Simplex Method (Linear Programming)

Learning Outcomes

At the end of this session, the students should be able to:

1. explain linear programming;
2. solve an optimization problem by implementing the Simplex method; and
3. interpret the solution resulting from the Simplex method.

Content

I. Linear Programming
II. Simplex Method
    A. Definition
    B. Implementation

Linear Programming

Linear programming (LP) is an optimization approach that deals with meeting a desired *objective* such as maximizing profit or minimizing cost in the presence of *constraints* such as limited resources. The term linear connotes that the mathematical functions representing both the objective and the constraints are linear. The term programming does not mean "computer programming," but rather, connotes "scheduling" or "setting an agenda".

A linear programming (LP) problem is a problem in which we are asked to find the maximum (or minimum) value of a **linear objective function**

$$p = a_1x_1 + a_2x_2 + \ldots + a_nx_n$$ ***Example:*** $p = 3x - 2y + z$

subject to a set of linear constraints of the form

$$b_1x_1 + b_2x_2 + \ldots + b_nx_n \leq c \quad \text{OR} \quad b_1x_1 + b_2x_2 + \ldots + b_nx_n \geq c$$ ***Example:*** $x + y - 3z \leq 12$

The desired largest (or smallest) value of the objective function is called the **optimal value**, and a collection of values of the unknowns $x_1, x_2, \ldots, x_n$ that gives the optimal value constitutes an **optimal solution**. The variables $x_1, x_2, \ldots, x_n$ are called the **decision variables**.

Simplex Method

Definition

The simplex method is a linear programming method which is based on the assumption that the optimum will be at an extreme point. The constraint inequalities in this method are reformulated as equalities by using slack variables.

Implementation (Steps)

Step 1: Set-up Constraints: Slack Variables

Slack variables dictate how much of a particular resource is available or how much of the available resource will not be used. Introducing a slack variable on a constraint will change the form of the constraint into $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n + S_i = b_i$.

If $S_i > 0$, then there is still some remaining resources after the activity. If $S_i = 0$, then all of the resources are being used up, meeting the constraint. Otherwise, if $S_i < 0$, the production of a resource needs to exceed the demands of the constraint.

The slack variables should also meet the ***positivity constraint*** by placing $S_i \geq 0$ as a separate constraint. It should be taken note that the number of inequalities should also be the number of slack variables made. After introducing the slack variables for the constraints, we shall end up a system of linear algebraic equations.

Step 2: Set-up: Initial Tableau

After the constraints have been lined up, the initial tableau can be set. Setting up the initial tableau column-wise includes all of the **coefficients** of the x_i's, the **slack** variables, the **variable** to be maximized and the **right hand side** of the equations in a matrix, in that order. Row-wise, the matrix shall be arranged **constraints** first, then the **objective function**.

Step 3: Solve

The **pivot column** PC will then be selected by choosing the <u>negative number</u> with the <u>highest magnitude</u> in the <u>bottom row, with the exception of the Solution Column</u>. Break ties at random.

After which, the **pivot element** PE will then be selected from the pivot column. For each positive (non-zero) entry b in the pivot column, we shall compute for the test ratio $\frac{a}{b}$, where a is the rightmost value in its particular row. The entry with the **smallest positive test ratio** is the pivot element.

After getting the pivot element, we shall **clear** the pivot column by employing the steps of the **Gauss-Jordan Elimination** method, wherein the rows of the pivot column will be zero except for the pivot element. That is, for all rows except for the pivot row, we shall find a constant that when multiplied to the whole row will make the difference to the pivot element be equal to zero, replacing the difference.

The whole process continues until the **bottom row does not have negative answers**. The last column is permitted to have negative numbers.

Step 4: Finding the Answers

We can find the answer in this process by dividing the last column by the <u>rightmost number</u> in the <u>same row</u> with the <u>largest positive magnitude</u>, except for the last row. The quotient will be the value of the variable heading that column.

For the last row, which is the row for the objective function, the result can be gathered on the immediate column on the left.

