Part 4: Governing Equations

Problem Statement

The governing equations for this problem are the two-dimensional Euler equations, complemented with the equation of state for an ideal gas with $\gamma = 1.4$.

In your report:

- Write down the governing equations in vector form. Identify clearly the state vector $\mathbf{U}$, the x and y components of the flux ($\mathbf{F}$ and $\mathbf{G}$, respectively), and any additional equation that relates the variables together, e.g. the equation of state.
- Include working definitions of the local Mach number of the flow and the local sound speed. You will need those definitions when plotting quantities.
- Apply basic normal shock and isentropic flow relations to estimate the pressure at the stagnation point on the surface of the step at $(x, y) = (0.6, 0)$ for free stream conditions of $M = 3$ and $p = 1$ Pa.

Solutions

4) Governing Equations

Development of Governing Equations in Vector Form

Let us begin with the differential form of the momentum equation:

$$\frac{\partial \rho U}{\partial t} + \nabla\cdot(\rho U \otimes U) = -\nabla P + \nabla\cdot\tau + \rho b \qquad (1)$$

Given that the governing equations for the problem are the two-dimensional Euler equations, complemented with the equation of state for an ideal gas with $\gamma = 1.4$, we can take the Inviscid (Euler) Flow limit of the equation neglecting viscous stresses and body forces

$$\frac{\partial \rho U}{\partial t} + \nabla\cdot(\rho U \otimes U) = -\nabla P \qquad (2)$$

The differential form of the continuity equation can be expressed as

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho U) = 0 \qquad (3)$$

The differential form of the Energy equation can be expressed as

$$\frac{\partial \rho E}{\partial t} + \nabla\cdot(\rho E U) = \nabla\cdot(\lambda \nabla T) + \nabla\cdot(\sigma\cdot U) + S + \rho b\cdot U \qquad (4)$$

Manipulating the equation by substituting for "σ" gives,

$$\frac{\partial \rho E}{\partial t} + \nabla\cdot(\rho H U) = \nabla\cdot(\lambda \nabla T) + \nabla\cdot(\tau\cdot U) + S + \rho b\cdot U \qquad (5)$$

Taking the Inviscid (Euler) Flow limit of the equation neglecting viscous stresses and assuming that there are no body forces or heat addition in the flow, $b=0$, $Q = S + \rho b\cdot U = 0$, $\nabla T = 0$, and $\tau = 0$ gives

$$\frac{\partial \rho E}{\partial t} + \nabla\cdot(\rho U H) = 0 \qquad (6)$$

Here, $H = E + \frac{P}{\rho} = h + \frac{U^2}{2}$ (7)

Let us return to the momentum equation (2) and expand the LHS of the expression giving,

$$\frac{\partial \rho U}{\partial t} + \nabla \cdot (\rho U \otimes U) = \rho \left\{ \frac{\partial U}{\partial t} + U \cdot \nabla U \right\} + U \underbrace{\left\{ \cancel{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho U} \right\}}_{0}$$ (from Continuity Equation (3))

$$\Rightarrow \frac{\partial \rho U}{\partial t} + \nabla \cdot (\rho U \otimes U) = \rho \left\{ \frac{\partial U}{\partial t} + U \cdot \nabla U \right\} \qquad (8)$$

Substituting the RHS of equation (8) into the LHS of equation (2) gives

$$\rho \left\{ \underbrace{\frac{\partial U}{\partial t}}_{\text{local acceleration}} + \underbrace{U \cdot \nabla U}_{\text{convective acceleration}} \right\} = -\nabla P \qquad (9)$$

But, $\frac{\partial U}{\partial t} + U \cdot \nabla U = \frac{DU}{Dt}$ giving,

$$\rho \frac{DU}{Dt} = -\nabla P \qquad (10)$$

Dividing equation (10) by "ρ" gives

$$\frac{DU}{Dt} = \frac{-\nabla P}{\rho}$$

Then, since the rescaled (normalized) pressure, $\pi = \frac{P}{\rho}$, we have,

$$\frac{DU}{Dt} = -\nabla \pi \qquad (11)$$

Expanding equation (11) using the identities for the total acceleration and rescaled pressure according to x and y components gives,

$$x: \quad \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \frac{-\partial \left(\frac{P}{\rho}\right)}{\partial x} \qquad (12)$$

$$y: \quad \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -\frac{\partial \left(\frac{P}{\rho}\right)}{\partial y} \qquad (13)$$

