Solución Hoja 1

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Ejercicio 1

$y_{n+1} = y_n + h f_n$ Euler

$$y' = \frac{3}{2} y^{1/3}$$
$$y(0) = 0$$

$y_0 = 0 \quad f_0 = f(t_0, y_0) = f(0,0) = f(0) = 0$

$y_1 = 0 \Rightarrow y_2 = 0 \Rightarrow y_3 = 0 \Rightarrow y_n = 0 \quad \forall n$

$\Rightarrow$ el metodo de Euler $y_n = 0$

$Y(t) = t^{3/2} \quad Y'(t) = \frac{3}{2} t^{1/2} = \frac{3}{2} Y^{1/3}$

$f(y) = \frac{3}{2} Y^{1/3} \qquad f'(Y) = \frac{1}{2} Y^{-2/3}$

Truncatura $y_n = Y(t_n)$, ver el error $y(t_{n+1}) - y_{n+1}$

$$Y'(t) = f(t, Y(t))$$

$$y_{n+1} = y_n + h_n f(t_n, y_n) + \frac{h_n^2}{2}\left(f_t(t_n, y_n) + f_y(t_n, y_n) f(t_n, y_n) \right)$$

$$= Y(t_n) + h_n f(t_n, Y(t_n)) + \frac{h_n^2}{2}\left(f_t(t_n, Y(t_n)) + f_y(t_n, y(t_n)) f(t_n, Y(t_n)) \right)$$

$$= Y(t_n) + h_n Y'(t_n) + \frac{h_n^2}{2} Y''(t_n)$$

$$Y''(t) = \frac{d}{dt}\left(f(t, Y(t)) \right) = \frac{\partial f}{\partial t}(t, Y(t)) + f_y(t, Y(t)) f(t, Y(t))$$

$$= Y(t_{n+1}) - \frac{h_n^3}{6} Y'''(s) \qquad s \in (t_n, t_{n+1})$$

$$y_{n+1} - Y(t_{n+1}) = -\frac{h_n^3}{6} Y'''(s)$$

$$\tau_{n+1} = \frac{|Y(t_{n+1}) - y_{n+1}|}{h_n} = \frac{h_n^2}{6} |Y'''(s)| \qquad s \in (t_n, t_{n+1}) \qquad \forall n = 0, \dots, N-1$$

$$\max_{n=1,\dots,N} \tau_n = \max_{n=1,\dots,N} \frac{h_{n-1}^2}{6} |Y'''(s)| \qquad s \in (t_{n-1}, t_n)$$

$$\leq \frac{h^2}{6} \max_{\substack{s \in [0,T] \\ (t_0, T)}} Y'''(s)$$

Ejercicio 3. Dos pasos

Adams-Bashforth

$y'(t) = f(t, y(t))$ integro entre t_{n+1}, t_{n+2} $\qquad t_{n+1} - t_n = h \ \forall n$

$$y(t_{n+2}) - y(t_{n+1}) = \int_{t_{n+1}}^{t_{n+2}} f(s, y(s))\, ds$$

$$g(s) \approx g[x_0] + g[x_0,x_1](s-x_0) \qquad g[x_0] = g(x_0) \qquad g[x_0,x_1] = \frac{g(x_1)-g(x_0)}{x_1-x_0}$$

$$\approx g[x_0] + g[x_0,x_1](s-x_0) + g[x_0,x_1,x_2](s-x_0)(s-x_1)$$

$$g[x_0,x_1,x_2] = \frac{g[x_1,x_2]-g[x_0,x_1]}{x_2-x_0}$$

$$f(s,y(s)) \approx f(t_n, y(t_n)) + \frac{f(t_{n+1}, y(t_{n+1})) - f(t_n, y(t_n))}{h}(s-t_n)$$

$$y(t_{n+2}) - y(t_{n+1}) \approx \int_{t_{n+1}}^{t_{n+2}} \left(f(t_n, y(t_n)) + \frac{f(t_{n+1}, y(t_{n+1})) - f(t_n, y(t_n))}{h}(s-t_n) \right) ds$$

$$\approx h f(t_n, y(t_n)) + \frac{f(t_{n+1}, y(t_{n+1})) - f(t_n, y(t_n))}{h} \left[\frac{(s-t_n)^2}{2} \right]_{t_{n+1}}^{t_{n+2}}$$

$$\approx -\frac{1}{2} h f(t_n, y(t_n)) + \frac{3}{2} h f(t_{n+1}, y(t_{n+1})) \qquad 2h^2 - \frac{h^2}{2} = \frac{3}{2} h^2$$

Ejercicio 3. Tres pasos

$$y_{n+2} - y_{n+1} = -\frac{1}{2} h f(t_n, y_n) + \frac{3}{2} h f(t_{n+1}, y_{n+1})$$

$$= h\left(\frac{3}{2} f_{n+1} - \frac{1}{2} f_n\right)$$

$$y_{n+2} = y_{n+1} + h\left(\frac{3}{2} f_{n+1} - \frac{1}{2} f_n\right) \Rightarrow$$ 2-pasos explícito

$t_n, t_{n+1}, t_{n+2}, \Rightarrow t_{n+3}$

$$Y(t_{n+3}) - Y(t_{n+2}) = \int_{t_{n+2}}^{t_{n+3}} f(s, Y(s))\, ds$$

$$f(s, Y(s)) = F(s) \approx F[t_n] + F[t_n, t_{n+1}](s - t_n) + F[t_n, t_{n+1}, t_{n+2}](s - t_n)(s - t_{n+1})$$

$$y_{n+3} - y_{n+2} = h\left(\frac{5}{12} f_n - \frac{4}{3} f_{n+1} + \frac{23}{12} f_{n+2}\right)$$

Ejercicio 4.a

$N \geq 0 \quad h = \frac{1}{N} \quad t_0, \ldots, t_N \quad t_n = nh$

PVI $\begin{cases} Y' = Y, & t \in (0,1) \\ Y(0) = 1 \end{cases}$

$\Rightarrow \underline{Y(t) = e^t}$

$y_0 = 1 = Y(0)$

Euler $\quad y_{n+1} = y_n + h\, f(t_n, y_n) = y_n + h f_n$

$f(t_n, y_n) = y_n$

$y_1 = y_0 + h f_0 = y_0 + h y_0 = 1+h$

$y_2 = y_1 + h f_1 = y_1 + h y_1 = (1+h)\, y_1 = (1+h)^2$

Por inducción supongo que $y_{n-1} = (1+h)^{n-1}$

$y_n = y_{n-1} + h f_{n-1} = y_{n-1} + h\, y_{n-1} = (1+h)\, y_{n-1} = (1+h)\cdot(1+h)^{n-1} = (1+h)^n$

$y_n = (1+h)^n \quad n = 0, \ldots, N$

¿ $e^{t_n} - (1+h)^n = e^{nh} - (1+h)^n \xrightarrow[h \to 0]{} 0 \quad \forall n = 1, \ldots, N$?

Ejercicio 4.b

$$y(t) = e^t$$
$$y'(t) = e^t$$
$$y''(t) = e^t$$

① $e^h - (1+h) = \frac{h^2}{2} e^s \quad s \in (0,h)$

$e^t = 1 + t + \frac{t^2}{2} + \dots$

$|e^h - (1+h)| \le \frac{h^2}{2} e^h \le \frac{e}{2} h^2$

② $e^{2h} - (1+h)^2 = y(t_2) - y_2 = y(2h) - y(h) - h f(h, y(h)) + y(h) + h f(h, y(h)) - (1+h)^2$

$= e^{2h} - e^h - h e^h + e^h + h e^h - (1+h)^2 = \underbrace{e^{2h} - (1+h) e^h}_{①} + \underbrace{(1+h) e^h - (1+h)^2}$

i) $y(2h) - y(h) - h y(h) = y(2h) - y(h) - h y'(h) = \frac{h^2}{2} y''(\xi_1) \quad \xi_1 \in (h, 2h)$

$|e^{2h} - (1+h)^2| \le \frac{h^2}{2} |y''(\xi_1)| + (1+h) \, |e^h - (1+h)|$

$\le \frac{e}{2} h^2 + (1+h) \frac{e}{2} h^2 = \frac{e}{2} h^2 \sum_{k=0}^{1} (1+h)^k$

Ejercicio 4.b

Inducción $\left|y(t_{n-1}) - (1+h)^{n-1}\right| \le \frac{e}{2} h^2 \cdot \sum_{k=0}^{n-2} (1+h)^k$

quiero demostrar que $\left|y(t_n) - (1+h)^n\right| \le \frac{e}{2} h^2 \sum_{k=0}^{n-1} (1+h)^k$

$$y(t_n) - (1+h)^n = y(t_n) - y(t_{n-1}) - h f(t_{n-1}, y(t_{n-1})) + y(t_{n-1}) + h f(t_{n-1}, y(t_{n-1})) - (1+h)^n$$

$$= y(t_n) - y(t_{n-1}) - \overset{t_n - t_{n-1}}{\overset{\parallel}{h}}\, y'(t_{n-1}) + (1+h)\, y(t_{n-1}) - (1+h)^n$$

Taylor $$= \frac{h^2}{2} y''(s_{n-1}) + (1+h)\left(y(t_{n-1}) - (1+h)^{n-1}\right)$$

$s_{n-1} \in (t_{n-1}, t_n)$

$$\left|y(t_n) - (1+h)^n\right| \le \frac{h^2}{2} \overbrace{\left|y''(s_{n-1})\right|}^{\le e} + (1+h) \overbrace{h^2 \frac{e}{2} \sum_{k=0}^{n-2} (1+h)^k}^{\text{inducción}}$$

$$\le \frac{h^2}{2} e \left(1 + (1+h) \sum_{k=0}^{n-2} (1+h)^k\right) = \frac{h^2}{2} e \sum_{k=0}^{n-1} (1+h)^k$$

$\forall n = 1, \dots, N$

Ejercicio 4.c

$$\max_{n=0,\dots,N} |y(t_n) - (1+h)^n| \le \max_{n=1,\dots,N} \left| \frac{e}{2} h^2 \sum_{k=0}^{n-1} (1+h)^k \right|$$

$$y(t_0) = 1 = (1+h)^0 = y_0$$

serie geométrica

$$\le \frac{e}{2} h^2 \sum_{k=0}^{N-1} (1+h)^k = \frac{e}{2} h^2 \frac{(1+h)^N - 1}{1+h-1}$$

$$\le \frac{e}{2} h \left((1+h)^N - 1\right)$$

$$\le \frac{e}{2} h \left(\left(1+\frac{1}{N}\right)^N - 1\right)$$

como

$$\left(1+\frac{1}{N}\right)^N \le e$$

$$\le \frac{e}{2}(e-1)\, h = O(h)$$

$\Rightarrow$ error de orden 1