GARCH-M model with an asymmetric risk premium: Distinguishing between “good” and “bad” volatility periods

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Abstract

This paper introduces a novel method, the GARCH-M-LV model, designed to capture asymmetry in both the variance and return equations. The model development is motivated by the stylized fact that investors tend to demand a higher risk premium during periods of “bad” volatility compared to “good” volatility. To examine the properties of the estimators, I performed a Monte Carlo analysis, using a data-generating process that reflects the asymmetric response of the risk premium to volatility shifts. The results provide robust statistical evidence supporting the superiority of the proposed model over existing alternatives. Furthermore, the GARCH-M-LV model was applied to the S&P 500 index, revealing significant asymmetry in the relationship between the risk premium and volatility changes across most periods.

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1. Introduction

Volatility is widely recognized as one of the most critical factors in financial markets, as it reflects the risk level of assets and is typically measured by the standard deviation of returns (Engle et al., 1987). Consequently, various market participants are deeply interested in volatility modeling to assess the risk of security portfolios and implement effective risk management strategies. Notably, volatility forecasting plays an essential role in hedging, option pricing models, and other financial applications (Miralles-Marcelo et al., 2013).

The family of GARCH models, first introduced by (Engle, 1982) and generalized by (Bollerslev, 1986), is one of the most widely used methods to model the conditional volatility of financial assets. These models relax the assumption of constant variance for one-period forecasts, a key assumption in classical regression models such as linear regression and ARFIMA models (Engle, 1982), (Bollerslev, 1986). The development of GARCH models enabled the modeling of conditional variance dynamics, spurring a new line of literature on financial time series modeling.

The introduction of GARCH models led to extensive research on variations of these models. Over the past few decades, numerous univariate and multivariate GARCH-type models have been proposed, demonstrating their effectiveness in financial time series analysis (Bollerslev, 1990), (Engle, 2002), (Engle and Kroner, 1995). These modifications were driven by the need to adapt the model to financial theory and the stylized facts observed in financial markets.

According to the portfolio theories developed by (Markowitz, 1952) and (Sharpe, 1964), asset returns and volatility are key determinants in pricing, with a positive causal relation between them. Specifically, investors require higher returns for riskier assets, a concept fundamental to financial economics. This idea underpins the notion that risky assets include a risk premium component in their returns. However, classic GARCH models do not account for the presence of a risk premium in the return equation, which led to the development of the GARCH-in-Mean (GARCH-M) model introduced by (Engle et al., 1987). The key innovation of this model is incorporating conditional variance into the mean equation to capture the risk premium's contribution to return dynamics. The GARCH-M model has since become widely adopted in empirical research (Bollerslev et al., 1988).

Despite its broad usage, (Bollerslev, 2022) highlights inconsistencies in the sign and robustness of the estimated risk premium in several studies (Hong and Linton, 2020), (Rossi and Timmermann, 2015), (Bollerslev et al., 2006). According to (Bollerslev, 2022), recent studies show that the risk premium's contribution varies across periods and assets, and these contradictory results may stem from the asymmetric relationship between volatility and returns. The classic GARCH-M model cannot capture this asymmetry, leading to inconsistent estimates of the risk premium's effect.

While the concept of an asymmetric relationship between returns and volatility is relatively new within the GARCH-M framework, it has been well-documented in traditional GARCH models, where conditional variance exhibits asymmetric responses to return shocks. A widely observed phenomenon in financial markets is that volatility responds more sharply to negative return shocks than to positive ones (Zhang, 2006) (Black, 1976). Since classical GARCH models assume symmetry, they fail to account for this behavior (Nelson, 1991). This limitation spurred the development of leverage models, such as EGARCH (Nelson, 1991) and GJR-GARCH (Glosten et al., 1993), which incorporate asymmetry in the variance equation. Recent research has focused on more flexible asymmetric variance specifications (Hansen et al., 2012) or multivariate extensions of asymmetric GARCH models (McAleer et al., 2009). However, the literature has paid little attention to capturing asymmetric responses in returns to conditional volatility, i.e., asymmetry in the mean equation. This paper addresses this gap by introducing the leverage effect in the mean part of the GARCH-M model.

This study proposes the GARCH-M-LV model, which accounts for asymmetry not only in the variance equation but also in the return equation. The model, developed within the GJR-GARCH framework, suggests that negative return shocks lead to a greater impact of conditional volatility on returns. Specifically, the model includes an additional parameter in the mean equation to capture the asymmetric responses of returns to changes in volatility.

The results of the Monte Carlo analysis provide statistical evidence highlighting the proposed model's advantages over existing GARCH-type models. Furthermore, applying the model to the S&P 500 index log-returns reveals significant asymmetry in the risk premium across different time periods.

The structure of this paper is as follows: The next section reviews the literature on symmetric and asymmetric autoregressive conditional heteroscedasticity models and discusses the challenges of modeling the time-varying risk premium and the leverage effect. The following section introduces the GARCH-M-LV model, explaining its underlying intuition and presenting the unconditional variance theorem. Section 4 presents the results of the simulated data analysis and discusses the model's properties. Section 5 applies the model to the log-returns of the S&P 500 index and evaluates the findings. Finally, the conclusion summarizes the main contributions and results of the study.

2. Literature Review

2.1. The GARCH-M Model

On the probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$, let $\mathcal{F}_{t-1}$ be defined as the σ-algebra at period $t-1$, representing the set of all information available up to period $t-1$, where $t \in \mathbb{N}$. Let y denote a $T \times 1$ vector of log returns, and σ a $T \times 1$ vector of conditional volatilities, where $T \in \mathbb{N}$. Without loss of generality, the GARCH-M(1,1) process can be specified with the following system of equations (Engle et al., 1987):

$$\begin{aligned} y_t &= \mu + \lambda\sigma_t + \varepsilon_t, \\ \varepsilon_t|\mathcal{F}_{t-1} &\sim \mathcal{N}\left(0, \sigma_t^2\right), \\ Var(y_t|\mathcal{F}_{t-1}) = \sigma_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \\ \xi_t &\sim \mathcal{N}\left(0, 1\right); i.i.d., \\ \varepsilon_t &= \sigma_t\xi_t, \end{aligned} \tag{2.1}$$

where the parameters ω, α, and β determine the dynamics of conditional variance, and $t \in \{1, \ldots, T\}$. The key feature of this model is the inclusion of σ_t in the mean equation, allowing conditional volatility to directly impact returns. Consequently, the term $\lambda\sigma_t$ is typically interpreted as a risk premium. According to the portfolio theories of (Markowitz, 1952) and (Sharpe, 1964), λ is expected to be positive, as higher volatility is associated with higher returns (Engle et al., 1987).

While this model captures the risk premium effect, it does not account for asymmetric responses. Specifically, the model assumes that returns react symmetrically to changes in volatility, regardless of the sign of the shocks that caused the volatility. As a result, volatility during both "bear" and "bull" markets has the same impact on returns.

2.2. The GJR-GARCH Model

The GJR-GARCH model (Glosten et al., 1993) was developed to capture the leverage effect in financial markets. This framework is adopted as the foundation for the proposed model, as its theoretical properties are well established, even in the multivariate case (McAleer et al., 2009). Another possible alternative is the EGARCH model proposed by (Nelson, 1991). According to the empirical findings of (Awartani and Corradi, 2005), EGARCH demonstrated the highest accuracy when applied to S&P 500 index data, outperforming various asymmetric GARCH-family models, including GJR-GARCH, TGARCH, and AGARCH. However, deriving the theoretical properties of the EGARCH model is challenging, as it is based on a complex random coefficient nonlinear moving average process whose stationarity conditions remain unknown (McAleer and Hafner, 2014). Furthermore, (Awartani and Corradi, 2005) found that GJR-GARCH performed second best, following EGARCH, when combining asymmetric GARCH models with a classical GARCH-M specification. Therefore, GJR-GARCH is preferred over other asymmetric GARCH models as the basis for this study. Its modifications, such as GARCH-M-LV, allow for relatively straightforward derivation of theoretical properties, and it has demonstrated solid performance when applied to the S&P 500 index, which is used in the empirical section of this research.

