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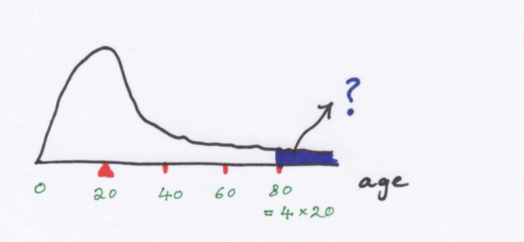
Andrey Markov (1856-1922) came up with a simple bound.

A tail bound



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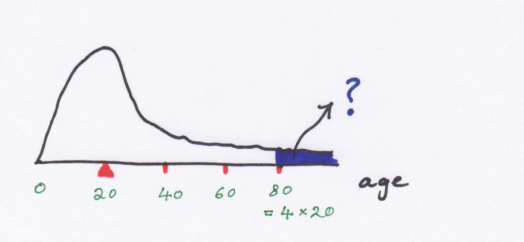
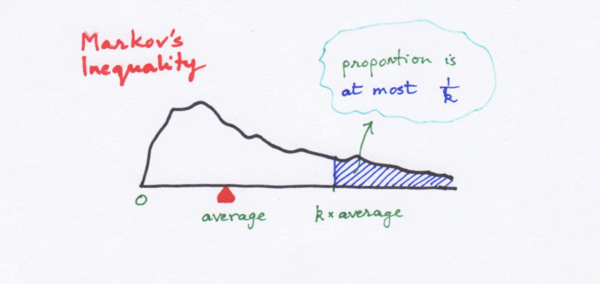


?
0
20
40
60
80
= 4 × 20
age



Markov's Inequality
proportion is at most $\frac{1}{k}$
0
average
k × average

A formal statement

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Example: In any list of non-negative numbers,
the proportion of entries that are at least as large as 4 times the average is **at most 1/4**; in other words, **no more than 25%**.

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The proportion could be exactly $1/10$;
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it could even be 0;
but it **cannot be more than 1/10.**

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The proportion that are more than 80 years old is at most the proportion that are greater than or equal to 80 years old.

And that proportion is at most $1/4$, by Markov's inequality.

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Markov's bound is most useful when k is large, that is, when you're interested in entries that are quite far above average.