MQF 620 Stochastic Modelling in Finance

Group Project Report

Prathmesh Desai
Rahul Sreeram
Srivatsa Mitragotri
Nandini Agarwal
Harshita Sachdev

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Part 1 Analytical Option Formula

1.1 Black Scholes Model

a. Advantages of the Black Scholes method:
1) It works entirely on objective figures rather than human judgment.
2) Another main advantage of the Black-Scholes model is speed -- it lets you calculate a very large number of option prices in a very short time.

b. Limitations of this method:
1) It can only calculate the price and valuation of European options. The stock price does not get present valued – it starts at its present value (a 'spot price') and drifts upwards over time at the risk-free rate.
2) Arbitrage Free Markets. Black Scholes formulas assume that traders try to maximize their personal profits and don't allow arbitrage opportunities (riskless opportunities to make a profit) to persist.
3) The level of volatility is constantly changing in real life. It assumes that the logarithm of the natural rate of return always has a normal distribution. But sometimes it can be leptokurtic.

$$
\begin{aligned}
C(0) &= S(0)N(d_1) - Ke^{-rT}N(d_2), \\
P(0) &= Ke^{-rT}N(-d_2) - S(0)N(-d_1),
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \\
d_2 &= d_1 - \sigma\sqrt{T}.
\end{aligned}
$$

Formula 1.1. Black Scholes option formula for call (C) and put (P) options.

1.2 Bachelier Model

a. Overall, there are three great things about this formula:
1) The clarification of the most important factors in the option model.
2) The use of the fair game idea.
3) The use of the concept of stochastic to depict market dynamics for the first time.

b. The flaws of this formula are also obvious:
1) It assumes that stock prices, rather than yields, are normally distributed, because in Bachelier's time, most of the data he used came from one of the Paris exchanges, but trading there was not very active, so prices had some appearance of a normal distribution.

$$
V_c = e^{-rT}[(S_0 - K)\Phi\left(\frac{S_0-K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\emptyset\left(\frac{S_0-K}{\sigma\sqrt{T}}\right)]
$$

$$
V_p = e^{-rT}[(K - S_0)\Phi\left(\frac{K-S_0}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\emptyset\left(\frac{K-S_0}{\sigma\sqrt{T}}\right)]
$$

Formula 1.2. Bachelier option formula for call (C) and put (P) options.

1.3 Black 76 Model

a. For the practitioner, the Black76 model is widely used to evaluate vanilla interest rate options like caps, floors or swaption.
b. The Black 76 model is for an option where the underlying commodity is traded based on a future price rather than a spot price.
c. Instead of dealing with a spot price that drifts upwards at the risk-free rate, this model deals with a forward price that needs to be present valued.

$$C = e^{-rT}[F\,N(d_1) - K\,N(d_2)]$$
$$P = e^{-rT}[K\,N(-d_2) - F\,N(-d_1)]$$
$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}},\ d_2 = d_1 - \sigma\sqrt{T}$$

Formula 1.3. Black76 option formula for call (C) and put (P) options.

1.4 Displaced-diffusion Model

a. Advantanges:

1) We propose a displaced-diffusion stochastic volatility model for swaption pricing. The displaced-diffusion model is also widely known in the industry as the shifted-lognormal model.
2) The key strength of the displaced-diffusion process lies in its ability to accommodate negative interest rates without any further additional adjustment.
3) The displaced-diffusion dynamic for the swap rate process can handle negative rates or strikes without any further ad hoc adjustment.

b. We derive a closed-form analytical expression for swaption pricing, and show that it can also match market prices with a high degree of accuracy. Consider the displaced-diffusion forward swap rate process as follows

$$\begin{aligned} C(0) &= \left(S(0) - \beta e^{-rT}\right) N(d_1) - (K-\beta)\, e^{-rT} N(d_2), \\ P(0) &= (K-\beta)\, e^{-rT} N(-d_2) - \left(S(0) - \beta e^{-rT}\right) N(-d_1), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S(0)-\beta e^{-rT}}{K-\beta}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Formula 1.4. Displaced-diffusion option formula for call (C) and put (P) options.

Part 2 Model Calibration

2.1 Calibration for European options:

2.1.1 SABR model:

Working Process: The whole process involves three steps. The first step is to clean the data. In this step we pull out all the data for the European options and then find out the T(time to maturity), interest rate by interpolating method and forward price for three options with different time to maturity. The second step is to drop out all the options with strike price smaller than forward price for call options and strike price bigger than forward price for put options, which means we only contain OTM options. The last step will be calculating the implied volatility based on the BS model and then we can calibrate the SABR model to fit the implied volatility by finding the parameters with the smallest mean squared error.