This process is similar to the classical GARCH model but incorporates asymmetric responses of conditional variance to shocks in returns. Using the same notation as in the previous model, the GJR-GARCH(1,1) process is specified as:

$$
\begin{aligned}
y_t &= \mu + \varepsilon_t, \\
\varepsilon_t|\mathcal{F}_{t-1} &\sim \mathcal{N}\left(0, \sigma_t^2\right), \\
Var(y_t|\mathcal{F}_{t-1}) = \sigma_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \gamma\mathbb{1}_{t-1}\varepsilon_{t-t}^2 + \beta\sigma_{t-1}^2, \\
\mathbb{1}_t &= \begin{cases} 0, \text{ if } \varepsilon_t \geq 0 \\ 1, \text{ if } \varepsilon_t < 0 \end{cases}, \\
\xi_t &\sim \mathcal{N}\left(0, 1\right) \ i.i.d., \\
\varepsilon_t &= \sigma_t\xi_t.
\end{aligned}
\tag{2.2}
$$

The key feature of this model is its ability to capture asymmetric responses of volatility to positive and negative return shocks. This is accomplished by incorporating the indicator variable $\mathbb{1}_t \coloneqq \mathbb{1}\{\varepsilon_t < 0\}$, which reflects the sign of the shock in period t. Specifically, if the shock is positive, the contribution to volatility comes only from the α parameter, while negative shocks result in a sharper increase in conditional volatility, equal to $\alpha + \gamma$.

It is straightforward to show that the unconditional mean of the process is $\mathbb{E}[y_t] = \mu$, while the unconditional variance is given by Lemma 1.

Lemma 1. *Assume y_t to be a weakly stationary process. Then the unconditional variance of y_t in the GJR-GARCH(1,1) model is given by the following expression (Glosten et al., 1993):*

$$
Var\left(y_t\right) = \sigma^2 = \frac{1}{1 - \alpha - \gamma/2 - \beta}.
$$

Proof of Lemma 1. Using the conditional variance specification in equation (2.2), we can expand the expectation as follows:

$$
\begin{aligned}
\mathbb{E}[\sigma_t^2] &= \mathbb{E}[\omega + \alpha\varepsilon_{t-1}^2 + \gamma\mathbb{1}_{t-1}\varepsilon_{t-t}^2 + \beta\sigma_{t-1}^2] \\
&= \omega + \alpha\mathbb{E}[\varepsilon_{t-1}^2] + \gamma\mathbb{E}[\mathbb{1}_{t-1}] \times \mathbb{E}[\varepsilon_{t-1}^2] + \beta\mathbb{E}[\sigma_{t-1}^2].
\end{aligned}
$$

Since $\xi_t \sim \mathcal{N}\left(0, 1\right)$, it follows that $\mathbb{E}(\mathbb{1}_{t-1}) = 1/2$ due to the symmetry of the standard normal distribution. Thus, under the stationarity assumption, we obtain:

$$
\mathbb{E}[\sigma_t^2] = \sigma^2 = \omega + \alpha\sigma^2 + \frac{1}{2}\gamma\sigma^2 + \beta\sigma^2.
$$

Therefore, the stationary solution exists and is given by:

$$
\sigma^2 = \frac{1}{1 - \alpha - \gamma/2 - \beta}.
$$

Note that the stationarity assumption holds only if:

$$
\begin{gathered}
\omega > 0, \\
\alpha + \gamma/2 + \beta < 1,
\end{gathered}
$$

since the variance must be positive by definition. Consequently, the parameter constraints necessary for ensuring the existence of a stationary solution can be easily imposed.

■

The model was designed to capture the stylized fact of an asymmetric relationship between volatility and shocks in asset returns, where negative shocks tend to have a greater impact on volatility than positive ones. As noted in (Zhang, 2006) and (Black, 1976), statistical evidence shows that volatility responds more strongly to negative shocks in returns than to positive ones. Thus, the sign of the γ parameter, which reflects the leverage effect, is expected to be positive. A positive estimate of γ indicates that volatility increases more significantly when the observed shocks are negative.

Several explanations for the asymmetry effect have been proposed in the literature. According to (Black, 1976) and (Christie, 1982), negative shocks result in an increase in the financial leverage of the issuing companies. As leverage rises, the risk level of the stock increases, leading to higher volatility. Additionally, the prospect theory developed by (Kahneman and Tversky, 1979) suggests that the asymmetry effect may also arise from cognitive biases of investors. Specifically, people tend to perceive losses more intensely, which may trigger widespread asset sales in response to negative return shocks, thereby causing an increase in volatility.

3. Methodology

3.1. The GARCH-M-GJR Model

Drawing from the literature, incorporating the leverage effect into the conditional variance equation within the GARCH-M model is a natural extension. This extended model is referred to as the GARCH-M-GJR. The development of this extension is not novel and has a straightforward representation. It is included in this manuscript to facilitate a comparative analysis with other methods. Specifically, the GARCH-M-GJR model employs the GJR-GARCH specification for the conditional variance equation, while the returns equation follows the GARCH-M process, as shown below:

$$\begin{gathered}
y_t = \mu + \lambda\sigma_t + \varepsilon_t, \\
\varepsilon_t|\mathcal{F}_{t-1} \sim \mathcal{N}\left(0, \sigma_t^2\right), \\
Var(y_t|\mathcal{F}_{t-1}) = \sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\mathbb{1}_{t-1}\varepsilon_{t-t}^2 + \beta\sigma_{t-1}^2, \\
\mathbb{1}_t = \begin{cases} 0, \text{ if } \varepsilon_t \geq 0 \\ 1, \text{ if } \varepsilon_t < 0 \end{cases}, \\
\xi_t \sim \mathcal{N}(0,1) \ \textit{i.i.d.}, \\
\varepsilon_t = \sigma_t\xi_t.
\end{gathered} \tag{3.1}$$

It is important to note that this model retains the same properties as the GJR-GARCH, particularly with respect to the unconditional variance, which has an identical representation. This extension of the GARCH-M family allows for the inclusion of the leverage effect in the variance equation. As a result, it captures the asymmetric responses of variance to return shocks while simultaneously modeling the time-varying risk premium through the GARCH-M mean equation specification. However, this representation imposes a symmetric structure on the risk premium itself, which is a rather rigid assumption. This issue is discussed in detail in the following sections, where an extension of the GARCH-M-GJR model is proposed to accommodate the asymmetric structure of the risk premium.

3.2. The GARCH-M-LV Model

The classic GARCH-M model captures the risk premium in asset returns. However, it does not account for the asymmetric responses of returns to volatility shifts. As a result, the model cannot distinguish between the signs of the shocks driving volatility, treating negative and positive shocks equivalently. Thus, the GARCH-M model assumes that investors demand equal risk premiums during both “good” and “bad” volatility periods, effectively considering only the absolute value of the shocks in determining the risk premium.

Nevertheless, as noted by (Bollerslev, 2022), statistical evidence suggests that the GARCH-M model may show insignificant risk premiums in asset returns. Moreover, some studies have even found negative risk premium estimates, which contradicts the core portfolio theories of (Markowitz, 1952) and (Sharpe, 1964), as these theories imply a positive risk-return relationship and a negative correlation between volatility and returns. According to (Bollerslev, 2022), this issue may be attributed to the need for distinguishing between “good” and “bad” volatility. Rational investors tend to demand a higher risk premium during periods of "bad" volatility (downside risk) compared to "good" volatility (upside potential).

This article introduces a new class of GARCH models that captures the asymmetric relationship between returns and volatility. The key feature of the proposed method is its ability to differentiate between "good" and "bad" volatility periods in the mean equation. The model is constructed within the GJR-GARCH framework, enabling it to capture leverage effects in both the return and variance equations.

Following the notation introduced in Section 2.1 and Section 2.2, the proposed model is specified as follows:

$$\begin{gathered} y_t = \mu + \lambda_1 \sigma^2_{t-1} + \lambda_2 \mathbb{1}_{t-1} \sigma^2_{t-1} + \varepsilon_t, \\ \varepsilon_t | \mathcal{F}_{t-1} \sim \mathcal{N}\left(0, \sigma^2_t\right), \\ Var(y_t|\mathcal{F}_{t-1}) = \sigma^2_t = \omega + \alpha \varepsilon^2_{t-1} + \gamma \mathbb{1}_{t-1} \varepsilon^2_{t-t} + \beta \sigma^2_{t-1}, \\ \mathbb{1}_t = \begin{cases} 0, \text{ if } \varepsilon_t \geq 0 \\ 1, \text{ if } \varepsilon_t < 0 \end{cases}, \\ \varepsilon_t = \sigma_t \xi_t, \\ \xi_t \sim \mathcal{N}\left(0, 1\right) \ i.i.d., \end{gathered} \tag{3.2}$$

where the parameters λ_1 and λ_2 determine the influence of conditional variance on returns. The term $\lambda_2 \mathbb{1}_{t-1} \sigma^2_{t-1}$ allows for distinguishing between "good" and "bad" volatility periods when defining the risk premium. As a result, volatility driven by negative shocks in returns may lead to a higher risk premium. Consequently, the magnitude of λ_2 is expected to be greater than λ_1. A positive sign of λ_2 reflects the notion that investors demand a higher risk premium when volatility is triggered by a "bear" market.

