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An Exact and Heuristic Approach for the Max-weight Popular Matching

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ABSTRACT

Consider a bipartite weighted graph $G = (A \cup B, E, w)$, where each vertex has a strict preference ranking of its neighbors, and $w : E \to \mathbb{R}$ is a function that assigns a value to the edges of the graph. The Max-weight Popular Matching problem (MWPM) aims to find the popular matching that maximizes the sum of the weights of the involved edges in the matching. A matching M is popular if there is no other matching M' on the graph, such that the number of vertices that prefer to be in M' is greater than those that choose to be in M. We propose the first integer linear programming formulation for the MWPM based on a modified characterization for popular matching. Moreover, we developed a local improvement operator to construct a heuristic for reaching good-quality solutions. We created several instances and performed experiments on them, and their results show the effectiveness of the proposed methods.

KEYWORDS. Popular matching, integer linear formulation, heuristic.

Paper topics (indicate in order of PRIORITY the paper topic(s)): OC-Combinatorial Optimization, MH-Metaheuristics

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1. Introduction

Given a graph $G = (V, E)$, a matching M in G is a set of pairwise non-adjacent edges where no two edges share common vertices.

A vast number of situations can be modeled as matching problems. For example, Roth [Roth, 1986] studied the problem involving the assignment of resident physicians to rural hospitals. Similarly, in [Abdulkadiroğlu et al., 2005], the issue of assigning students to schools in New York was formulated as a matching problem considering students' and schools' preferences. The algorithm to solve a matching problem was recently used in [Xiao et al., 2016] to design mobile networks efficiently.

This research studies a particular type of matching called popular matching [Abraham et al., 2007; Manlove and Sng, 2006]. A matching M of G is a subset of edges without common vertices, where a vertex u is matched if it is an endpoint of one of the edges in the matching M. Otherwise, the vertex u is unmatched. We denote the other endpoint of the edge joining matched vertex u by $M(u)$. For any two matchings M and M', we say that vertex u prefers M to M' if u is either matched in M and unmatched in M' or matched in both and prefers $M(u)$ to $M'(u)$ according to the preference list of u. A matching M is popular if there is no other matching M' on the graph, such that the number of vertices that prefer to be in M' is greater than those that prefer to be in M. In other words, the pairs created by the matching M lead to greater collective satisfaction concerning the preference of individuals for their neighbors.

Gärdenfors et al. [Gärdenfors, 1975] introduced the first notion of popular matching. Furthermore, Biró et al.[Biró et al., 2010] showed that the problem of verifying if a matching M is popular on bipartite graphs can be solved in polynomial computational time. A faster algorithm was proposed in [Huang and Kavitha, 2013], together with a polynomial-time algorithm to find a popular matching with a maximum number of edges.

Weights can be assigned to edges to model the preference for some edges in the graph over others [Cseh, 2017]. This scenario was to model situations where the objective is to maximize the benefit of creating a matching using favorites or non-favorite connections between vertices. The Max-weight Popular Matching problem for bipartite graphs (MWPM) considers a bipartite weighted graph $G = (A \cup B, E, w)$ where each vertex $u \in A \cup B$ ranks its neighbors in preference and $w : E \to \mathbb{R}$ is a function that assigns a value to each edge $e \in E$ of the graph. The MWPM aims to find the popular matching that maximizes the sum of the involved edges. Recently, Kavitha [Kavitha, 2018] proved that the MWPM problem on bipartite graphs is NP-hard, even when $w(e) \in \{1, 2\}$ for every $e \in E$. Moreover, the authors presented an algorithm with a complexity $O(2^{n/4})$ to find a Max-weight Popular Matching in bipartite graphs.

We studied exact and heuristic methods for solving the MWPM problem. We propose the first integer linear programming formulation to solve the MWPM using a branch-and-cut algorithm based on the characterization of a popular matching presented in [Huang and Kavitha, 2013] and reformulated in [Gupta et al., 2019]. Moreover, we present a heuristic approach based on a local improvement operator that performs successive refinements given an initial solution. We did not find any instances for this problem in the literature. Therefore, we created several random instances that are publicly available.

