16 August 2022

IOAA

Georgia

Theoretical Competition

Cover Sheet

1 Planck's units (10 points)

In Max Planck's system of natural units of measurements, commonly used in cosmology, all units are expressed in terms of 4 fundamental constants: speed of light (c), universal gravitational constant (G), reduced Planck constant ($\hbar = \frac{h}{2\pi}$) and Boltzmann constant (k_B). Also, ($4\pi\epsilon_0$) is included in this list of constants.
Using dimensional analysis, find expressions for

- Planck length (L_p),
- Planck time (t_p),
- Planck mass (m_p),
- Planck temperature (T_p) and
- the Planck charge (q_p).

(10pt)

2 Circumbinary planet (10 points)

A stellar system consists of two main-sequence stars each of $2M_\odot$ orbiting with a period of 4 years in a circular orbit. A circumbinary planet orbits the center of the binary system in the same orbital plane and direction at a fixed distance of 20 au.

(a) Calculate the amount of time it takes for the planet to reappear at the same position relative to the binary system. **(6pt)**

(b) At a given instant, what is the maximum fraction of the planet's surface area which receives light from at least one of the stars? Neglect any atmospheric effects and the sizes of the stars. **(4pt)**

3 Expanding ring nebula (10 Points)

A planetary nebula, located 100 pc from the Earth, has the shape of a perfect circular ring with an inner radius of $7.0'$ and an outer radius of $8.0'$. Its luminosity is powered by the UV radiation from the white dwarf remnant at the center of the nebula. From other observations, we know that 2000 years ago the inner radius of the nebula was $3.5'$ and outer radius was $4.0'$. We believe that throughout these 2000 years the evolution of the nebula obeys a free expansion scenario, so gravity is negligible and the expansion velocity remains constant with time. Assume that all material in the planetary nebula was ejected at the same instant, but different gas particles have different velocities.

(a) Estimate the range of velocities of the gas particles **(4pt)**

(b) Is the assumption of free expansion justified? Write **YES** or **NO** alongside appropriate calculations. **(3pt)**

(c) If this planetary nebula is bright enough, would an astronaut aboard the International Space Station be able to resolve the thickness of the shell clearly? Write **YES** or **NO** alongside with necessary calculations. **(3pt)**

4 Journey Between Galaxies (10 Points)

We start a long journey to a planet located at a distance of $d_0 = 10\,\text{Mpc}$ at the start of our journey. During our travel, the universe keeps expanding according to Hubble-Lemaître's law. (Assume that Hubble's constant (H) does not change.)

(a) Write an expression for the distance between Earth and the planet at a time t from the start of our journey. **(2pt)**

(b) What minimum constant speed v_0 should the rocket have to reach the destination? Assuming that the speed of our rocket is $v_0 = 1000\,\mathrm{km/s}$, can we reach the planet? Write **YES** or **NO**. If yes, how long will our journey take? **(8pt)**
Hint: A simplified way to account for the expansion of our universe is to introduce a "scale factor" $a(t)$, which relates distance between two objects $l(t)$ at a time t to a distance l_0 at a time $t = 0$ as $l(t) = a(t)l_0$

5 Flaring protoplanetary disk (10 Points)

Planets are the products of collapsing clouds forming circumstellar disks around the young stars. In this problem, we examine the thermal structure of a type of protoplanetary disks. We consider radiation from the central star to be the dominant heating process and assume that the disk is optically thick and the radiation can be only absorbed in the surface layers of the disk. This type of disk is called a *flaring* disk (see figure 1 below). In this case, the star illuminates the surface of the disk directly, and the shallow angle β between the light rays and the surface of the disk increases with distance from the star.