The model combines the GARCH-M process with the GJR-GARCH specification. Although, we use a slightly modified version of the GARCH-M model, imposing two specific differences. First, we specify the risk premiums through the conditional variances as $\lambda_i \sigma^2_{t-1},\ i = \{1, 2\}$, unlike the classic GARCH-M process that defines it using the conditional volatility, i.e., $\lambda \sigma_t$. We specify it this way since the unconditional variance of the proposed method may be explicitly derived only by including a conditional variance instead of conditional volatility. Second, while the classic GARCH-M model uses the current volatility to specify a risk premium, we use the variance of the previous period, i.e., σ^2_{t-1}. It is necessary since the last period of conditional variance is required to estimate the model. We may clearly see that ε_t defines the binary variable $\mathbb{1}_t$ in the same period t. In other words, if we specify the model with σ^2_t and $\mathbb{1}_t \sigma^2_t$ (instead of σ^2_{t-1} and $\mathbb{1}_{t-1} \sigma^2_{t-1}$), it will be impossible to define ε_t in the same period and, therefore, estimate the model via the maximum-likelihood method. Additionally, when the parameters λ_2 and γ are equal to zero, the model converges to a process that closely resembles the classical GARCH-M model.

Further, Theorem 1 provides an analytical expression for the unconditional variance of returns.

Theorem 1. *Assume y_t to be a weakly stationary process. Then, the expression for the unconditional variance of returns in the GARCH-M-LV model has the following form:*

$$\begin{aligned} Var\left(y_t\right) = \sigma^2 = \left(\lambda_1^2 + \lambda_1 \lambda_2\right) \times \left(\mathbb{E}\left[\sigma^4_t\right] - \mathbb{E}\left[\sigma^2_t\right]^2\right) \\ + \frac{1}{2}\lambda_2^2 \times \left(\mathbb{E}\left[\sigma^4_t\right] - \frac{1}{2}\mathbb{E}\left[\sigma^2_t\right]^2\right) + \mathbb{E}\left[\sigma^2_t\right], \end{aligned} \tag{3.3}$$

where $\mathbb{E}\left[\sigma^4_t\right]$ is an expectation of σ^4_t and is defined as:

$$\mathbb{E}\left[\sigma^4_t\right] = \frac{\omega^2 + \omega \mathbb{E}\left[\sigma^2_t\right] \times (2\alpha + 2\beta + \gamma)}{1 - 3\alpha^2 - \beta^2 - \frac{3}{2}\gamma^2 - 2\alpha\beta - 3\alpha\gamma - \beta\gamma}. \tag{3.4}$$

Also, $\mathbb{E}\left[\sigma^2_t\right]$ denotes the unconditional variance of ε_t:

$$\sigma^2_\varepsilon = Var(\varepsilon_t) = \frac{\omega}{1 - \alpha - \gamma/2 - \beta}. \tag{3.5}$$

Proof of Theorem 1. Remind that from Glosten et al. (1993):

$$Var\left(\varepsilon_t\right) = \mathbb{E}\left[\varepsilon^2_t\right] = \mathbb{E}\left[\sigma^2_t\right] = \frac{\omega}{1 - \alpha - \beta - \frac{1}{2}\gamma}. \tag{3.6}$$

Since $\mathbb{1}_t = \mathbb{1}_t^s \ \forall s \in \mathbb{N}$, and σ_t is independent of ξ_t, we obtain:

$$\mathbb{E}\left[\mathbb{1}_t^2\sigma_t^2\right] = \mathbb{E}\left[\mathbb{1}_t\sigma_t^2\right] = \mathbb{E}\underbrace{\left[\sigma_t^2|\xi_t<0\right]}_{\text{independent}} \times P\left(\xi_t<0\right) = \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right], \tag{3.7}$$

$$\mathbb{E}\left[\mathbb{1}_t^2\sigma_t^4\right] = \mathbb{E}\left[\mathbb{1}_t\sigma_t^4\right] = \mathbb{E}\underbrace{\left[\sigma_t^4|\xi_t<0\right]}_{\text{independent}} \times P\left(\xi_t<0\right) = \frac{1}{2}\mathbb{E}\left[\sigma_t^4\right]. \tag{3.8}$$

Using the first and fourth moments of the standard normal distribution, it follows:

$$\mathbb{E}\left[\varepsilon_t^2\right] = \mathbb{E}\left[\sigma_t^2\xi_t^2\right] = \mathbb{E}\left[\sigma_t^2\right] \times \mathbb{E}\left[\xi_t^2\right] = \mathbb{E}[\sigma_t^2], \tag{3.9}$$

$$\mathbb{E}\left[\varepsilon_t^4\right] = \mathbb{E}\left[\sigma_t^4\xi_t^4\right] = \mathbb{E}\left[\sigma_t^4\right] \times \mathbb{E}\left[\xi_t^4\right] = 3\mathbb{E}\left[\sigma_t^4\right]. \tag{3.10}$$

By expanding the expression of shocks:

$$\mathbb{E}\left[\varepsilon_t^2\sigma_t^2\right] = \mathbb{E}\left[\sigma_t^2\xi_t^2\sigma_t^2\right] = \mathbb{E}\left[\sigma_t^4\right]. \tag{3.11}$$

Since the distribution of ξ_t^2 does not depend on the event $\xi_t < 0$ if ξ_t is symmetric around zero (as is the case for the standard normal distribution):

$$\mathbb{E}\left[\mathbb{1}_t\varepsilon_t^2\right] = \mathbb{E}\underbrace{\left[\sigma_t^2\xi_t^2|\xi_t<0\right]}_{\text{independent}} \times P\left(\xi<0\right) = \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right] \times \mathbb{E}\left[\xi_t^2\right] = \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right], \tag{3.12}$$

$$\mathbb{E}\left[\mathbb{1}_t\varepsilon_t^4\right] = \mathbb{E}\underbrace{\left[\sigma_t^4\xi_t^4|\xi_t<0\right]}_{\text{independent}} \times P\left(\xi<0\right) = \frac{1}{2}\mathbb{E}\left[\sigma_t^4\right] \times \mathbb{E}\left[\xi_t^4\right] = \frac{3}{2}\mathbb{E}\left[\sigma_t^4\right], \tag{3.13}$$

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_t\sigma_t^2\varepsilon_t^2\right] = \mathbb{E}\left[\mathbb{1}_t\sigma_t^4\xi_t^2\right] &= \mathbb{E}\left[\sigma_t^4\xi_t^2|\xi_t<0\right] \times P\left(\xi_t<0\right) \\ &= \frac{1}{2}\mathbb{E}\left[\sigma_t^4\right] \times \mathbb{E}\left[\xi_t^2\right] = \frac{1}{2}\mathbb{E}\left[\sigma_t^4\right].\end{aligned} \tag{3.14}$$

By applying the above formulas, the following result is obtained:

$$\begin{aligned}\mathbb{E}\left[\sigma_t^4\right] &= \mathbb{E}\left[\left(\omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \gamma\mathbb{1}_{t-1}\varepsilon_{t-1}^2\right)^2\right] \\ &= \omega^2 + \alpha^2\mathbb{E}\left[\varepsilon_{t-1}^4\right] + \beta^2\mathbb{E}\left[\sigma_{t-1}^4\right] \\ &+ \gamma^2\mathbb{E}\left[\mathbb{1}_{t-1}^2\varepsilon_{t-1}^4 + 2\omega\alpha\mathbb{E}\left[\varepsilon_{t-1}^2\right] + 2\omega\beta\mathbb{E}\left[\sigma_{t-1}^2\right]\right] + 2\omega\gamma\mathbb{E}\left[\mathbb{1}_{t-1}\varepsilon_{t-1}^2\right] \\ &+ 2\alpha\beta\mathbb{E}\left[\varepsilon_{t-1}^2\sigma_{t-1}^2\right] + 2\alpha\gamma\mathbb{E}\left[\mathbb{1}_{t-1}\varepsilon_{t-1}^4\right] + 2\beta\gamma\mathbb{E}\left[\mathbb{1}_{t-1}\sigma_{t-1}^2\varepsilon_{t-1}^2\right] \\ &= \omega^2 + \left(3\alpha^2 + \beta^2 + \frac{3}{2}\gamma^2 + 2\alpha\beta + 3\alpha\gamma + \beta\gamma\right) \times \mathbb{E}\left[\sigma_t^4\right] \\ &+ \omega\mathbb{E}\left[\sigma_t^2\right] \times \left(2\alpha + 2\beta + \gamma\right).\end{aligned} \tag{3.15}$$

Solving for $\mathbb{E}\left[\sigma_t^4\right]$ yields:

$$\mathbb{E}\left[\sigma_t^4\right] = \frac{\omega^2 + \omega\mathbb{E}\left[\sigma_t^2\right] \times (2\alpha + 2\beta + \gamma)}{1 - 3\alpha^2 - \beta^2 - \frac{3}{2}\gamma^2 - 2\alpha\beta - 3\alpha\gamma - \beta\gamma}. \tag{3.16}$$