Best parameters for options with 17days to maturity: alpha = 1.212, beta = 0.7, rho = -0.301, nu = 5.460
Best parameters for options with 45days to maturity: alpha = 1.817, beta = 0.7, rho = -0.404, nu = 2.790
Best parameters for options with 80days to maturity: alpha = 2.140, beta = 0.7, rho = -0.575, nu = 1.842

2.1.2 Displaced-diffusion model

Working Process: We firstly calculate the T (time to maturity) and interest rate by interpolating method like we did for SABR model and then find out the ATM call and put options by setting the closest strike price equal to the current price. We can then find out the volatilities for ATM call and put options respectively and add them together divided by two to get our ATM sigma for three options. We set 4 betas which are 0.2, 0.4, 0.6 and 0.8. For each of them, we calculated the implied volatilities for different betas, market volatility, normal volatility, and lognormal volatility. Finally, we append them together to get the plots. Since when beta is equal to 0, the implied volatility will be closest to the market implied volatility for all options.

Best parameters for options with 17days to maturity: $\beta = 0, \sigma = 0.1751$
Best parameters for options with 45days to maturity: $\beta = 0, \sigma = 0.1854$
Best parameters for options with 80days to maturity: $\beta = 0, \sigma = 0.1910$

2.2 American Options

2.2.1 SABR model

Working process: same steps as how we calibrate for European options of SABR model except this time we need to compute the binomial tree model first and then find out the implied volatility by using the binomial tree pricing method.

Best parameters for options with 17days to maturity: alpha = 0.645, beta = 0.7, rho = -0.409, nu = 5.358
Best parameters for options with 45days to maturity: alpha = 0.896, beta = 0.7, rho = -0.484, nu = 2.779
Best parameters for options with 80days to maturity: alpha = 1.118, beta = 0.7, rho = -0.629, nu = 1.760

2.2.2 Displaced-diffusion model

Working process: Same as European options

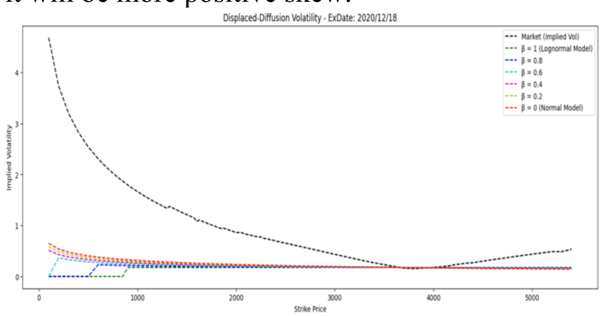
Best parameters for options with 17days to maturity: $\beta = 0, \ \sigma = 0.1867$

Best parameters for options with 45days to maturity: $\beta = 0, \ \sigma = 0.1848$

Best parameters for options with 80days to maturity: $\beta = 0, \sigma = 0.1911$

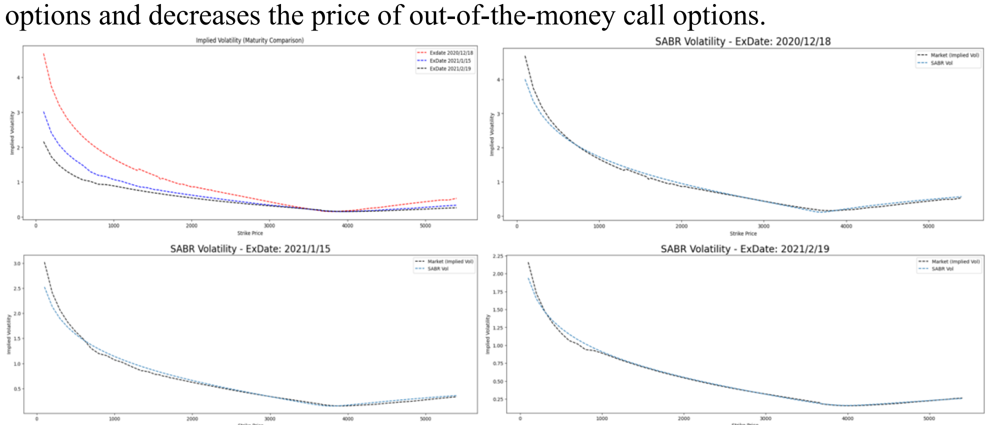
2.3 How changing β in the Displaced-diffusion model affect the shape of the implied volatility smile

Like the plot shows on the right, β will affect the skewness of the line. It increases the price of OTM put options and decreases the price of OTM call options. As beta decreases, it will be more positive skew.



2.4 How changing ν and ρ in the Displaced-diffusion model affect the shape of the implied volatility smile

Positive correlation between stock and volatility is associated with positive skew in return distribution. Negative correlation between stock and volatility is associated with negative skew in return distribution. Negative correlation increases the price of out-of-the-money put options and decreases the price of out-of-the-money call options.



Increasing volatility-of-volatility has the effect of increasing the kurtosis of return. When the volatility-of-volatility parameter is 0, volatility will be deterministic. Larger volatility-of-volatility ν increases the price of out-of-the-money call and put options

Part 3 Static Replication

3.1 Carr-Madan Formula:

Let $\boldsymbol{F = S_0 e^{rT}}$

$$\boldsymbol{V_0 = e^{-rT} E[h(S_T)] = \int_0^F h(K) \frac{6^2 P(K)}{6K^2} \, dK + \int_F^\infty h(K) \frac{6^2 C(K)}{6K^2} dK}$$

From put call parity:

$$V_0 = e^{-r} \, h(F) + \int_0^F h''(K) P(K) dK + \int_F^\infty h''(K) C(K) dK$$

3.2 Variance Swaps

3.2.1 Introduction

Variance swaps are contracts which allow us to gain explicit volatility (and variance) exposure. This frees us from the need to worry about delta or gamma hedging if we were to use vanilla options to gain volatility exposure.