(a) Find an expression for β as a function of r. Assume $h(r) \ll r$. **(4pt)**
Hint: the expression involves $h(r)$, r and $\frac{\Delta h(r)}{\Delta r}$

(b) Find an expression for the equilibrium temperature of the disk (T_D) as a function of $\beta(r)$, r and luminosity (L_s) of the star (Ignore the size of the star). **(3pt)**

(c) We parameterize the disc height as $h(r) = ar^b$, where a and b are constants. For which values of a and b is the condition for an isothermal layer (T_D =Constant) satisfied? Write your answer for constant a in terms of T_D and L_S **(3pt)**
Hint: If $\Delta r \ll r$, then $(1 + \frac{\Delta r}{r})^b \approx 1 + \frac{b\Delta r}{r}$.

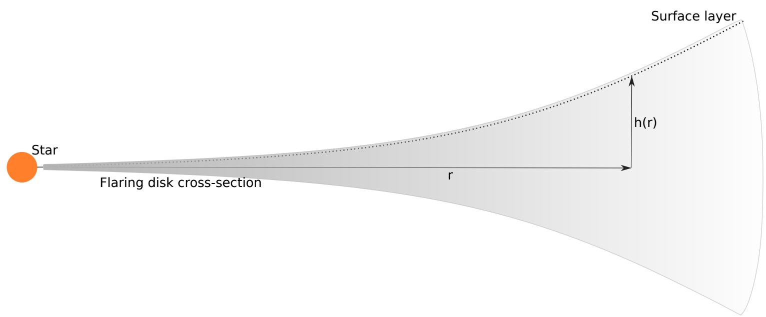


Figure 1: cross section of flaring disk

6 Photometry of Binary stars (20 Points)

We observe a binary system, at a distance $d = 89\,\mathrm{pc}$, with a circular, edge-on orbit around the common center of mass and period of 100 days. Our space-based telescope is of diameter $D = 10\,\mathrm{m}$ and operates at a wavelength $\lambda = 364\,\mathrm{nm}$. We notice that for a total of 38 days during each full period, the two stars cannot be resolved by this telescope as separate objects. The wavelength of peak emission for star A is $\lambda_A = 500\,\mathrm{nm}$ and that for star B is $\lambda_B = 600\,\mathrm{nm}$.

(a) What are the temperatures of the stars A and B, (T_A, T_B) **(2pt)**

(b) Calculate the distance l between the stars. **(6pt)**

(c) Calculate the sum of the masses of the stars (M_T). **(2pt)**

Combined photometry of the system:

	Configuration	U_0	$(U-B)$	$(B-V)$	BC
1	Stars next to each other	6.39	0.2	0.1	0.1
2	B transiting in front of A	6.86	0.25	0.12	0.175

The interstellar extinction per kpc of U is $a_U = 1.4\,\mathrm{mag/kpc}$. The ratio of the densities of the stars $\rho_A/\rho_B = 0.7$
note: For Bolometric correction we use convention:

$$BC = m_{bol} - m_V$$

(d) Calculate the masses of both the stars (M_A, M_B). **(10pt)**

7 Georgia to Georgia (20 points)

Astronomer Keto was flying west overnight along the shortest possible route from Tbilisi (the capital of Georgia) to Atlanta (the capital of the US state of Georgia). She noticed that she is able to observe the star Furud (ζ CMa) throughout the entire flight from one of the jet's windows (although it did touch the horizon at one point, when it also happened to be exactly due South).

Calculate the latitude ϕ_B and longitude λ_B, of Atlanta where she landed. **(20pt)**

It is given that:

- The journey had a duration of 11 hours 25 minutes, and the jet travelled at an average speed of $875\,\mathrm{km/h}$.
- Furud has a declination of $\delta_F = -30°4'$.
- The coordinates of Tbilisi are $\phi_A = 41°43'$N and $\lambda_A = 41°48'$E.
- You should ignore the effect of the rotation of the Earth during the flight, the altitude of the jet, atmospheric refraction, and any wind.

8 Ring of a planet (20 Points)

We are given a flat disk of mass M with inner radius r and the outer radius R.