2. The popular matching characterization

Consider a bipartite weighted graph $G = (A \cup B, E, w)$, where $V(G) = A \cup B$ is the set of vertices, $E(G)$ the set of edges, and $w : E \to \mathbb{R}$ is a function that assigns a value to each edge $e \in E$ of the graph. We denote the vertices adjacent to a vertex $u \in V$ as neighborhood $N_G(u)$. For any vertex $\in V(G)$, a preference list of v in G is a bijective function $l_v : N_G(v) \to \{1, 2, ..., | N_G(v) |\}$, where smaller values indicate a better rank of preference. A matching M in G is a subset of $E(G)$,

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whose edges are pairwise disjoint. A vertex u is matched in M if a unique vertex $v \in V(G)$ exists, such as $\{u, v\} \in M$. We denote by $V(M)$ the set of vertices matched in M.

Next, we present a characterization based on edge labeling for popular matching proposed by Huang and Kavitha [Huang and Kavitha, 2013] and rephrased in [Gupta et al., 2019].

Definition 1. *Let G be a bipartite weighted graph and M be a matching in G. The edge labeling function $label_M : (E(G) \setminus M) \to \{-2, 0, +2\}$ is defined as follows.*

$$label_M(\{u,v\}) = \begin{cases} -2 & \text{if } l_u(M(u)) < l_u(v) \text{ and } l_v(M(v)) < l_v(u) \\ +2 & \text{if } l_u(M(u)) > l_u(v) \text{ and } l_v(M(v)) > l_v(u) \\ 0 & \text{otherwise} \end{cases}$$

As defined in [Huang and Kavitha, 2013], if $u \notin V(M)$, then $label_M(\{u, v\}) = +2, \forall v \in N_G(u)$. Thus, a vertex will always prefer to be in the matching than to be isolated.

Definition 2. *Let G be a bipartite weighted graph and M be a matching in G. The graph G_M is the subgraph of G with $V(G_M) = V(G)$ and $E(G_M) = \{\{u, v\} \in E(G) : \{u, v\} \in M$ or $label_M(\{u, v\}) \neq -2\}$.*

An *alternating cycle* in G_M is a cycle such that each vertex in the cycle has exactly one incident edge that belongs to the matching M and one incident edge out of the matching. Similarly, an *alternating path* in G_M has its vertices with exactly one incident edge belonging to the matching and one incident edge out of the matching, except the first and last vertex. In addition, if the edge incident to the first or last vertex on the path is not in M, then that vertex is not matched in M. It is possible to characterize popular matchings as noted by Huang and Kavitha [Huang and Kavitha, 2013] and Gupta et al. [Gupta et al., 2019] using the graph G_M.

Theorem 1 ([Gupta et al., 2019])**.** *Let G be a bipartite weighted graph.*

A matching M in G is popular if and only if the following conditions hold in G_M.

1. *There is no alternating cycle in G_M that contains at least one edge labeled +2.*
2. *There is no alternating path starting at a vertex not matched in M that contains at least one edge labeled +2.*
3. *There is no alternating path with at least two edges labeled +2.*

We developed an equivalent characterization for popular matching based on the concept of *matched alternating path*.

Definition 3. *A matched alternating path is defined as an alternating path that starts and ends with edges in M.*

Using the concept of matched alternating path, Theorem 1 can be restated in the following equivalent way:

Proposition 2. *Let G be a bipartite weighted graph.*

A matching M in G is popular if and only if the following conditions hold in G_M.

1. *There is no alternating cycle in G_M that contains at least one edge labeled +2.*
2. *There is no vertex not matched in M, with an incident edge labeled +2 or adjacent to one of the endpoints of a matched alternating path that contains at least one edge labeled +2.*

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3. *There is no matched alternating path with at least two +2 edges.*

Proposition 3. *Theorem 1 and Proposition 2 are equivalent.*

Proof. We denote Theorem 1 and Proposition 2 as A and B to simplify the terminology. Moreover, each condition is denoted using a subindex, so condition 1 of Theorem 1 is denoted as A.1. First, let us prove that if G_M satisfies A, it satisfies B.

