CSCI 3104 Fall 2021
Instructors: Profs. Chen and Layer

Problem Set 10

Due Date .... April 26
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Contents

1 Instructions

- The solutions **must be typed**, using proper mathematical notation. We cannot accept hand-written solutions. Here's a short intro to LaTeX.
- You should submit your work through the **class Canvas page** only. Please submit one PDF file, compiled using this LaTeX template.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this document with no fewer pages than the blank template (or Gradescope has issues with it).
- You are welcome and encouraged to collaborate with your classmates, as well as consult outside resources. You must **cite your sources in this document. Copying from any source is an Honor Code violation. Furthermore, all submissions must be in your own words and reflect your understanding of the material.** If there is any confusion about this policy, it is your responsibility to clarify before the due date.
- Posting to **any** service including, but not limited to Chegg, Reddit, StackExchange, etc., for help on an assignment is a violation of the Honor Code.

2 Standard 26 - Showing problems belong to P

Problem 1. Consider the Shortest Path problem that takes as input a graph $G = (V, E)$ and two vertices $v, t \in V$ and returns the shortest path from v to t. The shortest path decision problem takes as input a graph $G = (V, E)$, two a vertices $v, t \in V$, and a value k, and returns True if there is a path from v to t that is at most k edges and False otherwise. Show that the shortest path decision problem is in P. You are welcome and encouraged to cite algorithms we have previously covered in class, including known facts about their runtime. [**Note:** To gauge the level of detail, we expect your solutions to this problem will be 2-4 sentences. We are not asking you to analyze an algorithm in great detail.]

Answer. To show a problem is in P, we must show that there is a polynomial time algorithm to solve the problem. Here, we can apply Dijkstra's algorithm to G starting at vertex v. If there is a path from v to t with weight $\leq k$, then the decision problem is YES. Otherwise, return NO.

Since Dijkstra's runs in $O(|E| + |V|log(|V|)$, it runs in polynomial time.

So we conclude that Shortest Path $\in$ P $\square$

3 Standard 27 - Showing problems belong to NP

Problem 2. Consider the Simple Shortest Path decision problem that takes as input a directed graph $G = (V, E)$, a cost function $c(e) \in \mathbb{Z}$ for $e \in E$, and two vertices $v, t \in V$. The problem returns True if there is a simple path from v to t with edge weights that sum to at most k, and False otherwise. Show this problem is in NP.

Answer. Suppose G is a graph with a path from v to t with edge weights that sum to at most k.
A viable certificate is a sequence of vertices that forms a path from v to t with weight $\leq k$
We then check that consecutive vertices are adjacent and that edge weights sum to $\leq k$
This algorithm takes $O(|V| + |E|)$ time to verify the certificate, and since this is polynomial time, we have it that the Simple Shortest Path $\in NP$ $\square$

Problem 3. Indiana Jones is gathering n artifacts from a tomb, which is about to crumble and needs to fit them into 5 cases. Each case can carry up to W kilograms, where W is fixed. Suppose the weight of artifact i is the positive integer w_i. Indiana Jones needs to decide if he is able to pack all the artifacts. We formalize the Indiana Jones decision problem as follows.

- Instance: The weights of our n items, $w_1, \ldots, w_n > 0$.
- Decision: Is there a way to place the n items into different cases, such that each case is carrying weight at most W?

Show that Indiana Jones $\in$ NP.

Answer. Suppose A is an arrangement of n artifacts packed into Indian Jones' 5 cases and each case is carrying $\leq W$.
A viable certificate is an arrangement of n artifacts in 5 cases such that weight of each case $\leq W$.
We then need to check whether the weight satisfies the limit. This takes $O(n)$ time. We also must check that all n items are packed in a case. Again this takes $O(n)$ times. Thus, it takes polynomial time to verify the certificate.
Therefore, Indiana Jones $\in$ NP.

□

4 Standard 27 - NP-compelteness: Reduction

Problem 4. A student has a decision problem L which they know is in the class NP. This student wishes to show that L is NP-complete. They attempt to do so by constructing a polynomial time reduction from L to SAT, a known NP-complete problem. That is, the student attempts to show that $L \leq_p \mathsf{SAT}$. Determine if this student's approach is correct and justify your answer.

Answer. No, the students approach is incorrect. By theorem 8.14 of Jon Klienberg, if Y is an NP complete problem and X is an NP problem and $Y \leq_p X$, then X is NP complete.
Thus, to prove problem L is NP-complete, the student must show how to reduce L from SAT. Thus, she must show that $\mathsf{SAT} \leq_p L$..

□

Problem 5. Consider the Simple Shortest Path decision problem that takes as input a directed graph $G = (V, E)$, a cost function $c(e) \in \mathbb{Z}$ for $e \in E$, and two vertices $v, t \in V$. The problem returns True if there is a simple path from v to t with edge weights that sum to at most k, and False otherwise. Show this problem is NP-compelte.

Answer. First, we know $SimpleShortestPath \in NP$ by proof above.
We know that $3SAT \in NPcomplete$
We now show that $3SAT \leq_p SimpleShortestPath$

Let ϕ be an instance of 3SAT with m three-literals clauses $C_j, 0 \leq j < m$ over n variables $x_i 0 \leq i < n$

We construct a weighted graph from each variable x_i and clause C_j and then join these subgraphs. Edge wegihts are either 0 or 1. The first vertex of subgraph x_0 is v and the last node of the subgraph is C_{m-1} which is t. Thus to get from v to t, the path will pass through all variables and clause subgraphs.

The constructed graph has $O(n + m)$ vertices and edjes, which is polynomial time.

It suffices to show that ϕ is satisfiable if and only if the shortest path from v to t is length n. Since any path v to t must pass through all vertices with an edge weight 1, length must be at least n.

So we conclude that $3SAT \leq_p SimpleShortestPath$ so $SimpleShortestPath$ is NP complete

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