Example

Problem statement:

Suppose that a gas processing plant receives a fixed amount of raw gas each week. The raw gas is processed into two grades of heating gas: regular and premium quality. These grades of gas are in high demand (that is, they are guaranteed to sell) and yield different profits to the company. However, their production involves both time and on-storage constraints. For example, only one of the grades can be produced at a time, and the facility is open for only 80 hours/week. Further, there is a limited on-site storage for each of the products. All these factors are listed below.

Resource (Unit)	**Product**		**Resource Availability**
	Regular	**Premium**	
Raw gas (m3/ton)	7	11	77 m^3/wk
Production time (hr/ton)	10	8	80 hr/wk
Storage (ton)	9	6	
Profit (price/ton)	150	175	

Develop/set up a linear programming formulation to maximize the profits for this operation.

We must decide how much of each gas to produce to maximize profits. If the amounts of regular and premium produced weekly are designated as x_1 and x_2 respectively, then the total weekly profit can be calculated as $P = 150x_1 + 175x_2$. Converting into a linear programming objective function, we write this as:

Maximize $Z = 150x_1 + 175x_2$.

Constraints are then being made. Since the total raw gas should not exceed 77 m3/week, the constraint should be written as $7x_1 + 11x_2 \leq 77$. The other constraints should also be made in the same order. Therefore, the linear programming problem will be formulated as follows:

Maximize

$$Z = 150x_1 + 175x_2$$

subject to

$$7x_1 + 11x_2 \leq 77,$$
$$10x_1 + 8x_2 \leq 80$$
$$x_1 \leq 9$$
$$x_2 \leq 6$$
$$x_1,\ x_2 \geq 0$$

Solution:

Step 1: Set-up Constraints: Slack Variables

After formulating our linear programming problem, the slack variables will produce the following to the constraints:

Maximize

$$Z = 150x_1 + 175x_2$$

subject to

$$7x_1 + 11x_2 + S_1 = 77$$
$$10x_1 + 8x_2 + S_2 = 80$$

$$x_1 + S_3 = 9$$
$$x_2 + S_4 = 6$$
$$x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$$

Or can be written as

$$\begin{array}{lllllll} 7x_1 + 11x_2 + S_1 & & & & = 77 \\ 10x_1 + 8x_2 & + S_2 & & & = 80 \\ x_1 & & + S_3 & & = 9 \\ x_2 & & & + S_4 & = 6 \end{array}$$
$$x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$$

Step 2: Set-up: Initial Tableau

Doing the setup on our current maximization problem, we shall obtain the initial tableau below.

Basic	x_1	x_2	S_1	S_2	S_3	S_4	Z	**Solution**
S_1	7	11	1	0	0	0	0	77
S_2	10	8	0	1	0	0	0	80
S_3	1	0	0	0	1	0	0	9
S_4	0	1	0	0	0	1	0	6
Z	-150	-175	0	0	0	0	1	0

Step 3: Solve

In our example, we shall iterate on the tableau with the iterations given below.

Iteration 0.

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	**Solution**	**TR**
S_1	7	11	1	0	0	0	0	77	7
S_2	10	8	0	1	0	0	0	80	10
S_3	1	0	0	0	1	0	0	9	cannot be
S_4	0	1	0	0	0	1	0	6	6
Z	-150	-175	0	0	0	0	1	0	

Note: We picked x_2 as Pivot Column since -175 has the highest magnitude among the negative number excluding the solution column on the bottom row. 1 is chosen as the Pivot Element since its test ratio or TR (computed by: right most column (Solution Column) / Pivot column (x_2)), is 6 which is the smallest positive test ratio. Then, we clear the Pivot Row x_2 by Gauss Jordan.

Iteration 1:

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	**Solution**	**TR**
S_1	7	0	1	0	0	-11	0	11	1.5174
S_2	10	0	0	1	0	-8	0	32	3.2
S_3	1	0	0	0	1	0	0	9	9
S_4	0	1	0	0	0	1	0	6	cannot be
Z	-150	0	0	0	0	175	1	1050	

Note: Since there is still a negative number on the bottom row, we picked x_1 as Pivot Column since -150 has the highest magnitude among the negative number excluding the solution column on the bottom row. 7 is chosen as the Pivot Element since its test ratio or TR (computed by: right most column (Solution Column) / Pivot column (x_1)) is 1.5174 which is the smallest positive test ratio. Then, we clear the Pivot Row x_1 by Gauss Jordan.