It is easy to see that G_M satisfies B.1 and B.3 because G_M satisfies A.1 and A.3. Moreover, if $u \notin V(M)$ is adjacent to one of the endpoints of a matched alternating path that violates B.2, G_M would violate A.2, resulting in a contradiction. Likewise, an incident edge labeled +2 cannot exist in u as it would lead to an alternating path in G_M with at least one edge +2 starting in u. So, if G_M satisfies A (Theorem 1), it satisfies B (Proposition 2).

Now, we prove that if G_M satisfies B, it satisfies A. As before, G_M satisfies A.1 because G_M satisfies B.1. Let P be an alternating path starting at the edge $\{w, t\}$, with $w \notin V(M)$ and $t \in V(M)$. Suppose that P violates A.2. Let us consider the case where $label_M(\{w, t\}) \neq +2$, and the alternating path P ends at the vertex $v \notin V(M)$ (otherwise, the matched alternating path $\{t, M(t)\}, ..., \{M(v), v\}$ would violate B.2 with w). In this situation, we have $v \notin V(M)$ and the previous vertex $u \in V(M)$ in the path, and two possible situations:

a) The edge labeled +2 precedes the edge $\{u, v\}$. In this case, the alternating path from t to u would be a matched alternating path that would violate condition B.2, with $w \notin V(M)$ being the vertex not matched in M, which results in a contradiction with the fact that G_M satisfies B.

b) The edge labeled +2 is $\{u, v\}$. In this case, v would be a vertex not matched by M, adjacent to an end of the matched alternating path formed only by the edge $\{u, M(u)\}$, violating condition B.2, resulting in contradiction, because G_M satisfies B, and concluding that G_M satisfies A.2.

Consider now an alternating path P in G_M violating A.3. If both endpoints are in $V(M)$, P would be a matched alternating path violating B.3. Therefore, the cases of paths with one of its endpoints not matched in M would be of interest. Consider that the ends of the path are the edges $\{w, t\}$ and $\{u, v\}$, with w and v being the path's endpoints. Suppose that, without losing generality, $w \notin V(M)$. In this case, $\{, t\}$ cannot be labeled +2 because that would violate B.2. A similar analysis is applied to the vertex v, concluding that the extremes of P cannot be labeled +2 because that would violate B.2. For this reason, edges labeled +2 could only be within the matched alternating path $\{t, M(t)\} \ldots \{M(u), u\}$, resulting in a violation of B.3 and therefore in a contradiction. So, G_M satisfies A.3. From the above, we can deduce that if G_M satisfies B (Proposition 2), it satisfies A (Theorem 1), resulting in the equivalence of the propositions. Finally, we point out that Theorem 1 and Proposition 2 hold for bipartite and general graphs, as presented in [Gupta et al., 2019]. □

2.1. Mathematical formulation

This section presents our integer linear formulation for the WMPM problem based on Proposition 2. To define the formulation, alternating paths and cycles are represented only by the edges that belong to the matching (i.e., we use only the matched edges to represent the structures).

We use the following additional notation:

$\mathcal{M}$: Family of all feasible matchings of the graph G.

$\Theta^{\mathcal{M}}$: Set of all alternating cycles in G_M, $\forall M \in \mathcal{M}$.

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$\Omega^{\mathcal{M}}$: Set of all matched alternating paths in G_M, $\forall M \in \mathcal{M}$.

$\Omega_i^{\mathcal{M}}$: Set of all matched alternating paths in G_M, $\forall M \in \mathcal{M}$, adjacent to the vertex i, where $i \notin V(M)$.

$f_{+2}(S)$: number of edges with label +2 on a matched alternating path or alternating cycle S.

