

Assignment: Black–Scholes Δ & Γ Analytic vs. Finite Difference vs. Complex-Step

OP Kiitorata Trainee Program

Goal. Implement *stable* analytic Greeks (Delta and Gamma) for the Black–Scholes call price and validate them by comparison against (i) classical forward finite differences and (ii) *complex-step* differentiation. Run the specified step-size sweep(s) and write results to CSV for both Δ and Γ . Based on the data you generate, briefly evaluate accuracy, stability, and step-size sensitivity of each method.

What you receive

- A stable free function for the *call* price

```
double bs_price_call(double S, double K, double r, double q, double sigma, double T);
```

implemented with numerically robust ingredients (e.g. $\Phi(z) = \frac{1}{2} \operatorname{erfc}(-z/\sqrt{2})$, `log1p` for $\log(F/K)$ near ATM, etc.). Treat this as the reference price.

Your tasks

1. **Analytic Greeks (exact).** Implement:

$$\Delta_{\text{call}} = e^{-qT} \Phi(d_1), \quad \Gamma = e^{-qT} \frac{\phi(d_1)}{S \sigma \sqrt{T}},$$

where $F = S e^{(r-q)T}$, $d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}$, $d_2 = d_1 - \sigma \sqrt{T}$. Compute $\phi(d_1)$ via $\log \phi(d_1) = -\frac{1}{2}d_1^2 - \log \sqrt{2\pi}$ to avoid underflow.

2. **Classical forward differences.** Implement:

$$\Delta_{\text{fwd}}(S; h) = \frac{C(S+h) - C(S)}{h}, \quad \Gamma_{\text{fwd}}(S; h) = \frac{C(S+2h) - 2C(S+h) + C(S)}{h^2},$$

where $C(\cdot) = \text{bs_price_call}(\cdot, K, r, q, \sigma, T)$.

3. Complex-step differentiation

Let f be analytic near $x \in \mathbb{R}$. A Taylor expansion with a purely imaginary step gives

$$f(x + ih) = f(x) + ih f'(x) - \frac{h^2}{2} f''(x) - \frac{ih^3}{6} f^{(3)}(x) + \frac{h^4}{24} f^{(4)}(x) + O(h^5).$$

First derivative. Taking imaginary parts on both sides and dividing by h yields

$$\frac{\Im f(x + ih)}{h} = f'(x) - \frac{h^2}{6} f^{(3)}(x) + O(h^4).$$

Therefore the estimator $f'(x) \approx \Im f(x + ih)/h$ has truncation error $O(h^2)$.

Second derivative. Taking real parts on both sides gives

$$\Re f(x + ih) = f(x) - \frac{h^2}{2} f''(x) + \frac{h^4}{24} f^{(4)}(x) + O(h^6),$$

hence

$$f''(x) = -\frac{2}{h^2} (\Re f(x + ih) - f(x)) - \frac{h^2}{12} f^{(4)}(x) + O(h^4).$$

Therefore the estimator $f''(x) \approx -2(\Re f(x + ih) - f(x))/h^2$ has truncation error $O(h^2)$.

45° imaginary-step alternative. Let $\omega = e^{i\pi/4} = (1 + i)/\sqrt{2}$. Using Taylor,

$$f(x + h\omega) + f(x - h\omega) = 2f(x) + (\omega^2 + \omega^{-2}) \frac{h^2}{2} f''(x) + (\omega^4 + \omega^{-4}) \frac{h^4}{24} f^{(4)}(x) + (\omega^6 + \omega^{-6}) \frac{h^6}{720} f^{(6)}(x) + O(h^8).$$

Since $\Im(\omega^2) = 1$, $\Im(\omega^4) = 0$, and $\Im(\omega^6) = -1$, taking imaginary parts on both sides and dividing by h^2 yields

$$\frac{\Im(f(x + h\omega) + f(x - h\omega))}{h^2} = f''(x) - \frac{h^4}{360} f^{(6)}(x) + O(h^6).$$

Therefore the estimator $f''(x) \approx \Im(f(x + h\omega) + f(x - h\omega))/h^2$ has truncation error $O(h^4)$.

What to implement

Templated CDF dispatch Φ_t . Define a type-dispatched normal CDF that works for both real and complex scalars:

$$\Phi_t(z) = \begin{cases} \frac{1}{2} \operatorname{erfc}(-z/\sqrt{2}), & z \in \mathbb{R}, \\ \Phi(\Re z) + i \Im z \phi(\Re z), & z \in \mathbb{C}. \end{cases}$$

This complex branch is just the first-order Taylor expansion of Φ about the real axis. Since $\Phi'(z) = \phi(z)$ and Φ is analytic,

$$\Phi(z_r + iz_i) = \Phi(z_r) + i z_i \phi(z_r) + O(z_i^2).$$

For complex-step differentiation you only need this first-order imaginary part. Reuse your high-quality *real* implementations of Φ and ϕ and synthesize

$$\Phi(z_r + iz_i) \approx \operatorname{complex}(\Phi(z_r), z_i \phi(z_r)),$$

which preserves the $O(h^2)$ truncation of the complex-step derivative while avoiding a full complex erf implementation. Use Φ_t inside your templated Black-Scholes price so the formula extends analytically to complex inputs (e.g. $S \rightarrow S + ih$) needed by complex-step.