The next step is to derive the expression for the unconditional variance of y_t:

$$
\begin{aligned}
\sigma^2 = Var(y_t) &= Var\left(\mu + \lambda_1\sigma_{t-1}^2 + \lambda_2 \mathbb{1}_{t-1}\sigma_{t-1}^2 + \varepsilon_t\right) \\
&= Var\left(\lambda_1\sigma_{t-1}^2 + \lambda_2 \mathbb{1}_{t-1}\sigma_{t-1}^2\right) + Var\left(\varepsilon_t\right) \\
&+ 2\underbrace{Cov\left(\lambda_1\sigma_{t-1}^2 + \lambda_2 \mathbb{1}_{t-1}\sigma_{t-1}^2, \varepsilon_t\right)}_{\text{zero because of independence}} \\
&= \lambda_1^2 Var\left(\sigma_{t-1}^2\right) + \lambda_2^2 Var\left(\mathbb{1}_{t-1}\sigma_{t-1}^2\right) \\
&+ 2\lambda_1\lambda_2 Cov\left(\sigma_{t-1}^2, \mathbb{1}_{t-1}\sigma_{t-1}^2\right) + \mathbb{E}\left[\sigma_t^2\right] = \\
&= \lambda_1^2 \times \left(\mathbb{E}\left[\sigma_{t-1}^4\right] - \mathbb{E}\left[\sigma_{t-1}^2\right]^2\right) \\
&+ \frac{1}{2}\lambda_2^2 \times \left(\mathbb{E}\left[\sigma_{t-1}^4\right] - \frac{1}{2}\mathbb{E}\left[\sigma_{t-1}^2\right]^2\right) \\
&+ \lambda_1\lambda_2 \times \left(\mathbb{E}\left[\sigma_{t-1}^4\right] - \mathbb{E}\left[\sigma_{t-1}^2\right]^2\right) + \mathbb{E}\left[\sigma_t^2\right].
\end{aligned} \tag{3.17}
$$

Under the stationarity assumption, $\mathbb{E}\left[\sigma_{t-1}^4\right] = \mathbb{E}\left[\sigma_t^4\right]$ and $\mathbb{E}\left[\sigma_{t-1}^2\right] = \mathbb{E}\left[\sigma_t^2\right]$. Applying these equalities and simplifying the expression, the final specification for the unconditional variance in the GARCH-M-LV model is given by:

$$
\begin{aligned}
Var\left(y_t\right) &= \left(\lambda_1^2 + \lambda_1\lambda_2\right) \times \left(\mathbb{E}\left[\sigma_t^4\right] - \mathbb{E}\left[\sigma_t^2\right]^2\right) + \\
&+ \frac{1}{2}\lambda_2^2 \times \left(\mathbb{E}\left[\sigma_t^4\right] - \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right]^2\right) + \mathbb{E}\left[\sigma_t^2\right],
\end{aligned} \tag{3.18}
$$

where $\mathbb{E}\left[\sigma_t^2\right]$ and $\mathbb{E}\left[\sigma_t^4\right]$ are computed using equations (3.6) and (3.16) respectively.

■

Remark 2. *Additionally, it can be shown that the unconditional variance of σ_t^2 is given by:*

$$
\begin{aligned}
Var(\sigma_t^2) = &\left(\alpha^2 \times \left(3\mathbb{E}\left[\sigma_t^4\right] - \mathbb{E}\left[\sigma_t^2\right]^2\right) + 2\alpha\beta \times \left(\mathbb{E}\left[\sigma_t^4\right] - \mathbb{E}\left[\sigma_t^2\right]^2\right)\right. \\
&+ \gamma^2 \times \left(\frac{3}{2}\mathbb{E}[\sigma_t^4] - \frac{1}{4}\mathbb{E}\left[\sigma_t^2\right]^2\right) + \alpha\gamma \times \left(3\mathbb{E}[\sigma_t^4] - \mathbb{E}\left[\sigma_t^2\right]^2\right) \\
&\left.+ \beta\gamma \times \left(\mathbb{E}[\sigma_t^4] - \mathbb{E}\left[\sigma_t^2\right]^2\right)\right) \times \left(1 - \beta^2\right)^{-1}.
\end{aligned} \tag{3.19}
$$

■

Note that the stationarity assumption holds only if $Var(\sigma_t^2)$ is positive, since the variance must be positive by definition. Thus, the stationary solution exists when:

$$
\sigma^2 = Var(y_t) > 0. \tag{3.20}
$$

Typically, GARCH models are estimated using the maximum likelihood method. The log-likelihood function for the GARCH-M-LV model, assuming a normal distribution of random shocks, is specified as follows:

$$
\ln L(\theta, \varepsilon) = -\frac{T}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\ln\sigma_t^2 - \frac{1}{2}\sum_{t=1}^{T}\frac{\varepsilon_t^2}{\sigma_t^2}, \tag{3.21}
$$

where $\theta = (\mu, \omega, \alpha, \beta, \lambda_1, \gamma, \lambda_2)'$ is a vector of estimated parameters, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_T)'$. The conditional volatilities σ_t and random shocks ε_t are recursively computed according to the GARCH-M-LV process:

$$
\begin{aligned}
\varepsilon_t &= y_t - \mu - \lambda_1\sigma_{t-1}^2 - \lambda_2 I_{t-1}\sigma_{t-1}^2, \\
\sigma_t^2 &= \omega + \alpha\varepsilon_{t-1}^2 + \gamma I_{t-1}\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2.
\end{aligned} \tag{3.22}
$$

Maximizing the log-likelihood function yields estimates for the unknown parameters $\mu, \omega, \alpha, \beta, \lambda_1, \gamma$, and λ_2.

4. Simulations

We conducted a Monte Carlo analysis to evaluate the properties of the estimators and compare them with alternative models. The primary focus was to assess whether accounting for asymmetric responses in the mean equation is essential or if other models can still provide accurate estimates of the parameters, volatilities, and returns, even when the data is generated by the GARCH-M-LV process. The parameters used in the simulation are detailed in Table 1.

Table 1: Simulation Parameters.

Parameter	Set I	Set II
μ	0.01	0.05
ω	0.1	0.05
α	0.1	0.05
β	0.7	0.8
λ_1	0.2	-0.05
γ	0.15	0.2
λ_2	0.5	0.2
Sample size	1000	
Number of simulations	100	

Two sets of parameter values were considered: Set I and Set II. In Set I, the parameter values were chosen to reflect those commonly observed in applications of GARCH models to stock returns. Parameters responsible for leverage effects, in both the mean and variance equations, were assigned values that significantly influence the data-generating process. In Set II, the parameters were set to replicate the results from the GARCH-M-LV model applied to the S&P 500 index stock returns (see Section 5).

A sample size of 1000 observations was selected to align the experiment with real-world applications, where long time series of returns are often available but larger samples are limited due to structural breaks. Lastly, the total number of simulations was set to 100, as this number is sufficient to demonstrate the significant advantage of the proposed method over its alternatives (see Tables 4-5).

The simulated data analysis proceeds as follows. First, a pseudo-random sample is generated based on the GARCH-M-LV data-generating process. Next, the generated data is used to estimate three models: GARCH-M, GARCH-M-GJR, and GARCH-M-LV[1]. Once the estimates for each model are obtained, their accuracy and properties are compared using information criteria and accuracy metrics.

RMSE values were calculated[2] to assess the estimation accuracy across all four models. The following formula was used to compute the RMSE for each coefficient estimate:

$$RMSE(\hat{\theta}) = \sqrt{\frac{1}{100}\sum_{m=1}^{100}\left(\theta_m - \hat{\theta}_m\right)^2}, \tag{4.1}$$

where $\theta \in \{\mu, \omega, \alpha, \beta, \lambda_1, \gamma, \lambda_2\}$ represents the set of estimated parameters, and m is the index for each simulation.

The RMSE values for each parameter estimate across all models are presented in Table 2 and Table 3 for Set I and Set II, respectively.

The results for Set I demonstrate that the GARCH-M-LV model consistently produces lower RMSE values for all coefficient estimates, with the exception of $\hat{\alpha}$, which is slightly more accurate in the GARCH-M-GJR model. This provides strong statistical evidence in favor of the proposed method's superiority in terms of estimation accuracy. As shown in Table 2, when the true data-generating process differs from the model's assumptions, the estimators may become inaccurate if the leverage effect is not accounted for.