Variable y_i equals 1 if vertex $i \in V(G)$ belongs to the matching; otherwise, it equals 0. Variable x_{ij} takes the value 1 if and only if edge $\{i, j\} \in E(G)$ belongs to the matching. The formulation for the WMPM problem is shown below:

$$\max \sum_{\{i,j\} \in E} x_{ij} \cdot w_{ij} \tag{1}$$

subject to:

$$\sum_{j \in N_G(i)} x_{ij} = y_i, \qquad \forall i \in V(G) \tag{2}$$

$$\sum_{\{i,j\} \in C} x_{ij} \leq \mid C \mid -1, \qquad \forall C \in \Theta^{\mathcal{M}},\ f_{+2}(C) >= 1 \tag{3}$$

$$\sum_{\{j,k\} \in P} x_{jk} + (1 - y_i) \leq \mid P \mid, \qquad \forall i \in V(G),\ \forall P \in \Omega_i^{\mathcal{M}},\ f_{+2}(P) >= 1 \tag{4}$$

$$\sum_{\{i,j\} \in P} x_{ij} \leq \mid P \mid -1, \qquad \forall P \in \Omega^{\mathcal{M}},\ f_{+2}(P) >= 2 \tag{5}$$

$$\sum_{k \in N_G(i)} x_{ik} + \sum_{r \in N_G(j)} x_{jr} \geq 1 - x_{ij}, \qquad \forall \{i, j\} \in E(G) \tag{6}$$

$$x_{ij} \in \{0, 1\}, \qquad \forall \{i, j\} \in E(G) \tag{7}$$

$$y_i \in \{0, 1\}, \qquad \forall i \in V(G) \tag{8}$$

The objective function (1) maximizes the total weight of the edges in the matching. Constraints (2) guarantee that the number of edges incident on a vertex is 1 only if the vertex belongs to the matching. Constraints (3) eliminate matchings with alternating cycles with at least one edge labeled +2. Note that cycle C is defined only by edges that belong to the matching. Moreover, Constraints (4) forbid matchings with not matched vertices adjacent to a matched alternating path with at least one edge labeled +2. In addition, Constraints (5) eliminate matchings with a matched alternating path with at least two edges labeled +2. Finally, Constraints (6) do not allow two adjacent vertices in the graph simultaneously out of the matching.

We start solving the WMPM problem using our formulation with only Constraints (2) and (6). Then, we execute the linear-time algorithm given by Huang and Kavitha [Huang and Kavitha, 2013] to determine if the found matching is popular in a bipartite graph. If the method indicates that the matching is not popular, we extract (by backtracking) structures that have violations of Proposition 2 (matched alternating paths or alternating cycles). Subsequently, we include violated Constraints 3, 4 or 5. One can notice that the presented formulation is valid not only for bipartite graphs, but also general ones, however the separation problem of violated constraints can be done in linear time considering bipartite graphs.

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3. Heuristic for the WMPM

Our heuristic starts from a matching provided by the algorithm proposed in [Huang and Kavitha, 2013] that finds a popular matching with a maximum number of edges and applies a series of improvements to increase the weight of the resulting matching. Let $M^c = E(G) \setminus M$ be the set that contains the edges of the graph that are not in the matching M. Given an alternating cycle C, $C(M)$ denotes the set of the edges of the cycle that belong to the matching, and $C(M^c)$ denotes the edges of the cycle that do not belong to the matching.

Definition 4. *The* neighborhood $N_M(C)$ *of an alternating cycle* C *is defined as*

$$N_M(C) = \{\{i,j\} \in E(G_M) \mid i \in V(C), j \in V(G)\}$$

Proposition 4. *Let* M *be a popular matching and* $C \subseteq E(G_M)$ *an alternating cycle of* G_M*, such that there is no path from any vertex of* $V(C)$ *to an edge labeled +2 in* G_M*. Consider the matching* $M' = (M \setminus C(M)) \cup C(M^c)$*. The matching* M' *is popular if the following conditions are satisfied:*

- *The neighborhood* $N_{M'}(C)$ *has no edge labeled +2.*
- *If there are edges with labels +2 in* G_M*, then* $N_{M'}(C) \subseteq N_M(C)$*.*

Proof. Let us prove that matching M' satisfies all conditions of Proposition 2. The only chance that a cycle C' in $G_{M'}$ can exist, violating Proposition 2.1, is that this cycle contains at least one of the new matched edges of $C(M^c)$. Otherwise, such a cycle would also exist in G_M, contradicting that M is popular. Since $N_{M'}(C)$ has no edge labeled +2, the cycle C' must contain an edge e, existing in G_M and labeled +2. Let u be a vertex of $V(C)$ with the shortest distance to one of the extremes of e among all the vertices of $V(C)$. As $N_{M'}(C) \subseteq N_M(C)$, the path from u to e in $G_{M'}$ would also exist in G_M, contradicting the fact that from the vertices of $V(C)$ there is no path to an edge labeled +2 in G_M.

Let us prove that M' satisfies Proposition 2.2.

As the neighborhood of $N_{M'}(C)$ has no edge labeled +2, there is no vertex $u \notin V(M')$ adjacent to an edge labeled +2 since this vertex would also exist in M. Furthermore, the only possibility that a path P exists in $G_{M'}$ violating Proposition 2.2 is that this path contains at least one of the new matched edges of $C(M^c)$. Since $N_{M'}(C)$ has no edge labeled +2, the path P must contain an edge e existing in G_M with label +2. Let u be a vertex of $V(C)$ with the shortest distance to one of the extremes of e among all the vertices of $V(C)$. As $N_{M'}(C) \subseteq N_M(C)$, the path from u to e in $G_{M'}$ would also exist in G_M, contradicting the fact that from the vertices of $V(C)$ there is no path to an edge labeled +2 in G_M. Doing the same analysis as above, we can conclude that it is not possible to exist paths violating Proposition 2.3, concluding that the current matching is also popular. □

Thus, let M be a popular matching over graph G. If $C \subseteq E(G_M)$ is an alternating cycle that satisfies Proposition 4, such that

$$\sum_{e \in C(M)} w_e < \sum_{e \in C(M^c)} w_e,$$

then $M' = (M \setminus C(M)) \cup C(M^c)$ represents a popular matching with greater weight than M. Thus, we denote $M' = \mathcal{SWAP}(M, C)$ as the popular matching M' obtained from M after applying the improvement operator $\mathcal{SWAP}$ over cycle C. When C does not satisfy Proposition 4, this operator returns the same matching M.

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To illustrate the operation of $\mathcal{SWAP}$, Table 1 presents the matrix of weights of each edge (i, j) of graph G. The symbol ”-” indicates no edge between the vertices. Table 2 shows the matrix of preferences where the value in (i, j) indicates the preference of vertex i for vertex j in the same graph G.

	0	**1**	**2**	**3**	**4**
0	-	40	-	-	25
1	40	-	-	35	-
2	-	-	-	60	-
3	-	35	60	-	30
4	25	-	-	30	-

Table 1: Edge weight

	0	**1**	**2**	**3**	**4**
0	-	1	-	-	2
1	2	-	-	1	-
2	-	-	-	1	-
3	-	2	3	-	1
4	1	-	-	2	-

Table 2: Vertex preference

Consider the initial popular matching shown in Figure 1 where bold lines represent edges in the matching. The weight of this matching is calculated as $w(\{1, 3\}) + w(\{0, 4\}) = 35 + 25 = 60$. The labeled graph G_M associated with that matching contains the alternating cycle $C = \{\{0, 1\}, \{1, 3\}, \{3, 4\}, \{4, 0\}\}$. This cycle satisfies Proposition 4 since there are no $+2$ edges in G_M. For this reason, it is possible to apply the $\mathcal{SWAP}$ operator to obtain the new popular matching shown in Figure 2, which presents a new popular matching with higher weight $w(\{0, 1\})+ w(\{3, 4\}) = 40 + 30 = 70$.

