Mathematics
L2 Final Exam
(2h)

French-AZerbaijani University
L2 - S1
2021-2022

1- **Write your answers very clearly and draw a line between the solutions of the different problems**,
2- **Explain all your calculation steps**,
3- **Calculators are allowed**.

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**Questions on the lecture : [4.5 pts]**

1. Let $E$ and $F$ be two Vector Spaces over $K$. Let $f : E \to F$ be an application.
   a) What are the conditions for $f$ to be linear. [0.5 pts]
   b) If $f$ is a linear application $f : E \to F$, define the kernel and image of $f$. [0.5 pts]

2. a) Let $f : E \to F$. Prove that if $Ker(f) = \{0_E\}$ then $f$ is injective. [1 pts]
   b) Let $A, B \in M_n(K)$. Show that if $A$ and $B$ are similar, then $det(A) = det(B)$. [1 pts]

3. a) If $A$ does not have $n$ distinct eigenvalues, under what conditions is $A$ diagonalizable? [0.5 pts]
   b) Recall the definition of an eigenvalue and eigenvector for an $n \times n$ matrix $A$. [1 pts]

**Exercise 1 : [3 pts]**

1. a) Determine whether the vectors $v_1 = (1, 2, 0, 3)$, $v_2 = (0, 0, 3, 5)$ and $v_3 = (1, 1, 1, 0)$ are linearly independent in $\mathbb{R}^4$. [1 pts]
b) Is the set $V = \{v_1, v_2, v_3\}$ a basis of $\mathbb{R}^4$? Justify. [0.5 pts]
c) If we add the vector $v_4 = (0, 0, 0, 1)$, is the new set $V' = \{v_1, v_2, v_3, v_4\}$ a basis of $\mathbb{R}^4$? Justify. [0.5 pts]
2. Find a basis of the subspace of $\mathbb{R}^3$ that is generated by the vectors $v_1 = (1, 0, 0)$, $v_2 = (1, 0, 1)$, $v_3 = (2, 0, 1)$, $v4 = (0, 1, -1)$. [1 pts]

**Exercise 2 : [5 pts]**

Let $\phi$ the application defined as

$$\phi : \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - 2y + 2z \\ 2x + y + z \\ x + y \end{pmatrix}$$

a) Show that $\phi$ is a linear application.[1 pts]
b) Determine a basis of the kernel of $\phi$ and its dimension.[1 pts]
c) Use rank theorem to determine the dimension of the image and deduce the rank of $\phi$. [1 pts]
d) Is $\phi$ injective? Is it surjective? Justify.[1 pts]
e) Determine the matrix of the linear application in the canonical basis of $\mathbb{R}^3$. [1 pts]

**Exercise 3 : [4 pts]**

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the endomorphism whose matrix in the canonical basis $\mathcal{C} = \{e_1, e_2, e_3\}$ is $A = \begin{pmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{pmatrix}$.

a) Determine all eigenvalues of $f$ and their multiplicities. [1 pts]
b) For each eigenvalue, find a basis of the corresponding eigenspace. [2 pts]
c) Is A diagonalizable? Justify your answer. [1 pts]

**Exercise 4 : [2 pts]**

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the endomorphism whose matrix in the canonical basis $\mathcal{C} = \{e_1, e_2, e_3\}$ is $A = \begin{pmatrix} 9 & -6 & 10 \\ -5 & 2 & -5 \\ -12 & 6 & -13 \end{pmatrix}$.

Let $B = \{u_1, u_2, u_3\}$ be another basis of $R^3$ with $u_1 = (2, -1, -2)$, $u_2 = (1, 0, -1)$ and $u_3 = (-2, 1, 3)$.

a) Calculate the transition matrices $T_{C,B}$ and $T_{B,C}$ [1 pts]
b) Calculate the matrix of $f$ in the new basis $B$ [1 pts].

**Exercise 5 : [1.5 pts]**

Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & t & t^2 \\ 1 & -1 & 1 \end{pmatrix}$ and vector $v = (1, 1, -1)$. Find all values of $t$ for which $v$ is an eigenvector of $A$ for an eigenvalue $\lambda$.

**END**