Let us now return to the Continuity Equation (3) and expand the LHS of the expression giving,

$$\frac{\partial \rho}{\partial t} + \left[\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (\rho u, \rho v)\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad (14)$$

Lastly, let us return to the Energy equation (6) and expand the LHS of the expression giving,

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho H u) = 0 \quad \left(\text{where } H = E + \frac{P}{\rho}\right)$$

$$\rightarrow \frac{\partial \rho E}{\partial t} + \nabla \cdot \left(\rho \left(E + \frac{P}{\rho}\right) u\right) = 0$$

$$\rightarrow \frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho E u + P u) = 0$$

$$\rightarrow \frac{\partial \rho E}{\partial t} + \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot \left(\rho E \begin{bmatrix} u_x \\ v_y \end{bmatrix} + P \begin{bmatrix} u_x \\ v \end{bmatrix}\right) = 0$$

$$\rightarrow \frac{\partial \rho E}{\partial t} + \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (\rho E u + P u, \rho E v + P v) = 0$$

$$\rightarrow \frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}(\rho E u + P u) + \frac{\partial}{\partial y}(\rho E v + P v) = 0 \qquad (15)$$

Now, the governing equations can be expressed in vector form with a single vector equation that can be interpreted as a conservation equation for the four final simplified N-S equations broken down into a vector with time-dependent terms, and two vectors with space-dependent terms as,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \qquad (16)$$

(where "U" is the state vector, "F" is the x-component of the flux ~~vector~~, and "G" is the y-component of the flux).

The State vector, "U" contains the time gradient terms from each of the four equations.

Continuity: ρ

x-Momentum: ρu

y-momentum: ρv

Energy: ρE

$$\Rightarrow \boxed{U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}}$$

the flux vector, "F", contains the x-gradient terms,

Continuity: ρu

x-momentum: $\rho u^2 + p$

y-momentum: $\rho uv + p$

Energy: $Eu + pu$

$$\Rightarrow \boxed{F_{(U)} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv + p \\ Eu + pu \end{bmatrix}}$$

the flux vector, "G", contains the y-gradient terms,

Continuity: ρv

x-momentum: $\rho uv + p$

y-momentum: $\rho v^2 + p$

Energy: $Ev + pv$

$$\Rightarrow \boxed{G_{(U)} = \begin{bmatrix} \rho v \\ \rho uv + p \\ \rho v^2 + p \\ Ev + pv \end{bmatrix}}$$

The equation of state for an ideal, calorically perfect gas is expressed as,

$$\underline{\rho R T = p} \qquad (17)$$

(where "R" is the gas constant in $J\,kg^{-1}K^{-1}$, "T" is the temperature in K, and "p" is the pressure in Nm^{-2}).

The gas constant is expressed as,

$$R = \frac{\mathcal{R}}{W} \qquad (18)$$

(where "$\mathcal{R}$" is the universal gas constant, in $J\,kmol^{-1}K^{-1}$, and "W" is the molar mass of the gas in $kg \cdot kmol^{-1}$).

The universal gas constant, "$\mathcal{R}$" equals,

$$\mathcal{R} = 8314\ J\,kmol^{-1}K^{-1}$$

Again, for an ideal, calorically perfect gas,

$$e = \frac{p}{\rho}\frac{1}{\gamma - 1} = \frac{R}{\gamma-1}T = C_p T \qquad (19)$$

(where $\gamma = \frac{C_p}{C_v}$, $C_p = \frac{R}{\gamma - 1}$, and $C_v = C_p - R$)

The speed of sound can be expressed as:

$$a = \sqrt{\gamma R T} = \sqrt{\frac{\gamma p}{\rho}} \qquad (20)$$

Now, we can recover pressure from the state vector by rearranging equation (19)

$$p = (\gamma-1)\rho e = (\gamma-1)(E - \rho |u|^2/2) \qquad (21)$$

The local mach number of the flow is quantified according to:

$$\boxed{Ma = \frac{\overbrace{|u|}^{\text{(magnitude of local velocity)}}}{\underbrace{a}_{\text{(local sound speed)}}}} \qquad (22)$$

Estimation of pressure at Stagnation Point on Surface of Step at $(x,y) = (0.6, 0)$