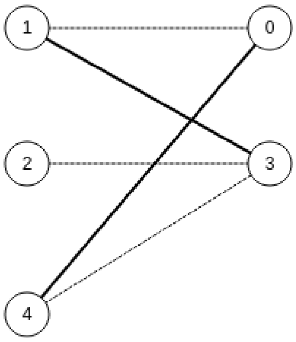

Figure 1: A popular matching

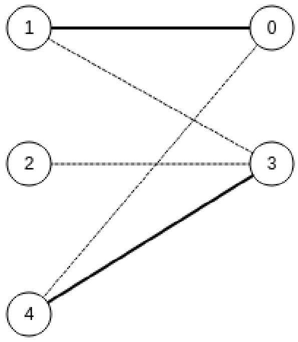

Figure 2: Popular matching obtained after applying the $\mathcal{SWAP}$ operator.

Algorithm 1 shows our proposed heuristic for the Max-weight Popular Matching (MAX-POP). Note that the possible number of alternating cycles is exponential; thus, we only considered alternating cycles C where $| C(M) | \leq \Gamma$.

4. Experiments and results

This section presents the experiments carried out to analyze the developed methods. We could not find instances in the literature, so we generated them using the Python *Networkx* library, using different probability values for having edges between vertices.

The preference lists of each $v \in V(G)$ were generated by assigning to each vertex neighbor a unique random in $\{1, ..., | N_G(v) |\}$.

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Algorithm 1: MaxPoP

$M \leftarrow$ a popular matching with maximum number of edges using [Huang and Kavitha, 2013];
while $stop \neq true$ **do**
    $stop \leftarrow true$;
    $\Delta \leftarrow \{$all alternating cycles $C \subseteq E(G_M)$ such that $\mid C(M) \mid \leq \Gamma\}$;
    foreach $C \in \Delta$ **do**
        $M' \leftarrow \mathcal{SWAP}(M, C)$;
        if $Weight(M') > Weight(M)$ **then**
            $stop \leftarrow false$;
            $M \leftarrow M'$;
            break
return M;

Finally, the edges' weights were randomly assigned in $\{1, ..., 100\}$. Following these criteria, we generated two sets with the following characteristics:

Small: Instances of bipartite graphs $G = (A \cup B, E, w)$, such that $|A|$ and $|B|$ ranging in $\{8, ..., 12\}$, and $|A| \leq |B|$. For each pair, (A, B) graphs were generated with different edge probability $p \in \{0.1, ..., 0.9\}$, resulting in 135 graphs.

Large: Instances of bipartite graphs $G = (A \cup B, E, w)$, such that $|A|$ and $|B|$ varying in the $\{100, 110, 120set, 130, 140, 150\}$ and edge probability p in the set $\{0.1, 0.3, 0.5, 0.7, 0.9\}$, resulting in 180 graphs.

Tests were performed on an Intel(R) Core(TM) i5-7200U CPU @ 2.50GHz with 8 GB of RAM using Ubuntu 16.04. All methods were programmed in C++ language using the gcc compiler. For the exact method, the IBM ILOG CPLEX 12.6 was used with a single thread of execution, and all other CPLEX parameters were left to their default values. The value $\Gamma = 4$ was selected as the maximum size for the analyzed cycles for the heuristic. Above this value, we noticed that the execution times considerably increased when executing the large instance set.

4.1. *Small* instances

Table 3 presents the results for the *Small* group of instances. The first column presents the instance's name, while the second column shows the weight of the matching obtained when applying the algorithm to determine the matching with the maximum number of edges [Huang and Kavitha, 2013]. The third and fourth columns show the weight and execution time for the popular matching obtained by the heuristic presented in Algorithm 1. The last two columns present the weight and time obtained by the exact method described in Section 2.1.

Of the 135 instances, 87 had the optimal value found by the initial solution provided by [Huang and Kavitha, 2013]. In these cases, improving the results using our heuristic was impossible. Moreover, of the 48 instances that had not initially reached the optimal solution, our heuristic improved the result of 15 instances (31.25%). In addition, for 11 of these 15 instances (73.33%), the optimal value was reached by Algorithm 1. We also note that for these small instances, our heuristic improved the results of graphs with edge probability greater than 0.4.

