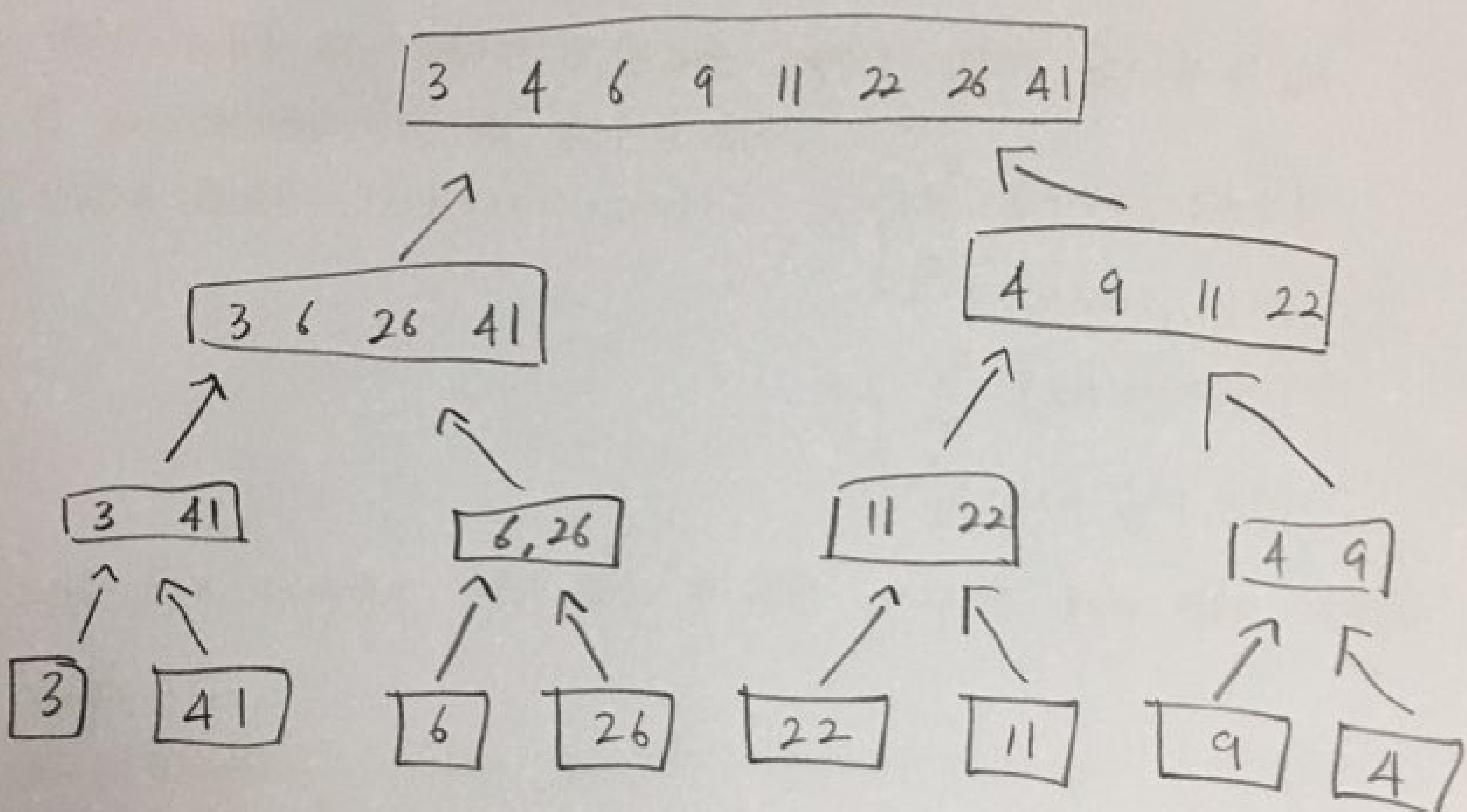
2019311188 김주원.

1. Merge sort (ascending order)

Array A = ⟨3, 41, 6, 26, 22, 11, 9, 4⟩.



2). exchange operations.

Best-case의 경우, 의사코드의 ①, ② 과정을 수행하지 않는다.
즉 이미 A가 오름차순으로 정렬되어 있는 경우이다.
따라서 exchange operations의 횟수는 0이다.

worst-case의 경우, 의사코드의 ①, ② 과정을 최대한으로 수행하는 경우로 볼 수 있다.
즉 A가 내림차순으로 정렬된 채로 입력이 된 경우이고,
의사코드의 ①에서

$$(n-1)+(n-3)+(n-5)+\cdots = \begin{cases} k^2 & (n=2k,\ k\in\mathbb{N}) \\ k^2+k & (n=2k+1,\ k\in\mathbb{N}) \end{cases}$$

$$= \begin{cases} \frac{n^2}{4} & (n\text{이 짝수인 경우}) \\ \frac{n^2-1}{4} & (n\text{이 홀수인 경우}) \end{cases}$$

그리고 ②의 swap에서 다음과 같이 한 번의 swap 당 3번의 연산이 수행된다.

```
{ temp = A[i];
  A[i] = A[min];
  A[min] = temp; }
```

따라서 $3\times(n-1)$의 연산이 요구되므로,

$$\frac{n(n-1)}{2} + \begin{Bmatrix} \frac{n^2}{4} \\ \frac{n^2-1}{4} \end{Bmatrix} + 3(n-1) \ge T(n) \ge \frac{n(n-1)}{2} \text{ unit times}$$

따라서 $T(n) = \Theta(n^2)$이라고 할 수 있다.

d. $A = \langle 13, 16, 12, 21, 7, 8, 25, 32 \rangle$.

a).

