Lyapunov Stability Theory: Linear Systems

M. S. Fadali
Professor of EE

Outline

- Lyapunov's (first, indirect) linearization method.
- Linear time-invariant case.
- Domain of attraction.

Lyapunov's Linearization Method

- Linearize nonlinear $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ system in vicinity of equilibrium $\boldsymbol{x}_e$:

$$\Delta\dot{\boldsymbol{x}} = \left.\frac{\partial f(x)}{\partial x}\right]_{\boldsymbol{x}_e} \Delta\boldsymbol{x}.$$

- Find the eigenvalues of the linearized system. The equilibrium $\boldsymbol{x}_e$ of the nonlinear system is:
  - Exponentially stable if **all** the eigenvalues are in the open LHP.
  - Unstable if one or more of its eigenvalues is in the open RHP.
- Inconclusive for LHP eigenvalues and one or more eigenvalues on the imaginary axis.

Example

- Determine the stability of the equilibrium of the mechanical system at the origin

$$m\ddot{y} + b\dot{y} + k_1 y + k_3 y^3 = f$$

- Equilibrium with $f = 0$

$$\dot{y} = 0, \ddot{y} = 0$$

$$k_1 y + k_3 y^3 = 0$$

$$y = 0$$

Nonlinear State Equations

- Physical state variables

$$x_1 = y, \qquad x_2 = \dot{y}$$

- State Equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(f - bx_2 - k_1 x_1 - k_3 x_1^3)$$

$$\frac{1}{m}\frac{\partial(f - bx_2 - k_1 x_1 - k_3 x_1^3)}{\partial x_1}\bigg]_{\boldsymbol{x}_e = \mathbf{0}} = -\frac{k_1}{m}$$

Linearization and Stability

- Equilibrium state $\boldsymbol{x} = [0 \quad 0]^T$
- Linearized model with $m = 1$

$$\Delta\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -k_1 & -b \end{bmatrix} \Delta\boldsymbol{x}$$

- Characteristic polynomial and stability

$$\lambda^2 + b\lambda + k_1 = 0$$

$$\lambda_{1,2} = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - k_1}$$

- Stable $Re\{\lambda_{1,2}\} < 0$

Linear Time-invariant Case

The LTI system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

is asymptotically stable **if and only** if for any positive definite matrix Q there exists a positive definite symmetric solution P to the Lyapunov equation

$$A^T P + PA = -Q$$

Proof: Sufficiency

- Use a quadratic Lyapunov function

$$V(\boldsymbol{x}) = \boldsymbol{x}^T P \boldsymbol{x}, \qquad P > 0$$

$$\dot{V}(\boldsymbol{x}) = \dot{\boldsymbol{x}}^T P \boldsymbol{x} + \boldsymbol{x}^T P \dot{\boldsymbol{x}}$$

$$= \boldsymbol{x}^T A^T P \boldsymbol{x} + \boldsymbol{x}^T P A \boldsymbol{x}$$

$$= \boldsymbol{x}^T [A^T P + PA] \boldsymbol{x} = -\boldsymbol{x}^T Q \boldsymbol{x}$$

$$A^T P + PA = -Q$$

$V(\boldsymbol{x}) > \mathbf{0}, \dot{V}(\boldsymbol{x}) < \mathbf{0} \Rightarrow$globally exp. stable.

Proof: Necessity

- Let $Q > 0, A$ Hurwitz $(Re[\lambda_i(A)] < 0)$

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

$$A^T P + PA = \int_0^\infty A^T e^{A^T t} Q e^{At} dt + \int_0^\infty e^{A^T t} Q e^{At} A dt$$

$$= \int_0^\infty \frac{d}{dt}\{e^{A^T t} Q e^{At}\} dt = -Q$$

$$\frac{d}{dt} e^{At} = A e^{At} = e^{At} A, \lim_{t\to\infty} e^{At} = [\mathbf{0}]$$

P Symmetric Positive Definite

$$P^T = \int_0^\infty [(e^{At})^T Q e^{At}]^T dt = P$$

$$\boldsymbol{x}^T P \boldsymbol{x} = \int_0^\infty \boldsymbol{x}^T e^{A^T t} Q_s^T Q_s \, e^{At} \boldsymbol{x} dt = \int_0^\infty \boldsymbol{y}(t)^T \boldsymbol{y}(t) dt$$

$\boldsymbol{y}(t) = Q_s \, e^{At} \boldsymbol{x} = \mathbf{0}, \forall t$ for some nonzero $\boldsymbol{x}$ iff (A, Q_s) is not an observable pair.

$P > 0$ for (A, Q_s) observable.

Note: Q can be positive semidefinite.

Uniqueness

$$A^T P + PA = -Q$$
$$A^T P_1 + P_1 A = -Q$$

Subtract

$$A^T(P - P_1) + (P - P_1)A = [\mathbf{0}]$$

$$e^{A^T t}\{A^T(P - P_1) + (P - P_1)A\} e^{At} = [\mathbf{0}]$$

$$= \frac{d}{dt}\left\{e^{A^T t}(P - P_1) e^{At}\right\}$$

$e^{A^T t}(P - P_1) e^{At}$ **constant if and only if**

$$P - P_1 = [\mathbf{0}]$$

Remarks

- Recall that the original Lyapunov theorem only gives a **sufficient** condition.
- If we start with P (i.e. with $V(\boldsymbol{x})$) and solve for Q, the condition the test may or may not work.
- If we start with Q (i.e. with the derivative and we find a P the condition is necessary and sufficient.

