Lecture 8: Fri Sept 19th

§12.5: Level surfaces $g(x,y,z)=c$

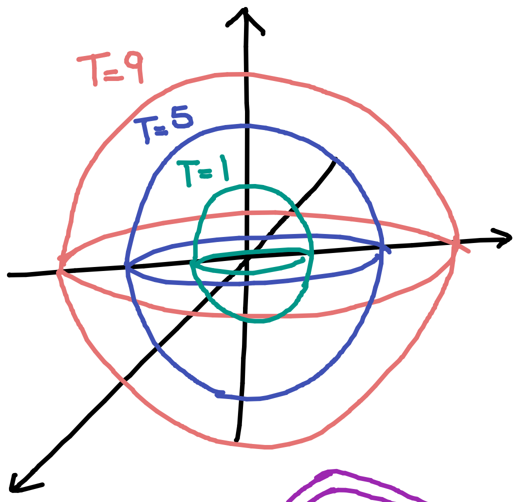
Example: A block of ice is located at $(0,0,0)$. Its temperature at (x,y,z) in °C is $T(x,y,z) = \frac{1}{4}(x^2+y^2+z^2)$, where x,y,z is in metres.

a) How many dimensions is required to describe the graph of T?

→ best way to think about this is a family of level surfaces in $\mathbb{R}^3$. The 4th variable w tells you how the level surfaces change.

A: 4 b/c $\Gamma_T = \{(x,y,z,w) \in \mathbb{R}^4 : w = T(x,y,z)\}$.

b) Pictured to the right are surfaces $T(x,y,z)=1$, 5 and 9. Find formulas for these surfaces & explain their significance in this context.



A: At $T(x,y,z)=1$, $\frac{1}{4}(x^2+y^2+z^2)=1$

$$x^2+y^2+z^2=4$$

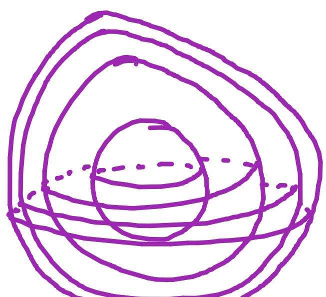
Sphere of centre $(0,0,0)$ and radius 2.

These are the points of the ice block where $T=1$°C.

At $T(x,y,z)=5$, $\frac{1}{4}(x^2+y^2+z^2)=5$

$$x^2+y^2+z^2=20$$

Sphere of centre $(0,0,0)$ and radius $\sqrt{20}$.
≈ 4.47



At $T(x,y,z)=9$, $\frac{1}{4}(x^2+y^2+z^2)=9$

$$x^2+y^2+z^2=36$$

Sphere of centre $(0,0,0)$ and radius 6.

These spheres are level surfaces of T. They are unequally spaced. Since the radii is $\sqrt{4c}$, the radii grows as a square root function as $c\to\infty$ ⇒ slower growth rate as $c\to\infty$.

Example: Describe the level surfaces of each function.

a) $f(x,y,z) = e^{-(x^2+y^2)}$

A: Set $e^{-(x^2+y^2)} = c$

$$-(x^2+y^2) = \ln(c)$$

$$x^2+y^2 = -\ln(c) = \ln(1/c).$$

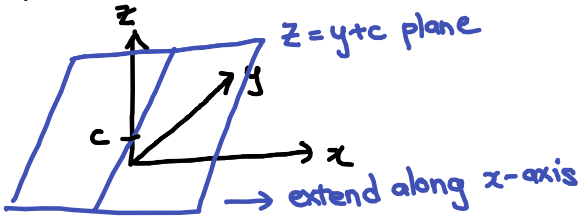
For $0<c<1$, the level surfaces of f are cylinders of radius $\sqrt{\ln(1/c)}$.

If $c=1$, then $x^2+y^2 = \ln(1/1) = 0$. So, the level surface is the z-axis.
$\Rightarrow x=y=0$.

If $c\le 0$ or $c>1$, the level surfaces are undefined.

b) $g(x,y,z) = z-y$. (x is free, i.e. $\forall x$, you see the line $z=y+c$)

A: The level surfaces $z-y=c$ (or $z=y+c$) are planes that are equally spaced out.



c) $h(x,y,z) = \ln(x^2+y^2+z^2)$

A: Set $\ln(x^2+y^2+z^2) = c$

$$x^2+y^2+z^2 = e^c$$

The level surfaces are spheres centred at $(0,0,0)$ with radius $\sqrt{e^c} = e^{c/2}$. The radii grow at an exponential rate.