The results for Set II (Table 3) further highlight the advantage of the proposed method, particularly in estimating the λ_1 parameter, by effectively capturing the leverage effect in the risk

[1] A modified version of the GARCH-M model is used, where the risk premiums are defined as $\lambda\sigma_{t-1}^2$ (see Section 4). This modification allows for a straightforward comparison with the GARCH-M-LV model, as the GARCH-M model is nested within the GARCH-M-LV process.

[2] For convenience, RMSE values are reported multiplied by 100. Alternative metrics can be found in Appendix B.

Table 2: Accuracy Metrics for Coefficient Estimates (Set I).

Metric/Model	**GARCH-M**	**GARCH-M-GJR**	**GARCH-M-LV**
$RMSE(\hat{\mu})$	7.091	6.056	4.616
$RMSE(\hat{\omega})$	3.390	3.111	2.609
$RMSE(\hat{\alpha})$	10.106	4.013	4.144
$RMSE(\hat{\beta})$	6.147	5.885	5.403
$RMSE(\hat{\lambda}_1)$	21.438	7.657	6.639
$RMSE(\hat{\gamma})$	-	22.917	6.513
$RMSE(\hat{\lambda}_2)$	-	-	8.934

Table 3: Accuracy Metrics for Coefficient Estimates (Set II).

Metric/Model	**GARCH-M**	**GARCH-M-GJR**	**GARCH-M-LV**
$RMSE(\hat{\mu})$	4.196	3.283	3.383
$RMSE(\hat{\omega})$	3.150	1.694	1.700
$RMSE(\hat{\alpha})$	12.824	3.092	3.168
$RMSE(\hat{\beta})$	7.106	4.095	3.965
$RMSE(\hat{\lambda}_1)$	14.072	26.764	25.596
$RMSE(\hat{\gamma})$	-	15.310	23.970
$RMSE(\hat{\lambda}_2)$	-	-	6.997

premium. Additionally, the RMSE value for $\hat{\lambda}_2$ suggests that the proposed model estimates λ_2 with high precision, even more accurately than λ_1. These findings justify the development of the GARCH-M-LV model, as it effectively addresses the asymmetric relationship between the risk premium and volatility. Without this adjustment, existing methods are likely to yield inefficient and inaccurate estimates when such asymmetry is present in the observed data.

While parameter estimates are crucial, the accuracy of predicted volatilities and returns is even more important for practical applications. Therefore, RMSE values were calculated for predicted conditional volatilities and returns.[3] RMSE metrics were evaluated for in-sample predictions for each simulation across all models. To aggregate the results, the mean RMSE values were calculated across all simulations.

The mean RMSE values for conditional volatilities and returns are presented in Tables 4-5 for Set I and Set II, respectively. In addition, the tables display the percentage of simulations in which each model achieved the lowest RMSE for conditional volatilities and returns (Victories %). Lastly, the mean values of the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are reported for each model.

Table 4: Accuracy metrics and information criteria (Set I).

Metric/Model	**GARCH-M**	**GARCH-M-GJR**	**GARCH-M-LV**
$RMSE(\hat{\sigma})$	12.345	10.295	6.142
$Victories_\sigma\%$	0%	4%	96%
$RMSE(\hat{y})$	94.581	94.210	90.568
$Victories_y\%$	0%	0%	100%
AIC	2573.072	2559.644	2500.384
BIC	2597.610	2589.091	2534.738

The results clearly demonstrate that the proposed method yields significantly lower RMSE values. For Set I, the GARCH-M-LV model produces RMSE values for conditional volatilities that are approximately half those of the alternative models. Furthermore, the share of simulations with lower RMSE values is considerably higher for the proposed method: the GARCH-M-LV model provides more accurate conditional volatility forecasts in 92% of cases and more accurate return forecasts in 98% of cases. Similarly, the results for Set II show that the developed model estimates volatilities and returns more precisely than its counterparts.

These findings underscore the advantage of the proposed method, further justifying its development and application. Specifically, when the data-generating process reflects an asymmetric

[3] MAE and MSE metrics were also considered, but they have been moved to the Appendix as they produced similar results.

Table 5: Accuracy metrics and information criteria (Set II).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-LV
$RMSE(\hat{\sigma})$	13.290	6.938	5.804
$Victories_{\sigma}\%$	0%	19%	81%
$RMSE(\hat{y})$	100.393	100.285	98.766
$Victories_{y}\%$	0%	4%	96%
AIC	2619.222	2598.067	2585.704
BIC	2643.761	2627.514	2620.058

relationship between risk premium and volatility, existing methods are likely to produce inefficient estimators and significantly less accurate forecasts for both volatilities and returns.

Finally, we have tested the robustness of the maximum likelihood estimator of the GARCH-M-LV model to distributional assumption violations.[4] It is well known that a classical GARCH process estimator (based on the normality assumption) may preserve consistency even if random shocks' distribution deviates from normality (Berkes and Horváth, 2004). Unfortunately, establishing a similar formal result for the GARCH-M-LV model is a technically complicated task beyond this paper's scope. So, instead, we test the finite sample robustness of the GARCH-M-LV estimator by replicating the analysis for Set I, alternately assuming that ξ_t follows the Student's t and Non-central Student's t-distributions (standardized to zero mean and unit variance). To ensure that tails are rather heavy, we use 5 degrees of freedom for both distributions. Non-centrality parameter for the Non-Central Student's t-distribution also has been set to 5, which makes this distribution positively skewed. The results are presented in the Appendix (Tables B.15-B.18) and suggest that distributional assumption violation provides just a slight increase in RMSE values. It indicates in favor of the finite sample robustness of the GARCH-M-LV estimator.

5. Empirical Illustration

The proposed method was applied to study the volatility of log-returns of the S&P 500 market index. The sample consists of 4,531 observations, covering the period from January 1, 2004, to December 31, 2021. The dynamics of the S&P 500 returns are illustrated in Figure 1. The data was sourced from the Bloomberg Terminal.

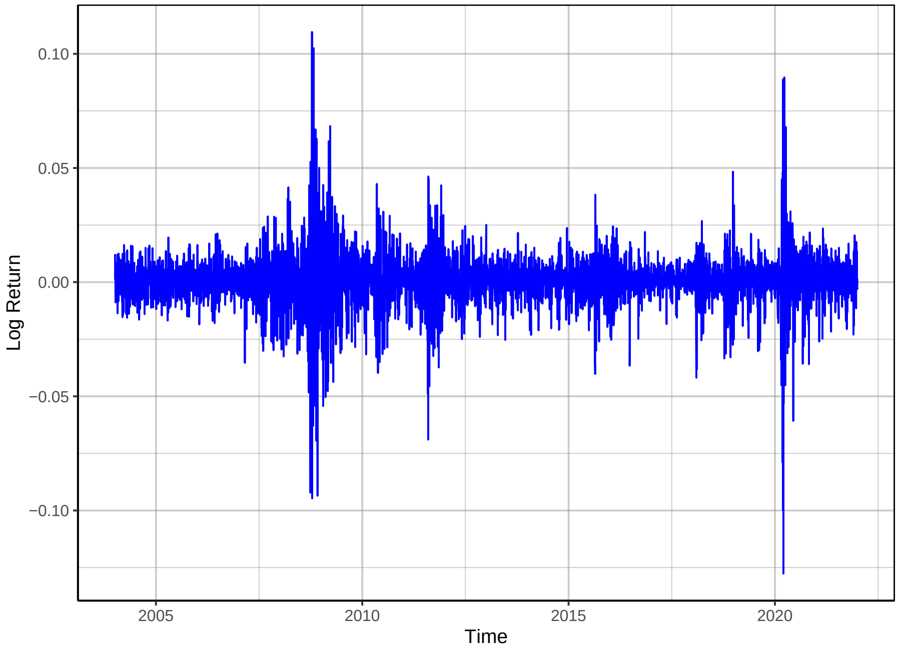

Figure 1: Log Returns of the S&P 500 Market Index Over Time

[4]In the case of distributional assumption violation, it becomes a quasi-maximum likelihood estimator (QMLE).

To ensure the robustness of the results, the models were estimated for both the entire sample and specific periods. Since long financial time series can exhibit structural breaks, the model was also estimated on samples containing 3-year periods. The primary objective of this analysis is to investigate the asymmetric responses of the risk premium to conditional volatility in the S&P 500 index and to compare the performance of the GARCH-M-LV model with alternative models. The estimation results for the full sample and specific periods are presented in Tables 6-10. The covariance matrix was estimated using the Gradient Outer Product (GOP) method.

5.1. Whole sample analysis

In this subsection, the estimation results for the entire sample are presented. To interpret these results meaningfully, it is essential to evaluate all three models: GARCH-M, GARCH-M-GJR, and GARCH-M-LV. The GARCH-M model captures the effect of the risk premium, while the GARCH-M-GJR model accounts for both the risk premium and the leverage effect in the volatility equation. Finally, the proposed GARCH-M-LV model incorporates asymmetry in both the volatility and the risk premium responses to shocks in returns.

