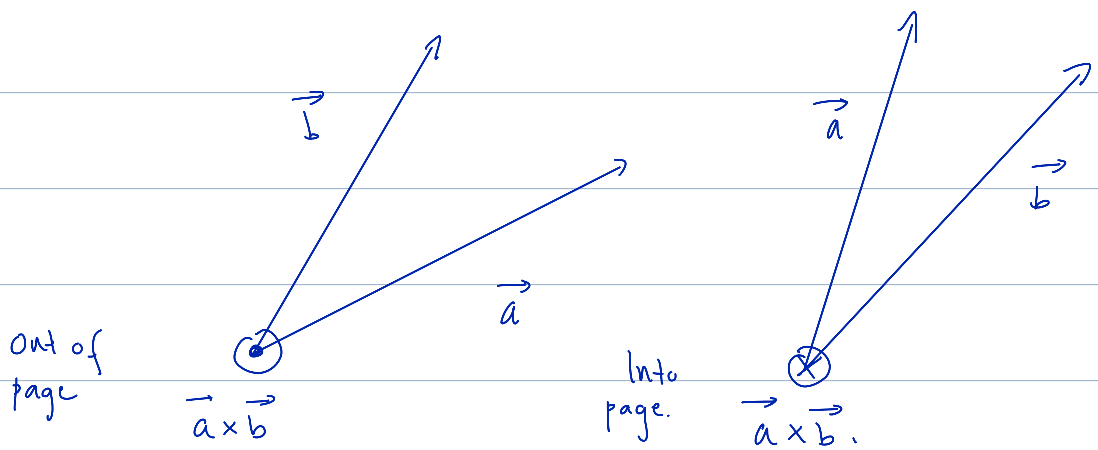
$$\det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} = \vec{0}.$$

Theorem: The vector $\vec{a} \times \vec{b}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$.

Proof: Show that $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ and similarly $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$.

Remark: The direction of $\vec{a} \times \vec{b}$ is given by the right hand rule. It states that if the fingers of your right hand curl in the direction of rotation from $\vec{a}$ to $\vec{b}$ (smaller angle), then your thumb pts in the direction of $\vec{a} \times \vec{b}$.



$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}($$