Exercises For Final Question

Friday, 18 July 2025 23:00

**8.** Carry out one iteration of Newton's method on the function $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x) = \begin{bmatrix} (x_1+3)(x_2^3-7) \\ \sin(x_2 e^{x_1}-1) \end{bmatrix},$$

starting with the initial guess $x_0 = (0,1)$.

# Problem 3

$X_{k+1} = X_k - \frac{f(X_k)}{f'(X_k)} \Rightarrow$ newton step

for multivariable functions

$X_{k+1} = X_k - J^{-1}_{(\bar{x}_k)} \cdot F(\bar{x})$

$J_{(\bar{x})} = \begin{vmatrix} \nabla^T f_1(\bar{x}) \\ \nabla^T f_2(\bar{x}) \end{vmatrix}$

$\boxed{\nabla f \Rightarrow \text{gradient vector}}$

$\boxed{\text{Jacobian matrix}}$

$$\underset{[n \times 1]}{\nabla f} = \begin{bmatrix} \frac{\partial f(\bar{x})}{\partial x_1} \\ \frac{\partial f(\bar{x})}{\partial x_2} \\ \frac{\partial f(\bar{x})}{\partial x_n} \end{bmatrix}$$

solution

$$J_{F(\bar{x})} = \begin{bmatrix} x_2^3 - 7 & 3x_2^2(x_1+3) \\ x_2 e^{x_1}\cos(x_2 e^{x_1}-1) & e^{x_1}\cdot\cos(x_2 e^{x_1}-1) \end{bmatrix}$$

$$J_{F(0,1)} = \begin{bmatrix} -6 & 9 \\ 1 & 1 \end{bmatrix}$$

$$X_{k+1} = X_k \underbrace{- J^{-1}_{F(\bar{x}_k)} \cdot F(\bar{x}_k)}_{P_d}$$

$P_d = -J^{-1}_{F(\bar{x}_k)} \cdot F(\bar{x}_k)$

$P_d \cdot J^{-1}_{F(\bar{x}_k)} = -F(\bar{x}_k)$

$$\begin{bmatrix} -6 & 9 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 9 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \end{bmatrix}$$

$-6P_1 + 9P_2 = 18$

$P_1 = -P_2$

$$P_d = \begin{bmatrix} -\frac{6}{5} \\ \frac{6}{5} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -\frac{6}{5} \\ \frac{6}{5} \end{bmatrix}$$

$X_{k+1} = X_k + P$

$$X_{k+1} = \begin{bmatrix} -\frac{6}{5} \\ \frac{11}{5} \end{bmatrix}$$

**9.** Suppose the function

$$f(x_1, x_2) = e^{x_1} x_2^2$$

is sought to be minimized starting from $x^{(0)} = (0,1)$ over $\mathbb{R}^2$.

Find the steepest descent and Newton search directions at $x^{(0)}$. Is the Newton direction a d
direction? Does the quadratic model used by Newton's method have a minimizer?

# Problem 9

$f(x_1, x_2) = e^{x_1} x_2^2$

sought to be minimized $x_0 = (0,1)$ over $\mathbb{R}^2$

1) Steepest descent direction $\to P^{sd}$

$P_{sd} = -\nabla f(\bar{x})$

$$\nabla f = \begin{bmatrix} x_2^2 \cdot e^{x_1} \\ 2x_2 \cdot e^{x_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$P_{sd} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \Rightarrow \nabla^2 f(\bar{x}) \cdot P_N = -\nabla f$

$$\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} P_N \\ P_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$P_1 + 2P_2 = -1$

$P_1 + P_2 = -1$

$P_1 = -1$

$P_2 = 0$

2

$$P_N = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

3

$\nabla f \cdot P_N = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \boxed{-1 < 0}$

Yes, dis in the direction of descent

4

$$m(p) = f(x) + \nabla f(x)^T p + \frac{1}{2} p^T \nabla^2 f(x)\, p$$

Quadratic model used by newton's method

For a function to be convex, its Hessian $\nabla^2 f$ must be positive semi-definite (PSD) for all possible $(x,y)$ in its domain.

**7.** Answer all parts for each of the quadratic polynomials

$$\begin{aligned} f(x_1,x_2) &= x_1^2 + 4x_2^2 - 4x_1x_2 + x_1 + x_2 + 4 \\ g(x_1,x_2) &= 2x_1^2 + 3x_2^2 + 2x_1x_2 + 4x_1 - x_2 + 3 \\ h(x_1,x_2,x_3) &= x_1^2 - 2x_2^2 + x_3^2 + 3x_1 + 4x_2 - x_3 + 8. \end{aligned}$$

(a) Write the polynomial in the form

$$\frac{1}{2}x^T A x + b^T x + c$$

where $A$ is a symmetric matrix, $x$ is the vector of variables, $b$ is a vector and $c$ is a scalar.

(b) Does the polynomial have a *stationary point* where the gradient is zero? If it has, find the stationary point.

(c) If a stationary point of the polynomial exists, is the stationary point a local minimizer? Justify your answer.

$f(x) = \frac{1}{2} x^T A x + b^T x + c$

$\nabla f = \frac{1}{2}(A + A^T)x + b$

If $A$ is a symmetrical matrix

$A = A^T$, so

$\nabla f = Ax + b$

$f(x) = \frac{1}{2} x^T A x + b^T x + c$

$\boxed{A_{n \times n} \text{ matrix}}$

$f(x) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + b^T x + c$

$\nabla f = \frac{1}{2}(A + A^T)x + b$

gradient