Example

Determine the stability of the system with state matrix

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

using the Lyapunov equation with $Q = I_2$.

Note: The system is clearly stable by inspection since A is in companion form.

Solution

$$A^T P + PA = -Q, \qquad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Multiply

$$\begin{bmatrix} -12p_{12} & -6p_{22} + p_{11} - 5p_{12} \\ -6p_{22} + p_{11} - 5p_{12} & 2p_{12} - 10p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Equate to obtain three equations in three unknowns.

Equivalent Linear System

$$\begin{bmatrix} -12p_{12} & -6p_{22} + p_{11} - 5p_{12} \\ -6p_{22} + p_{11} - 5p_{12} & 2p_{12} - 10p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 12 & 0 \\ 1 & -5 & -6 \\ 0 & -2 & 10 \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$p_{12} = 1/12$$

$$p_{22} = (1 + 2p_{12})/10 = 7/60$$

$$p_{11} = 6p_{22} + 5p_{12} = 7/10 + 5/12 = 67/60$$

$$P = \begin{bmatrix} 67/60 & 1/12 \\ 1/12 & 7/60 \end{bmatrix} = \begin{bmatrix} 1.1167 & 0.08333 \\ 0.08333 & 0.1167 \end{bmatrix}$$

Choose $P = I$

$$A^T + A = -Q$$

$$\begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = -\begin{bmatrix} 0 & 5 \\ 5 & 10 \end{bmatrix}$$

- Q not positive definite.
- No conclusion: sufficient condition only.
- Choose Q and solve for P.

MAPLE

Compute:

with(LinearAlgebra):

Transpose(A).P+P.A

Solve the equivalent linear system: M.p=-q

p is a vector whose entries are the entries of the P matrix, similarly define q

LinearSolve(M,B)

Equivalent Linear System

$$A^T P + PA = -Q, \qquad P = [\boldsymbol{p}_1 \quad \cdots \quad \boldsymbol{p}_n]$$

$$L = A^T \otimes I_n + I_n \otimes A^T$$

$$st(P) = col\{\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_n\}$$

$$L\, st(P) = -st(Q)$$

$$A \otimes B = [a_{ij} B]$$

MATLAB

$$A^T P + PA = -Q$$

- Solve a different equation.
- Identical to our equation with A replaced by A^T.

$$AP + PA^T = -Q$$

- Eigenvalues are the same!

MATLAB Example

```
>> A=[0,1;-6,-5];
>> Q=eye(2)
>> P=lyap(A,eye(2))
P =
   0.5333   -0.5000
  -0.5000    0.7000
>> eig(P)
ans =
    0.1098
    1.1236
```

To Get Earlier Answer

$$P = \begin{bmatrix} 1.1167 & 0.08333 \\ 0.08333 & 0.1167 \end{bmatrix}$$

```
>> P=lyap(A',eye(2))
P =
    1.1167    0.0833
    0.0833    0.1167
```


Domain (Ball, Region) of Attraction

- Region in which the trajectories of the system converge to an asymptotically stable equilibrium point.
- Difficult to estimate, in general.
- Can be estimated using the linearized system in the vicinity of the asymptotically stable equilibrium.

Example

$$\dot{x}_1 = 3x_2$$
$$\dot{x}_2 = -5x_1 + x_1^3 - 2x_2$$

Equilibrium $x_2 = 0, x_1(x_1^2 - 5) = 0$

$$\boldsymbol{x}_e = \boldsymbol{0}, (\pm\sqrt{5}, 0)$$

Lyapunov function candidate for $\boldsymbol{x}_e = \boldsymbol{0}$

$V(\boldsymbol{x}) = ax_1^2 - bx_1^4 + cx_1x_2 + dx_2^2$

$= \dfrac{c}{2}(x_1 + x_2)^2 + \left(a - \dfrac{c}{2} - bx_1^2\right)x_1^2 + \left(d - \dfrac{c}{2}\right)x_2^2$

Calculate $\dot{V}(\boldsymbol{x})$

$$V(\boldsymbol{x}) = ax_1^2 - bx_1^4 + cx_1x_2 + dx_2^2$$

$$\dot{V}(\boldsymbol{x}) = [2ax_1 - 4bx_1^3 + cx_2 \quad cx_1 + 2dx_2] \times \begin{bmatrix} 3x_2 \\ -5x_1 + x_1^3 - 2x_2 \end{bmatrix}$$

$$= (3c - 4d)x_2^2 + 2(d - 6b)x_1^3x_2 + 2(3a - 5d - c)x_1x_2 + cx_1^2(x_1^2 - 5)$$