13	16	12	21	7	8	25	32

b).

7	16	12	21	13	8	25	32

c)

7	8	12	21	13	16	25	32

d)

7	8	12	13	21	16	25	32

e

7	8	12	13	16	21	25	32

3. a). $2n^2 + 2\lg n$.

Claim : For large n, $\exists c_0$ s.t $\lg n \le c_0 n$.

proof : Since, $\lim_{n\to\infty} \frac{\lg n}{n} = \lim_{n\to\infty} \frac{(\lg n)'}{(n)'}$ by L'Hospital's Rule

and then $\lim_{n\to\infty} \frac{(\lg n)'}{(n)'} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$.

proof of $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$. By Archimedean principle,

$\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t $\frac{1}{n} < \varepsilon$, therefore $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ $\lim_{n\to\infty} \frac{\lg n}{n} = 0$ means that

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \ge N$. $\left|\frac{\lg n}{n}\right| < \varepsilon$. $\Rightarrow \lg n < \varepsilon \cdot n$.

and if we set ε as small as $\lg n = \varepsilon n$,

we prove $\lg n \le \varepsilon n$ for such ε.

($\because$ $f(n) = \varepsilon n$ is one to one and onto). ∎

$\therefore$ $2n^2 + 2\lg n \le 2n^2 + 2\varepsilon n \le (2+2\varepsilon) n^2$ for large $n \in \mathbb{N}$.

and $2n^2 + 2\lg n \ge 2n^2$ for $\forall n \in \mathbb{N}$.

$\therefore$ $2n^2 + 2\lg n = \Theta(n^2)$.

3-b). $3n^3+5n+5$.

$$3n^3+5n+5 \geq 3n^3 \geq 0 \quad \text{for } \forall n \in \mathbb{N}.$$

$$\begin{aligned} 3n^3+5n+5 &\leq 3n^3+5n+5n \\ &= 3n^3+10n \\ &\leq 3n^3+10n^3 \\ &= 13n^3 \quad \text{for } \forall n \in \mathbb{N}. \end{aligned}$$

$$\therefore \; 0 \leq 3n^3 \leq 3n^3+5n+5 \leq 13n^3 \quad \text{for } \forall n \in \mathbb{N}.$$

$$\therefore \; 3n^3+5n+5 = \Theta(n^3).$$

4. Show that the function $3n^5-n^3+2n^2-2n+2 = \Theta(n^5)$

$$\begin{aligned} 3n^5-n^3+2n^2-2n+2 &\geq (3n^5-n^5)+2n(n-1)+2 \\ &= 2n^5+2n(n-1)+2 \\ &\geq 2n^5 \quad \text{for } \forall n \in \mathbb{N}. \end{aligned}$$

$$\begin{aligned} 3n^5-n^3+2n^2-2n+2 &\leq 3n^5+n^3+2n^2+2n+2 \\ &\leq 3n^5+n^5+2n^5+2n^5+2n^5 \\ &= 10n^5 \quad \text{for } \forall n \in \mathbb{N}. \end{aligned}$$

$$\therefore \; 2n^5 \leq 3n^5-n^3+2n^2-2n+2 \leq 10n^5 \quad \text{for } \forall n \in \mathbb{N}.$$

$$\therefore \; 3n^5-n^3+2n^2-2n+2 = \Theta(n^5)$$

5. $\sum_{i=0}^{n} ar^i = a(r^{n+1}-1)/_{r-1}$ for $\forall n \geq 0$.

where $a \in \mathbb{R}$ and $r \in \mathbb{R} \setminus \{1\}$.

Using induction,

$n=0$, (left side) $= \sum_{i=0}^{0} ar^i = ar^0 = a \cdot 1 = a$.

(Right side) $= a(r^{0+1}-1)/_{r-1} = a(r-1)/_{(r-1)} = a$.

$\therefore$ (left side) $= a = a =$ (Right side).