3. Templated Black–Scholes call price. Implement

```
template<class T> T bs_price_call_t(T S,T K,T r,T q,T sigma,T Tmat)
```

with

$$F = S e^{(r-q)T}, \quad d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}, \quad C = e^{-rT} (F \Phi(d_1) - K \Phi(d_2)).$$

For $T = \text{double}$ this is the standard price; for $T = \text{std::complex<double>}$ it enables complex-step.

4. Complex-step Greeks from the price. With $C(\cdot) = \text{bs_price_call_t}(\cdot, \dots)$, implement

$$\Delta_{\text{cs}}(S) = \frac{\Im C(S + ih)}{h}.$$

Two Γ estimators:

$$\Gamma_{\text{cs,real}}(S) = -\frac{2}{h^2} \left(\Re C(S + ih) - C(S) \right), \quad \Gamma_{45^\circ}(S) = \frac{\Im(C(S + h\omega) + C(S - h\omega))}{h^2}, \quad \omega = \frac{1+i}{\sqrt{2}}.$$

(Your validation will sweep h and compare these estimators.)

Validation scenarios

Scenario 1 ATM reference:

$$S = 100, \quad K = 100, \quad r = q = 0, \quad \sigma = 0.20, \quad T = 1.$$

Scenario 2. Near-expiry, low-vol, ATM:

$$S = K = 100, \quad r = q = 0, \quad \sigma = 0.01, \quad T = \frac{1}{365}.$$

Sweep over step sizes

For each scenario, form a logarithmic grid of relative steps $h_{\text{rel}} \in [10^{-16}, 10^{-4}]$ (e.g. 24 points). For each h_{rel} , set $h = h_{\text{rel}} \cdot S$ and compute:

$$\Delta_{\text{analytic}}, \quad \Delta_{\text{fwd}}(h) = \frac{C(S + h) - C(S)}{h}, \quad \Delta_{\text{cs}} = \frac{\Im C(S + ih)}{h},$$

$$\Gamma_{\text{analytic}}, \quad \Gamma_{\text{fwd}}(h) = \frac{C(S + 2h) - 2C(S + h) + C(S)}{h^2},$$

$$\Gamma_{\text{cs,real}} = -\frac{2}{h^2} (\Re C(S + ih) - C(S)), \quad \Gamma_{45^\circ} = \frac{\Im(C(S + h\omega) + C(S - h\omega))}{h^2}, \quad \omega = \frac{1+i}{\sqrt{2}},$$

where $C(\cdot) = \text{bs_price_call_t}(\cdot, \dots)$.

CSV output (exact columns & meaning)

Write one line per step into a CSV (one file per scenario is fine, e.g. `bs_fd_vs_complex_scenario1.csv` and `..._scenario2.csv`), with the header:

```
h_rel,h,  
Delta_analytic,Delta_fd,Delta_cs,err_D_fd,err_D_cs,  
Gamma_analytic,Gamma_fd,Gamma_cs_real,Gamma_cs_45,  
err_G_fd,err_G_cs_real,err_G_cs_45
```

Columns:

- `h_rel`, `h`: relative and absolute step sizes.
- `Delta_analytic`; `Delta_fd`, `Delta_cs`: forward-difference and complex-step Δ .
- `err_D_fd` = $|\Delta_{fd} - \Delta_{analytic}|$, `err_D_cs` similarly.
- `Gamma_analytic`; `Gamma_fd`, `Gamma_cs_real` (real-part CS), `Gamma_cs_45` (45° CS).
- `err_G_fd`, `err_G_cs_real`, `err_G_cs_45`: absolute errors vs. analytic Γ .

What to report (in README or DESIGN.md)

Provide a brief validation report for *both* scenarios.

- (a) **Plots or small tables.** From each CSV, include an error summary vs. h_{rel} for Δ and Γ covering all implemented methods:

$$\Delta_{fd}, \Delta_{cs}; \quad \Gamma_{fd}, \Gamma_{cs,real}, \Gamma_{45^\circ}.$$

A simple line plot or a compact table (e.g. max/median/ p_{99} error across h_{rel}) is sufficient.

- (b) **Your observations.** In your own words, describe what you see in the data:
- Accuracy ranking between methods for Δ and Γ .
 - How results depend on the step size h (e.g. sensitivity, best range).
 - Differences between the happy vs. stress scenarios.
 - Any stability issues you encountered (e.g. NaNs, extreme values) and how you addressed them.
- (c) **Brief reasoning.** Explain, at a high level, how truncation and roundoff can interact for numerical differentiation, and relate this to the behavior you observed in your sweeps.
- (d) **Recommendation.** Based on your experiments, give a short practical recommendation for computing Δ and Γ in each scenario (method + step size guidance).

Submission checklist

- Source files containing: analytic Δ, Γ ; FD Δ, Γ ; templated price; complex-step Δ and both Γ estimators.

- Two CSVs (scenario1 & scenario2) with the exact header and columns above, generated from the sweeps.
- A brief README/DESIGN.md with the requested plots/tables and short explanations.