

# Estimating the Asymptotic Variance of the ATE Estimator under the Cluster-Level Treatment Assignment

Juri Trifonov

March 11, 2024

## Proof of Theorem 3.1 under the perfect compliance and clusters

Let us denote:

$$Q \equiv \mathbb{E}[N_g \bar{Y}_g(1) - N_g \bar{Y}_g(0)],$$

$$H \equiv \mathbb{E}[N_g],$$

$$\hat{Q} \equiv \frac{1}{G} \sum_{g=1}^G \left[ \frac{A_g(N_g \bar{Y}_g - \hat{\mu}(1, S_g, X_g, N_g))}{\hat{\pi}(S_g)} - \frac{(1 - A_g)(N_g \bar{Y}_g - \hat{\mu}(0, S_g, X_g, N_g))}{1 - \hat{\pi}(S_g)} + \hat{\mu}(1, S_g, X_g, N_g) - \hat{\mu}(0, S_g, X_g, N_g) \right],$$

$$\hat{H} \equiv \frac{1}{G} \sum_{g=1}^G N_g.$$

The ATE estimator,  $\hat{\tau}$ , takes the following form:

$$\begin{aligned} \hat{\tau} &= \frac{1}{\sum_{g=1}^G N_g} \sum_{g=1}^G \hat{\Xi}_g, \\ \hat{\Xi}_g &= \frac{A_g(N_g \bar{Y}_g - \hat{\mu}(1, S_g, X_g, N_g))}{\hat{\pi}(S_g)} - \frac{(1 - A_g)(N_g \bar{Y}_g - \hat{\mu}(0, S_g, X_g, N_g))}{1 - \hat{\pi}(S_g)} \\ &\quad + \hat{\mu}(1, S_g, X_g, N_g) - \hat{\mu}(0, S_g, X_g, N_g), \end{aligned}$$

where  $\hat{\mu}(a, S_g, X_g, N_g) = X_g' \hat{\theta}_{a,s} + N_g' \hat{\Delta}_{a,s}$ . Where  $\hat{\theta}_{a,s}$  and  $\hat{\Delta}_{a,s}$  are obtained from:

$$N_g \bar{Y}_g \sim \gamma_{a,s} + X_{g,s}' \theta_{a,s} + N_g' \Delta_{a,s}.$$

Then, we can write the estimator as follows:

$$\begin{aligned} \sqrt{G}(\hat{\tau} - \tau) &= \sqrt{G} \left( \frac{\hat{Q}}{\hat{H}} - \frac{Q}{H} \right) \\ &= \frac{1}{\hat{H}} \left[ \sqrt{G}(\hat{Q} - Q) - \tau \sqrt{G}(\hat{H} - H) \right]. \end{aligned}$$

**Step 1. Obtain the linear expansion of  $\sqrt{G}(\hat{Q} - Q)$  and  $\sqrt{G}(\hat{H} - H)$**

Consider  $\sqrt{G}(\hat{Q} - Q)$ :

$$\begin{aligned}
\sqrt{G}(\hat{Q} - Q) &= \sqrt{G} \left\{ \frac{1}{G} \sum_{g=1}^G \left[ \frac{A_g(N_g \bar{Y}_g - \hat{\mu}(1, S_g, X_g, N_g))}{\hat{\pi}(S_g)} - \frac{(1 - A_g)(N_g \bar{Y}_g - \hat{\mu}(0, S_g, X_g, N_g))}{1 - \hat{\pi}(S_g)} \right. \right. \\
&\quad \left. \left. + \hat{\mu}(1, S_g, X_g, N_g) - \hat{\mu}(0, S_g, X_g, N_g) \right] - Q \right\} \\
&= \frac{1}{\sqrt{G}} \sum_{g=1}^G \left[ \hat{\mu}(1, S_g, X_g, N_g) - \frac{A_g \hat{\mu}(1, S_g, X_g, N_g)}{\hat{\pi}(S_g)} \right] \\
&\quad + \frac{1}{\sqrt{G}} \sum_{g=1}^G \left[ \frac{(1 - A_g) \hat{\mu}(0, S_g, X_g, N_g)}{1 - \hat{\pi}(S_g)} - \hat{\mu}(0, S_g, X_g, N_g) \right] \\
&\quad + \frac{1}{\sqrt{G}} \sum_{g=1}^G \frac{A_g N_g \bar{Y}_g}{\hat{\pi}(S_g)} - \frac{1}{\sqrt{G}} \sum_{g=1}^G \frac{(1 - A_g) N_g \bar{Y}_g}{1 - \hat{\pi}(S_g)} - \sqrt{G} Q \\
&\equiv R_{n,1} + R_{n,2} + R_{n,3}
\end{aligned}$$