(a) A point mass m is located on the symmetry axis of the disk at a distance x from the plane of the disk (We assume throughout that mass m stays on the symmetry axis, and is free to move in the vertical direction.). What is the gravitational force exerted on the point mass? **(10pt)**

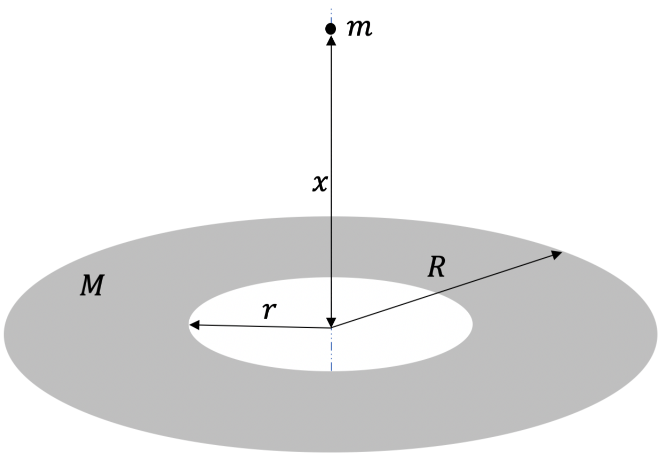


Figure 2: Flat disk around a point mass

Hint 1: You may denote surface density by the symbol σ and calculate force exerted by a small surface element ΔS subtending a solid angle $\Delta\Omega$ with m

Hint 2: The area cut by cone with opening angle 2θ on a sphere of radius R:

$$S = 2\pi R^2(1 - cos(\theta))$$

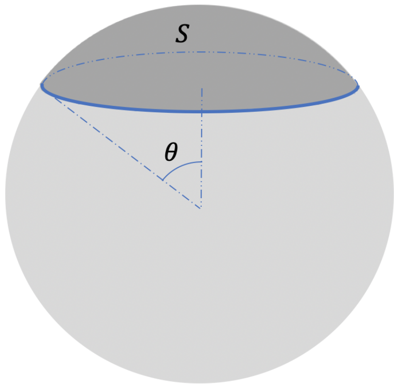


Figure 3: Area cut by a cone on a sphere

(b) what will be the frequency of the small oscillations of the system ($x \ll r$). **(10pt)**

9 Solar Retrograde Motion on Mercury (20 points)

Mercury's orbit has an unusually high eccentricity. Further, its sidereal rotational period is $\frac{2}{3}$ of its sidereal year. As a consequence of these factors, the Sun will exhibit retrograde motion in Mercury's sky, when Mercury is near perihelion.

Calculate the total duration of this apparent solar retrograde motion during one orbit of Mercury around the Sun. Express your answer in Earth days. **(20pt)**

10 Accretion (20 Points)

Consider a compact object (such as black hole, white dwarf or neutron star) with a spherically symmetric accretion of gas, assume that accreting gas is hydrogen. As particles fall into the object they heat up and radiate, thus creating radiation pressure acting on rest of the accreting material. This force is given by,

$$F_L = \sigma_e \frac{I}{c}$$
$$\text{where, } \sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2$$

is the Thompson cross section for electrons, c is the speed of light and I intensity of the light. Although F_L is calculated for electrons, it effectively acts on the whole atom.

(a) If the central compact object has a mass M, find an expression for the Eddington limit (L_E), which is the maximum possible luminosity for the accretion sphere. **(3pt)**

(b) For a particular compact object, the luminosity due to material accretion is the same as the Solar luminosity $L_\odot$. What is its minimum possible mass of this object to achieve this luminosity (in units of $M_\odot$)? **(4pt)**

Assume that the atoms in the accretion sphere originate far away from the compact object. When these atoms fall into the compact object, their gravitational energy is transformed into radiation.

(c) Derive an expression for accretion luminosity (L_{acc}) in terms of the compact object's mass (M), mass accretion rate ($\dot{M} = \frac{\Delta M}{\Delta t}$), and the compact object's radius (R). **(2pt)**

(d) Show that the maximum possible accretion rate in steady state is not directly dependent on the mass of the compact object. **(2pt)**

(e) For an object with $R = 12 \times 10^9$ m, calculate the maximum possible accretion rate $\dot{M}$ (in units of $M_\odot$ per year). **(2pt)**

In reality, the accretion geometry is disk shaped, where most particles trace an almost circular orbit around the compact object. Consider a binary system consisting of a compact object of mass M_1 and a hydrogen burning star of mass M_2 at a distance a from each other. The gas from the hydrogen burning star is accreted by the compact object and due to this mass transfer the period of the binary system changes.

