Lecture 11: Fri Sept 26th

Recap: Does the limit exist at $(0,0)$? At $(1,1)$?

$$\frac{2x^2}{4x^2+3y^2}$$

A: Limit at $(0,0)$:

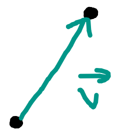
Path $x=0$: $\displaystyle\lim_{(0,y)\to(0,0)} \frac{0}{0+3y^2} = 0$

Path $y=0$: $\displaystyle\lim_{(x,0)\to(0,0)} \frac{2x^2}{4x^2+0} = 1/2$ $\quad\therefore$ Limit DNE at $(0,0)$.

Limit at $(1,1)$:

The function is ctns $\forall (x,y) \neq (0,0)$ b/c the denom $\neq 0$ and it is a quotient of ctns functions. $\therefore$ limit is $f(1,1) = \frac{2}{4+3} = \frac{2}{7}$.

§13.1 Displacement vectors

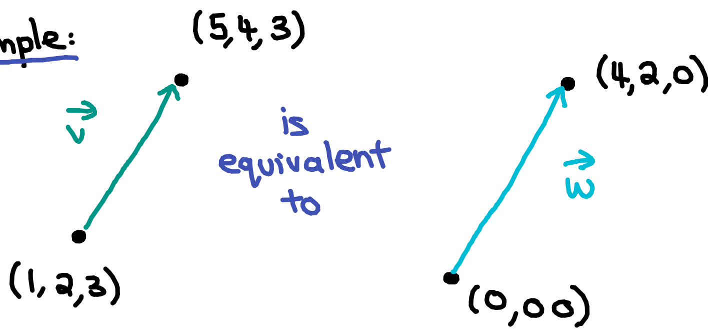


A displacement vector is an arrow from one point to another.

It has a magnitude/length $\|\vec{v}\|$ = distance between those points and a direction.

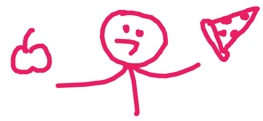
Two displacement vectors are equivalent if they have the same magnitude and direction.

Example:

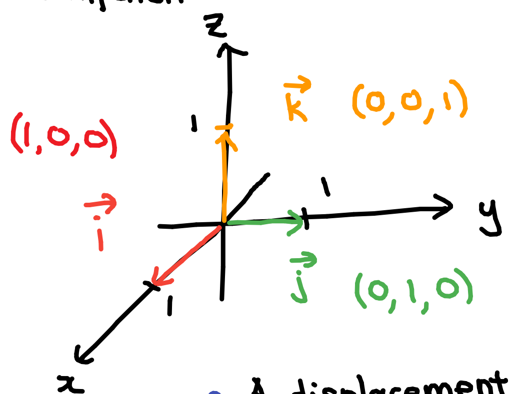


Scalars and vector are 2 different things.

$5+\vec{v}$ does not make sense.



Component vectors:



Every vector $\vec{v}$ can be "resolved into components".

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$
$$= v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1)$$
$$= (v_1, v_2, v_3)$$

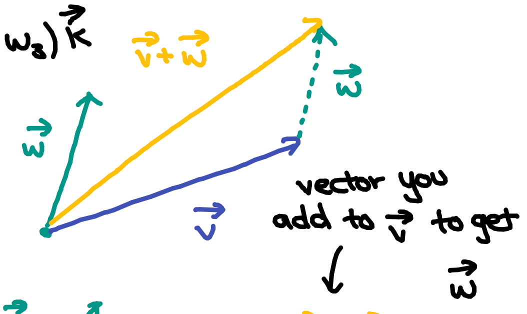
- A displacement vector b/twn $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is $\overrightarrow{P_1P_2} = (x_2-x_1)\vec{i} + (y_2-y_1)\vec{j} + (z_2-z_1)\vec{k}$

- **Position vector** = displacement vector from origin

$$\vec{OP} = a\vec{i} + b\vec{j} + c\vec{k} \quad \text{where } P=(a,b,c) \text{ is a point.}$$

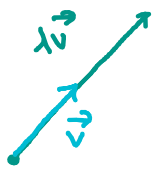
- The **zero vector** $\vec{0} = 0\vec{i} + 0\vec{j} + 0\vec{k}$.
- **Magnitude** of $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- Addition of vectors: $\vec{v} + \vec{w} = (v_1+w_1)\vec{i} + (v_2+w_2)\vec{j} + (v_3+w_3)\vec{k}$