Applying Lemma P.1, we have that:

$$\begin{aligned}
R_{n,1} &= \frac{1}{\sqrt{G}} \sum_{g=1}^G \left( 1 - \frac{1}{\pi(S_g)} \right) A_g \tilde{\mu}(1, S_g, X_g, N_g) + \frac{1}{\sqrt{G}} \sum_{g=1}^G (1 - A_g) \tilde{\mu}(1, S_g, X_g, N_g) + o_p(1) \\
R_{n,2} &= \frac{1}{\sqrt{G}} \sum_{g=1}^G \left( \frac{1}{1 - \pi(S_g)} - 1 \right) (1 - A_g) \tilde{\mu}(0, S_g, X_g, N_g) + \frac{1}{\sqrt{G}} \sum_{g=1}^G A_g \tilde{\mu}(0, S_g, X_g, N_g) + o_p(1) \\
R_{n,3} &= \frac{1}{\sqrt{G}} \sum_{g=1}^G \frac{A_g}{\pi(S_g)} \tilde{W}_g - \frac{1}{\sqrt{G}} \sum_{g=1}^G \frac{1 - A_g}{1 - \pi(S_g)} \tilde{Z}_g + \frac{1}{\sqrt{G}} \sum_{g=1}^G (\mathbb{E}[W_g - Z_g | S_g] - \mathbb{E}[W_g - Z_g]),
\end{aligned}$$

where  $W_g \equiv N_g \bar{Y}_g(1)$ ,  $Z_g \equiv N_g \bar{Y}_g(0)$ , and  $\tilde{W}_g = W_g - \mathbb{E}[W_g | S_g]$ ,  $\tilde{Z}_g = Z_g - \mathbb{E}[Z_g | S_g]$ .

Thus, it implies that:

$$\begin{aligned}
\sqrt{G}(\hat{Q} - Q) &= \frac{1}{\sqrt{G}} \sum_{g=1}^G \left[ \left( 1 - \frac{1}{\pi(S_g)} \right) \tilde{\mu}(1, S_g, X_g, N_g) - \tilde{\mu}(0, S_g, X_g, N_g) + \frac{\tilde{W}_g}{\pi(S_g)} \right] A_g \\
&\quad + \frac{1}{\sqrt{G}} \sum_{g=1}^G \left[ \left( \frac{1}{1 - \pi(S_g)} - 1 \right) \tilde{\mu}(0, S_g, X_g, N_g) + \tilde{\mu}(1, S_g, X_g, N_g) - \frac{\tilde{Z}_g}{1 - \pi(S_g)} \right] (1 - A_g) \\
&\quad + \left\{ \frac{1}{\sqrt{G}} \sum_{g=1}^G (\mathbb{E}[W_g - Z_g | S_g] - \mathbb{E}[W_g - Z_g]) \right\} + o_p
\end{aligned}$$

Now let us consider  $\sqrt{G}(\hat{H} - H)$ . Note that we can rewrite it as

$$\begin{aligned}\sqrt{G}(\hat{H} - H) &= \sqrt{G} \left( \frac{1}{G} \sum_{g=1}^G N_g - \mathbb{E}[N_g] \right) \\ &= \sqrt{G} \left\{ \frac{1}{G} \sum_{g=1}^G (N_g A_g - \mathbb{E}[N_g | S_g] A_g) \right. \\ &\quad + \frac{1}{G} \sum_{g=1}^G (N_g (1 - A_g) - \mathbb{E}[N_g | S_g] (1 - A_g)) \\ &\quad \left. + \frac{1}{G} \sum_{g=1}^G (\mathbb{E}[N_g | S_g] - \mathbb{E}[N_g]) \right\}\end{aligned}$$