(f) Find the condition on the two masses such that the separation between the stars is increasing. Ignore the rotation of the stars. **(7 pt)**

11 Dyson Sphere (50 Points)

The **Kardashev scale** distinguishes three stages of evolution of civilizations according to the criterion of access to and use of energy.
A type II civilization is capable of harnessing all the energy radiated by its own star. Currently, we are a type zero civilization (we are not even harnessing 100% of the energy that reaches the Earth). One of the ways to become a type II civilization is by building a **Dyson Sphere**. You can imagine it as a sphere, built around the Sun, having the inner surface covered with solar panels.

We assume that modern solar panels are used to build the sphere. First, let's find out at what distance from the Sun should it be built.

Emissivity of the back side of solar panels is $\epsilon = 0.8$.

(a) Solar panels absorb and transfer about $k = 30\%$ of the incident radiation into internal heat. Find the temperature of a Dyson Sphere of radius R. Express your answer in terms of k, R, ϵ and $L_\odot$ (solar luminosity). You may consider the Sun to be a black body. Ignore reflections from the solar panels. Ignore any possible effect of the energy not transferred to the internal heat of the solar panels or into electrical energy generated by the panels. **(3pt)**

Assume that the highest operational temperature for modern solar panels is about $T_{max} \approx 104.5\,°\mathrm{C}$. After that, efficiency drops significantly. To minimize the amount of material used, we should consider building the sphere as small as possible.

(b) Calculate the radius of the sphere for the panels to work properly. Does the Earth stay inside or outside the sphere? Write **IN** or **OUT** in the answer sheet. **(4pt)**

(c) Find the power harnessed by this Dyson Sphere, if the final power output of modern solar panels is about $\eta = 20\%$ of the incident energy. **(2pt)**

(d) Currently, the average power usage of the whole world is about 17 Terawatts. If this Dyson sphere collects energy for one second, for how long could that meet our energy needs? **(2pt)**

(e) In the case where the Dyson sphere completely blocks out the rays of the Sun, the temperature on Earth will drop significantly. Calculate the change in the average temperature of the Earth in this condition, if current average temperature is about $15\,°\mathrm{C}$. Assume that Earth is also a black body. **(5pt)**

(f) Building a rigid spherical object of that size is nearly impossible. Another way of "building" the sphere is by sending individual panels to orbit around the sun (in different inclined orbits) at the radius R found in part b. Calculate the period T of any object orbiting the sun at that radius. **(3pt)**

(g) Assume that each solar panel is a thin sheet of silicon, having unit surface mass $\rho = 1\,\mathrm{kg/m^2}$. The radiation pressure from the Sun, might interfere with the orbit of the panel. Calculate the ratio α of the gravitational and photon forces for unit surface area of panels at distance R. You assume that all incident light is absorbed. Will this radiation pressure have any measurable effect? Write **YES** or **NO** in the Answer Sheet. **(13pt)**

Now assume that the Dyson Sphere is a rigid body rotating with the period found in part 'f' and having the radius found in part 'b'.

(h) A major threat to the Dyson Sphere would be an asteroid whose orbit crosses the surface of the sphere. One way to solve this problem is by removing panels from the path of the asteroid. Obviously the size of the hole through which asteroid should pass is much smaller than radius of the sphere.

Astronomers discovered an asteroid on an orbit in the ecliptic plane in the same direction as the direction of rotation of the Dyson sphere. They calculated that this asteroid will enter the sphere on 14th of August and leave it on 20th of September. Calculate the angular distance between the two holes in the sphere needed to provide a safe passage for the asteroid. **(13pt)**
The trajectory of this asteroid in heliocentric cylindrical coordinate system can be described as

$$r = \frac{a}{1-\cos\theta}$$

where $a = 1.00\,\mathrm{au}$.