Let $\sum_{i=0}^{n} ar^i = a(r^{n+1}-1)/_{r-1}$ is true,

then $\sum_{i=0}^{n+1} ar^i = \left(\sum_{i=0}^{n} ar^i\right) + ar^{n+1}$

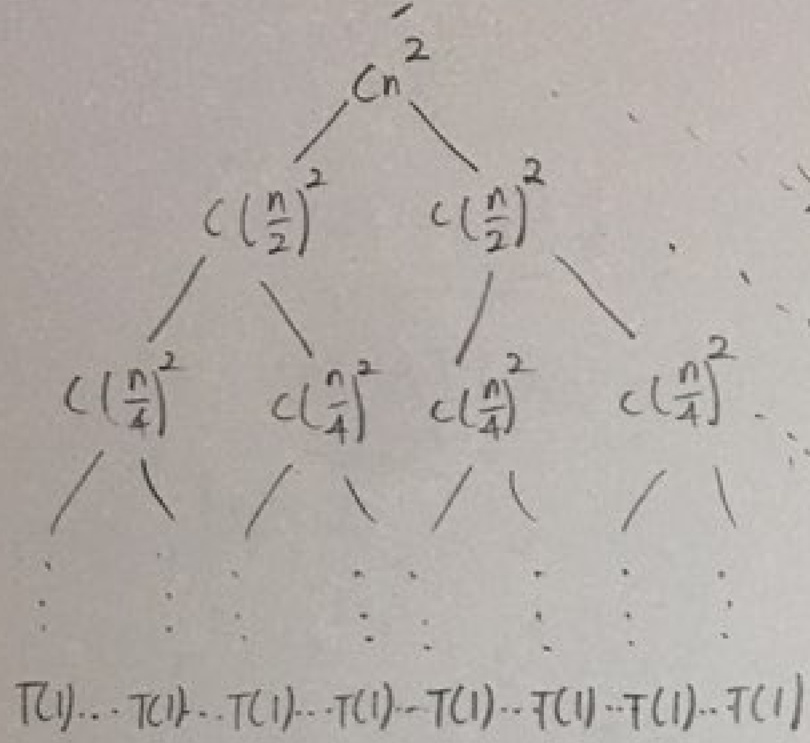
$$= a(r^{n+1}-1)/_{r-1} + ar^{n+1}$$

$$= \frac{a}{r-1}\{r^{n+1}-1+(r-1)\cdot r^{n+1}\}$$

$$= \frac{a}{r-1}(r^{n+2}-1)$$

Therefore, $\sum_{i=0}^{n} ar^i = a(r^{n+1}-1)/_{(r-1)}$ for $\forall n \geq 0$, $a \in \mathbb{R}$, $r \in \mathbb{R} \setminus \{1\}$.

6. $T(n) = 2T(\frac{n}{2}) + cn^2$, where c is constant.



$T(n)$	깊이	개수	cost
$T(\frac{n}{2^0})$	0	2^0	cn^2
$T(\frac{n}{2^1})$	1	2^1	$\frac{1}{2}cn^2$
$T(\frac{n}{2^2})$	2	2^2	$\frac{1}{4}cn^2$
⋮	⋮	⋮	⋮
$T(\frac{n}{2^{k-1}})$	$k-1$	2^{k-1}	$\frac{1}{2^{k-1}}cn^2$

$T(1) = \frac{n}{2^k}, k, 2^k$

$\therefore$ Total $= cn^2(1 + \frac{1}{2} + \frac{1}{4} + \cdots)$

$$= cn^2 \cdot \frac{1}{1-\frac{1}{2}}$$

$$= 2cn^2$$

$$= \Theta(n^2).$$

$$= O(n^2).$$

proof using substitution method.

Let $T(k) \le ak^2 - bk$ for $k < n$, $a, b \in \mathbb{R}^+$.

Then $T(n) = 2T(\frac{n}{2}) + cn^2$

$$\le 2(\frac{a}{4}n^2 - \frac{b}{2}n) + cn^2 \quad (\because \frac{n}{2} < n)$$

$$= (\frac{a}{2} + c)n^2 - bn$$

$$\le an^2 - bn.$$

$\frac{a}{2} + c \le a$

$\Rightarrow \boxed{a \ge 2c}$

$\therefore \underline{T(k) \le ak^2 - bk}$

7.

a). $T(n) = 9T(n/3) + n$.

$\log_3 9 = 2$.

$n = n^{2-1}$, $\varepsilon = 1$.

$\therefore$ $\underline{\underline{T(n) = \Theta(n^2)}}$.

b) $T(n) = 9T(n/3) + n^2$.

$\log_3 9 = 2$

$n^2 = n^{\log_3 9} = n^2$.

$\therefore T(n) = \Theta(n^{\log_3 9} \times \lg n)$

$\underline{\underline{= \Theta(n^2 \lg n)}}$.

c). $T(n) = 9T(n/3) + n^3$.

$\log_3 9 = 3$.

$n^3 = n^{2+1}$, $\varepsilon = 1 > 0$.

$9\left(\frac{n}{3}\right)^3 = \frac{1}{3}n^3 \leq \frac{1}{3}n^3$

$\therefore$ $\underline{\underline{T(n) = \Theta(n^3)}}$.