Now recall that  $\sqrt{G}(\hat{\tau} - \tau) = \frac{1}{\hat{H}} \left[ \sqrt{G}(\hat{Q} - Q) - \tau \sqrt{G}(\hat{H} - H) \right]$  and define  $\mathcal{D}_g \equiv \{W_g, Z_g, A_g, X_g, N_g\}$

$$\begin{aligned}\Xi_1(\mathcal{D}_g, S_g) &= \left[ \left( 1 - \frac{1}{\pi(S_g)} \right) \tilde{\mu}(1, S_g, X_g, N_g) - \tilde{\mu}(0, S_g, X_g, N_g) + \frac{\tilde{W}_g}{\pi(S_g)} \right] - \tau(N_g - \mathbb{E}[N_g | S_g]) \\ \Xi_0(\mathcal{D}_g, S_g) &= \left[ \left( \frac{1}{1 - \pi(S_g)} - 1 \right) \tilde{\mu}(0, S_g, X_g, N_g) + \tilde{\mu}(1, S_g, X_g, N_g) - \frac{\tilde{Z}_g}{1 - \pi(S_g)} \right] - \tau(N_g - \mathbb{E}[N_g | S_g]) \\ \Xi_2(\mathcal{D}_g, S_g) &= (\mathbb{E}[W_g - Z_g | S_g] - \mathbb{E}[W_g - Z_g]) - \tau(\mathbb{E}[N_g | S_g] - \mathbb{E}[N_g]).\end{aligned}$$

Then, we can define the variance estimand as follows:

$$\sqrt{G}(\hat{\tau} - \tau) = \frac{1}{\sum_{g=1}^G N_g} \left[ \frac{1}{\sqrt{G}} \sum_{g=1}^G \Xi_1(\mathcal{D}_g, S_g) A_g + \frac{1}{\sqrt{G}} \sum_{g=1}^G \Xi_0(\mathcal{D}_g, S_g) (1 - A_g) + \frac{1}{\sqrt{G}} \sum_{g=1}^G \Xi_2(\mathcal{D}_g, S_g) \right]$$

■

## Step 2. Obtain the asymptotic distribution of $\sqrt{G}(\hat{\tau} - \tau)$

Applying Lemma P.2, we get that three terms are asymptotically normally distributed and independent from each other:

$$\begin{aligned}\frac{1}{\sqrt{G}} \sum_{g=1}^G \Xi_1(\mathcal{D}_g, S_g) A_g &\xrightarrow{d} N(0, \sigma_1^2) \\ \frac{1}{\sqrt{G}} \sum_{g=1}^G \Xi_0(\mathcal{D}_g, S_g) (1 - A_g) &\xrightarrow{d} N(0, \sigma_0^2) \\ \frac{1}{\sqrt{G}} \sum_{g=1}^G \Xi_2(\mathcal{D}_g, S_g) &\xrightarrow{d} N(0, \sigma_2^2),\end{aligned}$$

where  $\sigma_1^2 = \mathbb{E}[\pi(S_g) \Xi_1^2(\mathcal{D}_g, S_g)]$ ;  $\sigma_0^2 = \mathbb{E}[(1 - \pi(S_g)) \Xi_0^2(\mathcal{D}_g, S_g)]$ ;  $\sigma_2^2 = \mathbb{E}[\Xi_2^2(\mathcal{D}_g, S_g)]$ . Finally, we can state the asymptotic normality

$$\sqrt{G}(\hat{\tau} - \tau) \xrightarrow{d} N \left( 0, \frac{\sigma_1^2 + \sigma_0^2 + \sigma_2^2}{\mathbb{E}[N_g]^2} \right), \sigma^2 = \frac{\sigma_1^2 + \sigma_0^2 + \sigma_2^2}{\mathbb{E}[N_g]^2}.$$

■

### Step 3. Obtain a consistent estimator

Note that we can write:

$$\begin{aligned}\Xi_2(\mathcal{D}_g, S_g) &= (\mathbb{E}[W_g - Z_g | S_g] - \mathbb{E}[W_g - Z_g]) - \tau(\mathbb{E}[N_g | S_g] - \mathbb{E}[N_g]) \\ &= (\mathbb{E}[W_g - Z_g | S_g] - \tau \mathbb{E}[N_g | S_g]) - (\mathbb{E}[W_g - Z_g] - \tau \mathbb{E}[N_g])\end{aligned}$$