(i) In what range of wavelengths should we be searching for a Dyson sphere created by a type II civilization in a distant galaxy, if its distance from Earth is d and the sphere can operate between temperatures T_1 and T_2 ($T_1 < T_2$). Assume only non-relativistic effects. **(5pt)**

12 Co-orbital satellites (50 Points)

This question applies a method of determining the masses of two approximately co-orbital satellites developed by Dermott and Murray in 1981.
Suppose that two small satellites of masses m_1 and m_2 are approximately co-orbital (moving on very similar orbits) around a large central body of mass M, with $m_1, m_2 \ll M$. At any instant, the orbits of the satellites may be approximated as circular Keplerian orbits with radii r_1 and r_2 respectively, although r_1 and r_2 will vary slightly over time due to the mutual gravitational interaction between the satellites.
Figure 4 depicts the shapes of the orbits in the rotating reference frame with zero angular momentum, centred on the central body. We denote by θ the angle $\measuredangle m_1 M m_2$, while R, x_1, and x_2 denote the mean orbital radius and radial deviations of the satellites.
Throughout this problem, write all answers in an inertial reference frame.
Hint: $(1+x)^\alpha \approx 1+\alpha x$ for $\alpha x \ll 1$
First we will determine the value of $\frac{m_1}{m_2}$.

(a) Write down the angular momentum L_i of the satellite with mass m_i when its circular orbit has radius r_i. **(3pt)**

(b) The satiletes total angular momentum $L_1 + L_2$ is conserved. Let $x_1,\ x_2 \ll R$ be the distances as shown in Figure 4. Find a simple relation between the ratios $\frac{m_1}{m_2}$ and $\frac{x_1}{x_2}$. **(8pt)**

Now, we will try and determine the value of $m_1 + m_2$. For next parts, we will use the actual barycenter of the system, which may not be exactly at the center of the planet.

(c) The individual angular momenta of the satellites m_1 and m_2 will vary over time due to their gravitational interactions. Show that the rate of change of the angular momentum of the second satellite is given by

$$\frac{\Delta L_2}{\Delta t} \approx -\frac{Gm_1m_2}{R}h(\theta) \quad \text{where} \quad h(\theta) = \left[\frac{\cos\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)} - \sin\theta\right]$$ **(18pt)**

(d) Show that $s = r_2 - r_1$ satisfies

$$\frac{\Delta s}{\Delta t} \approx -2\sqrt{\frac{G}{MR}}(m_1+m_2)h(\theta)$$ **(8pt)**

(e) For the angle $\theta = \measuredangle m_1 M m_2$ as indicated on Figure 4, find an expression for $\frac{\Delta\theta}{\Delta t}$ **(5pt)**

(f) Using the results above, find the relation between Δs and $\Delta\theta$. **(2pt)**

(g) After integrating the expression above, we will obtain the result,

$$\overline{x}^2 \approx \frac{4R^2}{3}\frac{m_1+m_2}{M}\left(\frac{1}{\sin\left(\frac{\theta_{\text{min}}}{2}\right)} - 2\cos\theta_{\text{min}} - 3\right)$$

where $\overline{x} = \dfrac{x_1+x_2}{2}$.

Epimetheus (m_1) and Janus (m_2) are two approximately co-orbital moons of Saturn. Detailed observations of their orbits have been performed by the Voyager 1 and Cassini spacecraft, which found that $R = 150\,000\,\text{km}$, $x_1 = 76\,\text{km}$ and $x_2 = 21\,\text{km}$. The minimum distance between Janus and Epimetheus is $13\,000\,\text{km}$. The mass of Saturn is known to be $5.7\times10^{26}\,\text{kg}$. Estimate the masses of Epimetheus and Janus. **(6pt)**