Example: A long thin rope is placed along the z-axis. The density at (x,y,z) is $\rho(x,y,z) = 5(x^2+y^2)$ kg/m^3 where x,y,z are in metres.

a) What is the density at $(1,1,2)$? Does density increase/decrease as you move away from the z-axis?

A: $\rho(1,1,2) = 5(1+1) = 10$ kg/m^3. The density increases as you move away from the z-axis because x^2 and $y^2 \to \infty$ as x and $y \to \infty$.

b) Describe the set of all points where the density is 20 kg/m^3.

A: $20 = 5(x^2+y^2)$

$4 = x^2+y^2 \Rightarrow$ cylinder of radius 2 has density 20 kg/m^3.

c) Describe the level surfaces of ρ.

A: Level surfaces $5(x^2+y^2) = c$ are cylinders centred along the z-axis with radius $\sqrt{c/5}$ m.

$$x^2+y^2 = c/5$$

Revisiting graphs of $f(x,y)$

We can get a surface as the graph of $f(x,y)$ or a single level surface $g(x,y,z)=c$.

Q: How are they related?

From graph of f to level surface: Start with $z=f(x,y)$ and set $g(x,y,z) = z-f(x,y)$.

Then, the graph of f is the level surface of g at $g(x,y,z)=0$.

Indeed, $\{(x,y,z,w)\in\mathbb{R}^4 : w = g(x,y,z) = 0\}$

$= \{(x,y,z,0)\in\mathbb{R}^4 : f(x,y) - z = 0\}$

$= \{(x,y,z)\in\mathbb{R}^3 : f(x,y) = z\}$.

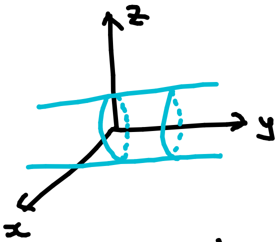
From level surface to graph of f:

Conversely, a level surface $g(x,y,z)=c$ can be written as a graph of $z=f(x,y)$ if it is possible to solve for z.

Example: a) The level surface $x^2+y^2+\sqrt{z}=2$ is the graph of $\sqrt{z}=2-x^2-y^2$
$z=(2-x^2-y^2)^2$.

b) The level surface $x^2+z^2=1$ cannot be written as a (single) graph because
$z=\pm\sqrt{1-x^2}$.

→ cylinder centred around y-axis



Example: Match the functions (a)–(f) with the descriptions of their level surfaces.

a) $f(x,y,z)=\sqrt{9-x^2-y^2}$

b) $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

c) $f(x,y,z)=\dfrac{1}{x^2+y^2+z^2}$

d) $f(x,y,z)=5+y^2+z^2$

e) $f(x,y,z)=z-y^2$

f) $f(x,y,z)=\sqrt{5x+3y+2z}$

I) Cylinders that get larger as c increases

II) Cylinders that get smaller as c increases

III) Spheres that get larger as c increases

IV) Spheres that get smaller as c increases

V) Parabolic cylinder along the x-axis with vertex $(x,0,c)$

VI) Unequally spaced planes

A: a) $c=\sqrt{9-x^2-y^2}$

$c^2=9-x^2-y^2$

$x^2+y^2=9-c^2$ cylinder of $r=\sqrt{9-c^2}$.
As $c\to\infty$, r gets smaller.

$\therefore$ II

b) $c=\sqrt{x^2+y^2+z^2}$

$c^2=x^2+y^2+z^2$ sphere of $r=c$
As $c\to\infty$, r gets bigger.

$\therefore$ III

c) $c=\dfrac{1}{x^2+y^2+z^2}$

$x^2+y^2+z^2=1/c$ sphere of $r=1/\sqrt{c}$
As $c\to\infty$, $r\to 0$

$\therefore$ IV

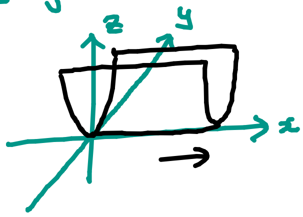
d) $c=5+y^2+z^2$

$c-5=y^2+z^2$ cylinder of $r=\sqrt{c-5}$
As $c\to\infty$, $r\to\infty$

$\therefore$ I

e) $c=z-y^2$

$z=y^2+c$ parabolic cylinders along x-axis



$\therefore$ V

f) $c=\sqrt{5x+3y+2z}$

$c^2=5x+3y+2z$

$z=\dfrac{c^2-5x-3y}{2}=\underbrace{\dfrac{c^2}{2}}-\dfrac{5}{2}x-\dfrac{3}{2}y$

This moves the plane up/down.
It grows quadratically as $c\to\infty$.
$\Rightarrow$ unequally spaced planes

$\therefore$ VI