Recalling that  $\tau = \frac{\mathbb{E}[W_g - Z_g]}{\mathbb{E}[N_g]}$ , it is clear that both terms are mean-zero. Since the second term is a constant, the variance of  $\Xi_2(\mathcal{D}_g, S_g)$  becomes:

$$\begin{aligned}\sigma_2^2 &= \text{Var}[(\mathbb{E}[W_g - Z_g | S_g] - \tau \mathbb{E}[N_g | S_g]) - (\mathbb{E}[W_g - Z_g] - \tau \mathbb{E}[N_g])] \\ &= \mathbb{E}[(\mathbb{E}[W_g - Z_g | S_g] - \tau \mathbb{E}[N_g | S_g])^2].\end{aligned}$$

Then, the consistent estimator for  $\sigma_2^2$  can be defined as:

$$\hat{\sigma}_2^2 = \left( \frac{1}{G_1(s)} \sum_{j \in I_1(s)} (N_j \bar{Y}_j - \hat{\tau} N_j) \right) - \left( \frac{1}{G_0(s)} \sum_{j \in I_0(s)} (N_j \bar{Y}_j - \hat{\tau} N_j) \right).$$

Let us define  $I_a(s) \equiv \{j \in [g] : A_j = a, S_j = s\}$ ,  $I(s) \equiv \{j \in [g] : S_j = s\}$ ,  $G(s) \equiv \sum_{j \in [g]} I\{S_j = s\}$ ,  $G_1(s) \equiv \sum_{j \in [g]} A_j I\{S_j = s\}$ ,  $G_0(s) \equiv G(s) - G_1(s)$ . Thus, following the results from (Jiang et al, 2023) and combining the terms, we can define the variance estimator,  $\hat{\sigma}^2$ , as follows:

$$\hat{\sigma}^2 = \frac{\frac{1}{G} \sum_{g=1}^G [A_g \hat{\Xi}_1^2(\mathcal{D}_g, S_g) + (1 - A_g) \hat{\Xi}_0^2(\mathcal{D}_g, S_g) + \hat{\Xi}_2^2(\mathcal{D}_g, S_g)]}{(\frac{1}{G} \sum_{g=1}^G N_g)^2},$$

where

$$\begin{aligned}\hat{\Xi}_1(\mathcal{D}_g, s) &= \tilde{\Xi}_1(s) - \frac{1}{G_1(s)} \sum_{j \in I_1(s)} \tilde{\Xi}_{1,j}(s) - \hat{\tau} \left( N_g - \frac{1}{G(s)} \sum_{j \in I(s)} N_j \right), \\ \hat{\Xi}_0(\mathcal{D}_g, s) &= \tilde{\Xi}_0(s) - \frac{1}{G_0(s)} \sum_{j \in I_0(s)} \tilde{\Xi}_{0,j}(s) - \hat{\tau} \left( N_g - \frac{1}{G(s)} \sum_{j \in I(s)} N_j \right), \\ \hat{\Xi}_2(s) &= \left( \frac{1}{G_1(s)} \sum_{j \in I_1(s)} N_j \bar{Y}_j \right) - \left( \frac{1}{G_0(s)} \sum_{j \in I_0(s)} N_j \bar{Y}_j \right) - \hat{\tau} \times \left( \frac{1}{G(s)} \sum_{j \in I(s)} N_j \right), \\ \tilde{\Xi}_1(\mathcal{D}_g, s) &= \left( 1 - \frac{1}{\hat{\pi}(s)} \right) \hat{\mu}(1, s, X_g, N_g) - \hat{\mu}(0, s, X_g, N_g) + \frac{N_g \bar{Y}_g}{\hat{\pi}(s)}, \\ \tilde{\Xi}_0(\mathcal{D}_g, s) &= \left( \frac{1}{1 - \hat{\pi}(s)} - 1 \right) \hat{\mu}(0, s, X_g, N_g) + \hat{\mu}(1, s, X_g, N_g) - \frac{N_g \bar{Y}_g}{1 - \hat{\pi}(s)}.\end{aligned}$$

■