3.2.2 Variance Swap formula:

$$\text{Var Swap} = \text{Notional} \times (\sigma_R^2 - \sigma_K^2)$$

where σ_R^2 is the realized variance of the stock and σ_K^2 is the strike variance

$$E\left[\int_0^T \sigma_t^2 dt\right] = 2e^{rT} \int_0^F \frac{P(K)}{K^2} dK + 2e^{rT} \int_F^\infty \frac{C(K)}{K^2} dK$$

3.3 Option Payoff Working Process

3.3.1 BSModel

First, we find the second derivative $h''(F)$ from payoff function, then calculate the price of call option and price of put options using Black Scholes Model respectively. Integrant function is call/put option formula multiply by $h''(F)$. The following is where parameter come:

a. S0: index level of SPX at start date (20201201), $F = S_0 e^{rT}$
b. T: 45/365, end date – start date
c. r is the 45 days risk-free rate from the second part
d. sigma: use SABR implied volatility for at the money option, I calculate at the money implied volatility by average implied volatility for call at 3665 and put at 3660.

Then plus this parameter into Carr-Madan final formula to get the expected price.

3.3.2 Bachelier Model

Everything is same as BSModel except using option valuation formula for bachelier model.

3.3.3 SABR Model

Everything is same as BSModel except using option valuation formula for bachelier model and parameters alpha, beta, rho and nu are come from SABR calibration.

3.3.4 Expected Price Result

BSModel: The expected price from BS Model is: 37.7049
Bachelier Model: The expected price from Bachelier Model is: 37.7153 SABR Model: The expected price from SABR model is: 37.5406

3.4 Variance Estimation

3.4.1 Formula:

$$\boldsymbol{E}\left[\int_0^T \sigma_t^2\, \boldsymbol{dt}\right] = \int_0^T \sigma^2\, \boldsymbol{dt} = \sigma^2 \boldsymbol{T}$$

3.4.2 Difference of processing

1) Change the calculation formula to coef * (price of call part + price of put part), where coef $= 2e^{rT}$

2) Change the integrant function multiplier from second derivative to $1/K^2$

3.4.3 Variance Estimation Result

BSModel: The expected variance from BS Model is: 0.0042
Bachelier Model: The expected variance from Bachelier Model is: 0.0000
SABR Model: The expected variance from SABR Model is: 0.0710

Part 4: Dynamic Hedging

4.1 Introduction of dynamic hedging:

- $\emptyset_t$ *and* ψ_t are units of security and bond we hold at time t, respectively.
- The pair $(\emptyset_t, \psi_t)$ is a dynamic trading strategy detailing the amount of each component to be held at each instant.
- Self-financing: the set of trading strategies that we are financially self-constrained, as known as self-financing. A portfolio is self-financing if and only if the change in its value only depends on the change of the asset prices.

4.2 Self-financing Test:

$$dV_t = \emptyset_t dS_t + \psi_t dB_t$$

Apply Ito's Formula:

$$dV_t = \emptyset_t dS_t + S_t d\emptyset_t + d\emptyset_t dS_t + \psi_t dB_t + B_t d\psi_t + d\psi_t dB_t (0)$$

4.3 Assumption Limitation:

Black-Scholes assumes a frictionless market, which indicates that S and B could be traded in arbitrary amounts with no transaction costs, and short positions are also allowed.

4.4 Self-financing PDE :

$$dX_t = \frac{\partial C}{\partial t} dS_t + \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 {S_t}^2 \frac{\partial^2 C}{\partial t^2}\right) dt - \emptyset_t dS_t - \psi_t dB_t$$

4.5 BS PDE:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial t^2} = rC$$

This partial differential equation tells us that under no-arbitrage assumption, fully hedged portfolio will receive a risk-free rate as portfolio return.

4.6 Five important observations of BS Formula:

- The options price does not depend on the empirical drift μ, which means that we could fully hedge against market move.
- The options price will be the same whether the market is bullish or bearish.
- It is easier for us to model without market movement parameter since the market movement parameter is hard to estimate.
- However, the risk due to stock price fluctuations is still there and has a huge impact on the price via the volatility parameter σ.
- We know how much stock we must hold as $\Delta(t, S_t) = \frac{\partial V}{\partial S_t}(t, S_t)$, and therefore, also how much cash we must hold in our replicating portfolio.

4.7 Core Statistics of hedging error:

N=21

```
Error Mean = 0.002430782730383348
Error Standard Deviation = 0.4247032058239902
Error StDev as a % of Option Premium = 16.906523244707927
```

N=84

```
Error Mean = 0.0009321509401988181
Error Standard Deviation = 0.21776058828491832
Error StDev as a % of Option Premium = 8.668581722799615
```

We could see from the above table that as we increase the number of steps, mean absolute deviation and variance become more converge for both securities. The following hedging error graph also show the same idea: