Disturbance rejection via parameter adaptation control

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Chapter 1

Introduction

Adaptive control is a control law paradigm that improves the performance of control systems in the presence of uncertainty. During the past decade, the adaptive control research has received widespread attention, where new adaptive control methods provided the closed-loop system with capabilities in terms of improved performance and robustness. Even though validation of adaptive control on full-scale aircraft and unmanned air vehicles has increased the confidence in model-reference adaptive control and researched for more that 50 years, as of yet no adaptive control system (ACS) has been used on safety-critical or human-rated systems due unresolved technical issues. Since ACS is a nonlinear control technique, the lack of stablished metrics for ACS is a major hurdle for certification, hence this is a major technical challenge for the adaptive control research community. Adaptive control is a mature research topic, with application, mainly, in aerospace systems, hence this chapter will provide the reader with:

- A brief overview of adaptive control research and t on-going research topics
- A discussion on design and deployment limitations

Adaptive control is a class of nonlinear control method that is able to address system with uncertainty. The uncertainty can be due to unknown changes in the system's dynamics or external disturbances. Adaptive control systems can be (loosely) defined as any control system that is able to adjust its control parameters based on inputs received from the plant and its divergence from some desired nominal behaviour. The adjustable parameters are called adaptive parameters and the adjusting mechanism is called an adaptive law. The nonlinear nature of the adaptive law makes adaptive control inherently difficult to design and analyse. ACS date back to the early 1950s and was introduced by researchers interested in designing advanced autopilots for high-performance aircraft [1]. Nonetheless, Gain-scheduling control overtook adaptive control due to the fact that gain-scheduling control uses classical control tools to design a selection of control gains that is later scheduled as a function of aircraft operating conditions.

1.1 Mathematical Preliminaries and Problem Statement

Consider the class of affine input nonlinear systems described by

$$\dot{x} = f(x,t) + g(x,t)u \tag{1.1}$$

The state vector is $x \in \mathbb{R}^{n_p}$ and $u \in \mathbb{R}^{n_u}$ is the control (input) vector. We will only consider autonomous nonlinear systems, hence the dependance of the nonlinear functions f and g on time will be dropped. The system output can be the states or some nonlinear function of them, i.e. $y = h(x)$. Characterising nonlinear systems is a cumbersome task and many complex behaviours may emerge. Due to this fact, it is desirable to analyse the system in the neighbourhood of some equilibrium point, i.e. study a linearisation of the system.

Notation

The set of real (complex) matrices of dimension $n \times m$ is represented by $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$). For a matrix $M \in \mathbb{R}^{n\times m}$, M^T and $M^\dagger$ denote its transpose and Moore–Penrose inverse, respectively. If $n = m$ and $M = M^T$, the matrix is called symmetric, where $\mathbb{S}^{n\times n}$ is the set of symmetric matrices of dimension $n \times n$; if in addition $M = M^T > 0$, then it is called symmetric positive definite (s.p.d.). The notation $\mathrm{He}\{M\} = M^T + M$ will be used for M being a square matrix. The symbol $\star$ within a matrix expression denotes a symmetric transpose term. For a square matrix, the symbols $\overline{\lambda}(M)$ and $\underline{\lambda}(M)$ denote its maximum and minimum eigenvalues, respectively. The notation $H(s) \in \mathcal{RH}_\infty$ indicates that the transfer function $H(s)$ is real, rational and is analytic in the right-half-complex plane. The $\mathcal{L}_2$ norm of a vector $x(t)$ is defined as

$$\|x\|_2 := \int_0^\infty x(t)^T x(t) dt$$

and the induced $\mathcal{L}_2$ norm (or $\mathcal{L}_2$ gain) of a nonlinear operator, $\mathcal{H}$ as

$$\|\mathcal{H}\|_{i,2} := \sup_{0 \neq x \in \mathcal{L}_2} \frac{\|Hx\|_2}{\|x\|_2}$$

In this paper, we are interested in seeking an adaptive control law for linear systems under disturbances effect, such that the response of the resulting closed-loop system enters a neighbourhood of an equilibrium point and remains within afterwards. It means that we are looking for Uniformly Bounded (UB) system behaviour. Next definition holds for non-perturbed linear systems:

Definition 1 *The solutions of $\dot{x} = f(x)$ are uniformly bounded if exists $c > 0$ and for every $0 < a < c$ and $b = b(a) > 0$ such that*

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \forall t \geq t_0 \geq 0$$

Now, it is necessary to define ultimate boundedness for linear systems with bounded non-vanishing disturbances:

Definition 2 *[1] Assume the origin is an exponentially stable equilibrium of $\dot{x} = f(x)$. Let $V(t,x)$ a Lyapunov function of the mentioned system that satisfies*

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$$
$$\frac{\partial V}{\partial x} f(x) \leq -c_3\|x\|^2, \qquad \left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|^2$$

Suppose the perturbation term $g(t,x)$...

$$\|x\| \leq \max\{k \exp[-\beta(t-t_0)]\|x(t_0)\|, b\}$$

where

$$k = \sqrt{\frac{c_2}{c_1}}, \quad \beta = \frac{(1-\theta)c_3}{2c_2}, \quad b = \frac{\delta c_4}{\theta c_3}\sqrt{\frac{c_2}{c_1}}$$

Chapter 2

Model Reference Adaptive Control

This chapter presents the underlying theory of model-reference adaptive control (MRAC), starting by addressing Single-Input Single-output (SISO) systems and then extending to Multiple-input Multiple-output (MIMO) systems. The main components of a MRAC system are presented and both direct and indirect adaptive control methods are discussed. Direct adaptive control methods adapt the control gains directly, whilst the indirect adaptive methods estimate unknown system parameters. MRAC guarantees that the tracking error tends to zero when time tends to infinity (Asymptotic Tracking is achieved), but adaptive parameters are only guaranteed to be bounded (i.e. ideal control parameters are not guaranteed to be achieved).

Modern control systems have contributed to improve the performance, safety and reliability of many engineering products, industrial processes and transportation systems, just to mention some applications. However, most of these controllers are designed under nominal operation conditions (an exact model of the system is available) and lack of external perturbations. Traditional control techniques typically cannot adapt themselves to adverse (or unaccounted) situations such as structural damages, failure of the actuators and/or sensors or exogenous inputs. For many real-world applications, modelling of systems is not perfect due to factors as nonlinearities, system degradation, parameter uncertainty due to modelling contaminated measurements, to mention some.

MRAC can be effective in improving the system's robustness properties when uncertainty becomes significant beyond a level of desired tolerance. Adaptive control may be beneficial when system failures or highly uncertain operating conditions are present, or when system complexity increases the cost of the modelling stage. This chapter will:

- Develop basic understanding of system uncertainty
- Present the basic components model-reference adaptive control system and their functionality
- Apply MRAC techniques for direct and indirect adaptation for SISO and MIMO systems
- Present Lyapunov stability arguments of MRAC using Barbalat's lemma
- Study tracking and parameter boundedness properties

Adaptive controllers can be divided into two clases: (1) direct adaptive control and (2) indirect adaptive control. Combinations of both types of adaptive control clases are also used and are referred to as composite, combined, or hybrid direct-indirect adaptive control.

A typical direct adaptive controller may be expressed as

$$u = k_x(t)x + k_r(t)r$$

where $k_x(t)$ and $k_r(t)$ are adjustable control gains. A direct adaptive control scheme adjusts feedback control gains directly to mitigate any unwanted system uncertainty. In contrast, an indirect adaptive controller

$$u = k_x(p(t))x + k_r(p(t))r$$

seeks the same objective by adjusting the control gains via an estimate of the system parameters $p(t)$.

2.1 Basic Components and Architecture of MRAC

Plant Model

The plant considered is assumed to be uncertain (linear or nonlinear), where the uncertainty can be classified as:

- Structured uncertainty has uncertain parameters but known functional characteristics. It is also referred to as parametric uncertainty.
- Unstructured uncertainty has both parameters and functional characteristics uncertain.
- Unmodeled dynamics represent system exogenous or endogenous dynamics that are not included in a plant model.

- Matched uncertainty is a type of structured uncertainty that can be matched by the control input. Consider the uncertain system

$$\dot{x} = f(x) + B[u + \Theta^{*T}\phi(x)]$$

  where $x(t) \in \mathbb{R}^{n_p}$ is a state vector, $u(t) \in \mathbb{R}^{n_u}$ is a control vector, the matrix B is a control input matrix, Θ^* is a matrix of uncertain parameters, and $\phi(x) \in \mathbb{R}^m$ is a known bounded regressor function. The quantity $\Theta^{*T}\phi(x)$ is called a parametric matched uncertainty since it appears in the range space of the control input matrix B (i.e. the range of B consists of all products Bu), hence the uncertainty can be cancelled out by a the linear affine-in-control system.

- Unmatched uncertainty is a type of uncertainty that cannot be matched by the control input. Consider the uncertain system

$$\dot{x} = f(x) + Bu + \Theta^{*T}\phi(x)$$

  The parametric uncertainty cannot be matched if the control input matrix B is row rank deficient, i.e. $n_p > n_u$, or if B is a rank deficient square matrix. In this case, the control input is not able to cancel out the uncertainty. On the other hand, if $n_p < n_u$ and B is full row rank, the uncertainty may be rewritten as a matched uncertainty by the using the pseudo-inverse, this is:

$$\dot{x} = f(x) + B[u + B^T(BB^T)^{-1}\Theta^{*T}\phi(x)]$$

- Control input uncertainty may be represented as

$$\dot{x} = f(x) + B\Lambda u$$

  where Λ is a positive diagonal matrix that represents control input effectiveness uncertainty.

Reference Model

The role of the reference model is to specify the desired system response that must be achieved by the adaptive control system in closed-lop. In other words, it is a command shaping filter intended to imprint the desired command tracking. The adaptive control driven by the tracking error between the reference model and the system output. It is highlighted that the reference model must be properly designed in order for the adaptive control system to be able to follow the commanded reference. The reference model may be formulated as a nonlinear or LTI model, but the latter is usually selected as a nonlinear reference model adds complexity to the design process. The LTI reference model must capture performance specifications such as response times and robustness specifications such as phase and gain stability margins.

Controller Architecture

A controller architecture must be selected such that it provides overall system performance and stability for a nominal plant-controller closed-loop (i.e. no uncertainty or controller adaptation are considered). The type of controller to be used considers the objective of a control design and typed of plant. The adaptive controller can be a nominal controller augmented with an adaptive controller or a fully adaptive controller. It has been observed that adaptive augmentation control architectures are more common since they generally are more robust than fully adaptive controllers.

Adaptive Law

An adaptive law must guarantee that controller adjustment achieves a small tracking error. Adaptive laws are expressed as mathematical equations that relate the tracking error and the controller adjustment and can be either LTI or nonlinear. Even though many adaptive laws have been proposed over the years, Lyapunov stability theory is the most commonly used tool to determine stability of an adaptive control system. As with any control scheme, designing an adaptive control system must address the inherent trade-off between performance and robustness guarantees, hence tuning parameters must be incorporated into an adaptive law.

Chapter 3

Direct Model Reference Adaptive Control

3.1 Direct MRAC: the first-order case

Consider a nonlinear SISO system

$$\dot{x} = ax + b[u + h(x)] \tag{3.1}$$

with initial conditions $x(0) = x_0$, where $h(x)$ is a structured matched uncertainty parametrised as

$$h(x) = \sum_{i=1}^{m} \theta_i^* \phi_i(x) = \Theta^{*T} \phi(x)$$

where $\Theta^* = [\theta_1^*, \ldots, \theta_m^*]^T$ is an unknown constant vector, and $\phi(x) = [\phi_1, \ldots, \phi_m]^T$ is a vector of known bounded basis functions.

Consider the case where a is unknown and b is partially known, i.e. the sign b is known. Define the reference model as

$$\dot{x}_m = a_m x_m + b_m r \tag{3.2}$$

with initial condition $x_m(0)$, where $a_m < 0$ (i.e. that plant is stable) and $r(t)$ is bounded, i.e. $r(t) \in \mathbb{L}_\infty$ is a piecewise continuous bounded reference command signal. Hence the model reference states are uniformly bounded signals. Consider that there exists a nominal controller

$$u^*(t) = k_x^* x(t) + k_r^* r(t) - \Theta^{*T} \phi(x)$$

that perfectly cancels out the uncertainty and guarantees that $x(t)$ follows $x_m(t)$, where the superscript $(*)$ denotes the unknown nominal constant values. Substituting the nominal controller into the plant model substituting into the plant model, the nominal closed loop behaviour is obtained as

$$\dot{x} = (a + bk_x^*)x + bk_r^* r$$

It can be observed that the ideal closed-loop can track perfectly the reference model if the nominal gains k_x^* and k_r^* can be determined by the so-called model matching conditions, i.e.

$$a_m = a + bk_x^* \qquad b_m = bk_r^* \tag{3.3}$$

For a first order system these conditions are always guaranteed since there are two equations and two unknowns; for higher order systems guaranteeing the model matching conditions is not always trivial, although they are generally met. Adaptive control can be viewed as an estimator of the ideal control parameters,. To this end let

$$u = k_x(t)x + k_r(t)r - \Theta^T(t)\phi(x) \tag{3.4}$$

be the adaptive controller, where $k_x(t)$, $k_r(t)$, and $\Theta(t)$ are estimates the ideal values denoted with the superscript $(*)$. The adaptive controller is a direct adaptive controller since the gains are estimated directly without knowledge of the unknown system parameters a, b and Θ. Define the estimation errors as

$$\tilde{k}_x(t) = k_x(t) - k_x^* \tag{3.5}$$
$$\tilde{k}_r(t) = k_r(t) - k_r^* \tag{3.6}$$
$$\tilde{\Theta}(t) = \Theta(t) - \Theta^* \tag{3.7}$$

Substituting equation (3.4) into the plant model and using the definition of the controller gain error,

$$\dot{x} = (\underbrace{a + bk_x^*}_{a_m})x + \underbrace{bk_r^*}_{b_m} r + \tilde{k}_x(t)x + \tilde{k}_r(t)r - \tilde{\Theta}^T(t)\phi(x) \tag{3.8}$$

Now the tracking error may be defined as

$$\dot{e} = \dot{x}_m - \dot{x} = a_m e - b\tilde{k}_x(t)x - b\tilde{k}_r(t)r + b\tilde{\Theta}^T(t)\phi(x) \tag{3.9}$$

The task of designing a suitable adaptive law such that $\lim_{t\to\infty} x_m - x = 0$ is addressed next.

Proposition 1 *Consider the plant* (3.1)*, controller* (3.4) *and model reference* (3.2)*. Assume that there exist constant parameters k_x^*, k_r^* such that the model matching* *conditions are met. Then, if the plant parameter a is unkown and b is partially known, i.e. $sgn(b)$ is known, then the following adaptive law guaranteed asymptotic tracking of the model reference, i.e.* $\lim_{t\to\infty} x_m - x = 0$*:*

$$\begin{aligned} \dot{k}_x(t) &= \gamma_x x e sgn(b) && (3.10) \\ \dot{k}_r(t) &= \gamma_r r e sgn(b) && (3.11) \\ \dot{\Theta}(t) &= -\Gamma\phi(x) e sgn(b) && (3.12) \end{aligned}$$

□

Proof: Stability can be determined via Lyapunov stability theory. Define a Lyapunov candidate function as

$$V = e^2 + |b|(\gamma_x^{-1}\tilde{k}_x^2 + \gamma_r^{-1}\tilde{k}_r^2 + \tilde{\Theta}^T\Gamma^{-1}\tilde{\Theta})$$

The design parameters γ_x, γ_r and Γ are termed learning rates and must be positive. Now consider the time derivative of V:

$$\begin{aligned} \dot{V} &= 2e\dot{e} + 2|b|(\gamma_x^{-1}\tilde{k}_x\dot{\tilde{k}}_x + \gamma_r^{-1}\tilde{k}_r\dot{\tilde{k}}_r + \tilde{\Theta}^T\Gamma^{-1}\dot{\tilde{\Theta}}) && (3.13) \\ &= 2a_m e^2 + 2\tilde{k}_x(-bex + |b|\gamma_x^{-1}\dot{\tilde{k}}_x) + 2\tilde{k}_r(-ber + |b|\gamma_r^{-1}\dot{\tilde{k}}_r) \\ &\quad +2\tilde{\Theta}^T(-be\phi(x) + |b|\Gamma^{-1}\dot{\tilde{\Theta}}) && (3.14) \end{aligned}$$

Noting that $b = |b|sgn(b)$, that $\dot{\tilde{k}}_x = \dot{k}_x$, $\dot{\tilde{k}}_r = \dot{k}_r$, $\dot{\tilde{\Theta}} = \dot{\Theta}$ (since the ideal controller gains are assumed to be constant) and choosing $\dot{k}_x$, $\dot{k}_r$ and $\dot{\Theta}$ as in (3.12), then

$$\dot{V} = 2a_m e^2$$

Since $a_m < 0$, then $\dot{V} \leq 0$ and the signals $e(t)$, $\tilde{k}_x(t)$, $\tilde{k}_r(t)$ and $\tilde{\Theta}(t)$ are guaranteed to be bounded. Even more, it can be concluded that since V has a finite limit at $t \to \infty$, then $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, but $\dot{e}$ is only guaranteed to lie in $\mathcal{L}_\infty$, hence uniformity cannot be straightforwardly concluded.

It is possible to show uniform continuity of $\dot{V}$ by showing boundedness of its time derivative, i.e. $\ddot{V}$ must be bounded. To this end observe that

$$\ddot{V} = 4a_m e\dot{e} = 4a_m e[a_m e - b\tilde{k}_x x - b\tilde{k}_r r + b\tilde{\Theta}^T\phi(x))]$$

Since $e(t)$, $k_x(t)$, $k_r(t)$ and $\Theta(t)$ are bounded (since $\dot{V} \leq 0$), $x(t)$ is bounded because $x_m(t)$ is bounded, $r(t)$ is a bounded by assumption, and $phi(x)$ is bounded because $x(t)$ is bounded, then $\ddot{V}$ and $\dot{V}$ is uniformly continuous. It follows from the Barbalat's lemma that $\dot{V} \rightarrow 0$, hence $e(t) \rightarrow 0$ as $t \rightarrow \infty$. Observe that the tracking error is asymptotically stable, but $k_x(t)$, $k_r(t)$ and $\Theta(t)$ can only be shown to be bounded, hence the system is not asymptotically stable.

3.2 Direct MRAC: the second order SISO case

Consider the second order system

$$\ddot{y} + a_2\dot{y} + a_1 y = b[u + h(y, \dot{y})]$$

Defining $x_1 = y$, $x_2 = \dot{y}$ and considering $h(y, \dot{y}) = \Theta^{*T}\phi(x)$,a state-space representation of the plant is given by

$$\dot{x} = Ax + Bu + B\Theta^{*T}\phi(x) \tag{3.15}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

Define the model reference as

$$\dot{x}_m = A_m x_m + B_m r \tag{3.16}$$

The ideal controller is

$$u^* = K_x^* x + k_r^* r - \Theta^{*T}\phi(x)$$

where K_x^*, k_r^* and Θ^{*T} are constant unknown nominal gains. Similar to the first order case, the model matching are given by

$$A_m = A + BK_x^* \qquad B_m = Bk_r^* \tag{3.17}$$

In general, there does not exist ideal gains such that these conditions are met. For a second order SISO system a solution exists if A, A_m, B and B_m must have the same structure, hence some knowledge of the system is required in order to appropriately design the reference model. If A and B are known, then K_x^*, k_r^* can be designed using some non-adaptive control technique and disregarding the matched uncertainty. Then, A_m and B_m can be computed from this solution. The state-feedback controller is given by

$$u = K_x(t)x + k_r(t)r - \Theta^T\phi(x) \tag{3.18}$$

Proposition 2 *Consider the plant* (3.15), *controller* (3.18) *and model reference* (3.16). *Assume that there exist constant parameters* K_x^*, k_r^* *such that the model matching conditions are met. Then, if* A *is unknown and* b *is partially known, i.e.* $sgn(b)$ *is known, then the following adaptive law guarantees asymptotic tracking of the model reference, i.e.* $\lim_{t\to\infty} x_m - x = 0$*:*

$$\begin{aligned} \dot{K}_x^T(t) &= \Gamma_x x e^T \bar{P} sgn(b) && (3.19)\\ \dot{k}_r(t) &= \gamma_r r e^T \bar{P} sgn(b) && (3.20)\\ \dot{\Theta}(t) &= -\Gamma_\theta \phi(x) e^T \bar{P} sgn(b) && (3.21) \end{aligned}$$

□

Proof: Stability can be determined via Lyapunov stability theory. The proof follows closely that of the first-order case. The error system can be computed as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e - B\tilde{K}_x(t)x - B\tilde{k}_r(t)r + B\tilde{\Theta}^T(t)\phi(x) \qquad (3.22)$$

Define a Lyapunov candidate function as

$$V = e^T P e + |b|(\tilde{K}_x^T \Gamma_x^{-1} \tilde{K}_x + \gamma_r^{-1}\tilde{k}_r^2 + \tilde{\Theta}^T \Gamma_\theta^{-1} \tilde{\Theta})$$

The positive definite matrix $P = P^T$ is obtained by solving the Lyapunov equation

$$A_m^T P + P A_m + Q = 0$$

where $Q = Q^T > 0$ is a design parameter. The design parameters $\Gamma_x \in \mathbb{R}^{2\times 2}$, γ_r and Γ_θ a chosen as positive constants. Now consider the time derivative of V:

$$\begin{aligned} \dot{V} &= e^T P\dot{e} + \dot{e}^T P e + 2|b|(\tilde{K}_x^T \Gamma_x^{-1} \dot{\tilde{K}}_x + \gamma_r^{-1}\tilde{k}_r \dot{\tilde{k}}_r + \tilde{\Theta}^T \Gamma_\theta^{-1} \dot{\tilde{\Theta}}) && (3.23)\\ &= e^T(A_m^T P + P A_m)e + 2e^T PB[-\tilde{K}_x x - \tilde{k}_r r + \tilde{\Theta}^T \phi(x)] \\ &\quad +2|b|(\tilde{K}_x^T \Gamma_x^{-1} \dot{\tilde{K}}_x + \gamma_r^{-1}\tilde{k}_r \dot{\tilde{k}}_r + \tilde{\Theta}^T \Gamma_\theta^{-1} \dot{\tilde{\Theta}}) && (3.24) \end{aligned}$$

By construction

$$2e^T PB = 2e^T \bar{P} b$$

where $\bar{P} = [p_{12}\ p_{22}]^T$ and p_{ij} refers to the (i, j) element of P. Equation (3.24) can be rewritten as

$$
\begin{aligned}
\dot{V} &= -e^T Qe + 2|b|\tilde{K}_x(-xe^T\bar{P}sgn(b) + \Gamma_x^{-1}\dot{\tilde{K}}_x^T) && (3.25)\\
& \quad 2|b|\tilde{k}_r(-re^T\bar{P}sgn(b) + \gamma_r^{-1}\dot{\tilde{k}}_r) && (3.26)\\
& \quad +2|b|\tilde{\Theta}^T(\phi(x)e^T\bar{P}sgn(b) + \Gamma_\theta^{-1}\dot{\tilde{\Theta}}) && (3.27)
\end{aligned}
$$

Noting $\dot{\tilde{K}}_x = \dot{K}_x$, $\dot{\tilde{k}}_r = \dot{k}_r$, $\dot{\tilde{\Theta}} = \dot{\Theta}$ (since the ideal controller gains are assumed to be constant) and choosing $\dot{K}_x$, $\dot{k}_r$ and $\dot{\Theta}$ as in (3.21), then

$$\dot{V} = -e^T Qe \leq -\lambda_{min}(Q)\|e\|^2$$

Since $Q > 0$, then $\dot{V} \leq 0$ and the signals $e(t)$, $K_x(t)$, $k_r(t)$ and $\Theta(t)$ are guaranteed to be bounded. Even more, it can be concluded that since V has a finite limit at $t \to \infty$, then $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, but $\dot{e}$ is only guaranteed to lie in $\mathcal{L}_\infty$. It is possible to show uniform continuity of $\dot{V}$ by showing boundedness of its time derivative, i.e. $\ddot{V}$ must be bounded. To this end observe that

$$\ddot{V} = -e^T(QA + A^TQ)e - 2e^TQ[A_m e - B\tilde{K}_x x - B\tilde{k}_r r + B\tilde{\Theta}^T\phi(x))]$$

Using the same argument with the Barbalat's lemma as in the previous sections, one can conclude that $\dot{V}$ is uniformly continuous, i.e. $\dot{V} \to 0$ as $t \to \infty$. Therefore, the tracking error is asymptotically stable.

3.3 Direct MRAC: the MIMO case

Consider the system with a matched uncertainty

$$\dot{x} = Ax + B\Lambda[u + h(x)] \tag{3.28}$$

where $x(t) \in \mathbb{R}^{n_p}$ is a state vector, $u(t) \in \mathbb{R}^{n_u}$. The matrix $\Lambda = diag(\lambda_1, \ldots, \lambda_{n_u}) \in \mathbb{R}^{n_u \times n_u}$ represents an input uncertainty and $h(x)$ is a matched uncertainty that can be linearly parametrised as $f(x) = \Theta^*\phi(x)$. The matrix $\Theta^* \in \mathbb{R}^{n_u \times l}$ is an unknown constant matrix, and $\phi(x) \in \mathbb{R}^l$ is a vector of known, bounded basis functions. It is assumed that the pair $(A, B\Lambda)$ is controllable.The controllability condition ensures that the control input $u(t)$ has a sufficient control authority to stabilise all modes of a plant. The controllability condition can be checked by the rank condition of the controllability, which has long been used in control theory (see for example [**?**]).

Define the reference function as

$$\dot{x}_m = A_m x_m + B_m r \tag{3.29}$$

where A_m is Hurwitz and the reference signal $r(t) \in \mathbb{R}^{n_r}$ is a piecewise constant, bounded signal. Also define the state feedback controller as

$$u = K_x(t)x + K_r(t)r - \Theta^T \phi(x) \tag{3.30}$$

As in previous sections, we assume that there exist ideal gains $K_x^* x$, K_r^* such that the model matching conditions are satisfied

$$A_m = A + B\Lambda K_x^* \qquad B_m = B\Lambda K_r^* \tag{3.31}$$

Again, the model matching are only satisfied under specific structural similarities between matrices, for example if $B\Lambda$ is a square and invertible matrix, then the ideal gains $K_x^* x$, K_r^* exist.

In the same way as be, by defining the parameter errors $\tilde{K}_x^(t) = K_x(t) - K_x^*$, $\tilde{K}_r^(t) = K_r(t) - K_r^*$, $\tilde{\Theta}^(t) = \Theta(t) - \Theta^*$ the error system can be expressed as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e - B\Lambda \tilde{K}_x(t)x - B\Lambda \tilde{K}_r(t)r + B\Lambda \tilde{\Theta}^T(t)\phi(x) \tag{3.32}$$

Proposition 3 *Consider the plant* (3.28)*, controller* (3.30) *and model reference* (3.29)*. Assume that there exist constant parameters K_x^*, K_r^* such that the model matching conditions are met. Then, if A is unknown, B is known and Λ is partially known, i.e. $sgn(\Lambda)$ is known, then adaptive law* (3.35) *guarantee asymptotic tracking of the model reference, i.e.* $\lim_{t\to\infty} x_m - x = 0$.

$$\begin{aligned} \dot{K}_x^T(t) &= \Gamma_x x e^T P B sgn(\Lambda) && (3.33)\\ \dot{K}_r^T(t) &= \Gamma_r r e^T P B sgn(\Lambda) && (3.34)\\ \dot{\Theta}(t) &= -\Gamma_\theta \phi(x) e^T P sgn(\Lambda) && (3.35) \end{aligned}$$

Matrix $P = P^T > 0$ is obtained by solving the Lyapunov equation $A_m^T P + P A_m + Q = 0$, for some $Q = Q^T > 0$ □

Proof: Consider the Lyapunov candidate function as

$$V = e^T P e + trace\left(|\Lambda| \tilde{K}_x \Gamma_x^{-1} \tilde{K}_x^T\right) + trace\left(|\Lambda| \tilde{K}_r \Gamma_r^{-1} \tilde{K}_r^T\right) + trace\left(|\Lambda| \tilde{\Theta}^T \Gamma_\theta^{-1} \tilde{\Theta}\right)$$

Taking its time derivative,

$$\begin{aligned}
\dot{V} &= \dot{e}^T P e + e^T P \dot{e} + 2trace\left(|\Lambda|\tilde{K}_x(t)\Gamma_x^{-1}\dot{\tilde{K}}_x^T(t)\right) + 2trace\left(|\Lambda|\tilde{K}_r(t)\Gamma_r^{-1}\dot{\tilde{K}}_r^T(t)\right) \\
&\quad +2trace\left(|\Lambda|\tilde{\Theta}^T(t)\Gamma_\theta^{-1}\dot{\tilde{\Theta}}(t)\right) \\
&= e^T[A_m^T P + PA_m]e - 2e^T PB\Lambda(\tilde{K}_x(t)x + \tilde{K}_r(t)r - \Theta(t)^T\phi(x)) \\
&\quad +2trace\left(|\Lambda|\tilde{K}_x\Gamma_x^{-1}\dot{\tilde{K}}_x^T(t)\right) + 2trace\left(|\Lambda|\tilde{K}_r(t)\Gamma_r^{-1}\dot{\tilde{K}}_r^T(t)\right) \\
&\quad +2trace\left(|\Lambda|\tilde{\Theta}^T(t)\Gamma_\theta^{-1}\dot{\tilde{\Theta}}(t)\right)
\end{aligned}$$

Using the property $trace(A^T B) = BA^T$, $\Lambda = |\Lambda|sgn(\Lambda)$ and the definition $A_m^T P + PA_m = -Q$ the previous equation can be rewritten as

$$\begin{aligned}
\dot{V} = &-e^T Q e + 2trace\left(-|\Lambda|\tilde{K}_x x e^T PBsgn(\Lambda) + |\Lambda|\tilde{K}_x\Gamma_x^{-1}\dot{\tilde{K}}T_x\right) \\
&+2trace\left(-|\Lambda|\tilde{K}_r r e^T PBsgn(\Lambda) + |\Lambda|\tilde{K}_r\Gamma_r^{-1}\dot{\tilde{K}}_r^T\right) \\
&+2|\Lambda|trace\left(|\Lambda|\tilde{\Theta}^T\phi(x)e^T PBsgn(\Lambda) + |\Lambda|\tilde{\Theta}^T(t)\Gamma_\theta^{-1}\dot{\tilde{\Theta}}(t)\right)
\end{aligned}$$

Observing that $\dot{\tilde{K}}_x = \dot{K}_x$, $\dot{\tilde{K}}_r = \dot{K}_r$, $\dot{\tilde{\Theta}} = \dot{\Theta}$, and by choosing $\dot{K}_x$ and $\dot{K}_r$ and $\dot{\Theta}$ as in (3.35), the following is obtained

$$\dot{V} = -e^T Q e \leq 0$$

This means that the tracking error and the parameter estimation errors are bounded. To complete the proof and show that asymptotic tracking is achieved, it must be shown that $\dot{V}$ is uniformly continuous, i.e. $\ddot{V}$ is bounded. Observe that $\ddot{V} = -2e^T Q\dot{e}$, so this function is bounded if e and $\dot{e}$ are bounded. The former condition is already guaranteed since $\dot{V} \leq 0$; the latter requieres $\dot{x}$ and $\dot{x}_r$ to be bounded.

By assumption, r is bounded, hence x_m and $\dot{x}_m$ are bounded. The signal $\dot{x}$ is bounded if x and u are bounded. Since $e = x - x_m$ is bounded, then x is bounded; u is bounded if the estimated gains, x and r are bounded. Since the parameter estimation error and the ideal value of the parameters are bounded, we can conclude that $\dot{x}$ is bounded. We can conclude that $\dot{e}$ is bounded, hence $\dot{V}$ is uniformly continuous and $\lim_{t\to\infty}\dot{V} = 0$. This fact, ensures that $\lim_{t\to\infty} e = 0$ and $x \to x_m$. This completes the proof. $\square$

Chapter 4

Robust Adaptive Control

This chapter will start by discussing the limitations MRAC. The limitations of adaptive control are mainly due to uncertainty (both exogenous or endogenous), structural characteristics of the plant and nonlinearities present in the system. These limitation all end up producing parameter drift. Systems with bounded external disturbances and MRAC can experience unlimited growth of the a control gain even though both the state and control signals remain bounded. This signal growth can cause instability of adaptive systems. Non-minimum phase systems represent a major challenge for MRAC since control gains are limited; additionally, the ideal property of asymptotic tracking incurs in an unstable pole-zero cancelation leads to instability. Model-reference adaptive control is generally sensitive to unmodeled dynamics, resulting in loss of robustness. The use of a large adaptation rate to improved tracking performance is termed Fast Adaptation. Unfortunately, as the adaptation rate increases, the time-delay margin of an adaptive control system decreases, hence the adaptation rate has a strong influence on the MRAC's robustness properties. Actuator saturation is a common nonlinearity that may cause systems to become unstable. Saturation has been recognised as a problematic issue in MRAC application and solutions to this limitation is a current research topic.

Model-reference adaptive control (MRAC) can be used to achieve asymptotic tracking if the uncertainty is structured. However, if the uncertainty is unstructured, MRAC is generally non-robust and the adaptive parameters cannot be proven to be bounded. In conclusion, parameter drift can cause an adaptive control to *blow-up*. As the complexity of a plant increases, robustness of MRAC becomes more important and the effects of unmodeled dynamics, unstructured uncertainty, and exogenous disturbances that may exist in a real plant must be explicitly considered. Robustness issues with adaptive control are discussed

The main objective of this chapter is to present some techniques proposed to improved robustness of model-reference adaptive control. These techniques, called robust modification, rely on two general principles: (i) limiting adaptive parameters and (ii) adding damping to the adaptive laws. There exist more modifications than those discussed in this chapter, so the interested reader can consult [2] for a review on additional robust modifications; in this chapter we will discuss a modification devised to address saturation issues. The σ modification is a well-known robust modification that seeks to add damping in order to bound adaptive parameters. The optimal control modification is a method that also adds damping, but is devised using optimal control theory. The main characteristic of the optimal control modification is that it explicitly seeks a bounded tracking; this is in contrast to the asymptotic tracking sought by standard MRAC. Bounded tracking is formulated as a minimisation problem with and unknown bound on the tracking error. An "optimal" trade-off between bounded tracking and robustness can therefore be achieved. The optimal control modification is linearly asymptotic under fast adaptation, causing the closed-loop system to resemble a linear systems in the limit. This property can be leveraged for the design and analysis of adaptive control systems using many existing well-known linear control techniques.

4.1 Robust Adaptive Modifications

Robust adaptive control research and development was fuelled by the instability onset observed when adaptive control was in implemented real life applications. Even more, the instability phenomena present in standard adaptive control lead to crash of the NASA X-15 Hypersonic in the 1960s. The fact that this event was unexpected and little understood at the time, lead to the diminishing confidence in adaptive-type controllers. In the 1980s, Rohrs *et. al.* [**?**] investigated the role of unmodeled dynamics in the on-set of instabilities in adaptive control. As a result, various robust modification schemes have been proposed, some of which will be discussed in this section. Two well stablished robustification schemes are the σ-modification and the e-modification [**?**, **?**], which add damping to the adaptive law. Techniques such as the dead-zone method and the projection method are non-linear modifications that restrict adaptive parameter growth [5]. One of the most interesting modification that have been recently proposed is the optimal control modification [**?**], which relies on optimal control theory and explicitly formulate bounded tracking instead of asymptotic tracking. Regarding actuator saturation, the work has been rather modest, with the most notable mechanisms is the positive μ-modification initially prosed by Lavretsky *et. al.* [**?**] and recently recasted as an Anti-windup scheme [3, **?**].

Robustness issues that produce parameter drift (i.e. non-minimum phase behaviours, time delay, unmodeled dynamics, actuator saturation and fast adaptation) are largely mitigated by using some robust modification schemes but in general cannot be entirely eliminated, hence there is always some robustness and performance trade-off. In this section, some of the mentioned robust adaptive control methods will be presented. At the end of this section, the reader is expected to be able to:

- Know how to apply "traditional" robust modification techniques for MRAC, namely the σ-modification
- Understand the underlying principle of modern robust adaptive control modification, namely the optimal control modification
- Implement adaptive robust modifications, i.e. the positive μ-modification, to increase the resilience of MRAC against the deleterious effects of actuator saturation

The σ-modification

Consider the case where the system is subject to non-zero external bounded perturbations, i.e $\|w\| \leq d_0$ in system (4.1).

$$\dot{x} = Ax + Bu + w \tag{4.1}$$

Note that we have removed the matched uncertainty; this is done for the sake of compactness, but an extension to include this term is considered to be trivial. The following proposition considers the problem of guaranteeing bounded gains under bounded disturbances. The main strategy presented is termed the σ-modification and guarantees boundedness of the estimation gains under non-zero tracking error.