Given free-stream conditions, $Ma = 3$, $p = 1$, $\gamma = 1.4$ and that density equals

$$\rho = \frac{\gamma p}{a^2} = \gamma = \underline{1.4}$$

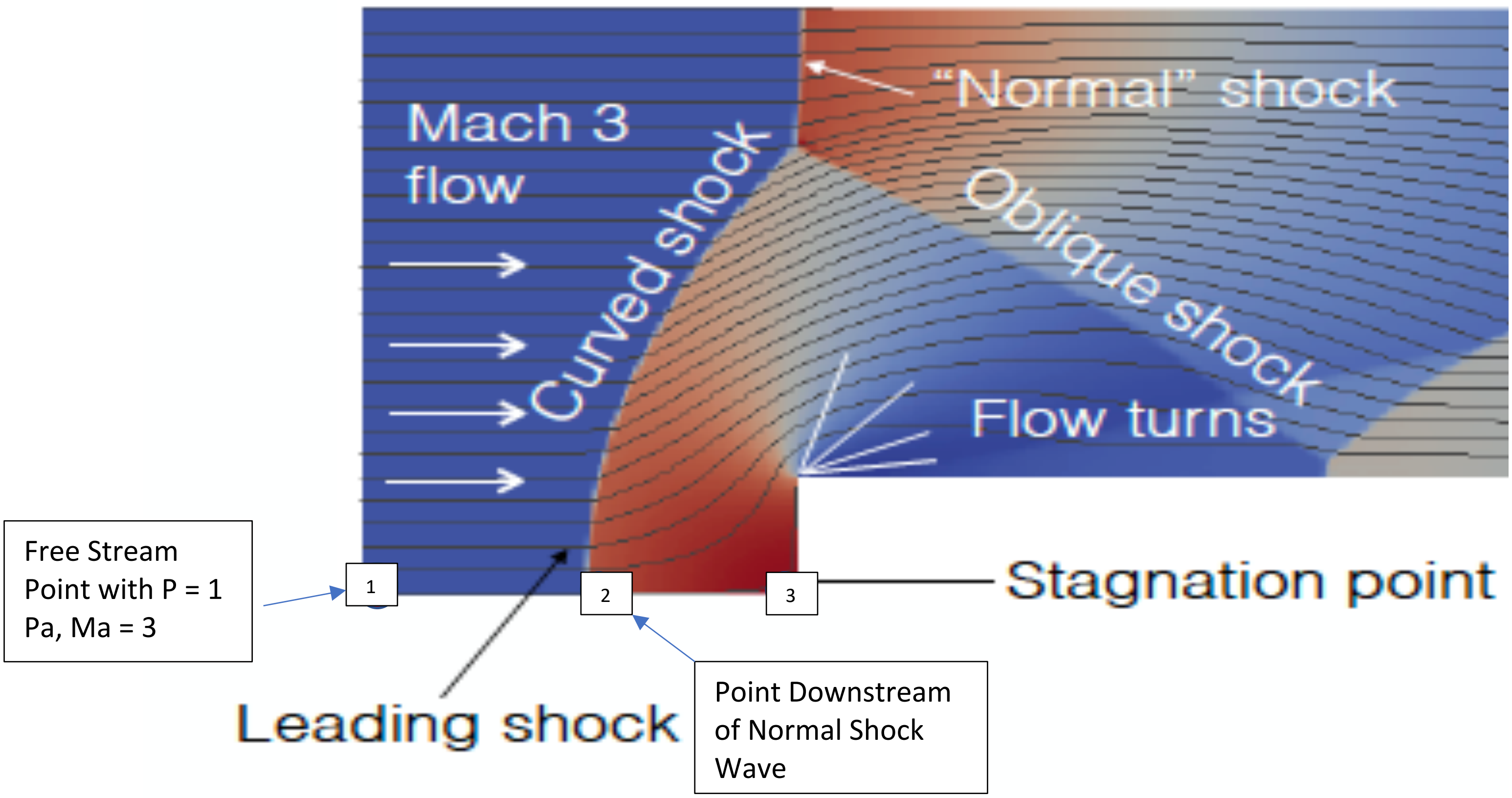
equation (20) can be used to obtain the speed of sound

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{(1.4)(1\,Pa)}{1.4\,kg/m^3}} = \underline{1\ m/s}$$

Substitution of $a = 1\,m/s$ and $Ma = 3$ into equation (22) gives the magnitude of local velocity in the free-stream,

$$|U| = Ma \cdot a = (3)(1\,m/s) = \underline{3\ m/s}$$

In referencing the flow streamlines over the computational grid in the slide "Flow Streamlines" in lecture16.pdf, it is shown that expected behavior contains a shock wave generated normal to the streamlines between the free stream and the stagnation point over $(x,y) = (0,0)$ to $(x,y) = (0.6, 0)$. This region is highlighted in the figure below.