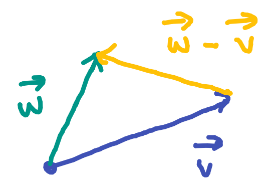
$$(v_1, v_2, v_3) + (w_1, w_2, w_3) = (v_1+w_1, v_2+w_2, v_3+w_3)$$



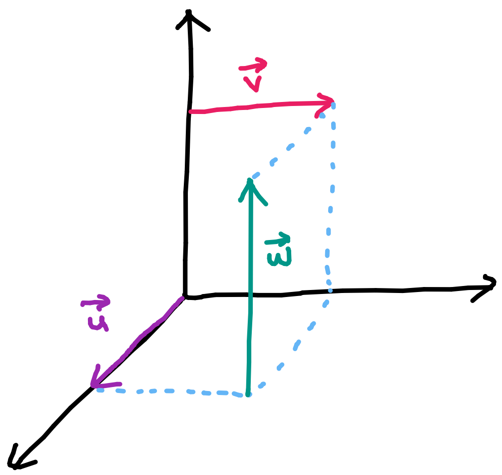
- Scalar multiplication: $\lambda\vec{v} = \lambda v_1\vec{i} + \lambda v_2\vec{j} + \lambda v_3\vec{k}$

$$\lambda(v_1, v_2, v_3) = (\lambda v_1, \lambda v_2, \lambda v_3)$$

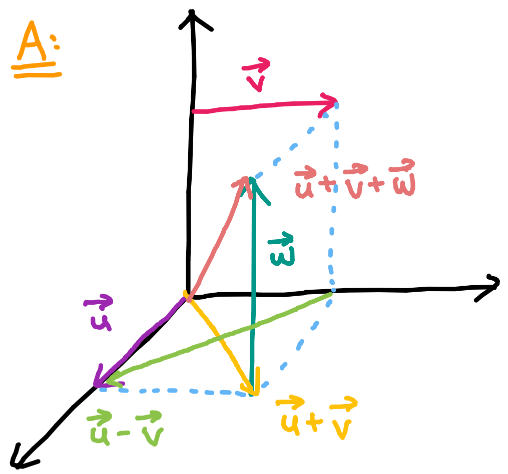




Example: Draw $\vec{u} + \vec{v}$, $\vec{u} + \vec{v} + \vec{w}$ and $\vec{u} - \vec{v}$



A:



A unit vector is a vector with unit length, i.e. $\|\vec{u}\| = 1$.

A unit vector pointing in the same direction of $\vec{v}$ is $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$.

Example: For $P = (1, -1, 3)$ and $Q = (-5, 2, 1)$,

a) Find $\vec{PQ}$.

A: $\vec{PQ} = (-5, 2, 1) - (1, -1, 3) = (-6, 3, -2)$

b) Find $\|\vec{PQ}\|$

A: $\|\vec{PQ}\| = \sqrt{(-6)^2 + 3^2 + (-2)^2} = \sqrt{36 + 9 + 4} = \sqrt{49} = 7$

c) Find the unit vector pointing in the same direction of $\vec{PQ}$.

A: $\vec{u} = \dfrac{\vec{PQ}}{\|\vec{PQ}\|} = \dfrac{1}{7}(-6, 3, -2) = \left(\dfrac{-6}{7}, \dfrac{3}{7}, \dfrac{-2}{7}\right)$

d) Find the value a so that $(9,a,3)$ is parallel to $\vec{PQ}$.

A: Parallel vectors point in the same direction, i.e. $(9,a,3) = \lambda(-6,3,-2)$.

$\Rightarrow 9 = \lambda(-6)$
$a = \lambda(3)$
$3 = \lambda(-2)$

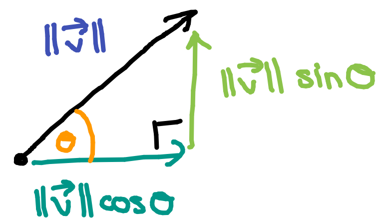
First, find λ (the scaling factor).