Table 6: S&P 500 application results for the period 2004-2021.

Parameters	**GARCH-M**	**GARCH-M-GJR**	**GARCH-M-LV**
μ	0.0499***	0.0225	0.0347**
	(0.0141)	(0.0138)	(0.0140)
ω	0.0286***	0.0287***	0.0291***
	(0.0024)	(0.0020)	(0.0012)
α	0.1347***	0.0129***	0.0162***
	(0.0084)	(0.0042)	(0.0030)
β	0.8389***	0.8589***	0.8553***
	(0.0092)	(0.0080)	(0.0005)
λ_1	0.0265*	0.0135	-0.0237***
	(0.0137)	(0.0128)	(0.0091)
γ	-	0.1903***	0.1871***
		(0.0120)	(0.0091)
λ_2	-	-	0.0760***
			(0.0003)
AIC	**11791.596**	**11643.245**	**11629.996**

*Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.*

Based on the results presented in Table 6, all estimates in the GARCH-M-LV model are statistically significant at any reasonable level. The parameters of primary interest are λ_1 and λ_2. The estimate for λ_1 has a negative sign, which is counterintuitive, as it suggests that returns react negatively to increases in volatility. Although this finding is consistent with previous research by (Bollerslev, 2022), it may be influenced by structural breaks in the data. To investigate this further, the effect will be examined more closely by estimating the model on smaller samples in the following subsections. In contrast, the estimate for λ_2 is positive, indicating that the risk premium increases more sharply when volatility rises due to negative shocks in returns.

The asymmetric relationship between the risk premium and previous shocks in returns is visualized in Figure 2. Given the negative estimate for λ_1 and the positive estimate for λ_2, the risk premium increases only when negative shocks in returns are observed. This result highlights the previously discussed contradictory findings, showing that during periods of "good" volatility, investors tend to demand a discount rather than a premium.

Comparing the results with the other two models, the estimate of λ in the GARCH-M model is positive and statistically significant, though its absolute value is much smaller than in the GARCH-M-GJR model. As a result, using the GARCH-M model instead of the GARCH-M-LV process may lead to an underestimation of the risk premium. Additionally, when applying the GARCH-M-GJR model, a significant estimate for the leverage effect in the volatility equation (γ) is identified, while the risk premium effect becomes insignificant and small. This suggests that by accounting for the leverage effect solely in the variance equation, researchers may overlook the presence of a risk premium in returns, as the leverage coefficient in the volatility equation absorbs much of the influence.

Therefore, to correctly identify both the risk premium and leverage effects, the proposed GARCH-M-LV model should be employed, as it captures the asymmetric risk premium effect

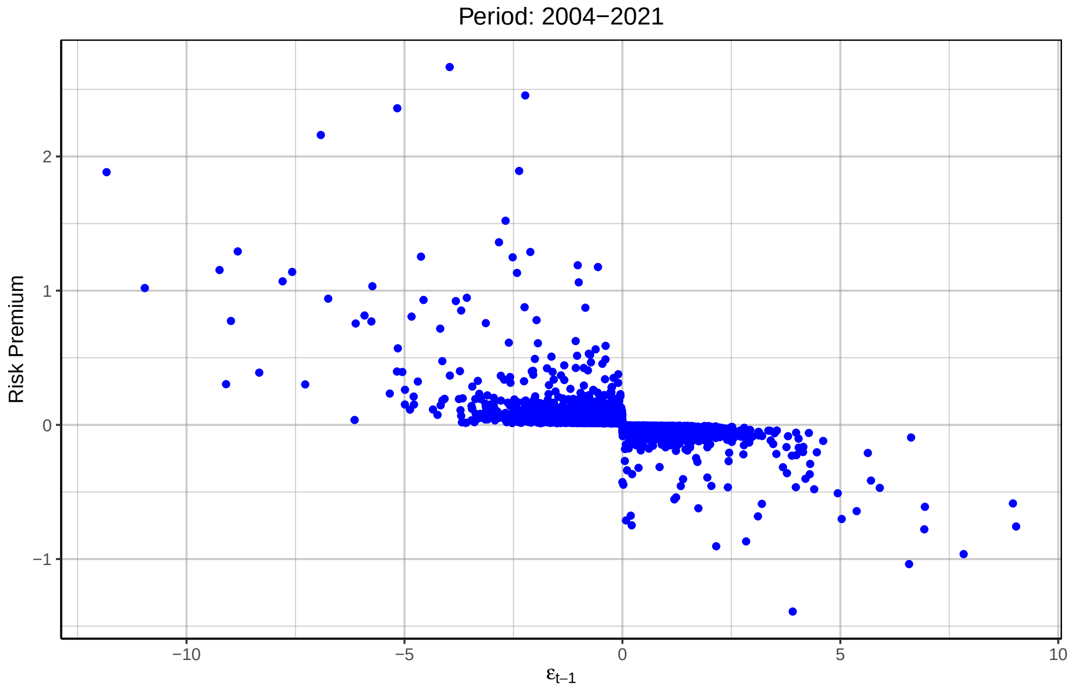

Figure 2: Risk premium responses to shocks in returns.

present in the data. Finally, the Akaike Information Criterion (AIC) confirms the GARCH-M-LV model as the best option among the three.

5.2. Analysis of 3-year samples: 2016-2018, 2015-2017, 2014-2016, 2013-2015

This subsection discusses the results from the analysis of four sequential 3-year samples, covering the period from 2013 to 2018.[5] The results for each period are presented in Tables 7-10.

The results show that all estimated periods exhibit statistically significant estimates for the λ_2 parameter in the GARCH-M-LV model. This provides evidence of an asymmetric relationship between the risk premium and shocks in returns, with λ_2 consistently positive across all periods. The findings indicate that the risk premium increases more sharply when volatility rises due to negative shocks rather than positive ones. This supports the "bad" and "good" volatility differentiation hypothesis of (Bollerslev, 2022), suggesting that investors tend to demand a higher risk premium during "bear" markets driven by "bad" volatility compared to "good" volatility periods.

According to Table 7, the estimate of λ_1 in the GARCH-M model is statistically insignificant, whereas the GARCH-M-LV model provides a significant estimate for λ_2. This is a critical finding, as it highlights the inability of the GARCH-M model to capture a risk premium, while the GARCH-M-LV model successfully identifies it. Given that the estimate of λ_1 in the proposed model is also statistically insignificant, it can be concluded that the risk premium in S&P 500 returns during this period reacts only to "bad" volatility. In contrast, fluctuations during "bull" markets do not increase the risk premium.

In other words, investors demand a risk premium only during downturns in the financial markets, while heightened market turbulence during periods of sharp growth does not lead to higher risk premium magnitudes. This pattern suggests a degree of investor irrationality, where negative news is perceived more dramatically than positive news. Such evidence aligns with the prospect theory of (Kahneman and Tversky, 1979) and the findings of (Zhang, 2006), (Black, 1976), and (Nelson, 1991).

Additionally, it is important to note that the data exhibits asymmetric volatility responses to shocks in returns, as evidenced by the significant and positive estimates of γ in both the GARCH-M-GJR and GARCH-M-LV models. This is further supported by the lower AIC values in the asymmetric models compared to the GARCH-M model.

[5]The results for the 2017–2019 subsample are provided in the appendix, as they do not exhibit any risk premium effect, either symmetric or asymmetric.

Table 7: S&P 500 application results for the period 2016-2018.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-LV
μ	0.0598**	0.0301	0.0470
	(0.0297)	(0.0304)	(0.0301)
ω	0.0394***	0.0370***	0.0344***
	(0.0057)	(0.0054)	(0.0051)
α	0.2146***	0.0507***	0.0581***
	(0.0213)	(0.0143)	(0.0171)
β	0.7382***	0.7634***	0.7701***
	(0.0297)	(0.0284)	(0.0288)
λ_1	0.0424	0.0319	-0.0749
	(0.0573)	(0.0544)	(0.0525)
γ	-	0.2556***	0.2527***
		(0.0298)	(0.0398)
λ_2	-	-	0.1914***
			(0.0483)
AIC	**1575.468**	**1557.305**	**1553.541**

*Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.*

Ultimately, the proposed GARCH-M-LV model proves to be the best option based on the Akaike Information Criterion (AIC). This model successfully captures both leverage effects, ensuring higher estimation accuracy.[6] Moreover, since the GARCH-M model did not capture a significant risk premium effect, failing to apply the GARCH-M-LV model could lead to a misidentification of the risk premium in returns.

Tables 8-10 show similar results, with one key difference: the λ_1 coefficient in the GARCH-M model is statistically significant across all three periods, indicating that the GARCH-M model successfully captured a risk-premium effect, unlike during the 2016-2018 period.