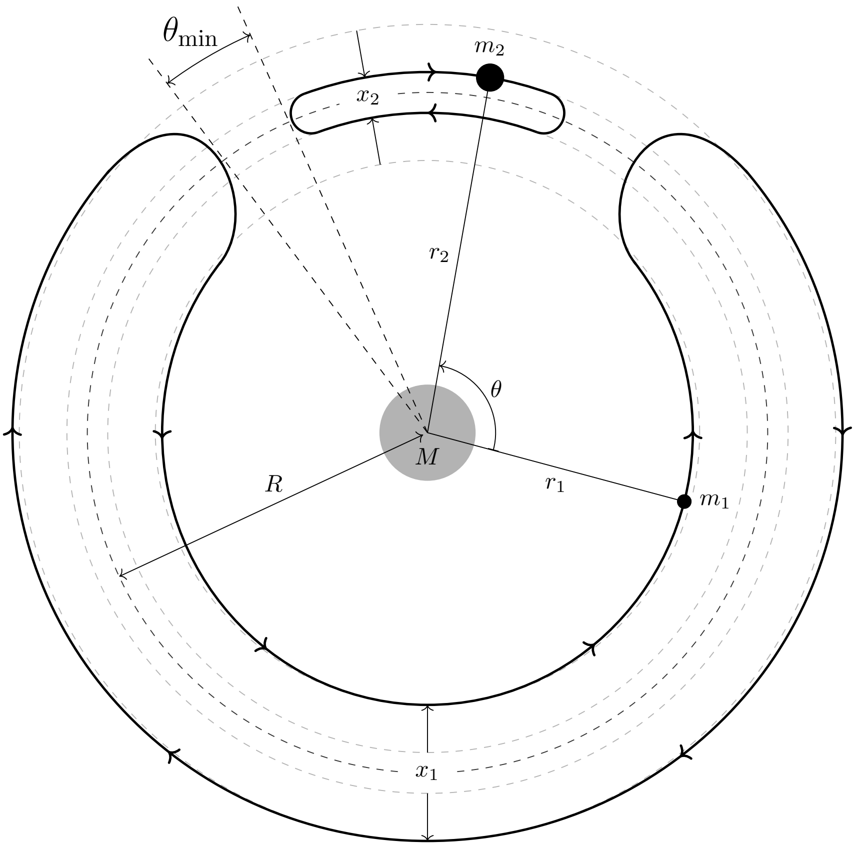


Figure 4: This figure schematically depicts the shapes of the orbits in the rotating reference frame, selected such that in this frame the total angular momentum of the two satellites is zero.

13 Relativistic Beaming (50 Points)

Consider an isotropic light source of frequency f_R in a frame which is fixed to the source (i.e. rest frame). In this rest frame, consider a light ray emitted from the source that makes an angle θ_R with the X-axis. The light source is moving along positive X direction with (relativistic) speed v as measured in the lab frame.

(a) Find an expression for the frequency f_L of this ray in the lab frame, and the cosine of the angle that this ray makes with the X-axis in the lab frame. **(11pt)**
Hint: In relativistic mechanics, energy E and momentum p of a particle between rest and lab frame are related in the following way:

$$\frac{E_L}{c} = \gamma\left(\frac{E_R}{c} + p_{x_R}\frac{v}{c}\right)$$
$$p_{x_L} = \gamma\left(p_{x_R} + \frac{E_R v}{c^2}\right)$$
$$p_{y_L} = p_{y_R}$$
$$p_{z_L} = p_{z_R}$$

where:

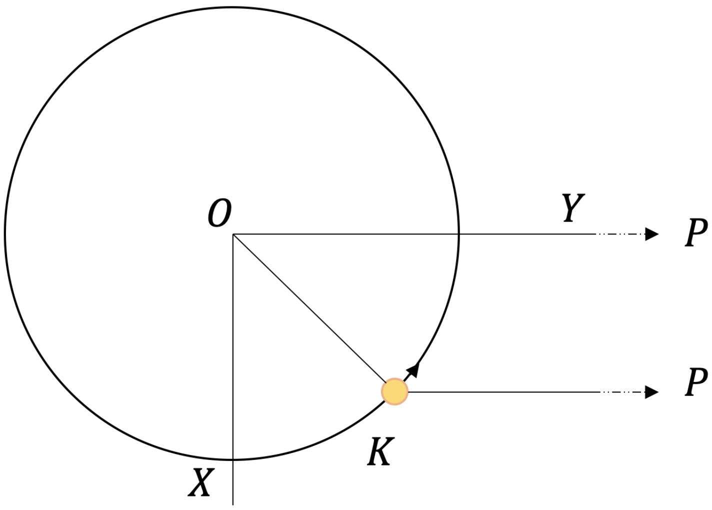
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(b) For the following cases:

i) $\theta_R = 0°$
ii) $\theta_R = \cos^{-1}(-v/c)$
iii) $\theta_R = 90°$
iv) $\theta_L = 180°$

draw direction vectors of the beam in XY plane of the rest frame as well as separately in $X'Y'$ plane of the lab frame. **(4 pt)**

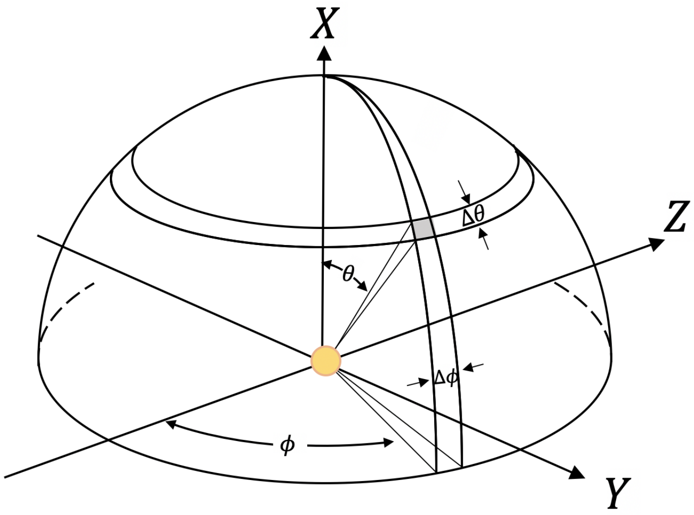
In accretion disks around black holes, the charged particles are orbiting at relativistic speeds and in their rest frames may be considered as isotropic point sources of light. Consider such a particle K in a circular orbit of radius r and angular speed ω around a central object located at O (see figure).



Let us assume that our lab frame is fixed to an observer located at a point P on the OY axis, which is stationary with respect to O. $OP = R \gg r$. Let $t_{L0} = t_{R0} = 0$ correspond to the moment when K is seen crossing the OX axis. As K is moving with relativistic speed, the duration Δt_R measured by an observer in the rest frame of the source K is related to the duration measured in the lab frame Δt_L at P by the expression $\Delta t_L = \gamma \Delta t_R$.

(c) Derive an expression for f_L as a function of t_L ($t_L > R/c$)? **(7pt)**

Let us consider a fraction of the light from the source that is emitted in an infinitesimal solid angle $\Delta\Omega_R = -\Delta(\cos\theta_R)\cdot\Delta\phi$ in the direction making an angle θ_R with respect to the X axis in the rest frame, as it is shown on the figure below.



(d) Show that, as measured in the lab frame

$$\Delta\Omega_L = \frac{\Delta\Omega_R}{\gamma^2\left(1+\dfrac{v}{c}\cos\theta_R\right)^2}$$ **(10pt)**

(e) If the intrinsic luminosity of the light source is L, what is the energy flux F_L observed by the observer at point P at the moment t_L ($t_L > R/c$)? **(15pt)**
Hint: In the rest frame of the source, you may assume N_R number of photons are directed within the solid angle $\Delta\Omega_R$ during the time interval Δt_R.

(f) Charged particles in the relativistic beam shot from the supermassive black hole at the centre of the galaxy M87 have speeds up to $0.95c$. What would be the maximum and minimum amplification factor for the energy flux for a relativistic beam from M87? **(3pt)**