ICMA214: Ordinary Differential Equations

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Contents

1 Series Solutions of Second-Order Linear Equations

1.1 Review of Power Series

Before we do a *deep dive* into how to use power series to solve differential equations, let us first recall what power series are. Firstly, recall what power series look like.

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

Additionallly, some useful concepts relating to power series are the following.

1. Power series $\sum_{n=0}^{m} a_n(x-x_0)^n$ is said to converge at given point x if

   $$\lim_{m\to\infty}\sum_{n=0}^{m} a_n(x-x_0)^n$$

   exists for said x. The series clearly converges for when $x = x_0$. However, it may converge for all x or only some, depending on the actual values.

2. Power series $\sum_{n=0}^{m} a_n(x-x_0)^n$ *converges absolutely* at point x associated power series of its absolutes

   $$\sum_{n=0}^{\infty} |a_n(x-x_0)|^n = \sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

   converges. If a power series converges absolutely, the it also converges. The converse, however, is not always true.

3. *Ratio Test* – For some fixed value of x and $a_n \neq 0$,

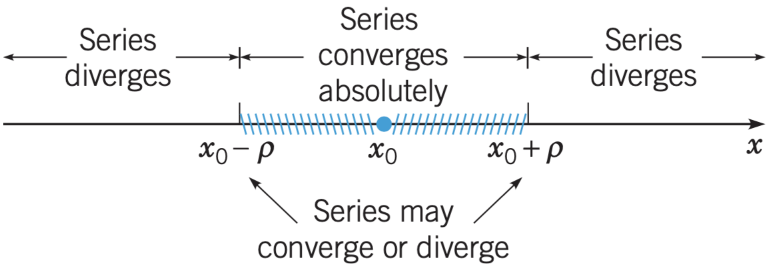
   $$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L$$

   then the power series

   - converges absolutely if $|x-x_0|L < 1$
   - diverges if $|x-x_0|L > 1$
   - is inconclusive if $|x-x_0|L = 1$

4. If given power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at $x = x_1$, it converges absolutely for $|x-x_0| < |x_1-x_0|$. If it diverges at $x = x_1$, it diverges for $|x-x_0| > |x_1-x_0|$.

5. For a typical power series, there exists a positive number ρ, called the **radius of convergence**, such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ always converges absolutely for all $|x-x_0| < \rho$ and diverges for all $|x-x_0| > \rho$. This interval where the series converges is called the **interval of convergence**. At $|x-x_0| = \rho$, the series may converge or diverge.



Now, suppose that $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x - x_0)^n$ converges to $f(x)$ and $g(x)$, respectively, for $|x - x_0| < \rho$ and $\rho > 0$.

6. The two series can be added or subtracted termwise, meaning

$$f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x - x_0)^n$$

and it will converge at least for $|x - x_0| < \rho$.

7. The two series can be multiplied by

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

where $c_n = a_0 b_n + a_1 b_{n-2} + \cdots + a_n b_0$ and it will converge at least for $|x - x_0| < \rho$.
Additionally, given that $b_0 \neq 0$ and $g(x_0) \neq 0$, series for $f(x)$ is divisible by series for $g(x)$, and

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n(x - x_0)^n$$

d_n can be most easily obtained by solving the following relation

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(x - x_0)^n &= \left(\sum_{n=0}^{\infty} d_n(x - x_0)^n\right)\left(\sum_{n=0}^{\infty} d_n(x - x_0)^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0} d_k b_{n-k}\right)(x - x_0)^n\end{aligned}$$

In the case of division, the radius of convergence of resulting series may be less than ρ.

8. Function $f(x)$ is continuous and has derivatives of all orders for $|x - x_0| < \rho$. Additionally, derivatives of all orders can be computed by differentiating the series termwise and each series (in the derivatives) converges absolutely for $|x - x_0| < \rho$.

9. Value of a_n is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

This is essentially the Taylor series for the function f about $x = x_0$.

10. If both series are equivalent for each x in some open interval with center x_0, then $a_n = b_n \forall\ n = 0, 1, 2, \cdots$. If $\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0$ for each such x, then $a_0 = a_1 = \cdots = a_n = \cdots = 0$.

A function f with Taylor Series Expansion about $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

with radius of convergence $\rho > 0$ is said to be **analytic** at $x = x_0$. Given that f is analytic at x_0, then $f \pm g$, $f \cdot g$, and f/g (for $g \neq 0$) are also analytic at $x = x_0$.

1.1.1 Shift of Index Summation

Index of summation in an infinite series is a dummy variable, implying that whatever letter we use to represent it, it will not matter. For instance,

$$\sum_{n=0} \frac{2^n x^n}{n!} \equiv \sum_{j=0}^{\infty} \frac{2^j x^j}{j!}$$

This means that we can shift the summation indices to compute series solutions.

> **Example.** We want to write $\sum_{n=2}^{\infty} a_n x^n$ as a series that starts from $n = 0$ instead of $n = 2$.
> A way to do this is to let $m = n - 2$, then $n = m + 2$. Thus, $n = 2$ corresponds to $m = 0$. Hence,
>
> $$\sum_{n=2}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+2} x^{m+2}$$
>
> What we have done here is essentially shifting the index upward by two while making the initial value of $n = 0$.