It is also important to examine the results from the GARCH-M-GJR model. In all three tables, after accounting for asymmetry in the variance equation using the GARCH-M-GJR model, the λ_1 estimate becomes statistically insignificant. This suggests that the risk premium may not be captured when using the GJR specification, potentially leading to a significant misinterpretation of the results.

As with the 2016-2017 period, the λ_1 estimate in the GARCH-M-LV model remains statistically insignificant (Tables 8-10). This further reinforces the conclusion that investors demand a risk premium only during "bad" volatility periods, when variance increases due to sharp declines in returns.

Table 8: S&P 500 application results for the period 2015-2017.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-LV
μ	0.0004	0.0011	0.0065
	(0.0313)	(0.0312)	(0.0164)
ω	0.0464***	0.0492***	0.0497***
	(0.0068)	(0.0086)	(0.0085)
α	0.2401***	0.0543***	0.0589***
	(0.0218)	(0.0134)	(0.0125)
β	0.6996***	0.7107***	0.7003***
	(0.0329)	(0.0403)	(0.0398)
λ_1	0.1664**	0.0892	0.0134
	(0.0650)	(0.0594)	(0.0324)
γ	-	0.3065***	0.3098***
		(0.0352)	(0.0351)
λ_2	-	-	0.1792***
			(0.0486)
AIC	**1553.335**	**1531.684**	**1523.906**

*Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.*

[6] Leverage effects in both the return and volatility equations.

Table 9: S&P 500 application results for the period 2014-2016.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-LV
μ	0.0004	0.0004	0.0004
	(0.0422)	(0.0397)	(0.0419)
ω	0.0691***	0.0703***	0.0679***
	(0.0123)	(0.0137)	(0.0139)
α	0.2499***	0.0233	0.0058
	(0.0262)	(0.0152)	(0.0174)
β	0.6874***	0.6996***	0.7172***
	(0.0399)	(0.0454)	(0.0461)
λ_1	0.1435**	0.0578	-0.0049
	(0.0656)	(0.0577)	(0.0635)
γ	-	0.3630***	0.3711***
		(0.0429)	(0.0402)
λ_2	-	-	0.1826**
			(0.0747)
AIC	**1753.369**	**1720.883**	**1715.978**

*Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.*

The estimate of the leverage effect (γ) in the variance equation is significant across all periods. This result is consistent between the GARCH-M-GJR and GARCH-M-LV models and remains stable throughout the analyzed periods. Consequently, the S&P 500 index returns exhibit a pronounced asymmetry effect, which must be properly accounted for to ensure higher estimation quality.

Table 10: S&P 500 application results for the period 2013-2015.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-LV
μ	0.0042	0.0041	0.0174
	(0.0514)	(0.0387)	(0.0282)
ω	0.0750***	0.0683***	0.0631***
	(0.0194)	(0.0159)	(0.0144)
α	0.1973***	0.0002	0.0003
	(0.0374)	(0.0379)	(0.0069)
β	0.6854***	0.6954***	0.7039***
	(0.0524)	(0.0493)	(0.0461)
λ_1	0.1671*	0.0794	0.0123
	(0.0877)	(0.0625)	(0.0715)
γ	-	0.3996***	0.3915***
		(0.0588)	(0.0411)
λ_2	-	-	0.1477***
			(0.0476)
AIC	**1700.135**	**1651.677**	**1650.606**

*Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.*

Figure 3 illustrates the dependence of the risk premium on previous shocks in returns for each period. All periods demonstrate an asymmetric relationship between the risk premium and volatility changes, driven by the sign of the shocks causing the volatility increase. The graph for the 2016-2018 period shows a pattern similar to the overall sample (Figure 2), while the other three periods display a different structure. This variation is primarily due to the sign of the λ_1 estimate. For 2016-2018, the estimate of λ_1 was negative and relatively high, indicating that investors demand a risk premium only during "bad" volatility periods, while "good" volatility may even lead to a discount.

The 2015-2017 period also shows a negative risk premium (or risk discount) when shocks in returns are positive, although the absolute value of the discount is significantly lower compared to 2016-2018, due to the smaller λ_1 estimate. The remaining two periods reveal that the risk premium increases during both "good" and "bad" volatility periods, but negative shocks in returns influence the premium more sharply—investors demand a higher risk premium during "bear" markets.

In summary, there is strong statistical evidence that the observed data exhibit both asymmetry

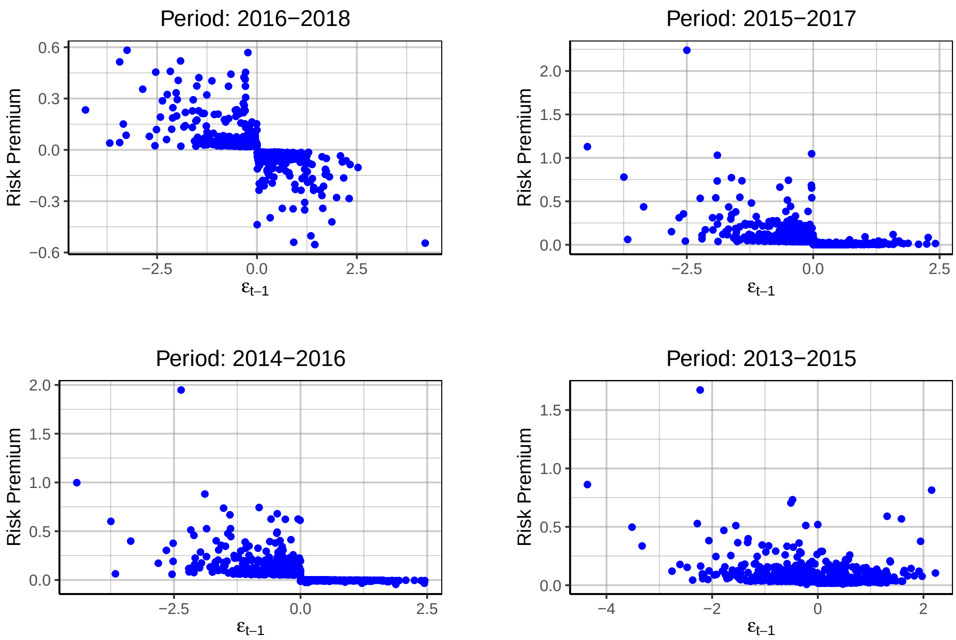

Figure 3: Risk premium responses to shocks in returns.

in the variance equation and asymmetric responses of the risk premium to volatility changes. The proposed GARCH-M-LV model effectively captures both effects simultaneously. This capability leads to improved estimation quality, as evidenced by the lower AIC values.

6. Conclusion

This study introduced the GARCH-M-LV model, distinguished by its key feature of simultaneously capturing asymmetric responses in both the return and variance equations. According to (Bollerslev, 2022), investors perceive "good" and "bad" volatility periods differently, demanding varying risk premiums. While well-established asymmetric GARCH models (e.g., EGARCH and GJR-GARCH) capture asymmetric responses of variance to shocks in returns, they either neglect the risk premium or assume it reacts equally to positive and negative fluctuations. In contrast, the proposed method distinguishes between volatility driven by "bear" and "bull" markets when modeling the risk premium.

To impose parameter restrictions ensuring the existence of a stationary solution, the analytical expression for the unconditional variance was derived. The results of Monte Carlo simulations show that the GARCH-M-LV model provides a significant advantage over other methods when the data-generating process exhibits an asymmetric relationship between the risk premium and volatility changes. As a result, applying this model is crucial for obtaining accurate estimates of GARCH model parameters, conditional volatility, and returns.

The model was also applied to study the volatility of the S&P 500 market index. The analysis provided statistical evidence supporting the leverage effect in the mean equation over various periods, indicating that investors demand a higher risk premium when volatility is driven by negative shocks in returns, rather than positive ones. Moreover, the findings suggest that during some intervals, only negative shocks contribute to the formation of a risk premium, meaning that investors demanded a risk premium solely during "bad" volatility periods. These results align with prior empirical findings of Bollerslev et al. (2006) and Rossi and Timmermann (2015).

Lastly, the analysis revealed that without applying the GARCH-M-LV model, researchers may be unable to correctly estimate the risk premium if the data exhibit a leverage effect in the return equation, potentially leading to significant misinterpretation of results.

* * *

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Appendix A. Alternative metrics for model comparison

In this section, the values of MAE and MSE accuracy metrics for parameter estimates, conditional volatilities, and returns are presented. As with the RMSE results, Tables A.11-A.14 demonstrate the clear advantage of the GARCH-M-LV model over alternative methods.

Appendix B. Robustness to normality assumption violation

This section presents the results for the QMLE estimators. Tables B.15-B.16 show the results assuming the Student's t-distribution, while Tables B.17-B.18 reflect the results based on the noncentral Student's t-distribution.