Define the reference model as in (3.29) and consider the controller

$$u = K_x(t)x + K_r(t)r \tag{4.2}$$

Again, we assume that the model matching conditions are satisfied and ideal gains K_x^*, K_r^* exist. In this section the model matching condition considered is that of (3.31) with $\Lambda = I$ (where I is the identity matrix of suitable dimensions). By defining the parameter errors $\tilde{K}_x(t) = K_x(t) - K_x^*$, $\tilde{K}_r(t) = K_r(t) - K_r^*$, and considering the exogenous perturbation $w(t)$the error system can be expressed as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e - B\tilde{K}_x(t)x - B\tilde{K}_r(t)r + w(t) \tag{4.3}$$

Proposition 4 *Consider the plant* (4.1)*, controller* (4.2) *, model reference* (3.29) *($\Lambda = I$) and bounded external disturbance $\|w(t)\| < w_0$. Assume that there exist constant parameters K_x^*, K_r^* such that the model matching conditions are met* *and (for some positive constants M_x, M_r)*

$$\begin{aligned} \|K_x^* K_x^{*T}\|_2 &\leq M_x \\ \|K_r^* K_r^{*T}\|_2 &\leq M_r \end{aligned}$$

Then, if A is unknown, B is known, and additionally

$$\|\Gamma_x^{-1}\|_2 \leq (\frac{w_0}{\sigma M_x})^2 \tag{4.4}$$

$$\|\Gamma_r^{-1}\|_2 \leq (\frac{w_0}{\sigma M_r})^2 \tag{4.5}$$

$$\sigma \leq \frac{Q_{min}}{2P_{max}} \tag{4.6}$$

then adaptive law (4.9)*, for some positive constant $\sigma > 0$, guarantees bounded tracking of the model reference, i.e.* $\lim_{t\to\infty} \|x_m - x\| \leq \epsilon$ *for some small, positive constant ϵ, and bounded adaptive parameters.*

$$\dot{K}_x^T(t) = -\sigma K_x^T(t) + \Gamma_x x e^T P B \tag{4.7}$$

$$\dot{K}_r^T(t) = -\sigma K_r^T(t) + \Gamma_r r e^T P B \tag{4.8}$$

$$\tag{4.9}$$

Matrix $P = P^T > 0$ is obtained by solving the Lyapunov equation $A_m^T P + P A_m + Q = 0$, for some $Q = Q^T > 0$; The terms Q_{min} and P_{max} are minimum eigenvalue of Q and the maximum eigenvalue of P respectively. Even more, the closed-loop tracking error will be uniformly ultimately bounded, i.e.

$$\lim_{t\to\infty} \|e\| \leq \frac{1 + \sqrt{P_{max}} w_0}{\sigma \sqrt{P_{min}}} \tag{4.10}$$

$$\lim_{t\to\infty} \|\tilde{K}_x\| \leq (1 + \sqrt{P_{max}}) M_x \tag{4.11}$$

$$\lim_{t\to\infty} \|\tilde{K}_r\| \leq (1 + \sqrt{P_{max}}) M_r \tag{4.12}$$

$\square$

Proof: Observe that the control law has been modified with an additional term $-\sigma K_{(.)}$. This is the so-called σ-modification, where the term σ is a forgetting factor that guarantees boundedness of the estimated gains even under non-zero tracking errors, at the expense of only guaranteeing bounded tracking. Define a Lyapunov candidate function as

$$V = e^T Pe + trace\left(\tilde{K}_x \Gamma_x^{-1} \tilde{K}_x^T\right) + trace\left(\tilde{K}_r \Gamma_r^{-1} \tilde{K}_r^T\right)$$

Taking its time derivative,

$$\dot{V} = e^T P\dot{e} + \dot{e}^T Pe + 2trace\left(\tilde{K}_x \Gamma_x^{-1} \dot{\tilde{K}}_x^T\right) + 2trace\left(\tilde{K}_r \Gamma_r^{-1} \dot{\tilde{K}}_r^T\right) + 2e^T Pw$$

Substituting for $\dot{e}$, using the adaptation law proposed, the property $trace(A^T B) = BA^T$ and the solution to the Lyapunov equation $A_m^T P + PA_m = -Q$, this can be rewritten as

$$\dot{V} \quad = \quad -e^T Qe - 2\sigma trace\left(\tilde{K}_x \Gamma_x^{-1} K_x^T\right) - 2\sigma trace\left(\tilde{K}_r \Gamma_r^{-1} K_r^T\right) + 2e^T Pw$$

Observe that $\tilde{K}_x = K_x(t) - K_x^*$, $\tilde{K}_r = K_r(t) - K_r^*$, hence

$$\dot{V} = -e^T Qe - 2\sigma trace\left(\tilde{K}_x \Gamma_x^{-1}(K_x^{*T} + \tilde{K}_x^T)\right) - 2\sigma trace\left(\tilde{K}_r \Gamma_r^{-1}(K_r^{*T} + \tilde{K}_r^T)\right) + 2e' Pw$$

Noting that $-e'Qe \leq -\frac{Q_{min}}{P_{max}} e'Pe$ and enforcing the condition $\sigma \leq \frac{Q_{min}}{P_{max}}$, we obtain

$$\begin{aligned} \dot{V} \quad & \leq -2\sigma \left(\underbrace{e^T Pe + trace\left(\tilde{K}_x \Gamma_x^{-1} \tilde{K}_x^T\right) + trace\left(\tilde{K}_r \Gamma_r^{-1} \tilde{K}_r^T\right)}_{V} \right) \\ & +2\sigma \left| trace\left(\tilde{K}_x \Gamma_x^{-1} K_x^{*T}\right) \right| + 2\sigma \left| trace\left(\tilde{K}_r \Gamma_r^{-1} K_r^{*T}\right) \right| + 2e' Pw \end{aligned}$$

Observe that

$$e^T Pd \leq \|P^{1/2} e\| \|P^{1/2} w\| \leq V^{1/2} P_{max}^{1/2} w_0$$

Using the property $|trace(AB)| \leq (trace(AA'^T)^{1/2}(trace(BB^T))^{1/2}$ and the bounds proposed for Γ_x and K_x in equations (**??**), it is possible to obtain a bound on the cross term

$$\left| trace\left\{\tilde{K}_x \Gamma_x^{-1} K_x^{*T}\right\} \right| \quad \leq \quad V^{1/2}(\tfrac{w_0}{\sigma})$$

Equating a similar expression for $\left| trace\left(\tilde{K}_r \Gamma_r^{-1} K_r^{*T}\right) \right|$, the following expression is obtained

$$\dot{V} \quad \leq \quad -2V^{1/2}\left[\sigma V^{1/2} - 2(1 + P_{max}^{1/2}) w_0\right]$$

In order to guarantee $\dot{V} \leq 0$, the expression in the squared brackets of the r.h.s of the previous equation must be positive, hence there exist an ultimate bound as presented in the proposition (4.12). $\square$

The Optimal Control Adaptive Modification

Robust adaptive control methods that add damping trade-off the ideal asymptotic tracking and robustness (boundedness) of the adaptive parameters. The optimal control modification addresses this trade-off within the optimal control context. More specifically, the optimal control modification is the optimal control solution that minimises the tracking error norm bounded by some unknown lower bound. By not obliging the tracking error to go to the origin asymptotically, increased robustness can be achieved. This method was developed by Nguyen in 2008 [**?**, **?**].