For $d = 6b, c = 3a - 5d, b = 1, a = 12$

$\Rightarrow d = 6, c = 6$

$\dot{V} = -6x_2^2 + 6x_1^2(x_1^2 - 5) < 0, |x_1| < \sqrt{5}$

$V(\boldsymbol{x}) = 3(x_1 + x_2)^2 + (9 - x_1^2)x_1^2 + 3x_2^2 > 0, |x_1| < 3$

Simulation Results

- The ball of attraction can be estimated to be $\mathrm{B} = \left\{ \boldsymbol{x} \in \mathcal{R}^2 : \|\boldsymbol{x}\| < \sqrt{5} \right\}$
- Although for $D = \{\boldsymbol{x} \in \mathcal{R}^n : |x_1| < 1.6\}$ we have $V(\boldsymbol{x}) > 0, \dot{V}(\boldsymbol{x}) < 0$, this region includes divergent trajectories because D is not an invariant set. For example, the trajectory starting at $\boldsymbol{x}_0 = [0,4]^T$ crosses $x_1 = \sqrt{5}$ then diverges.

Theorem 3.9

- Equilibrium $\boldsymbol{x}_e$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), V: D \to \mathcal{R}$, $\boldsymbol{f}: D \to \mathcal{R}^n$

I. $M \subset D$ compact set containing $\boldsymbol{x}_e$, invariant w.r.t. the solutions of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$

II. $\dot{V}(\boldsymbol{x}) < \boldsymbol{0}, \forall \boldsymbol{x} \in M, \boldsymbol{x} \neq \boldsymbol{x}_e$,

$\dot{V}(\boldsymbol{x}) = \boldsymbol{0}$, $\boldsymbol{x} = \boldsymbol{x}_e$

Then $M \subset R_A$ the region of attraction of $\boldsymbol{x}_e$

Proof

- Under the assumptions $E = \left\{ \boldsymbol{x} \in M : \dot{V}(\boldsymbol{x}) = 0 \right\} = \boldsymbol{x}_e$
- $N = \boldsymbol{x}_e$ is the largest invariant set in E
- By La Salle's Theorem, every solution starting in M approaches N as $t \to \infty$, i.e. approaches $\boldsymbol{x}$ as $t \to \infty$
- M is an estimate of the domain of attraction.

Example

$$\dot{x}_1 = 3x_2$$

$$\dot{x}_2 = -5x_1 + x_1^3 - 2x_2$$

$$\dot{V} = -6x_2^2 + 6x_1^2(x_1^2 - 5) < 0, |x_1| < \sqrt{5}$$

$$V(\boldsymbol{x}) = 3(x_1 + x_2)^2 + (9 - x_1^2)x_1^2 + 3x_2^2 > 0, |x_1| < 3$$

For $x_1 = \pm\sqrt{5}$

$$V(x_2) = 6x_2^2 \pm 13.42x_2 + 35$$

$$\frac{dV(x_2)}{dx_2} = 12x_2 \pm 13.42 = 0,\ x_2 = \mp 1.1183$$

Invariant Set

$$V(x_2) = 6x_2^2 \pm 13.42x_2 + 35$$

Minimum value at edge

$$\frac{dV(x_2)}{dx_2} = 12x_2 \pm 13.42 = 0,\ x_2 = \mp 1.1183$$

$$V(\boldsymbol{x}) = 27.5,\ \boldsymbol{x} = \left[\sqrt{5}, -1.1183\ \right]^{\boldsymbol{T}}$$

$$\boldsymbol{x} = \left[-\sqrt{5}, 1.1183\ \right]^{\boldsymbol{T}}$$

$$M = \{\boldsymbol{x} \in \mathcal{R}^2 : V(\boldsymbol{x}) \leq 27.5 - \epsilon\}$$

Estimate Using Linearized system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}_e = \boldsymbol{0}$$

$$\dot{\boldsymbol{x}} = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right]_{\boldsymbol{0}} \boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x})$$

$$V(\boldsymbol{x}) = \boldsymbol{x}^T P \boldsymbol{x}$$

Solve

$$A^T P + PA = -Q$$

$$\dot{V}(\boldsymbol{x}) = \dot{\boldsymbol{x}}^T P \boldsymbol{x} + \boldsymbol{x}^T P \dot{\boldsymbol{x}}$$

$$= \left(A\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x})\right)^T P\boldsymbol{x} + \boldsymbol{x}^T P\left(A\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x})\right)$$

$$= -\ \boldsymbol{x}^T Q \boldsymbol{x} + 2\boldsymbol{g}^T P \boldsymbol{x} < 0$$

Example

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}$$

Equilibrium $\boldsymbol{x}_e = \boldsymbol{0}$

Solve $\quad A^T P + PA = -2I_2 \Rightarrow P = I_2$

$$V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{x}$$

$$\dot{V}(\boldsymbol{x}) = -\ \boldsymbol{x}^T Q \boldsymbol{x} + 2\boldsymbol{g}^T P \boldsymbol{x}$$

$$= -2(x_1^2 + x_2^2) + 2x_2^3$$

$$= -2x_1^2 - 2x_2^2(1 - x_2) < 0$$

for $\|\boldsymbol{x}\| < 1$

Contours