Iteration 2:

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	**Solution**	**TR**
S_1	1	0	0.142857	0	0	-1.57143	0	1.57143	-1
S_2	0	0	-1.42857	1	0	7.71429	0	16.2857	2.1111
S_3	0	0	-0.142857	0	1	1.57143	0	7.42857	4.7273
S_4	0	1	0	0	0	1	0	6	6
Z	0	0	21.4286	0	0	-60.7143	1	1285.71	

Note: Since there is still a negative number on the bottom row, we picked s_4 as Pivot Column since -60 has the highest magnitude among the negative number excluding the solution column on the bottom row. 7.71429 is chosen as the Pivot Element since its test ratio or TR (computed by: right most column (Solution Column) / Pivot column (S_2)) is 2.1111 which is the smallest positive test ratio. Then, we clear the Pivot Row x_1 by Gauss Jordan.

Iteration 3:

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	**Solution**	**TR**
S_1	1	0	-0.148148	0.203704	0	0	0	4.88889	
S_2	0	0	-0.185185	0.12963	0	1	0	2.11111	
S_3	0	0	0.148148	-0.203704	1	0	0	4.11111	
S_4	0	1	0.185185	-0.12963	0	0	0	3.88889	
Z	0	0	10.1852	7.87037	0	0	1	1413.89	

Note: Since there are no more negative values on the bottom row, we stop the iteration

Step 4: Finding the Answers

In our example, the final tableau will be used to compute for the values of the variables.

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	**Solution**	**Value of**
S_1	1	0	-0.148148	0.203704	0	0	0	4.88889	x_1
S_2	0	0	-0.185185	0.12963	0	1	0	2.11111	S_4
S_3	0	0	0.148148	-0.203704	1	0	0	4.11111	S_3
S_4	0	1	0.185185	-0.12963	0	0	0	3.88889	x_2
Z	0	0	10.1852	7.87037	0	0	1	1413.89	Z

Therefore we could conclude that we could reap maximum profit of 1413.89 if we produce 4.88889 of regular gas and 3.88889 of premium gas.

Learning Experiences

1. Students will manually solve a real-world problem (maximization) using ***Simplex method.***
2. Students will interpret the solution to the problem.
3. Students will attempt to accomplish ***sample exercises for self-learning*** provided below.

Sample Exercises for Self-learning

1. *Required Competencies*: Linear Programming using Simplex method
   Using Simplex method, solve the problem below.

   The Cannon Hill furniture Company produces tables and chairs. Each table takes four hours of labor from the carpentry department and two hours of labor from the finishing department. Each chair requires three hours of carpentry and one hour of finishing. During the current week, 240 hours of carpentry time are available and 100 hours of finishing time. Each table produced gives a profit of $70 and each chair a profit of $50. How many chairs and tables should be made?

Resource	Tables (X1)	Chairs (X2)	Constraints
Carpentry (hr)	4	3	240
Finishing (hr)	2	1	100
Unit Profit	$70	$50	

Objective Function Carpentry Constraint Finishing Constraint Non-negativity conditions	$P = 70x_1 + 50$ $4x_1 + 3x_2 \leq 240$ $2x_1 + x_2 \leq 100$ $x_1, x_2 \geq 0$

RESULT
This simplex tableau represents the optimal solution to the LP problem and is interpreted as:

$x_1 = 30,\ x_2 = 40,\ S_1 = 0,\ S_2 = 0$ **and profit or P=$4100**

The optimal solution (maximum profit to be made) is to company 30 tables and 40 chairs for a profit of $4100.

Assessment Tool

A ***hand-written exercise*** to solve the optimization problem using the Simplex method. Interpret the results of the method in the context of the real-world problem provided.

References

[1] Chapra, S. C., & Canale, R. P. (2015). ***Numerical methods for engineers*** (7th ed.). Boston ; New Delhi: McGraw-Hill Higher Education.
[2] Rick Jason Obrero. (2015). ***Interpolation: Newton's Divided Difference***. CMSC 150 Handout 5.
[3] (Online material) N.A. ***Comp. oriented optimization technique.*** BCA 504 Handout. Maharishi Markandeshwar (Deemed to be University).