The derivation of the optimal control modification follows optimal control theory (see for example [**?**]) and uses a Gradient Descent algorithm to obtain the optimal solution. Define a quadratic cost function as

$$\mathcal{J} = \int_0^\infty \frac{1}{2}(e-\Delta)^T Q(e-\Delta)dt$$

where the tracking error is defined as in (4.3) and $Q = Q^T > 0$. Defining $z = [x(t)^T \ r(t)^T]^T$, $\mathcal{K}(t) = [K_x(t) \ K_r(t)]$, $\mathcal{K}^* = [K_x^* \ K_r^*]$ and $\tilde{\mathcal{K}} = [\tilde{K}_x(t) \ \tilde{K}_r(t)]$. Now define the Hamiltonian as

$$\begin{aligned} \mathcal{H} &= \tfrac{1}{2}(e-\Delta)^T Q(e-\Delta) + \lambda^T[A_m e - B\tilde{K}_x(t)x - B\tilde{K}_r(t)r + w(t)] \\ &= \tfrac{1}{2}(e-\Delta)^T Q(e-\Delta) + \lambda^T[A_m e - B\tilde{\mathcal{K}(t)}z + w(t)] \end{aligned}$$

where λ is the adjoint vector. The adjoint equation is give by

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial e}^T = -Q(e-\Delta) - (A_m)^T\lambda \tag{4.13}$$

The transversality condition is $\lambda(t \to \infty) = 0$ since $e(0)$ is known. Taking $\tilde{\mathcal{K}} = [\tilde{K}_x(t) \ \tilde{K}_r(t)]$ as the control signal, the optimality condition is

$$\frac{\partial \mathcal{H}}{\partial \tilde{\mathcal{K}}} = 0$$

Using a the gradient descent algorithm to provide the optimal solution, the adaptive law is obtained as

$$\dot{\tilde{\mathcal{K}}}^T = -\Xi\frac{\partial}{\partial \tilde{\mathcal{K}}}^T = -\Xi z\lambda^T B \tag{4.14}$$

where Ξ is a diagonal, positive matrix defining the adaptation rates. In order to obtain λ, the "sweep" method is used. To this end, define

$$\lambda = Pe + S(\mathcal{K}(t)z) \tag{4.15}$$

The time derivative of λ is

$$\begin{aligned}
\dot{\lambda} &= P\dot{e} + \dot{P}e + S\frac{d(\mathcal{K}(t)z)}{dt} + \dot{S}(\mathcal{K}(t)z) && (4.16)\\
&= P[A_m e - B\tilde{\mathcal{K}}(t)z + w] + \dot{P}e + S\frac{d(\mathcal{K}(t)z)}{dt} + \dot{S}(\mathcal{K}(t)z) && (4.17)
\end{aligned}$$

Substituting (4.15) into $\dot{\lambda}$ in (4.13), and substituting this result in the previous equation, the following condition is obtained

$$\begin{aligned}
0 &= [\dot{P} + PA_m + A_m P^T + Q]e && (4.18)\\
&\quad +[\dot{S} + A_m^T S - PB]\mathcal{K}(t)z && (4.19)\\
&\quad +S\frac{d(\mathcal{K}(t)z)}{dt} - PB\mathcal{K}^*(t)z + Pw - Q\Delta && (4.20)
\end{aligned}$$

Matrices P and S are chosen by solving the following equations for variables P and V:

$$\begin{aligned}
\dot{P} + A_m^T P + PA_m + Q &= 0 && (4.21)\\
\dot{S} - A_m^T S + PB &= 0 && (4.22)
\end{aligned}$$

subject to the transversality conditions $P(t \to \infty) = 0$ and $S(t \to \infty) = 0$. The additional terms will determine the bound Δ following the condition

$$S\frac{d(\mathcal{K}(t)z)}{dt} + PB\mathcal{K}^*(t)z + Pw - Q\Delta \qquad (4.23)$$

Note that the error will not converge to the origin if disturbance w are present. For the finite horizon case, i.e. $t \to \infty$, matrices P, S will tend to its constant initial value ($P(t) = P(0)$, $S(t) = S(0)$). Hence, conditions (4.20) reduce to

$$A_m^T P + PA_m + Q = 0 \qquad (4.24)$$

$$S = A_m^{-T} PB \qquad (4.25)$$

Finally, the scheduling parameter adaptive parameter is defined given by

$$\dot{\hat{\mathcal{K}}}^T = -\Xi z[-e^T PB - \nu z^T \mathcal{K}^T B^T P A_m^{-1} B]$$

The firs term in the parenthesis of the r.h.s of equation is the standard (non-robust) MRAC adaptive law, whilst the second term represents the optimal control modification. It is necessary to adjust the controller in order to off-set performance and robustness, therefore a modification parameter $\nu \geq 0$ is introduced to weight the influence of the optimal modification term. This added flexibility is at the expense of only achieving suboptimal solutions. Note that at the limit, i.e. $\nu = 0$, the standard MRAC adaptation law is recovered. Observe that the optimal modification introduces a damping term, where PA_m^{-1} is required to be negative definite. This is true since $A_m^{-T}Q'A_m^{-1} > 0$. Finally, the bound on the signal Δ, which in turn defines the bound on the tracking error, is given by

$$\|\Delta\| \leq \frac{1}{Q_{min}}\left[\nu\|S\|\|\frac{d(\mathcal{K}(t)z)}{dt}\| + \|PB\|\|\mathcal{K}^*z\| + w_0\right]$$

The μ-modification for Actuator Constrained MRAC

The problem considered is that of designing a feedback controller that stabilises the system dynamics given by (4.26) using the measured angles and angular velocities. The problem of attitude feedback stabilisation has been widely addressed and several solutions exist, where the most common approach has been the use of P+D controllers (i.e. a static controller with velocity feedback). This is common in flight control systems and several well documented tuning approaches exist. Nonetheless, this solution only produces locally stabilising controllers which may not be well suited for all spacecraft missions and/or situations, where globally stabilizing controllers are sought. The main complication (as mentioned in [4]) is that the equilibrium point is not unique, hence no time-invariant controller exists such that it renders the system globally stable. Several solutions have been proposed including discontinuous (e.g. sliding mode), switched or hybrid controllers. It has been proven that that there exists no continuous time-invariant (static or dynamic) controller that globally asymptotically stabilizes the system (see [5])

Anti-windup Model Reference Adaptive control (MRAC) for systems with input saturation was formally studied in [**?**] and recently extended to systems with controller rate-saturation; the latter extension is part the novel research presented in this report. For completeness, the actuator saturation problem will be considered first and will pave the way for the rate-limit problem. A complete stability and convergence analysis for the input saturation case may be found in [**?**]. The main objective of this section is to state the problem of saturation for MRAC controllers. Consider the (linear) plant

$$\dot{x} = Ax + Bsat(u) \tag{4.26}$$

The state vector is represented by $x \in \mathbb{R}^{n_p}$ and $u \in \mathbb{R}^{n_u}$ is the control vector. The vector saturation (nonlinear) function is defined as $sat(u) = [sat(u_1), \ldots, sat(u_{n_u})]$, where

$$sat(u_i) = sign(u_i)\max\{u_i, \bar{u}_i\}$$

The dead-zone function $Dz(u)$ will be used frequently and is defines as

$$u_i = sat(u_i) + Dz(u_i)$$

The problem addressed differs from the standard MRAC, hence the standard definitions of reference model and control gain must be modified in order to guaranteed stability of the (input) saturated closed-loop system. The modification used was first presented by Lavretsky $et.al.$ and has been named μ-modification. The main philosophy with this MRAC approach is that both the reference model and the control signal are modified when saturation is present. Although this is similar to the AW approach, the μ-modification does not aim to recover (unsaturated) "nominal" dynamics; in fact, in Lavertsky's work it is not clear what this return to nominal dynamics means. Next, the μ-modification is presented and some new analysis results are sketched.

