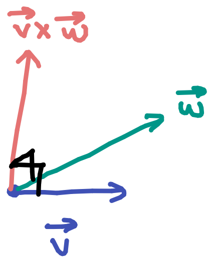
Lecture 14: Fri Oct 3rd

Recap: $\vec{v} \times \vec{w} = (\text{Area of parallelogram})\,\vec{n}$ ← unit vector $\perp \vec{v}, \vec{w}$

$= (\|\vec{v}\|\,\|\vec{w}\| \sin\theta)\,\vec{n}$

$\boxed{\text{Geometric Defn}}$



For the algebraic definition, we need determinants.

① Determinant of a 2x2 matrix

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

E.g. $\begin{vmatrix} 4 & -3 \\ 9 & 7 \end{vmatrix} = 4\cdot 7 - (-3)\cdot 9$
$= 28 - (-27)$
$= 55$

② Determinant of a 3x3 matrix

Note the sign!

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - b \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + c \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$\boxed{\text{Algebraic defn}}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - \vec{j} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \vec{k} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (v_2 w_3 - v_3 w_2)\,\vec{i} - (v_1 w_3 - v_3 w_1)\,\vec{j} + (v_1 w_2 - v_2 w_1)\,\vec{k}.$$

Example: Find $(2, -3, 5) \times (4, 1, -3)$. (Will probably skip this in the lecture)

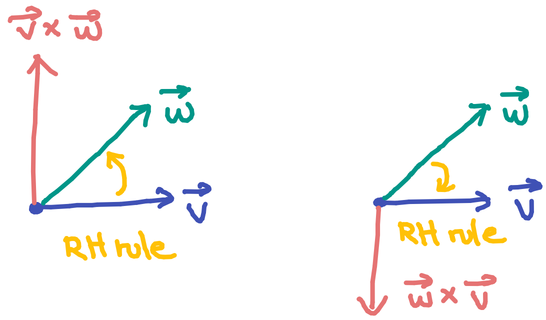
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ 4 & 1 & -3 \end{vmatrix} = \vec{i} \begin{vmatrix} -3 & 5 \\ 1 & -3 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 5 \\ 4 & -3 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix}$$

$$= (9 - 5)\,\vec{i} - (-6 - 20)\,\vec{j} + (2 + 12)\,\vec{k}$$

$$= (4, 26, 14)$$

Properties of $\times$

1) $\vec{v}\times\vec{w} = -(\vec{w}\times\vec{v})$
2) $(\lambda\vec{v})\times\vec{w} = \lambda(\vec{v}\times\vec{w}) = \vec{v}\times(\lambda\vec{w})$
3) $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

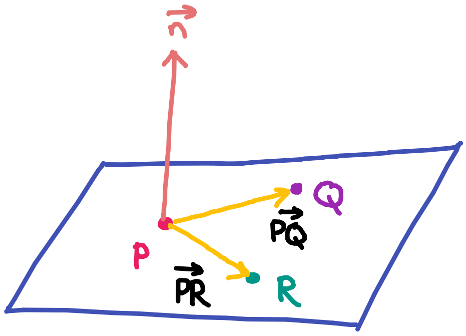


Example: Find unit vector perpendicular to both $\vec{v} = 3\vec{i}+3\vec{j}+4\vec{k}$ and $\vec{w} = 3\vec{i}+4\vec{k}$.

A://
$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & 4 \\ 3 & 0 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 4 \\ 0 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 4 \\ 3 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 3 \\ 3 & 0 \end{vmatrix}$$
$$= (12, 0, -9).$$

$$\text{Unit vector} = \frac{(12, 0, -9)}{\sqrt{144+81}} = \frac{(12, 0, -9)}{\sqrt{225}} = \left(\frac{12}{15}, 0, \frac{-9}{15}\right) = \left(\frac{4}{5}, 0, \frac{-3}{5}\right)$$

Example: Find the equation of the plane containing 3 points: $P=(1,2,-1)$, $Q=(2,3,0)$ and $R=(3,-1,2)$.



A:// Choose P as a basepoint.