Appendix B.1. Analysis of 3-year samples: 2019-2021, 2018-2020, 2017-2019

This subsection presents the results for the last three sequential 3-year samples. The estimation results for each period are provided in Tables B.19-B.21. These three periods are combined into a single subsection as they yield similar results.

All three periods are characterized by statistically insignificant and small estimates for the λ_1 and λ_2 coefficients in the GARCH-M-LV model. As a result, no statistical evidence of a risk premium effect in S&P 500 returns was found for the analyzed periods. This finding aligns with

Table A.11: Additional accuracy metrics of coefficient estimates (Set I).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-LV
$MAE(\hat{\mu})$	5.047	4.247	2.608
$MAE(\hat{\omega})$	2.501	2.281	2.092
$MAE(\hat{\alpha})$	9.239	3.014	2.908
$MAE(\hat{\beta})$	4.481	4.120	4.143
$MAE(\hat{\lambda}_1)$	19.959	5.473	5.340
$MAE(\hat{\gamma})$	-	21.676	5.341
$MAE(\hat{\lambda}_2)$	-	-	5.879
$MSE(\hat{\mu})$	0.500	0.364	0.211
$MSE(\hat{\omega})$	0.114	0.096	0.067
$MSE(\hat{\alpha})$	1.020	0.159	0.170
$MSE(\hat{\beta})$	0.374	0.343	0.289
$MSE(\hat{\lambda}_1)$	4.589	0.581	0.438
$MSE(\hat{\gamma})$	-	5.246	0.421
$MSE(\hat{\lambda}_2)$	-	-	0.790

Table A.12: Additional accuracy metrics of coefficient estimates (Set II).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-LV
$MAE(\hat{\mu})$	3.537	2.766	2.754
$MAE(\hat{\omega})$	1.725	1.357	1.377
$MAE(\hat{\alpha})$	12.245	2.528	2.580
$MAE(\hat{\beta})$	4.772	3.174	3.299
$MAE(\hat{\lambda}_1)$	13.063	26.252	25.080
$MAE(\hat{\gamma})$	-	14.681	23.254
$MAE(\hat{\lambda}_2)$	-	-	5.472
$MSE(\hat{\mu})$	0.176	0.108	0.114
$MSE(\hat{\omega})$	0.099	0.029	0.029
$MSE(\hat{\alpha})$	1.645	0.096	0.100
$MSE(\hat{\beta})$	0.505	0.168	0.157
$MSE(\hat{\lambda}_1)$	1.980	7.163	6.552
$MSE(\hat{\gamma})$	-	2.344	5.745
$MSE(\hat{\lambda}_2)$	-	-	0.490

Table A.13: Additional accuracy metrics of conditional volatilities and return predictions (Set I).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-LV
$MAE(\hat{\sigma})$	7.655	6.237	3.694
$MSE(\hat{\sigma})$	1.524	1.060	0.377
$MAE(\hat{y})$	71.677	71.453	69.117
$MSE(\hat{y})$	89.456	88.755	82.025

Table A.14: Additional accuracy metrics of conditional volatilities and return predictions (Set II).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-LV
$MAE(\hat{\sigma})$	9.161	4.478	3.957
$MSE(\hat{\sigma})$	1.766	0.481	0.337
$MAE(\hat{y})$	74.851	74.763	74.032
$MSE(\hat{y})$	100.787	100.572	97.548

the results from both the GARCH-M and GARCH-M-GJR models, which also fail to capture a significant risk premium effect. The slight difference in the AIC criteria between the GARCH-M-GJR and GARCH-M-LV models further reflects this outcome.

However, it remains essential to account for the leverage effect in the variance equation, as both the GARCH-M-GJR and GARCH-M-LV models yield significant and positive estimates for γ. This indicates that volatility reacts more sharply to negative shocks in returns than to positive ones. This conclusion is reinforced by the notable difference in the AIC criteria between the GARCH-M model and the other two methods.

Table B.15: Accuracy metrics of coefficient estimates (t-distribution).

Metric/Model	**GARCH-M**	**GARCH-M-GJR**	**GARCH-M-LV**
$RMSE(\hat{\mu})$	8.961	7.596	5.076
$RMSE(\hat{\omega})$	5.962	3.881	3.275
$RMSE(\hat{\alpha})$	10.866	4.709	3.935
$RMSE(\hat{\beta})$	8.679	6.944	6.245
$RMSE(\hat{\lambda}_1)$	20.727	8.462	7.262
$RMSE(\hat{\gamma})$	-	22.989	7.666
$RMSE(\hat{\lambda}_2)$	-	-	11.212

Table B.16: Accuracy metrics and information criteria (t-distribution).

Metric/Model	**GARCH-M**	**GARCH-M-GJR**	**GARCH-M-LV**
$RMSE(\hat{\sigma})$	14.342	11.423	7.229
$Victories_{\sigma}\%$	2%	11%	87%
$RMSE(\hat{y})$	92.934	92.082	87.910
$Victories_{y}\%$	0%	1%	99%
AIC	2484.250	2468.050	2413.541
BIC	2508.789	2497.497	2447.896

In summary, although the three analyzed periods show no evidence of a risk premium, they do exhibit a significant leverage effect in the volatility equation. Consequently, the GARCH-M-GJR and GARCH-M-LV models yield similar results.

Table B.17: Accuracy metrics of coefficient estimates (Non-central t-distribution).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-LV
$RMSE(\hat{\mu})$	9.309	8.144	5.804
$RMSE(\hat{\omega})$	7.633	7.206	6.827
$RMSE(\hat{\alpha})$	13.954	11.561	10.734
$RMSE(\hat{\beta})$	14.588	13.559	13.519
$RMSE(\hat{\lambda}_1)$	24.855	24.556	21.107
$RMSE(\hat{\gamma})$	-	28.285	9.677
$RMSE(\hat{\lambda}_2)$	-	-	15.128

Table B.18: Accuracy metrics and information criteria (Non-central t-distribution).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-LV
$RMSE(\hat{\sigma})$	13.147	12.676	10.230
$Victories_{\sigma}\%$	19%	16%	65%
$RMSE(\hat{y})$	83.456	83.335	80.786
$Victories_{y}\%$	5%	0%	95%
AIC	2318.284	2311.030	2270.023
BIC	2342.823	2340.477	2304.377

Table B.19: S&P 500 estimation results for the period 2019-2021.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-LV
μ	0.0994***	0.0762**	0.0801**
	(0.032)	(0.0319)	(0.0327)
ω	0.0656***	0.0718***	0.0711***
	(0.0108)	(0.0115)	(0.0115)
α	0.3034***	0.1623***	0.1616***
	(0.0321)	(0.0231)	(0.023)
β	0.6777***	0.6569***	0.6574***
	(0.0296)	(0.0297)	(0.0298)
λ_1	0.0325	0.0149	-0.0114
	(0.0219)	(0.019)	(0.0271)
γ	-	0.3176***	0.3205***
		(0.0682)	(0.0682)
λ_2	-	-	0.0644
			(0.0405)
AIC	**2058.326**	**2046.602**	**2046.535**

*Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.*

Table B.20: S&P 500 estimation results for the period 2018-2020.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-LV
μ	0.1017***	0.0780**	0.0829**
	(0.0314)	(0.0328)	(0.0328)
ω	0.0550***	0.0533***	0.0522***
	(0.0092)	(0.0094)	(0.0093)
α	0.2467***	0.1390***	0.1399***
	(0.0289)	(0.0208)	(0.0210)
β	0.7227***	0.7228***	0.7242***
	(0.0274)	(0.0261)	(0.0260)
λ_1	0.0185	0.0060	-0.0156
	(0.0242)	(0.0224)	(0.0097)
γ	-	0.2116***	0.2077***
		(0.0437)	(0.0426)
λ_2	-	-	0.0489
			(0.0317)
AIC	**2145.537**	**2132.454**	**2133.173**

*Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.*

Table B.21: S&P 500 estimation results for the period 2017-2019.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-LV
μ	0.0911***	0.0616**	0.0566*
	(0.0300)	(0.0291)	(0.0301)
ω	0.0306***	0.0287***	0.0298***
	(0.0051)	(0.0045)	(0.0049)
α	0.2022***	0.0106	0.0134
	(0.0258)	(0.0204)	(0.0218)
β	0.7561***	0.7910***	0.7808***
	(0.0303)	(0.0277)	(0.0291)
λ_1	0.0390	0.0064	0.0243
	(0.0579)	(0.0594)	(0.0753)
γ	-	0.2856***	0.2928***
		(0.0357)	(0.0387)
λ_2	-	-	-0.0266
			(0.0789)
AIC	**1538.754**	**1500.615**	**1502.444**

*Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.*