$\lambda = 9/-6 = 3/-2 = -1.5$. So, $-1.5 = a/3$

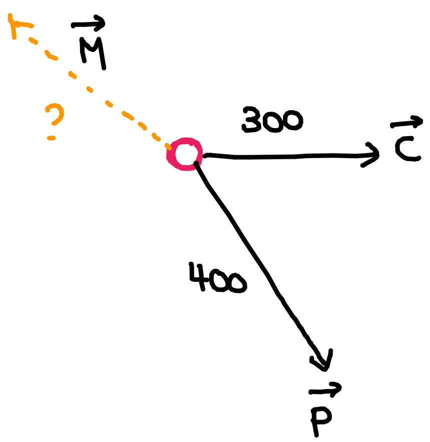
$\underline{-4.5 = a}$

Resolving 2d vectors into components:

$$\vec{v} = (\|\vec{v}\| \cos\theta, \|\vec{v}\| \sin\theta)$$



Example: Three ropes are attached to an indestructible donut. A group of chemists pull one rope with a force of 300 pounds east. A group of physicists pull one rope southeast with a force of 400 pounds. With what force would a group of mathematicians have to pull the third rope so that the donut stays stationary?



A: For the donut to be stationary, $\vec{M} + \vec{P} + \vec{C} = 0$

$\vec{C} = (300\cos(0), 300\sin(0)) = (300, 0)$

$\vec{P} = (400\cos(-\pi/4), 400\sin(-\pi/4)) = (400 \cdot 1/\sqrt{2}, 400 \cdot -1/\sqrt{2})$

For $\vec{M} + \vec{P} + \vec{C} = 0$,

$\vec{M} = -\vec{C} - \vec{P} = (-300, 0) - (400/\sqrt{2}, -400/\sqrt{2})$

$\approx (-582.84, 400/\sqrt{2})$

BONUS MATERIAL (Not examined!)

A vector space is a set with vector addition $\vec{v}+\vec{w}$ and scalar multiplication $\lambda\vec{v}$ such that the operations satisfy:

1) Associativity of addition:
$$\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$$

2) Commutativity of addition:
$$\vec{u}+\vec{v} = \vec{v}+\vec{u}$$

3) Existence of additive identity:
$$\exists\, \vec{0} \text{ s.t. } \vec{v}+\vec{0} = \vec{0}+\vec{v}$$

4) Existence of additive inverses:
For each $\vec{v}$, $\exists$ a $(-\vec{v})$ s.t. $\vec{v}+(-\vec{v}) = \vec{0}$

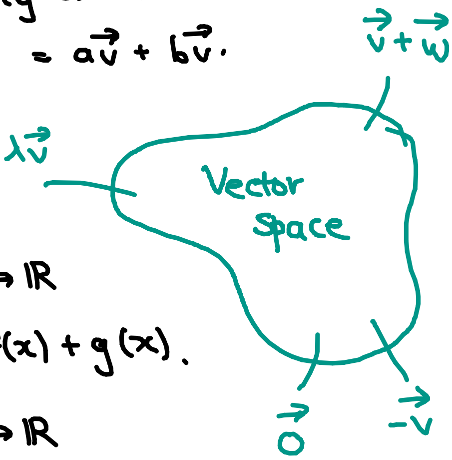
5) Compatibility of scalar mult.
$$(ab)\vec{v} = a(b\vec{v})$$

6) Identity of scalar mult.
$$1\cdot\vec{v} = \vec{v}$$

7) Distributivity of scalar mult. with respect to vector addition
$$a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$$

8) Distributivity of scalar mult.
$$(a+b)\vec{v} = a\vec{v}+b\vec{v}.$$

Elements are called vectors.



Example: $V = \{\text{ctns functions } f:[a,b]\to\mathbb{R}\}$

- Vector addition: $f+g$ is the function $f+g:[a,b]\to\mathbb{R}$
$$(f+g)(x) = f(x)+g(x).$$
- Scalar multiplication: λf is the function $\lambda f:[a,b]\to\mathbb{R}$
$$(\lambda f)(x) = \lambda f(x)$$
- zero function: This is the constant function $0:[a,b]\to\mathbb{R}$
$$0(x) = 0$$

Onto the vector space of functions, you can define many different notions of length $\|\cdot\|$. A vector space with $\|\cdot\|$ is called a normed vector space.

E.g. $$\|f\|_1 = \int_a^b |f(x)|\,dx$$
$$\|f\|_p = \sqrt[p]{\int_a^b |f(x)|^p\,dx}$$

$(V,\|\cdot\|_1)$ is a different space to $(V,\|\cdot\|_p)$.
They have different "rulers".