$\vec{PQ} = (1,1,1)$

$\vec{PR} = (2,-3,3)$

$$\vec{n} = \vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & -3 & 3 \end{vmatrix}$$
$$= \vec{i}\begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$
$$= \vec{i}(3+3) - \vec{j}(3-2) + \vec{k}(-3-2)$$
$$= 6\vec{i} - \vec{j} - 5\vec{k}$$
$$= (6,-1,-5)$$

Let $P_0 = (1,2,-1)$. By Point-normal form,

$$\vec{n}\cdot\vec{PP_0} = (6,-1,-5)\cdot(x-1, y-2, z+1) = 0$$
$$6(x-1) - (y-2) - 5(z+1) = 0$$
$$6x - 6 - y + 2 - 5z - 5 = 0$$
$$6x - y - 5z = 9$$

Example: Find the area of the parallelogram with vertices

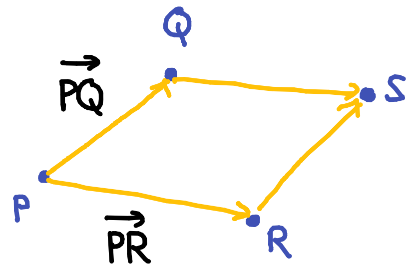
$P=(2,1,1)$, $Q=(3,3,0)$, $R=(4,0,2)$, $S=(5,2,1)$.

A://

$\vec{PQ} = (3,3,0)-(2,1,1) = (1,2,-1)$

$\vec{PR} = (4,0,2)-(2,1,1) = (2,-1,1)$

Check: $\vec{PQ}+\vec{PR} = \vec{PS}$



$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

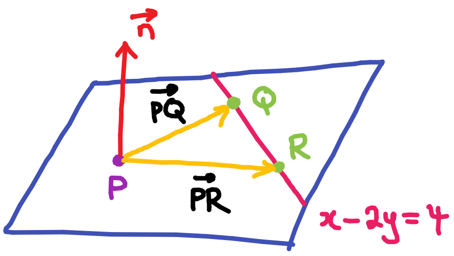
$$= (2-1)\vec{i} - \vec{j}(1+2) + \vec{k}(-1-4)$$

$$= (1,-3,-5)$$

Area $= \|(1,-3,-5)\| = \sqrt{1+9+25} = \sqrt{35}$

Example: Find the plane that contains $(1,0,-1)$ and the line $x-2y=4$ in the xy-plane.

A:// Method 1: Using the cross product (More steps but works for general lines)



Set $P=(1,0,-1)$.

Find 2 other points on the line: $Q=(4,0,0)$

$R=(0,-2,0)$

Other points work like $(6,1,0)$.

$\vec{PQ} = (4-1, 0-0, 0--1) = (3,0,1)$

$\vec{PR} = (0-1, -2-0, 0--1) = (-1,-2,1)$

$$\vec{n} = \vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 1 \\ -1 & -2 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 0 \\ -1 & -2 \end{vmatrix}$$

$$= \vec{i}(0+2) - \vec{j}(3+1) + \vec{k}(-6)$$

$$= (2,-4,-6)$$

Set $P_0 = P = (1,0,-1)$.

$$\vec{n}\cdot\vec{PP_0} = (2,-4,-6)\cdot(x-1, y, z+1) = 0$$

$$2(x-1) - 4y - 6(z+1) = 0$$

$$2x - 4y - 6z - 2 - 6 = 0$$

$$2x - 4y - 6z = 8$$

For more examples like this, see Questions 9-12 of 13.3-13.4 practice worksheet.

Method 2: Using the fact that the line is in the xy-plane

In xy-plane, $z=0$ and the plane is $x+2y+0=4$.

In general, the plane is $x+2y+cz=4$.

Since $(1,0,-1) \in$ plane, $1+0-c=4 \Rightarrow c=-3$.

$\therefore$ The plane is $x+2y-3z=4$.

Bonus: Determinant of 4×4 matrices

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} & \cancel{d} \\ \cancel{e} & f & g & h \\ \cancel{i} & j & k & l \\ \cancel{m} & n & o & p \end{vmatrix} - b\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} & \cancel{d} \\ e & \cancel{f} & g & h \\ i & \cancel{j} & k & l \\ m & \cancel{n} & o & p \end{vmatrix} + c\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} & \cancel{d} \\ e & f & \cancel{g} & h \\ i & j & \cancel{k} & l \\ m & n & \cancel{o} & p \end{vmatrix}$$

$$- d\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} & \cancel{d} \\ e & f & g & \cancel{h} \\ i & j & k & \cancel{l} \\ m & n & o & \cancel{p} \end{vmatrix}$$

$$= a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

$=$ .... Now compute each 3×3 determinant