This improvement makes sense since denser graphs have larger numbers of popular matchings. Finally, based on the formulation in Section 2.1, we noted that our exact algorithm quickly found the optimal value for all instances in the group.

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Table 3: *Small* instances improved by MaxPop

Instances	[Huang and Kavitha, 2013]	MaxPop	Time	ILP	Time
g_10_10_7	429	494	0.00	533	2.81
g_10_11_6	581	621	0.00	621	0.25
g_11_11_8	568	744	0.00	746	4.33
g_11_11_9	870	904	0.00	904	0.73
g_11_12_5	238	389	0.00	389	1.08
g_12_12_7	342	404	0.00	404	35.18
g_12_12_9	637	681	0.00	748	125.74
g_8_10_7	454	476	0.00	476	0.21
g_8_11_5	385	419	0.00	419	0.06
g_8_8_8	487	524	0.00	524	0.21
g_8_8_9	171	344	0.00	344	0.60
g_8_9_4	431	472	0.00	472	0.04
g_8_9_6	398	479	0.00	479	0.11
g_8_9_8	253	309	0.00	309	0.11
g_9_12_5	403	409	0.00	409	0.32

4.2. *Large* instances

As the exact method could not find integer solutions for instances from the *Large* group, Table 4 shows only the results of the heuristics methods. The columns of this table have the same meaning as the columns in Table 3.

Out of the 180 instances, our heuristic improved (compared to [Huang and Kavitha, 2013]) the result of 35 instances (19.44%). In this group, our heuristic improved all instances independently of the density of the graph (value of p), with no significant differences between them. Therefore, we verified that for larger graphs, the chances of having a great variety of matchings increase, and the probability of our heuristic improving the initial matching also increases.

5. Conclusion

We proposed the first integer linear programming formulation for MWPM based on the concept of matched alternating paths, which is defined in this work and used to characterize popular matchings. The formulation yielded optimal solutions for all instances with small graphs.

We also proposed a heuristic approach to reach good quality solutions for larger instances. Random instances were generated to test the proposed methods. Our experiments showed that, on average, our heuristic improved the initial weight value of the matching with the maximum number of edges in around 20% of the analyzed instances. However, this percentage increases if we only consider instances that the initial matching provided by [Huang and Kavitha, 2013] did not reach the optimal solution (e.g., 73.33% in the *Small* instance group).

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Table 4: *Large* instances improved by MaxPop

Instances	[Huang and Kavitha, 2013]	MaxPop	Time
g_100_110_3	5287	5295	0.24
g_100_120_7	4804	4816	10.01
g_100_130_1	4863	4903	0.04
g_100_140_1	4894	4903	0.03
g_110_100_5	5055	5068	1.16
g_110_120_3	5387	5447	0.70
g_110_150_3	5435	5512	1.04
g_120_100_1	4670	4739	0.04
g_120_100_5	4850	4929	2.09
g_120_100_9	5378	5380	13.60
g_120_120_3	6156	6257	0.52
g_120_140_9	6740	6853	36.01
g_130_120_9	5852	5894	24.49
g_130_140_3	6457	6534	1.55
g_130_140_5	6838	6908	3.24
g_130_140_9	6533	6542	79.22
g_140_110_1	5265	5415	0.03
g_140_110_5	5649	5691	3.14
g_140_120_7	6010	6083	8.50
g_140_120_9	6351	6418	30.82
g_140_130_7	6508	6569	9.43
g_140_140_3	5911	5923	1.22
g_140_140_5	6976	7086	4.13
g_140_150_5	6932	6944	5.50
g_150_110_1	5894	5896	0.05
g_150_120_3	6348	6354	1.11
g_150_120_5	6637	6725	5.73
g_150_130_1	6139	6164	0.10
g_150_130_3	7120	7180	1.38
g_150_130_9	6476	6506	59.32
g_150_140_1	6652	6660	0.09
g_150_140_3	6796	6886	1.47
g_150_140_9	6701	6783	44.20
g_150_150_5	7579	7690	1.95
g_150_150_9	8070	8194	25.21

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