Mach 3
flow
Curved shock
"Normal" shock
Oblique shock
Flow turns
Stagnation point
Leading shock
Free Stream Point with P = 1 Pa, Ma = 3
1
2
3
Point Downstream of Normal Shock Wave

According to NASA documentation on Normal shock wave properties in compressible flow, "For compressible flows with little or small flow turning, the flow process is reversible and the entropy is constant. the change in flow properties are then given by the isentropic relations. the region in the freestream that is compressible flow with little flow turning is then compatible with this definition. the documentation then proceeds to state the following regarding shock waves, "If the shock wave is perpendicular to the flow direction it is called normal shock.". Again, referencing the illustration above, it can be assumed for simplicity that the behavior of the flow at the streamline intersecting the bottom of the curved shock field and the stagnation point exhibits normal shock since the streamline is normal to the shock region along this line. Operating off of this assumption, equations 23) and 24) below are used describe the change in flow properties across a normal shock wave:

$$Ma_2^2 = \frac{(\gamma-1)Ma_1^2 + 2}{2\gamma Ma_1^2 - (\gamma-1)} \qquad (23)$$

(downstream)

(where "Ma_2" is the mach number to the right of the Normal shock wave "Ma_1" is the mach number of the free stream)

$$\frac{P_2}{P_1} = \frac{2\gamma Ma_1^2 - (\gamma-1)}{\gamma+1} \qquad (24)$$

(where "P_2" is the static pressure downstream Normal shock wave, "P_1" is the static pressure in the freestream).

Substitution of $\gamma = 1.4$, $Ma_1 = 3$ into equation (23) and solving for "Ma_2" gives,

$$Ma_2 = \sqrt{\frac{(1.4-1)(3)^2+2}{2(1.4)(3)^2-(1.4-1)}} \cong \underline{0.4752}$$

Rearranging equation (24) and solving for the static pressure gives,

$$p_2 = p_1\left(\frac{2\gamma Ma_1^2-(\gamma-1)}{\gamma+1}\right)$$

$$= (1Pa)\left(\frac{2(1.4)(3)^2-(1.4-1)}{1.4+1}\right)$$

$$\cong \underline{10.3333\ Pa}$$

Now, the region of the streamline between points 2 (downstream of shock wave) and 3 (stagnation point) is isentropic, meaning that the change in flow variables is small and gradual. In referencing NASA's article on "Isentropic Flow" whose citation is provided at the end of the report, equation 25) below depicts the ratio of pressure change between two points in an isentropic flow region,

$$\frac{p}{p_t} = \left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{-\frac{\gamma}{\gamma-1}} \qquad (25)$$

where "p" is the static pressure downstream of shock wave in isentropic flow and "p_t" is the total pressure at any point in isentropic flow region (and at stagnation point)

Rearranging the equation to solve for the stagnation pressure gives,

$$\frac{p_2}{p_3} = \left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{-\frac{\gamma}{\gamma-1}}$$

$$p_3 = \left[\left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{\frac{\gamma}{\gamma-1}}\right]p_2 = \left[\left(1+\frac{1.4-1}{2}(0.475^2)\right)^{\frac{1.4}{1.4-1}}\right](10.333\ Pa)$$

$$\cong \underline{12.06\ Pa}$$

(total)
therefore, the pressure at the stagnation point on the surface of the step at $(x,y) = (0.6, 0)$ for free stream conditions of $M = 3$ and $p = 1$ Pa is approximately 12.06 Pa

References

1) "Normal Shock Wave Equations." Edited by Nancy Editor, *NASA*, NASA, https://www.grc.nasa.gov/WWW/k-12/airplane/normal.html.
2) "Isentropic Flow Equations." Edited by Nancy Hall, *NASA*, NASA, https://www.grc.nasa.gov/WWW/K-12/airplane/isentrop.html.