Define the new reference model as

$$\dot{x}_m = A_m x_m + B_m r + K_u(1+\mu)Dz(u) \tag{4.27}$$

and consider the new control signal

$$u = K_x x + K_r r - \mu Dz(u) \tag{4.28}$$

with adaptation law

$$\begin{array}{rcl} \dot{K}_x^T & = & \Gamma_x x(e^T PB) \\ \dot{K}_r^T & = & \Gamma_r r(e^T PB) \\ \dot{K}_u^T & = & \Gamma_u Dz(u)(e^T PB) \end{array} \tag{4.29}$$

Theorem 1 *Levretsky* et. al. *[6] Consider system* (4.26) *with controller* (4.28)*, reference model* (**??**)*. Adaptation law* (**??**) *guarantees that*

$$\lim_{t\to\infty} e(t) \quad = \quad 0$$

□

Proof: Observe that the control law has been modified with an additional term $-\mu Dz(u)$. This is part of the so-called μ-modification, where the term is an Anti-windup like feedback that seeks stability of the saturated system. It must be noted that the reference model has also been modified to include the term $K_u(1+\mu)Dz(u)$.

This term is used to cancel the effect of saturation and include an additional adaptive gain K_u. This gain is of useful when input uncertainty is present and adds damping to the closed-loop by modifying the reference model. The price paid for obtaining asymptotic tracking is the fact that *ideal* tracking is only achieved when no saturation is present (i.e. $Dz(u) = 0$).

To calculate the error system, observe that

$$\dot{x} = Ax + Bsat(u) = Ax + Bu - BDz(u) = Ax + BK_x x + BK_r r - (1+\mu)BD(z)$$

Defining $\tilde{K}_x = K_x - K_x^*$, $\tilde{K}_r = K_r - K_r^*$, $\tilde{K}_u = K_u - K_u^*$, the error is given by

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e - B\tilde{K}_x(t)x - B\tilde{K}_r(t)r - B\tilde{K}_u(t)r \tag{4.30}$$

where it is assumed that $K_u^* = I$; in this case there is no input uncertainty, but it may be considered with no additional effort, i.e. $K_u \neq I$.Define a Lyapunov candidate function as

$$V = e^T Pe + trace\left(\tilde{K}_x \Gamma_x^{-1} \tilde{K}_x^T\right) + trace\left(\tilde{K}_r \Gamma_r^{-1} \tilde{K}_r^T\right) + trace\left(\tilde{K}_u \Gamma_u^{-1} \tilde{K}_u^T\right)$$

The proof follows the same line as previous derivations, where it is easy to demonstrate that, for some $Q = Q^T > 0$, the proposed adaptive laws render

$$\dot{V} \leq -e^T Qe$$

At this point it is necessary to demonstrate that the conditions in Barbalat's lemmas are achieved. Observe that the control signal is given by

$$u = \hat{K}_x x + \hat{K}_r r - \mu Dz(u)$$

Writing $Dz(u) = \beta(u)u$ where $\beta(.) : \mathbb{R} \leftarrow [0, 1)$ gives

$$u = \kappa(u)[\hat{K}_x x + \hat{K}_r r]$$

where $\kappa(u) = (1 + \beta(u)\mu) - 1$. Because the system is assumed to be well-posed, $\kappa(u)$ exists and is unique. The control signal is be bounded because x is bounded (since A is Hurwitz and $sat(u)$ is a bounded function, the x is bounded), $\hat{K}_x$ and $\hat{K}_x$ are bounded (as proved above), and r is bounded (by assumption). It follows that $\dot{e}$ is bounded since all terms on the right hand side of (4.30) are bounded. Then, since

$$\ddot{V} = -e^T Q\dot{e}$$

is bounded, $\dot{V}$ is uniformly continuous and thus that V converges to zero by Barbalat's lemma. This then implies that $e(t)$ converges asymptotically since Q is positive definite.

The results in [6] are clearly an advance over other adaptive strategies for constrained systems, but convergence of the system states to the modified dynamics is a rather unsuitable goal. Note that $\lim_{t\to\infty} e = 0$ implies that the system will follow some non-ideal reference model; in fact, ideal behaviour

$$\dot{x}_r = A_m x_r + B_m r \tag{4.31}$$

is recovered only if $Dz(u) = 0$ for all t and the controller gains converge to their ideal value. The goal of the MRAC can be modified as follows.

Proposition 5 *M.C. Turner, [**?**] Define $e_m = x_m - x_r$, $u^* = K_x^* x + K_r^* r$. System* (4.26) *with controller* (4.28)*, reference model* (4.27)*, ideal reference model* (4.31) *and adaptation law* (4.29) *guarantees that*

$$\begin{array}{rcl} \lim_{t\to\infty} e(t) & = & 0 \\ \lim_{t\to\infty} e_m(t) & = & 0 \end{array}$$

if

$$\begin{array}{c} Dz(u^*) \in \mathcal{L}_2 \\ \Delta u \in \mathcal{L}_2 \end{array}$$

where $\Delta u = \tilde{K}_x r + \tilde{K}_r r$ □

This is a clear advance: the MRAC-AW tries to recover the ideal model reference under AW-like conditions (i.e. the steady state value of the control signal does not saturate). Nonetheless, the conditions required are difficult to acquire, in particular the condition $\Delta u \in \mathcal{L}_2$, which states that the gains converge to the optimal (matching condition) gains; as mentioned previously, this is only the case under persistent excitation of the reference model input. As an alternative, the previous lemma can be replaced by the following

Lemma 2 *Define e_m, u^*, Δu as in Proposition 5. Define the steady state values*

$$\begin{array}{c} \Delta u_{ss} = \Delta K_{x,ss} x + \Delta K_{r,ss} r \\ \lim_{t\to\infty} \hat{K}_{(.)} = \Delta \hat{K}_{(.),ss} \\ \lim_{t\to\infty} \Delta K_{(.)} = \Delta K_{(.),ss} \end{array}$$

System (4.26) *with controller* (4.28)*, reference model* (4.27)*, ideal reference model* (**??**) *and adaptation law* (4.29) *guarantees that*

$$\begin{array}{rcl} \lim_{t\to\infty} e(t) & = & 0 \\ \lim_{t\to\infty} e_m(t) & = & 0 \end{array}$$

if

$$
\begin{aligned}
Dz(u^* + \Delta u_{ss}) &\in \mathcal{L}_2 \\
\Delta u - \Delta u_{ss} &\in \mathcal{L}_2
\end{aligned}
$$

□

An interesting consequence of the previous Proposition is that convergence of the ideal reference model is achieved if the steady state control signal does not saturate and the controller gains converge to some steady state value (not necessarily the ideal matching condition gains).

4.2 Conclusions

Adaptive control systems are schemes which can handle changes and uncertainty of a plant. Model Reference Adaptive Control (MRAC) is a common approach in adaptive systems [7, 8, 9, 10]. It allows to track reference signals by estimating the control gains or adaptive parameters in order to cancel undesired effects like uncertainties or disturbances. If uncertainties are structured and/or disturbances are bounded, asymptotic tracking can be achieved. Nonetheless, in real life applications, a model cannot fully describe the effects due to exogenous disturbances, unmodeled dynamics and unstructured uncertainties that may exist in a real system. Therefore, MRAC may become non-robust since the bounds on the adaptive parameters cannot be established by using traditional Lyapunov stability analysis [11]. Parameter drift may cause catastrophic failures in adaptive control algorithms [12]. Even more, unmodeled dynamics and disturbances may cause closed-loop instability in systems with MRAC [13]. In order to compensate this issue, adaptive control modification can be used to handle the onset of parameter drift.

Robust adaptive control is a well-researched topic. Robust modifications use two main ideas: (1) limit adaptive parameters and (2) add damping to the adaptive law. This course has concentrated on exhibiting some of the most representative modification of the latter type. The performance deterioration associated to parameter drift (i.e. non-minimum phase behaviour, unmodeled dynamics, and fast adaptation) is mitigated using these robust modification schemes, but is not eliminated unless the structure of the uncertainty is known. In particular, the optimal control modification is an adaptive optimal control method that seeks to minimise a the induced 2-norm of the tracking error norm, bounded away from the origin.

4.3 References

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